Mean value formulas for Euler-Zagier multiple zeta functions

Kaneaki Matsuoka
Graduate School of Mathematics, Nagoya University

Advisor : Prof. Kohji Matsumoto

Contents

Chapter 1

Introduction

This thesis is concerned with analytic properties of multiple zeta functions. The aim of this thesis is to study mean value formulas for the multiple zeta functions of Euler-Zagier type.

In Section 1 we introduce some basic properties of the Riemann zeta function and show the mean square value formula for the Riemann zeta function on the critical line in two different ways.

In Section 2 we collect some analytic properties of the multiple zeta function.

1.1 Analytic properties of the Riemann zeta function

Let $s = \sigma + it$ be a complex variable with $\sigma,\ t \in \mathbb{R}$. The Riemann zeta function is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Let $B_n(x)$ be the Bernoulli polynomial defined by

$$\frac{ze^{xz}}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} z^n.$$

Let $B_n(0) = B_n$. By using the Euler-Maclaurin summation formula we get the following lemma (see p. 114 in Edwards [6]).

Lemma 1.1.1. *Let $s = \sigma + it \in \mathbb{C}$, $m, N \in \mathbb{N}$ and $M = 2m + 1$. We have*

$$\zeta(s) = \sum_{n \le N} \frac{1}{n^s} + \frac{N^{1-s}}{s-1} - \frac{N^{-s}}{2} + \sum_{k=1}^{2m} \frac{B_{k+1}}{(k+1)!}(s)_k N^{-(s+k)} + \\ + R_{M,N}(s),$$

where

$$R_{M,N}(s) = -\frac{(s)_M}{M!} \int_N^\infty B_M(x - [x]) x^{-s-M} dx.$$

If $\sigma > -M - 1$ then the integral is absolutely convergent. Hence we get an analytic continuation of the Riemann zeta function to the region $\{s = \sigma + it \in \mathbb{C} | \sigma > -M - 1\}$. Riemann [26] first showed that $\zeta(s)$ is continued meromorphically to the whole complex plane and discovered the functional equation for the Riemann zeta function

$$\zeta(s) = \chi(s)\zeta(1 - s)$$

where

$$\chi(s) = 2^s \pi^{s-1} \sin(s\pi/2)\Gamma(1 - s).$$

It is well-known that the distribution of zeros of the Riemann zeta function is closely connected with that of prime numbers. In fact Koch [16] showed that the Riemann Hypothesis is equivalent to

$$\pi(x) = li(x) + O(x^{1/2} \log x)$$

where

$$li(x) = \int_2^x \frac{dt}{\log t}$$

and $\pi(x)$ is the number of primes less than x. The Riemann Hypothesis is not proved or disproved, but there are some results supporting the Riemann Hypothesis. Let $N(T)$ be the number of complex zeros of the Riemann zeta function with multiplicity having imaginary part between 0 and T, and let $N_0(T)$ be the number of those zeros having real part equal to 1/2. The Riemann Hypothesis states that all complex zeros of the Riemann zeta function are on the critical line $\sigma = 1/2$ i.e. $N(T) = N_0(T)$. Selberg [27] proved that there exists a constant A such that $N_0(T) > AN(T)$. Recently Bui, Conrey and Young [5] proved

$$\liminf_{T\to\infty} \frac{N_0(T)}{N(T)} \ge 0.4105.$$

An important problem in the theory of the Riemann zeta function is to determine the rate of growth of the Riemann zeta function. Let $\mu(\sigma)$ denote the infimum of a number c such that

$$\zeta(\sigma+it) \ll t^c.$$

The notation $f \ll g$ means that there exists some constant $c > 0$ such that $|f| \leq cg$. The Lindelöf Hypothesis states that $\mu(1/2) = 0$. The Lindelöf Hypothesis is not proved or disproved, but it is well known that the Riemann Hypothesis implies the Lindelöf Hypothesis. If the Lindelöf Hypothesis is true, then by the Phragmén-Lindelöf theorem we have

$$\mu(\sigma) = \begin{cases} 0 & (\sigma \geq 1/2) \\ 1/2-\sigma & (\sigma < 1/2). \end{cases}$$

Hence, if the Riemann Hypothesis is true then we get the exact rate of growth of the Riemann zeta function. Take $M = [t^2]$ in Lemma 1.1.1, then we have

$$\zeta(s) = \sum_{n \leq t^2} n^{-s} + O(t^{1-2\sigma}).$$

Divide the sum over the range $t < n \leq t^2$ into $\ll \log t$ subsums of the form

$$\sum_{N<n\leq N_1} \frac{1}{n^s},$$

where $N_1 = \min(2N, t^2)$ and use partial summation, we have

$$\zeta(1/2+it) \ll \left| \sum_{n \leq t} n^{-1/2-it} \right| + \log t.$$

Hence, if we get the growth rate of $\sum_{n\leq t} n^{-1/2-it}$, we know that of the Riemann zeta function on the critical line. But it is very difficult to evaluate the function $\sum_{n\leq t} n^{-1/2-it}$. The rate $\mu(1/2)$ has been studied by many mathematicians. Hardy and Littlewood showed $\mu(1/2) \leq 1/6$, Kolesnik [17] showed $\mu(1/2) \leq 139/858$, and recently Bourgain [3] showed $\mu(1/2) \leq 53/342$.

Consider mean values of the Riemann zeta function. Define

$$I_{\sigma,2k}(T) = \int_2^T |\zeta(\sigma+it)|^{2k} dt.$$

It is well known that if $I_{1/2,2k}(T) \ll T^{1+\epsilon}$ for any $k \in \mathbb{N}$ then the Lindelöf Hypothesis holds. This is one of the reasons why the study of mean values is important. In the case $k = 1, 2$, it is classically well known that

$$I_{\sigma,2}(T) \sim \begin{cases} \zeta(2\sigma)T & (\sigma > 1/2) \\ T \log T & (\sigma = 1/2) \end{cases}$$

and

$$I_{\sigma,4}(T) \sim \begin{cases} \frac{\zeta^4(2\sigma)}{\zeta(4\sigma)}T & (\sigma > 1/2) \\ \frac{T \log^4 T}{2\pi^2} & (\sigma = 1/2). \end{cases}$$

It is believed that $I_{1/2,2k}(T) \sim C_k T \log^{k^2} T$ where C_k is a positive constant. Ramachandra [24] [25] proved $I_{1/2,2k}(T) \gg T(\log T)^{k^2}$ and recently Harper [7] proved $I_{1/2,2k}(T) \ll T(\log T)^{k^2}$, assuming the truth of the Riemann Hypothesis. Littlewood [18] showed

$$I_{1/2,2}(T) = T \log T + (2\gamma - 1)T + E(T),$$

where γ is the Euler constant defined by

$$\gamma = \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \log n \right)$$

and $E(T) \ll T^{3/4+\epsilon}$. The function $E(T)$ was further studied by Atkinson [2] who proved that

$$\begin{aligned} E(T) = {} & \left(\frac{2T}{\pi}\right)^{1/4} \sum_{n\le X} (-1)^n \frac{d(n)}{n^{3/4}} e(T,n) \cos(f(T,n)) \\ & - 2 \sum_{n \le l(T,X)} \frac{d(n)}{n^{1/2}} \left(\log \frac{T}{2\pi n}\right)^{-1} \cos(g(T,n)) + O(\log^2 T) \end{aligned}$$

under the condition $T \ll X \ll T$, where $d(n)$ is the number of divisors of n,

$$e(T,u) = \left(1 + \frac{\pi u}{2T}\right)^{-1/4} \left(\sqrt{\frac{2T}{\pi u}} \mathrm{arsinh} \sqrt{\frac{\pi u}{2T}} \right)^{-1},$$

$$f(T,u) = 2T \mathrm{arsinh} \sqrt{\frac{\pi u}{2T}} + \sqrt{2\pi u T + \pi^2 u^2} - \frac{\pi}{4},$$

$$g(T,u) = T\log\frac{T}{2\pi u} - T + 2\pi u + \frac{\pi}{4},$$

and

$$l(T,u) = \frac{T}{2\pi} + \frac{u}{2} - \sqrt{\frac{u^2}{4} + \frac{uT}{2\pi}}.$$

In order to get this result, Atkinson used the double zeta function of Euler-Zagier type (the k-ple zeta function of Euler-Zagier type is defined in the next section). Atkinson also derived the formula

$$\zeta(s_1)\zeta(s_2) = \zeta_2(s_1,s_2) + \zeta_2(s_2,s_1) + \zeta(s_1+s_2). \tag{1.1.1}$$

This formula is sometimes called the harmonic product formula for the double zeta function. By using Atkinson's formula Heath-Brown [8] showed

$$\int_2^T E(t)^2 dt = \frac{2\zeta^4(3/2)}{3\zeta(3)\sqrt{2\pi}} T^{3/2} + F(T)$$

with $F(T) = O(T^{5/4}\log^2 T)$.

Here we sketch two different proofs of $I_{1/2,2}(T) \sim T\log T$.

The first method :
This method is based on the following equation (see pp. 81-84 in [12]).

Theorem 1.1.1. *Let $0 < \sigma < 1$ and $2\pi xy = t$ with $x, y > h > 0$. Then we have*

$$\zeta(s) = \sum_{n<x}\frac{1}{n^s} + \chi(s)\sum_{n<y}\frac{1}{n^{1-s}} + O(x^{-\sigma}) + O(t^{\frac{1}{2}-\sigma}y^{\sigma-1}), \tag{1.1.2}$$

where the constants in the O-terms depend on h and σ.

Take $\sigma = 1/2$, $t > 2$, $x = t/(2\pi\sqrt{\log t})$ and $y = \sqrt{\log t}$ in (1.1.2). Since $\chi(1/2+it) = O(1)$ we have

$$\zeta(1/2+it) = \sum_{n<x} n^{\frac{1}{2}-it} + O(\log^{\frac{1}{4}} t).$$

Define T_1 such that

$$\max(m,n) = \frac{T_1}{2\pi\sqrt{\log T_1}}$$

and $X = T/(2\pi\sqrt{\log T})$. We have

$$\begin{aligned}\int_0^T |\sum_{n<x} n^{\frac{1}{2}-it}|^2 dt &= \sum_{m,n<X} \int_{T_1}^T m^{-\frac{1}{2}-it} n^{-\frac{1}{2}+it} dt \\ &= \sum_{n<X} \frac{T - T_1(n,n)}{n} + \sum_{m\neq n} \frac{1}{\sqrt{mn}} \int_{T_1}^T (\frac{n}{m})^{it} \\ &= T\log T + O(T\log\log T) + O(\sum_{n<X} \frac{T_1(n,n)}{n}) + \\ &+ O(\sum_{m<n<X} \frac{1}{\sqrt{mn\log(n/m)}}).\end{aligned}$$

Since $T_1(n,n) = 2\pi n\sqrt{\log n}$, third term on the right side is $O(X\sqrt{\log X})$. By using the following lemma we get $I_{1/2,2}(T) \sim T\log T$.

Lemma 1.1.2.

$$\sum_{m<n<X} \frac{1}{\sqrt{mn\log(n/m)}} = O(X\log X).$$

This lemma is proved by dividing the sum into two parts $m \leq \frac{n}{2}$ and $\frac{n}{2} < m < n$, and evaluate each part carefully.

The second method :
This method is based on the following equation.

Lemma 1.1.3 (Theorem 5.2 in [12])**.** *Let $a_1, \cdots, a_N$ be arbitrary complex numbers. Then*

$$\int_0^T \left|\sum_{n\leq N} a_n n^{it}\right|^2 dt = T\sum_{n\leq N} |a_n|^2 + O\left(\sum_{n\leq N} n|a_n|^2\right), \tag{1.1.3}$$

and the above formula remains also valid if $N = \infty$, provided that the series on the right-hand side of (1.1.3) converges.

Take $m = 2$ in (1.1.1) we have

$$\zeta(1/2 + it) = \sum_{n\leq T} \frac{1}{n^{\frac{1}{2}+it}} + O(T^{-\frac{1}{2}}). \tag{1.1.4}$$

Take $a_n = n^{-1/2}$ and $N = [T]$ in (1.1.3) we have

$$\int_2^T |\sum_{n\leq T} \frac{1}{n^s}|^2 dt = T \sum_{n\leq T} n^{-1} + O(T)$$
$$= T \log T + O(T). \qquad (1.1.5)$$

By (1.1.4), (1.1.5) and Cauchy's inequality we get $I_{1/2,2}(T) \sim T \log T$.

We can observe that the second proof is much simpler. This clearly shows the importance of Lemma 1.1.3 in the mean value theorem.

1.2 Analytic properties of the multiple zeta function

Let $s_j = \sigma_j + it_j (j = 1, 2, ..., k)$ be complex variables with $\sigma_j,\ t_j \in \mathbb{R}$. The k-ple zeta function of Euler-Zagier type is defined by

$$\zeta_k(s_1, s_2, ..., s_k) = \sum_{1\leq n_1 < \cdots < n_k} \frac{1}{n_1^{s_1} n_2^{s_2} \cdots n_k^{s_k}},$$

which is absolutely convergent for $\sigma_k > 1$, $\sigma_k + \sigma_{k-1} > 2$, ..., $\sigma_k + \sigma_{k-1} + \cdots \sigma_1 > k$. The multiple zeta functions of Euler-Zagier type at positive integers are closely connected with various fields of mathematics and physics (for example, knot theory [29] [19] and perturbative quantum field theory [4]). There are many relations among these values (see [23] [32], for example). Recently analytic properties of multiple zeta functions have been studied extensively. Akiyama, Egami and Tanigawa [1] studied the analytic continuation of multiple zeta functions. Zhao [33] also obtained the continuation independently.

Theorem 1.2.1 (Theorem 1 in [1])**.** *The multiple zeta function $\zeta_k(s_1, .., s_k)$ continues meromorphically to $\mathbb{C}^k$ and has singularities on*

$$s_k = 1, \qquad s_{k-1} + s_k = 2, 1, 0, -2, -4, ...,$$

and

$$\sum_{i=1}^{j} s_{k-i+1} \in \mathbb{Z}_{\leq j}$$

where $\mathbb{Z}_{\leq j}$ is the set of integers less than or equal to j.

Matsumoto [21] discovered the functional equation for the double zeta function. Let

$$g(s_1, s_2) = \zeta_2(s_1, s_2) - \frac{\Gamma(1-s_1)}{\Gamma(s_2)}\Gamma(s_1+s_2-1)\zeta(s_1+s_2-1).$$

Let

$$\Psi(a, c; x) = \frac{1}{\Gamma(a)}\int_0^{\infty e^{i\phi}} e^{-xy}y^{a-1}(1+y)^{c-a-1}dy$$

be the confluent hypergeometric function, where $\Re a > 0$, $-\pi < \phi < \pi$, $|\phi + \arg x| < \pi/2$. We use the notation $\sigma_l(k) = \sum_{d|k} d^l$.

Theorem 1.2.2 (Matsumoto [21])**.** *We have*

$$\frac{g(s_1, s_2)}{(2\pi)^{s_1+s_2-1}\Gamma(1-s_1)} = \frac{g(1-s_2, 1-s_1)}{i^{s_1+s_2-1}\Gamma(s_2)} + 2i\sin\left(\frac{\pi}{2}(s_1+s_2-1)\right)F_+(s_1, s_2), \tag{1.2.1}$$

where $i = \sqrt{-1} = \exp(\pi i/2)$ *and* $F_+(u, v)$ *is the series defined by*

$$F_+(u, v) = \sum_{k=1}^{\infty}\sigma_{u+v-1}(k)\Psi(v, u+v; 2\pi ik). \tag{1.2.2}$$

The series (1.2.2) is convergent only in the region $\Re u < 0$, $\Re v > 1$, *but it can be continued meromorphically to the whole* $\mathbb{C}^2$ *space.*

We do not know the equation (1.2.1) has significant meaning as the functional equation for the Riemann zeta function, but this equation is an analogue of the functional equation for the Riemann zeta function.

Ishikawa and Matsumoto [11] showed that for a fixed α and any $\epsilon > 0$,

$$\zeta_2(it, i\alpha t) \ll (1+|t|)^{3/2+\epsilon}.$$

This result was improved by Kiuchi and Tanigawa [13].

Theorem 1.2.3 (Theorem 1.1 in [13])**.** *Let* $|t_1|$ *and* $|t_2| \geq 2$ *be real numbers such that*

$$|t_1| \asymp |t_2| \qquad \textit{and} \qquad |t_1+t_2| \gg 1.$$

In the case $\sigma_1 = \sigma_2 = 0$, *we have*

$$\zeta_2(it_1, it_2) \ll |t_1| \log^2 |t_1|.$$

Suppose that $0 \leq \sigma_j < 1 (j = 1, 2)$ *and* $\sigma_1 + \sigma_2 > 0$. *Then we have*

$$\zeta_2(\sigma_1 + it_1, \sigma_2 + it_2) \ll \begin{cases} |t_1|^{1-\frac{2}{3}(\sigma_1+\sigma_2)} \log^2 |t_1| & (0 \leq \sigma_1 \leq \frac{1}{2}, 0 \leq \sigma_2 \leq \frac{1}{2}) \\ |t_1|^{\frac{5}{6}-\frac{1}{3}(\sigma_1+2\sigma_2)} \log^3 |t_1| & (\frac{1}{2} < \sigma_1 < 1, 0 \leq \sigma_2 \leq \frac{1}{2}) \\ |t_1|^{\frac{5}{6}-\frac{1}{3}(2\sigma_1+\sigma_2)} \log^3 |t_1| & (0 \leq \sigma_1 \leq \frac{1}{2}, \frac{1}{2} < \sigma_2 < 1) \\ |t_1|^{\frac{2}{3}-\frac{1}{3}(\sigma_1+\sigma_2)} \log^4 |t_1| & (\frac{1}{2} < \sigma_1 < 1, \frac{1}{2} < \sigma_2 < 1). \end{cases}$$

Note that Kiuchi and Tanigawa [14] studied the triple zeta function and obtained the upper bound of this function. In the case $s_1 = s_2 = 1/2 + it$, by (1.1.1) we have

$$2\zeta_2(1/2 + it, 1/2 + it) = \zeta^2(1/2 + it) - \zeta(1 + 2it).$$

Hence we obtain

$$\zeta_2(1/2 + it, 1/2 + it) \ll |t|^{2\mu(1/2)+\epsilon}.$$

But from Theorem 1.2.3 we only get

$$\zeta_2(1/2 + it, 1/2 + it) \ll t^{1/3+\epsilon}.$$

From the above equations, Theorem 1.2.3 is far from the correct order of the double zeta function. Kiuchi and Tanigawa considered that under the condition $|t_1| \asymp |t_2|$ and $|t_1 + t_2| \gg 1$,

$$\zeta_2(s_1, s_2) \ll |t_1|^{\mu(\sigma_1)+\mu(\sigma_2)} \log^A |t_1|$$

holds, where A is some constant (see Remark 1.3 in [13]).

Recently Matsumoto and Tsumura [22] first studied a new type of some mean value formulas for $\int_2^T |\zeta_2(s_1, s_2)|^2 dt_2$ for a fixed complex number s_1 and any large positive number T. They derived two approximate formulas for $\zeta_2(s_1, s_2)$ and three mean value formulas for $\zeta_2(s_1, s_2)$. Let

$$\zeta_2^{[2]}(s_1, s_2) = \sum_{k=2}^{\infty} \left| \sum_{m=1}^{k-1} \frac{1}{m^{s_1}} \right|^2 \frac{1}{k^{s_2}}. \tag{1.2.3}$$

Since

$$\sum_{m=1}^{k-1} \frac{1}{m^{s_1}} \ll \begin{cases} 1 & (\sigma_1 > 1) \\ \log k & (\sigma_1 = 1) \\ k^{1-\sigma_1} & (\sigma_1 < 1) \end{cases}$$

we have

$$\left|\sum_{m=1}^{k-1}\frac{1}{m^{s_1}}\right|^2\frac{1}{k^{2\sigma_2}} \ll \begin{cases} k^{-2\sigma_2} & (\sigma_1>1) \\ k^{-2\sigma_2}\log^2 k & (\sigma_1=1) \\ k^{2-2\sigma_1-2\sigma_2} & (\sigma_1<1). \end{cases}$$

Hence the right hand side of (1.2.3) is convergent when $\sigma_1+\sigma_2>3/2$ and $\sigma_2>1/2$.

Theorem 1.2.4 (Theorem 1.1 in [22]). *For $s_0=\sigma_0+it_0\in\mathbb{C}$ with $\sigma_0>1$ and $s=\sigma+it\in\mathbb{C}$ with $\sigma>1$, $t\geq 2$, we have*

$$\int_2^T|\zeta_2(s_0,s)|^2dt=\zeta_2^{[2]}(s_0,2\sigma)T+O(1).$$

Theorem 1.2.5 (Theorem 1.2 in [22]). *For $s_0=\sigma_0+it_0\in\mathbb{C}$ with $\sigma_0>1$ and $s=\sigma+it\in\mathbb{C}$ with $\frac{1}{2}<\sigma\leq 1$, $t\geq 2$ and $\sigma_0+\sigma>2$, we have*

$$\int_2^T|\zeta_2(s_0,s)|^2dt=\zeta_2^{[2]}(s_0,2\sigma)T+O(T^{2-2\sigma}\log T)+O(T^{1/2}).$$

Theorem 1.2.6 (Theorem 1.3 in [22]). *Let $s_0=\sigma_0+it_0\in\mathbb{C}$ with $\sigma_0>1$ and $s=\sigma+it\in\mathbb{C}$ with $\frac{1}{2}<\sigma\leq 1$, $t\geq 2$ and $\frac{3}{2}<\sigma_0+\sigma\leq 2$, Assume that when t moves from 2 to T, the point (s_0,s) does not encounter the hyperplane $s_0+s=2$ (which is a singular locus of ζ_2). Then*

$$\int_2^T|\zeta_2(s_0,s)|^2dt=\zeta_2^{[2]}(s_0,2\sigma)T$$

$$+\begin{cases} O(T^{4-2\sigma_0-2\sigma})\log T+O(T^{1/2}) & (\frac{1}{2}<\sigma_0\leq 1,\frac{1}{2}<\sigma<1) \\ O(T^{2-2\sigma_0})(\log T)^2+O(T^{1/2}) & (\frac{1}{2}<\sigma_0\leq 1,\sigma=1) \\ O(T^{2-2\sigma_0})(\log T)^3+O(T^{1/2}) & (\sigma_0=1,\frac{1}{2}<\sigma<1) \\ O(T^{1/2}) & (\sigma_0=1,\sigma=1) \\ O(T^{2-2\sigma_0})\log T+O(T^{1/2}) & (1<\sigma_0<\frac{3}{2},\frac{1}{2}<\sigma<1). \end{cases}$$

They conjectured that when $\sigma_1+\sigma_2=3/2$, the form of the main term of the mean square formula would not be CT (with a constant C; most probably, some log-factor would appear) (see their conjecture (ii) in [22]). They also conjectured that we could reduce the error terms $O(T^{1/2})$ by more elaborate analysis (see Remark 1.4 in [22]).

In Chapter 2 we improve upon Theorem 1.2.4, Theorem 1.2.5 and Theorem 1.2.6, and prove that their conjectures are true. Furthermore we study two mean value formulas for $\int_2^T |\zeta_2(\sigma_1+it_1, \sigma_2+it_2)|^2 dt_1$ and $\int_2^T |\zeta(\sigma+it, \sigma+it)|^2 dt$. This is a joint work with Soichi Ikeda and Yoshikazu Nagata.

In Chapter 3 we study $\int_2^T |\zeta_k(s_1, s_2, .., s_k)|^2 dt_1$. This is a joint work with Soichi Ikeda.

Chapter 2

Certain mean values of the double zeta function

In this chapter we discuss three types of mean values of the Euler double zeta function. In order to get results we introduce three approximate formulas for this function. This is a joint work with Soichi Ikeda and Yoshikazu Nagata and the contents of this chapter are based on the paper [10].

2.1 Introduction

Matsumoto and Tsumura studied the mean values

$$\int_2^T |\zeta_2(s_1, s_2)|^2 dt_2, \tag{2.1.1}$$

where s_1 is a fixed complex number. This is the first study of the mean values of $\zeta_2(s_1, s_2)$. In this chapter we study (2.1.1) in the regions which are not covered in the work of Matsumoto and Tsumura and introduce new types of mean values of $\zeta_2(s_1, s_2)$.

In this chapter we prove the following theorems.

Theorem 2.1.1. *Let* $s_1 = \sigma_1 + it_1, s_2 = \sigma_2 + it_2 \in \mathbb{C}$, $T \geq 2$ *and*

$$I^{[1]}(T) = \int_2^T |\zeta_2(s_1, s_2)|^2 dt_1.$$

Assume that when t_1 moves from 2 to T, the point $(s_1, s_2) \in \mathbb{C}^2$ does not encounter the singularities of $\zeta_2(s_1, s_2)$. In the case $\sigma_1 + \sigma_2 > 2$, we have

$$I^{[1]}(T) = \zeta_2^{[1]}(2\sigma_1, s_2)T + O(1),$$

where, here and below, the implied constants depend on σ_1, σ_2, t_2 and $\zeta_2^{[1]}(2\sigma_1, s_2)$ is a series which converges $\sigma_1 + \sigma_2 > 3/2$ (we define $\zeta_2^{[1]}(\sigma_1, s_2)$ in the next section). In the case $3/2 < \sigma_1 + \sigma_2 \leq 2$, we have

$$I^{[1]}(T) = \zeta_2^{[1]}(2\sigma_1, s_2)T + \begin{cases} O(T^{4-2\sigma_1-2\sigma_2}) & (3/2 < \sigma_1 + \sigma_2 < 2), \\ O((\log T)^2) & (\sigma_1 + \sigma_2 = 2). \end{cases}$$

In the case $\sigma_1 + \sigma_2 = 3/2$, we have

$$I^{[1]}(T) = |s_2 - 1|^{-2} T \log T + O(T).$$

Theorem 2.1.2. *Let $s_1 = \sigma_1 + it_1, s_2 = \sigma_2 + it_2 \in \mathbb{C}$, $T \geq 2$ and*

$$I^{[2]}(T) = \int_2^T |\zeta_2(s_1, s_2)|^2 dt_2.$$

Assume that when t_2 moves from 2 to T, the point $(s_1, s_2) \in \mathbb{C}^2$ does not encounter the singularities of $\zeta_2(s_1, s_2)$. In the case $\sigma_2 > 1$ and $\sigma_1 + \sigma_2 > 2$, we have

$$I^{[2]}(T) = \zeta_2^{[2]}(s_1, 2\sigma_2)T + O(1),$$

where, here and below, the implied constants depend on σ_1, σ_2, t_1 and $\zeta_2^{[2]}(s_1, 2\sigma_2)$ is a series which converges $\sigma_1 + \sigma_2 > 3/2$ and $\sigma_2 > 1/2$ ($\zeta_2^{[2]}(s_1, \sigma_2)$ is used in [22] and we show the definition of $\zeta_2^{[2]}(s_1, \sigma_2)$ in the next section). In the case $\sigma_1 > 1$ and $1/2 < \sigma_2 \leq 1$, we have

$$I^{[2]}(T) = \zeta_2^{[2]}(s_1, 2\sigma_2)T + \begin{cases} O(T^{2-2\sigma_2}) & (\sigma_2 \neq 1), \\ O((\log T)^2) & (\sigma_2 = 1). \end{cases}$$

In the case $\sigma_1 \leq 1$, $3/2 < \sigma_1 + \sigma_2 \leq 2$ and $s_1 \neq 1$, we have

$$I^{[2]}(T) = \zeta_2^{[2]}(s_1, 2\sigma_2)T + \begin{cases} O(T^{4-2\sigma_1-2\sigma_2}) & (\sigma_1 + \sigma_2 \neq 2), \\ O((\log T)^2) & (\sigma_1 + \sigma_2 = 2). \end{cases}$$

In the case $s_1 = 1$ and $1/2 < \sigma_2 \le 1$, we have

$$I^{[2]}(T) = \zeta_2^{[2]}(s_1, 2\sigma_2)T + \begin{cases} O(T^{2-2\sigma_2}(\log T)^2) & (\sigma_2 \neq 1), \\ O((\log T)^4) & (\sigma_2 = 1). \end{cases}$$

In the case $\sigma_1 > 1$ and $\sigma_2 = 1/2$, we have

$$I^{[2]}(T) = |\zeta(s_1)|^2 T \log T + O(T).$$

In the case $\sigma_1 + \sigma_2 = 3/2$ and $\sigma_2 > 1/2$, we have

$$I^{[2]}(T) = |s_1 - 1|^{-2} T \log T + O(T).$$

In the case $\sigma_2 = 1/2$, $\sigma_1 = 1$ and $s_1 \neq 1$, we have

$$I^{[2]}(T) = (|s_1 - 1|^{-2} + |\zeta(s_1)|^2) T \log T + O(T).$$

In the case $\sigma_2 = 1/2$ and $s_1 = 1$, we have

$$I^{[2]}(T) = \frac{T(\log T)^3}{3} + O(T(\log T)^2).$$

Theorem 2.1.3. *Let $s_1 = \sigma_1 + it, s_2 = \sigma_2 + it \in \mathbb{C}$, $T \ge 2$ and*

$$I^{\square}(T) = \int_2^T |\zeta_2(s_1, s_2)|^2 dt.$$

In the case $\sigma_2 > 1$ and $\sigma_1 + \sigma_2 > 2$, we have

$$I^{\square}(T) = \zeta_2^{\square}(\sigma_1, \sigma_2)T + O(1),$$

where, here and below, the implied constants depend on σ_1, σ_2 and $\zeta_2^{\square}(\sigma_1, \sigma_2)$ is a series which converges if and only if $\sigma_2 > 1/2$ and $\sigma_1 + \sigma_2 > 1$ (we define $\zeta_2^{\square}(\sigma_1, \sigma_2)$ in the next section). In the case $\sigma_1 > 1$ and $1/2 < \sigma_2 \le 1$, we have

$$I^{\square}(T) = \zeta_2^{\square}(\sigma_1, \sigma_2)T + O(T^{2-2\sigma_2+\epsilon}) + O(T^{1/2})$$

for sufficiently small $\epsilon > 0$. In the case $\sigma_1 \le 1$ and $3/2 < \sigma_1 + \sigma_2 \le 2$, we have

$$I^{\square}(T) = \zeta_2^{\square}(\sigma_1, \sigma_2)T + O(T^{4-2\sigma_1-2\sigma_2+\epsilon}) + O(T^{1/2})$$

for sufficiently small $\epsilon > 0$. In the case $\sigma_1 > 1$ and $\sigma_2 = 1/2$, we have

$$I^{\square}(T) \sim \frac{\zeta(2\sigma_1)\zeta(\sigma_1 + 1/2)^2}{\zeta(2\sigma_1 + 1)} T \log T.$$

Note that we can obtain $I^{[2]}(T) \sim |\zeta(s_1)|^2 T \log T$ ($\sigma_1 > 1$, $\sigma_2 = 1/2$) by (1.1.1) and Theorem 2.1.1.

Matsumoto and Tsumura introduced $I^{[2]}(T)$ and studied the cases

1. $\sigma_1 > 1$ and $\sigma_2 > 1$ (Theorem 1.1 of [22]),

2. $\sigma_1 + \sigma_2 > 2$ and $1/2 < \sigma_2 \leq 1$ (Theorem 1.2 of [22]),

3. $1/2 < \sigma_1 < 3/2$, $1/2 < \sigma_2 \leq 1$ and $3/2 < \sigma_1 + \sigma_2 \leq 2$ (Theorem 1.3 of [22]).

They conjectured that when $\sigma_1 + \sigma_2 = 3/2$, the form of the main term of the mean square formula would not be CT (with a constant C; most probably, some log-factor would appear)(see their conjecture (ii) in [22]). Our results include the regions which Matsumoto and Tsumura did not study and give an improvement on the error estimate. Moreover by Theorem 1.2 we see that their conjecture (ii) is true.

Outlines of the proof of our theorems are as follows. We can obtain Theorem 2.1.1 and Theorem 2.1.2 by using the mean value theorems for Dirichlet polynomials and suitable approximate formulas in each theorem (cf. Theorem 3.1 and Theorem 6.3 in Matsumoto and Tsumura [22]). The approximate formulas used in the proof of Theorem 2.1.1 and Theorem 2.1.2 are derived from the Euler-Maclaurin formula and the simplest approximate formula to $\zeta(s)$ due to Hardy and Littlewood. On the other hand we need a more elaborate method to get the proof of Theorem 2.1.3. In order to obtain the suitable approximate formula for $\zeta_2(\sigma_1+it, \sigma_2+it)$ we need the technique of Kiuchi and Tanigawa [13], which enables us to get good estimates of the error terms in the Euler-Maclaurin formula.

In Theorem 2.1.1 (resp. Theorem 2.1.2) we regard s_2 (resp. s_1) as a constant term. On the other hand, from the study of Kiuchi, Tanigawa and Zhai [15], we know that the behavior of $|\zeta_2(s_1, s_2)|$ depends on both s_1 and s_2 strongly. Therefore it is also important to consider a mean value which depends on both s_1 and s_2.

From Theorem 2.1.1 and Theorem 2.1.2 we may expect that the behavior of $\zeta_2(s_1, s_2)$ in the region $\sigma_1 + \sigma_2 = 3/2$ is special (Matsumoto and Tsumura conjectured that $\sigma_1 + \sigma_2 = 3/2$ might be the double analogue of the critical line of the Riemann zeta-function (see Remark 1.6 in [22])). The error terms in Theorem 2.1.3 support their conjecture. However, we can take a different point of view. For the Riemann zeta function $\zeta(\sigma+it)$, we know that

$$\int_2^T |\zeta(\sigma+it)|^2 dt \sim \zeta(2\sigma)T$$

for $\sigma > 1/2$ and

$$\int_2^T |\zeta(1/2+it)|^2 dt \sim T\log T$$

hold (see, for example, Theorem 7.2 and Theorem 7.3 in [31]). The line $\sigma = 1/2$ is the critical line for $\zeta(\sigma+it)$ and the series

$$\zeta(2\sigma) = \sum_{n=1}^{\infty} \frac{1}{n^{2\sigma}}$$

diverges on $\sigma = 1/2$. On the other hand, $\zeta_2^{\Box}(\sigma_1, \sigma_2)$ converges if and only if $\sigma_2 > 1/2$ and $\sigma_1 + \sigma_2 > 1$. Moreover, if $\sigma_1 = \sigma_2 > 1/2$ then $I^{\Box}(T) \sim \zeta_2^{\Box}(\sigma_1, \sigma_2)T$ holds by

$$\int_2^T |\zeta(\sigma+it)|^4 dt = O(T)$$

for $\sigma > 1/2$ (see Theorem 7.5 in [31]) and Carlson's mean value theorem (see p. 304 in [30]). Hence we can expect that $I^{\Box}(T) \sim \zeta_2^{\Box}(\sigma_1, \sigma_2)T$ holds for $\sigma_2 > 1/2$ and $\sigma_1 + \sigma_2 > 1$ and the boundary of the region $\sigma_2 > 1/2$ and $\sigma_1 + \sigma_2 > 1$ is an analogue of the critical line for $\zeta_2(\sigma_1 + it, \sigma_2 + it)$.

2.2 Lemmas for the proof of theorems

In this section, we collect some auxiliary results and definitions.

First, we give the definition of $\zeta_2^{[1]}(\sigma_1, s_2)$, $\zeta_2^{[2]}(s_1, \sigma_2)$ and $\zeta_2^{\Box}(\sigma_1, \sigma_2)$.

We define

$$\zeta_2^{[1]}(\sigma_1, s_2) = \sum_{m=1}^{\infty} \frac{1}{m^{\sigma_1}} \left| \zeta(s_2) - \sum_{n=1}^{m} \frac{1}{n^{s_2}} \right|^2$$

for $s_2 \neq 1$. Since we have

$$\zeta_2^{[1]}(2\sigma_1, s_2) \ll \sum_{m=1}^{\infty} \begin{cases} m^{2-2\sigma_1-2\sigma_2} & (\sigma_2 > 1) \\ m^{-2\sigma_1}(\log m)^2 & (\sigma_2 = 1) \\ m^{2-2\sigma_1-2\sigma_2} & (\sigma_2 < 1), \end{cases} \tag{2.2.1}$$

the series $\zeta_2^{[1]}(2\sigma_1, s_2)$ converges in the region $\sigma_1 + \sigma_2 > 3/2$.

We define

$$\zeta_2^{[2]}(s_1, \sigma_2) = \sum_{n=2}^{\infty} \left| \sum_{m=1}^{n-1} \frac{1}{m^{s_1}} \right|^2 \frac{1}{n^{\sigma_2}}$$

(this definition is the same as [22]). Since we have

$$\zeta_2^{[2]}(s_1, 2\sigma_2) \ll \sum_{n=2}^{\infty} \begin{cases} n^{-2\sigma_2} & (\sigma_1 > 1) \\ n^{-2\sigma_2}(\log n)^2 & (\sigma_1 = 1) \\ n^{2-2\sigma_1-2\sigma_2} & (\sigma_1 < 1), \end{cases} \tag{2.2.2}$$

the series $\zeta_2^{[2]}(s_1, 2\sigma_2)$ converges in the region $\sigma_2 > 1/2$ and $\sigma_1 + \sigma_2 > 3/2$.

We define

$$\zeta_2^{\square}(\sigma_1, \sigma_2) = \sum_{k=2}^{\infty} \left(\sum_{\substack{mn=k \\ m<n}} \frac{1}{m^{\sigma_1} n^{\sigma_2}} \right)^2.$$

We note that $\#\{(m,n) | mn = k, m < n\} \ll k^{\epsilon}$ for any $\epsilon > 0$. Since

$$\begin{aligned} \zeta_2(2\sigma_1, 2\sigma_2) &< \zeta_2^{\square}(\sigma_1, \sigma_2) \\ &= \sum_{k=2}^{\infty} k^{-2\sigma_2} \left(\sum_{\substack{m|k \\ m<\sqrt{k}}} \frac{1}{m^{\sigma_1-\sigma_2}} \right)^2 \\ &\ll \sum_{k=2}^{\infty} \begin{cases} k^{-2\sigma_2+\epsilon} & (\sigma_1 \geq \sigma_2) \\ k^{-\sigma_1-\sigma_2+\epsilon} & (\sigma_1 < \sigma_2) \end{cases} \end{aligned} \tag{2.2.3}$$

for any $\epsilon > 0$, the series $\zeta_2^{\square}(\sigma_1, \sigma_2)$ converges if and only if $\sigma_2 > 1/2$ and $\sigma_1 + \sigma_2 > 1$. From Lemma (1.1.1), we get the following corollary and lemma.

Corollary 2.2.1. *Let $s = 1 + it$. For fixed $t > 0$ we have*

$$\zeta(s) - \sum_{n \leq N} \frac{1}{n^s} = \frac{N^{1-s}}{s-1} + O(N^{-1}) = O(1),$$

where the implied constants do not depend on N.

Lemma 2.2.1. *Let $s = \sigma + it \in \mathbb{C}$. We have*

$$\zeta(s) = \sum_{1 \leq n \leq x} \frac{1}{n^s} - \frac{x^{1-s}}{1-s} + O(x^{-\sigma})$$

uniformly for $\sigma \geq \sigma_0 > 0$, $x \geq 1$, $|t| \leq 2\pi x/C$, *where* C *is a given constant greater than* 1.

We use the following evaluations in this chapter.

Remark 2.2.1. Let $T \geq 1$ and $M \geq 1$ with $M \ll \log T$. For fixed $\alpha, \beta \geq 0$ we have

$$\begin{aligned}\sum_{k \leq M} \left(\frac{T}{2^k}\right)^{\alpha} \left(\log\left(\frac{T}{2^k}\right)\right)^{\beta} &\ll T^{\alpha} \sum_{k \leq M} \left(\frac{1}{2^{\alpha}}\right)^{k} \left((\log T)^{\beta} + k^{\beta}\right) \\ &\ll \begin{cases} T^{\alpha}(\log T)^{\beta} & (\alpha \neq 0) \\ (\log T)^{\beta+1} & (\alpha = 0). \end{cases}\end{aligned}$$

2.3 Proof of Theorem 2.1.1

In this section, we regard σ_1, s_2 as constants. We divide the proof into two cases.

Proof of Theorem 2.1.1 for $\sigma_1 + \sigma_2 > 2$. We set

$$a_m = \frac{1}{m^{\sigma_1}} \left(\zeta(s_2) - \sum_{n=1}^{m} \frac{1}{n^{s_2}}\right)$$

for $m \in \mathbb{N}$. If we assume $\sigma_2 > 1$ then we have

$$\begin{aligned}\zeta_2(s_1, s_2) &= \sum_{m=1}^{\infty} \frac{1}{m^{\sigma_1 + it_1}} \sum_{n=m+1}^{\infty} \frac{1}{n^{s_2}} \\ &= \sum_{m=1}^{\infty} a_m m^{-it_1}.\end{aligned}$$

The last series converges absolutely in $\sigma_1 + \sigma_2 > 2$. Since

$$\sum_{m=1}^{\infty} m|a_m|^2 = \sum_{m=1}^{\infty} \frac{1}{m^{2\sigma_1 - 1}} \left|\zeta(s_2) - \sum_{n=1}^{m} \frac{1}{n^{s_2}}\right|^2$$

converges by (2.2.1), we have

$$I^{[1]}(T) = \zeta_2^{[1]}(2\sigma_1, s_2)T + O(1)$$

by Lemma 1.1.3. □

In the case $3/2 \leq \sigma_1 + \sigma_2 \leq 2$, we use the following lemma.

Lemma 2.3.1. *Let $s_1 = \sigma_1 + it_1, s_2 = \sigma_2 + it_2 \in \mathbb{C}$ with $t_1 \geq 1$ and $N \in \mathbb{N}$. Let $C > 1$ be a given constant. Assume that the point $(s_1, s_2) \in \mathbb{C}^2$ does not encounter the singularities of $\zeta_2(s_1, s_2)$. If $1 < |t_1 + t_2| < 2\pi N/C$, then we have*

$$\zeta_2(s_1, s_2) = \sum_{m \leq N} \frac{1}{m^{s_1}} \left(\zeta(s_2) - \sum_{n=1}^{m} \frac{1}{n^{s_2}} \right) + O(t_1^{-1} N^{2-\sigma_1-\sigma_2})$$

for $\sigma_1 + \sigma_2 > 1$ and any fixed σ_1, s_2.

Proof. Let $l \in \mathbb{N}$ with $\sigma_2 > -2l$. In order to obtain the analytic continuation of $\zeta_2(s_1, s_2)$, we regard s_1 and s_2 as complex variables and assume $\sigma_1, \sigma_2 > 1$ temporarily. For any $N \in \mathbb{N}$, we have

$$\zeta_2(s_1, s_2) = \sum_{m=1}^{N} \frac{1}{m^{s_1}} \sum_{n=m+1}^{\infty} \frac{1}{n^{s_2}} + \sum_{m=N+1}^{\infty} \frac{1}{m^{s_1}} \sum_{n=m+1}^{\infty} \frac{1}{n^{s_2}} = V_1 + V_2,$$

say. Since

$$V_1 = \sum_{m=1}^{N} \frac{1}{m^{s_1}} \left(\zeta(s_2) - \sum_{n=1}^{m} \frac{1}{n^{s_2}} \right),$$

V_1 is continued meromorphically to $\mathbb{C}^2$. By setting $M = 2l + 1$ in Lemma 1.1.1, we have

$$\begin{aligned}
V_2 &= \sum_{m=N+1}^{\infty} \frac{1}{m^{s_1}} \left(\frac{m^{1-s_2}}{s_2 - 1} - \frac{m^{-s_2}}{2} + \sum_{k=1}^{M-1} \frac{B_{k+1}}{(k+1)!} (s_2)_k m^{-s_2-k} + R_{M,m}(s_2) \right) \\
&= \frac{1}{s_2 - 1} \sum_{m=N+1}^{\infty} \frac{1}{m^{s_1+s_2-1}} - \frac{1}{2} \sum_{m=N+1}^{\infty} \frac{1}{m^{s_1+s_2}} + \\
&\quad + \sum_{k=1}^{M-1} \frac{B_{k+1}}{(k+1)!} (s_2)_k \sum_{m=N+1}^{\infty} \frac{1}{m^{s_1+s_2+k}} + \sum_{m=N+1}^{\infty} \frac{1}{m^{s_1}} R_{M,m}(s_2) \\
&= \frac{1}{s_2 - 1} \left(\zeta(s_1 + s_2 - 1) - \sum_{m=1}^{N} \frac{1}{m^{s_1+s_2-1}} \right) - \frac{1}{2} \sum_{m=N+1}^{\infty} \frac{1}{m^{s_1+s_2}} + \\
&\quad + \sum_{k=1}^{M-1} \frac{B_{k+1}}{(k+1)!} (s_2)_k \sum_{m=N+1}^{\infty} \frac{1}{m^{s_1+s_2+k}} + \sum_{m=N+1}^{\infty} \frac{1}{m^{s_1}} R_{M,m}(s_2) \\
&= I_1 + I_2 + I_3 + I_4,
\end{aligned}$$

say. Since I_4 converges absolutely for $\sigma_2 > -M+1 = -2l$ and $\sigma_1+\sigma_2 > -1$, V_2 is continued meromorphically to $\sigma_2 > -2l$ and $\sigma_1+\sigma_2 > 1$. Now, we regard σ_1, s_2 as constants. By Lemma 2.2.1, we have $I_1 \ll t_1^{-1}N^{2-\sigma_1-\sigma_2}$. Also we can easily obtain $I_2, I_3, I_4 \ll t_1^{-1}N^{2-\sigma_1-\sigma_2}$. This implies the lemma. □

Proof of Theorem 2.1.1 for $3/2 \le \sigma_1+\sigma_2 \le 2$. Let

$$a_m = m^{-\sigma_1}(\zeta(s_2) - \sum_{n=1}^{m} n^{-s_2})$$

and

$$m_0 = \max\{m \in \mathbb{N} \mid \frac{T}{2^m} > |t_2|+1\}.$$

Note that

$$\sum_{m=1}^{\infty} |a_m|^2 = \zeta_2^{[1]}(2\sigma_1, s_2)$$

in the case $\sigma_1+\sigma_2 > 3/2$ and

$$m_0 < \frac{\log T - \log(|t_2|+1)}{\log 2} \le m_0+1$$

hold. We take $T \ge 2$ and $N \in \mathbb{N}$ with $|t_2|+1 < T$ and $3T < 2\pi N/C$, where $C > 1$, and we assume $T < t_1 < 2T$. Then we have

$$1 < t_1 - |t_2| < |t_1+t_2| < |t_1| + |t_2| < 3T < \frac{2\pi N}{C}.$$

Therefore we can use Lemma 2.3.1, and we have

$$\zeta_2(s_1,s_2) = \sum_{m=1}^{N} a_m m^{-it_1} + O(t_1^{-1}N^{2-\sigma_1-\sigma_2}) = I_1 + I_2,$$

say. Since $a_m \ll m^{-\sigma_1-\sigma_2+1}$ by Corollary 2.2.1, we obtain

$$\sum_{m=1}^{N} m a_m^2 \ll \sum_{m=1}^{N} m^{3-2\sigma_1-2\sigma_2} \ll \begin{cases} \log N & (\sigma_1+\sigma_2=2) \\ N^{4-2\sigma_1-2\sigma_2} & (\sigma_1+\sigma_2<2) \end{cases}$$

and

$$I_1 \ll \sum_{m=1}^{N} a_m \ll \sum_{m=1}^{N} m^{1-\sigma_1-\sigma_2} \ll \begin{cases} \log N & (\sigma_1+\sigma_2=2) \\ N^{2-\sigma_1-\sigma_2} & (\sigma_1+\sigma_2<2). \end{cases}$$

Therefore we have

$$\int_T^{2T} |I_1|^2 dt_1 = T \sum_{m=1}^{N} |a_m|^2 + \begin{cases} O(\log N) & (\sigma_1+\sigma_2=2) \\ O(N^{4-2\sigma_1-2\sigma_2}) & (\sigma_1+\sigma_2<2) \end{cases}$$

by Lemma 1.1.3 and

$$\int_T^{2T} |I_1 I_2| dt_1 \ll N^{2-\sigma_1-\sigma_2} \max_{T<t_1<2T} |I_1| \ll \begin{cases} \log N & (\sigma_1+\sigma_2=2) \\ N^{4-2\sigma_1-2\sigma_2} & (\sigma_1+\sigma_2<2). \end{cases}$$

On the other hand, we have

$$\int_T^{2T} |I_2|^2 dt_1 \ll N^{4-2\sigma_1-2\sigma_2} \int_T^{2T} \frac{dt_1}{t_1^2} \ll T^{-1} N^{4-2\sigma_1-2\sigma_2}.$$

Therefore we have

$$\int_T^{2T} |\zeta_2(s_1,s_2)|^2 dt_1 = T \sum_{m=1}^{N} |a_m|^2 + \begin{cases} O(\log N) & (\sigma_1+\sigma_2=2) \\ O(N^{4-2\sigma_1-2\sigma_2}) & (\sigma_1+\sigma_2<2). \end{cases}$$

By setting $N = [T]+1$, we obtain

$$\int_T^{2T} |\zeta_2(s_1,s_2)|^2 dt_1 = T \sum_{m\le T} |a_m|^2 + \begin{cases} O(\log T) & (\sigma_1+\sigma_2=2) \\ O(T^{4-2\sigma_1-2\sigma_2}) & (\sigma_1+\sigma_2<2). \end{cases} \tag{2.3.1}$$

Therefore, in the case $\sigma_1+\sigma_2 > 3/2$, we have

$$\int_T^{2T} |\zeta_2(s_1,s_2)|^2 dt_1 = \zeta_2^{[1]}(2\sigma_1, s_2) T + \begin{cases} O(\log T) & (\sigma_1+\sigma_2=2) \\ O(T^{4-2\sigma_1-2\sigma_2}) & (3/2<\sigma_1+\sigma_2<2). \end{cases}$$

By this relation and Remark 2.2.1, we obtain

$$
\begin{aligned}
&\int_{|t_2|+1}^{T} |\zeta_2(s_1,s_2)|^2 dt_1 \\
&= \int_{T/2^{m_0}}^{T} |\zeta_2(s_1,s_2)|^2 dt_1 + O(1) \\
&= \sum_{1\le k\le m_0} \int_{T/2^k}^{T/2^{k-1}} |\zeta_2(s_1,s_2)|^2 dt_1 + O(1) \\
&= \zeta_2^{[1]}(2\sigma_1,s_2)T \sum_{1\le k\le m_0} \frac{1}{2^k} + \begin{cases} O\Big(\sum_{1\le k\le m_0} \log \frac{T}{2^k}\Big) & (\sigma_1+\sigma_2=2) \\ O\Big(\sum_{1\le k\le m_0} \Big(\frac{T}{2^k}\Big)^{4-2\sigma_1-2\sigma_2}\Big) & (3/2<\sigma_1+\sigma_2<2) \end{cases} \\
&= \zeta_2^{[1]}(2\sigma_1,s_2)T + \begin{cases} O((\log T)^2) & (\sigma_1+\sigma_2=2) \\ O(T^{4-2\sigma_1-2\sigma_2}) & (3/2<\sigma_1+\sigma_2<2). \end{cases}
\end{aligned}
$$

This implies the theorem for $3/2 < \sigma_1+\sigma_2 \le 2$.

In the case $\sigma_1+\sigma_2 = 3/2$, since

$$a_m = m^{-\sigma_1}\left(\zeta(s_2) - \sum_{n=1}^{m} n^{-s_2}\right) = \frac{m^{1-\sigma_1-s_2}}{s_2-1} + O(m^{-\sigma_1-\sigma_2})$$

by Lemma 1.1.1, we have

$$|a_m|^2 = \frac{m^{-1}}{|s_2-1|^2} + O(m^{-2}).$$

Therefore we have

$$\int_{T}^{2T} |\zeta_2(s_1,s_2)|^2 dt_1 = \frac{T\log T}{|s_2-1|^2} + O(T)$$

by (2.3.1). By Remark 2.2.1 and this relation we obtain the theorem. □

2.4 Proof of Theorem 2.1.2

In this section, we regard σ_2, s_1 as constants. We divide the proof into three cases.

Proof of Theorem 2.1.2 for $\sigma_2 > 1$ and $\sigma_1 + \sigma_2 > 2$. We set

$$a_n = \frac{1}{n^{\sigma_2}} \sum_{m=1}^{n-1} \frac{1}{m^{s_1}}$$

for $n \in \mathbb{N}$. We have

$$\zeta_2(s_1, s_2) = \sum_{n=2}^{\infty} \left(\sum_{m=1}^{n-1} \frac{1}{m^{s_1}} \right) \frac{1}{n^{\sigma_2 + it_2}} = \sum_{n=2}^{\infty} a_n n^{-it_2}.$$

Since

$$\sum_{n=2}^{\infty} n|a_n|^2 = \sum_{n=2}^{\infty} \left(\sum_{m=1}^{n-1} \frac{1}{m^{s_1}} \right) \frac{1}{n^{2\sigma_2 - 1}}$$

converges by (2.2.2), we have

$$I^{[2]}(T) = \zeta_2^{[2]}(s_1, 2\sigma_2)T + O(1)$$

by Lemma 1.1.3. □

We use the following lemma in the cases either $\sigma_1 > 1$, $1/2 < \sigma_2 \le 1$ or $\sigma_1 \le 1$, $3/2 \le \sigma_1 + \sigma_2 \le 2$.

Lemma 2.4.1. *Let $s_1 = \sigma_1 + it_1, s_2 = \sigma_2 + it_2 \in \mathbb{C}$ with $t_2 \ge 1$ and $N \in \mathbb{N}$ with $N > e^2$. Let $C > 1$ be a given constant. Assume that the point $(s_1, s_2) \in \mathbb{C}^2$ does not encounter the singularities of $\zeta_2(s_1, s_2)$. If $1 < t_2 < 2\pi N/C$ and $1 < |t_1 + t_2| < 2\pi N/C$, then we have*

$$\zeta_2(s_1, s_2) = \sum_{2 \le n \le N} \left(\sum_{m=1}^{n-1} \frac{1}{m^{s_1}} \right) \frac{1}{n^{s_2}} + \begin{cases} O(t_2^{-1} N^{1-\sigma_2} + t_2^{-1} N^{2-\sigma_1-\sigma_2}) & (s_1 \ne 1) \\ O(t_2^{-1} N^{1-\sigma_2} \log N) & (s_1 = 1) \end{cases}$$

for $\sigma_2 \ge 1/2$, $\sigma_1 + \sigma_2 > 1$ and any fixed σ_2, s_1.

Proof. Let $l \in \mathbb{N}$ with $\sigma_1 > -2l$. In order to obtain the analytic continuation of $\zeta_2(s_1, s_2)$, we regard s_1 and s_2 as complex variables and assume $\sigma_1, \sigma_2 > 1$ temporarily. For any $N \in \mathbb{N}$, we have

$$\zeta_2(s_1, s_2) = \sum_{2 \le n \le N} \left(\sum_{m=1}^{n-1} \frac{1}{m^{s_1}} \right) \frac{1}{n^{s_2}} + \sum_{n > N} \left(\sum_{m=1}^{n-1} \frac{1}{m^{s_1}} \right) = U_1 + U_2, \qquad (2.4.1)$$

say. The first term U_1 is obviously holomorphic in $\mathbb{C}^2$. By setting $M=2l+1$ in Lemma 1.1.1, we have

$$
\begin{aligned}
U_2 &= \sum_{n>N}\left(\sum_{m=1}^{n}\frac{1}{m^{s_1}}-\frac{1}{n^{s_1}}\right)\frac{1}{n^{s_2}}\\
&= \sum_{n>N}\left(\zeta(s_1)-\frac{n^{1-s_1}}{s_1-1}-\frac{n^{-s_1}}{2}-\sum_{k=1}^{M-1}\frac{B_{k+1}}{(k+1)!}(s_1)_k n^{-s_1-k}-R_{M,n}(s_1)\right)\frac{1}{n^{s_2}}\\
&= \zeta(s_1)\sum_{n>N}\frac{1}{n^{s_2}}+\frac{1}{1-s_1}\sum_{n>N}\frac{1}{n^{s_1+s_2-1}}-\frac{1}{2}\sum_{n>N}\frac{1}{n^{s_1+s_2}}-\\
&\quad -\sum_{k=1}^{M-1}\frac{B_{k+1}}{(k+1)!}(s_1)_k\sum_{n>N}\frac{1}{n^{s_1+s_2+k}}-\sum_{n>N}\frac{1}{n^{s_2}}R_{M,n}(s_1)\\
&= \zeta(s_1)\left(\zeta(s_2)-\sum_{n=1}^{N}\frac{1}{n^{s_2}}\right)+\frac{1}{1-s_1}\left(\zeta(s_1+s_2-1)-\sum_{n=1}^{N}\frac{1}{n^{s_1+s_2-1}}\right)-\\
&\quad -\frac{1}{2}\sum_{n>N}\frac{1}{n^{s_1+s_2}}-\sum_{k=1}^{M-1}\frac{B_{k+1}}{(k+1)!}(s_1)_k\sum_{n>N}\frac{1}{n^{s_1+s_2+k}}-\sum_{n>N}\frac{1}{n^{s_2}}R_{M,n}(s_1)\\
&= I_1+I_2+I_3+I_4+I_5,
\end{aligned}
\tag{2.4.2}
$$

say. Since I_5 converges absolutely for $\sigma_1>-M+1=-2l$ and $\sigma_1+\sigma_2>-1$, U_2 is continued meromorphically to $\sigma_2>0$, $\sigma_1>-2l$ and $\sigma_1+\sigma_2>1$. Now, we regard σ_2, s_1 as constants.

In the case $s_1\neq 1$, by Lemma 2.2.1, we have $I_1\ll |s_1-1|^{-1}t_2^{-1}N^{1-\sigma_2}$ and $I_2\ll |s_1-1|^{-1}t_2^{-1}N^{2-\sigma_1-\sigma_2}$. Also we can easily obtain $I_3,I_4,I_5\ll t_2^{-1}N^{2-\sigma_1-\sigma_2}$. This implies the lemma for $s_1\neq 1$. In the case $s_1=1$, we obtain the lemma by using the maximum modulus principle. □

We prove Theorem 2.1.2 for $\sigma_1>1$, $1/2\le\sigma_2\le 1$ or $\sigma_1\le 1$, $3/2\le\sigma_1+\sigma_2\le 2$. We divide the proof into the case $s_1\neq 1$ and the case $s_1=1$.

Proof of Theorem 2.1.2 for $s_1\neq 1$. We prove the theorem by the same argument as in the proof of Theorem 2.1.1.

Let

$$a_n=n^{-\sigma_2}\sum_{m=1}^{n-1}m^{-s_1}.$$

Note that

$$\sum_{n=2}^{\infty} |a_n|^2 = \zeta_2^{[2]}(s_1, 2\sigma_2)$$

in the case $\sigma_1 + \sigma_2 > 3/2$ and $\sigma_2 > 1/2$. We take $T \geq 2$ and $N \in \mathbb{N}$ with $N > e^2$, $|t_1| + 1 < T$ and $3T < 2\pi N/C$, where $C > 1$, and we assume $T < t_2 < 2T$. Then we can use Lemma 2.4.1, and we have

$$\zeta_2(s_1, s_2) = \sum_{n=2}^{N} a_n n^{-it_2} + O(t_2^{-1} N^{1-\sigma_2} + t_2^{-1} N^{2-\sigma_1-\sigma_2}) = I_1 + I_2,$$

say. Since

$$a_n \ll \begin{cases} n^{-\sigma_2} & (\sigma_1 \geq 1) \\ n^{-\sigma_1-\sigma_2+1} & (\sigma_1 < 1) \end{cases}$$

by Corollary 2.2.1, we obtain

$$\sum_{n=2}^{N} n a_n^2 \ll \begin{cases} \log N & (\sigma_2 = 1, \sigma_1 \geq 1) \\ N^{2-2\sigma_2} & (\sigma_2 < 1, \sigma_1 \geq 1) \\ \log N & (\sigma_1 + \sigma_2 = 2, \sigma_1 < 1) \\ N^{4-2\sigma_1-2\sigma_2} & (\sigma_1 + \sigma_2 < 2, \sigma_1 < 1) \end{cases}$$

and

$$I_1 \ll \sum_{n=2}^{N} a_n \ll \begin{cases} \log N & (\sigma_2 = 1, \sigma_1 \geq 1) \\ N^{1-\sigma_2} & (\sigma_2 < 1, \sigma_1 \geq 1) \\ \log N & (\sigma_1 + \sigma_2 = 2, \sigma_1 < 1) \\ N^{2-\sigma_1-\sigma_2} & (\sigma_1 + \sigma_2 < 2, \sigma_1 < 1). \end{cases}$$

Therefore we have

$$\int_T^{2T} |I_1|^2 dt_2 = T \sum_{n=2}^{N} |a_n|^2 + \begin{cases} O(\log N) & (\sigma_2 = 1, \sigma_1 \geq 1) \\ O(N^{2-2\sigma_2}) & (\sigma_2 < 1, \sigma_1 \geq 1) \\ O(\log N) & (\sigma_1 + \sigma_2 = 2, \sigma_1 < 1) \\ O(N^{4-2\sigma_1-2\sigma_2}) & (\sigma_1 + \sigma_2 < 2, \sigma_1 < 1) \end{cases}$$

by Lemma 1.1.3 and

$$\int_T^{2T} |I_1 I_2| dt_2 \ll \begin{cases} (N^{1-\sigma_2} + N^{2-\sigma_1-\sigma_2}) \log N & (\sigma_2 = 1, \sigma_1 \geq 1) \\ (N^{1-\sigma_2} + N^{2-\sigma_1-\sigma_2}) N^{1-\sigma_2} & (\sigma_2 < 1, \sigma_1 \geq 1) \\ (N^{1-\sigma_2} + N^{2-\sigma_1-\sigma_2}) \log N & (\sigma_1 + \sigma_2 = 2, \sigma_1 < 1) \\ (N^{1-\sigma_2} + N^{2-\sigma_1-\sigma_2}) N^{2-\sigma_1-\sigma_2} & (\sigma_1 + \sigma_2 < 2, \sigma_1 < 1) \end{cases}$$

$$\ll \begin{cases} \log N & (\sigma_2 = 1, \sigma_1 \geq 1) \\ N^{2-2\sigma_2} & (\sigma_2 < 1, \sigma_1 \geq 1) \\ \log N & (\sigma_1 + \sigma_2 = 2, \sigma_1 < 1) \\ N^{4-2\sigma_1-2\sigma_2} & (\sigma_1 + \sigma_2 < 2, \sigma_1 < 1). \end{cases}$$

On the other hand, we have

$$\int_T^{2T} |I_2|^2 dt_2 \ll T^{-1}(N^{2-2\sigma_2} + N^{4-2\sigma_1-2\sigma_2}).$$

Therefore we have

$$\int_T^{2T} |\zeta_2(s_1, s_2)|^2 dt_2 = T \sum_{n=2}^{N} |a_n|^2 + \begin{cases} O(\log N) & (\sigma_2 = 1, \sigma_1 \geq 1) \\ O(N^{2-2\sigma_2}) & (\sigma_2 < 1, \sigma_1 \geq 1) \\ O(\log N) & (\sigma_1 + \sigma_2 = 2, \sigma_1 < 1) \\ O(N^{4-2\sigma_1-2\sigma_2}) & (\sigma_1 + \sigma_2 < 2, \sigma_1 < 1). \end{cases}$$

By setting $N = [T] + 1$, we obtain

$$\int_T^{2T} |\zeta_2(s_1, s_2)|^2 dt_2 = T \sum_{n \leq T} |a_n|^2 + \begin{cases} O(\log T) & (\sigma_2 = 1, \sigma_1 \geq 1) \\ O(T^{2-2\sigma_2}) & (\sigma_2 < 1, \sigma_1 \geq 1) \\ O(\log T) & (\sigma_1 + \sigma_2 = 2, \sigma_1 < 1) \\ O(T^{4-2\sigma_1-2\sigma_2}) & (\sigma_1 + \sigma_2 < 2, \sigma_1 < 1). \end{cases} \tag{2.4.3}$$

Therefore, in the case $\sigma_1 + \sigma_2 > 3/2$ and $\sigma_2 > 1/2$, we have

$$\int_T^{2T} |\zeta_2(s_1, s_2)|^2 dt_2 = \zeta_2^{[2]}(s_1, 2\sigma_2) T + \begin{cases} O(\log T) & (\sigma_2 = 1, \sigma_1 \geq 1) \\ O(T^{2-2\sigma_2}) & (\sigma_2 < 1, \sigma_1 \geq 1) \\ O(\log T) & (\sigma_1 + \sigma_2 = 2, \sigma_1 < 1) \\ O(T^{4-2\sigma_1-2\sigma_2}) & (\sigma_1 + \sigma_2 < 2, \sigma_1 < 1). \end{cases} \tag{2.4.4}$$

In the case $\sigma_1 > 1, \sigma_2 = 1/2$, since

$$a_n = n^{-\sigma_2} \sum_{m=1}^{n-1} m^{-s_1} = n^{-\sigma_2}(\zeta(s_1) + O(n^{-\sigma_1+1}))$$

by Lemma 1.1.1, we have

$$|a_n|^2 = n^{-1}|\zeta(s_1) + O(n^{-\sigma_1+1})|^2 = n^{-1}|\zeta(s_1)|^2 + O(n^{-\sigma_1}).$$

Therefore we obtain

$$\int_T^{2T} |\zeta_2(s_1, s_2)|^2 dt_2 = |\zeta(s_1)|^2 T \log T + O(T) \tag{2.4.5}$$

by (2.4.3). In the case $\sigma_1 < 1$ and $\sigma_1 + \sigma_2 = 3/2$, since

$$a_n = n^{-\sigma_2}\Big(\frac{n^{-s_1+1}}{s_1 - 1} + O(n^{-\sigma_1}) + O(1)\Big)$$

by Lemma 1.1.1, we have

$$\begin{aligned} |a_n|^2 &= n^{-2\sigma_2}\left(\left|\frac{n^{-s_1+1}}{s_1-1}\right|^2 + O(n^{-2\sigma_1+1}) + O(n^{-\sigma_1+1})\right) \\ &= \frac{n^{-1}}{|s_1-1|^2} + O(n^{-2}) + O(n^{-2+\sigma_1}). \end{aligned}$$

Therefore we obtain

$$\int_T^{2T} |\zeta_2(s_1, s_2)|^2 dt_2 = \frac{T \log T}{|s_1 - 1|^2} + O(T) \tag{2.4.6}$$

by (2.4.3). In the case $\sigma_1 = 1$ and $\sigma_2 = 1/2$, since

$$a_n = n^{-\sigma_2}\left(\zeta(s_1) - \frac{n^{-s_1+1}}{s_1 - 1} + O(n^{-\sigma_1})\right)$$

by Lemma 1.1.1, we have

$$\sum_{n \le T} |a_n|^2 = \sum_{n \le T}\left(n^{-1}\left|\zeta(s_1) - \frac{n^{-s_1+1}}{s_1-1}\right|^2 + O(n^{-2})\right)$$

by Corollary 2.2.1. Since we have

$$\begin{aligned}
&\sum_{n\le T} n^{-1}\left|\zeta(s_1)-\frac{n^{-s_1+1}}{s_1-1}\right|^2\\
&=(|\zeta(s_1)|^2+|s_1-1|^{-2})\log T-2\sum_{n\le T}\Re\left(\overline{\zeta(s_1)}\frac{n^{-s_1}}{s_1-1}\right)+O(1)\\
&=(|\zeta(s_1)|^2+|s_1-1|^{-2})\log T+O(1)
\end{aligned}$$

by Corollary 2.2.1, we have

$$\sum_{n\le T}|a_n|^2=(|\zeta(s_1)|^2+|s_1-1|^{-2})\log T+O(1).$$

Therefore we obtain

$$\int_T^{2T}|\zeta_2(s_1,s_2)|^2dt_2=(|\zeta(s_1)|^2+|s_1-1|^{-2})T\log T+O(T) \tag{2.4.7}$$

by (2.4.3). By (2.4.4), (2.4.5), (2.4.6) and (2.4.7), we can obtain the theorem by the same argument as in the proof of Theorem 2.1.1. □

Proof of Theorem 2.1.2 for $s_1=1$. We prove the theorem by the same argument as in the proof of Theorem 2.1.1.

Hereafter we use the same notations as in the previous proof. Note that, in this case, we have $I_2=O(t_2^{-1}N^{1-\sigma_2}\log N)$ by using Lemma 2.4.1. Since $a_n\ll n^{-\sigma_2}\log n$, we obtain

$$\sum_{n=2}^N n|a_n|^2\ll\sum_{n=2}^N n^{1-2\sigma_2}(\log n)^2\ll\begin{cases}O((\log N)^3) & (\sigma_2=1)\\ O(N^{2-2\sigma_2}(\log N)^2) & (\sigma_2<1)\end{cases}$$

and

$$I_1\ll\sum_{n=2}^N|a_n|\ll\begin{cases}(\log N)^2 & (\sigma_2=1)\\ N^{1-\sigma_2}\log N & (\sigma_2<1).\end{cases}$$

Therefore we have

$$\int_T^{2T}|I_1|^2dt_2=T\sum_{n=2}^N|a_n|^2+\begin{cases}O((\log N)^3) & (\sigma_2=1)\\ O(N^{2-2\sigma_2}(\log N)^2) & (\sigma_2<1)\end{cases}$$

by Lemma 1.1.3 and

$$\int_T^{2T} |I_1 I_2| dt_2 \ll \begin{cases} O((\log N)^3) & (\sigma_2 = 1) \\ O(N^{2-2\sigma_2}(\log N)^2) & (\sigma_2 < 1). \end{cases}$$

On the other hand, we have

$$\int_T^{2T} |I_2|^2 dt_2 \ll T^{-1} N^{2-2\sigma_2} (\log N)^2.$$

Therefore, by setting $N = [T] + 1$, we obtain

$$\int_T^{2T} |\zeta_2(s_1, s_2)|^2 dt_2 = T \sum_{n \le T} |a_n|^2 + \begin{cases} O((\log T)^3) & (\sigma_2 = 1) \\ O(T^{2-2\sigma_2}(\log T)^2) & (\sigma_2 < 1). \end{cases} \tag{2.4.8}$$

In the case $\sigma_2 > 1/2$, we have

$$\int_T^{2T} |\zeta_2(s_1, s_2)|^2 dt_2 = \zeta_2^{[2]}(s_1, 2\sigma_2) T + \begin{cases} O((\log T)^3) & (\sigma_2 = 1) \\ O(T^{2-2\sigma_2}(\log T)^2) & (\sigma_2 < 1). \end{cases} \tag{2.4.9}$$

In the case $\sigma_2 = 1/2$, since

$$|a_n|^2 = n^{-1} \Big(\sum_{m=1}^{n-1} m^{-1} \Big)^2 = \frac{(\log n)^2}{n} + O\Big(\frac{\log n}{n}\Big)$$

and

$$\sum_{n=2}^{N} \frac{(\log n)^2}{n} = \int_1^N x^{-1} (\log x)^2 dx + O(1) = \frac{(\log N)^3}{3} + O(1)$$

hold, we have

$$\int_T^{2T} |\zeta_2(s_1, s_2)|^2 dt_2 = \frac{T(\log T)^3}{3} + O(T(\log T)^2) \tag{2.4.10}$$

by (2.4.8). By (2.4.9) and (2.4.10), we can obtain the theorem by the same argument as in the proof of Theorem 2.1.1. □

2.5 Proof of Theorem 2.1.3

We divide the proof into four cases.

Proof of Theorem 2.1.3 for $\sigma_2 > 1$ and $\sigma_1 + \sigma_2 > 2$. We set

$$a_k = \Bigg(\sum_{\substack{m|k \\ m<\sqrt{k}}} \frac{1}{m^{\sigma_1-\sigma_2}} \Bigg) \frac{1}{k^{\sigma_2}}$$

for $k \in \mathbb{N}$. We have

$$\begin{aligned}
\zeta_2(s_1, s_2) &= \sum_{1\le m<n} \frac{1}{m^{\sigma_1} n^{\sigma_2} (mn)^{it}} \\
&= \sum_{k\ge 2} \Bigg(\sum_{\substack{mn=k \\ m<n}} \frac{1}{m^{\sigma_1} n^{\sigma_2}} \Bigg) \frac{1}{k^{it}} \\
&= \sum_{k\ge 2} \Bigg(\sum_{\substack{m|k \\ m<\sqrt{k}}} \frac{1}{m^{\sigma_1-\sigma_2}} \Bigg) \frac{1}{k^{\sigma_2+it}} \\
&= \sum_{k\ge 2} a_k k^{-it}.
\end{aligned}$$

Since

$$\sum_{k\ge 2} k|a_k|^2 = \sum_{k\ge 2}^{\infty} \Bigg(\sum_{\substack{m|k \\ m<\sqrt{k}}} \frac{1}{m^{\sigma_1-\sigma_2}} \Bigg)^2 \frac{1}{k^{2\sigma_2-1}}$$

converges by (2.2.3), we have

$$I^{\square}(T) = \zeta_2^{\square}(\sigma_1, \sigma_2) T + O(1)$$

by Lemma 1.1.3. $\square$

We use the following lemma in the cases either $\sigma_1 > 1$, $1/2 < \sigma_2 \le 1$ or $\sigma_1 \le 1$, $3/2 < \sigma_1 + \sigma_2 \le 2$.

Lemma 2.5.1. *Let $\sigma_1 + \sigma_2 > 1$, $\sigma_2 > 0$, $s_1 = \sigma_1 + it$ and $s_2 = \sigma_2 + it$. Then*

$$\zeta_2(s_1, s_2) = \sum_{n\le t} n^{-s_2} \sum_{m=1}^{n-1} m^{-s_1} + \begin{cases} O(t^{-\sigma_2}) & (\sigma_1 > 1) \\ O(t^{-\sigma_2+\epsilon}) & (\sigma_1 = 1) \\ O(t^{1-\sigma_1-\sigma_2}) & (\sigma_1 < 1) \end{cases}$$

holds for $t \ge 2$, where the implied constants depend on σ_1, σ_2.

In order to prove Lemma 2.5.1, we use the following lemma and corollary.

Lemma 2.5.2 (Lemma 2.2 in [13]). *Let $s = \sigma + it$, $|t| > 1$. For $N > \frac{1}{4}|t|$, $m \geq 1$ and $\sigma > -2m - 1$, we have*

$$\zeta(s) = \sum_{n \leq N} \frac{1}{n^s} + \frac{N^{1-s}}{s-1} - \frac{N^{-s}}{2} + \sum_{k=1}^{2m} \frac{B_{k+1}}{(k+1)!}(s)_k N^{-(s+k)} + \\ + O\big(|t|^{2m+1} N^{-\sigma-2m-1}\big),$$

where the implied constant does not depend on t.

Corollary 2.5.1 (Corollary 2.3 in [13]). *Let $s = \sigma + it$, $|t| > 1$. For $N > \frac{1}{4}|t|$ and $\sigma > -3$, we have*

$$\zeta(s) = \sum_{n \leq N} \frac{1}{n^s} + \frac{N^{1-s}}{s-1} - \frac{N^{-s}}{2} + \frac{s}{12} N^{-s-1} + O\big(|t|^3 N^{-\sigma-3}\big),$$

where the implied constant does not depend on t.

The following proof is similar to that in [13] (section 4.1 Evaluation of $S_2(s_1, s_2)$).

Proof of Lemma 2.5.1. Let $l \in \mathbb{N}$ with $\sigma_1 > -2l$. We use (2.4.1) and (2.4.2). Hence we obtain the analytic continuation of $\zeta_2(s_1, s_2)$ for $\sigma_2 > 0$, $\sigma_1 > -2l$ and $\sigma_1 + \sigma_2 > 1$. Now, we set $s_1 = \sigma_1 + it$, $s_2 = \sigma_2 + it$ with $t \geq 1$ and $N = [t]$. Then we have

$$\begin{aligned} I_1 &= \zeta(s_1)\left(\frac{N^{1-s_2}}{s_2 - 1} - \frac{N^{-s_2}}{2} + \frac{s_2}{12} N^{-s_2-1} + O\big(|t|^3 N^{-\sigma_2-3}\big)\right) \\ &\ll |\zeta(s_1)| t^{-\sigma_2} \\ &\ll \begin{cases} t^{-\sigma_2} & (\sigma_1 > 1) \\ t^{-\sigma_2+\epsilon} & (\sigma_1 = 1) \\ t^{1-\sigma_1-\sigma_2} & (\sigma_1 < 1) \end{cases} \end{aligned}$$

for $\sigma_2 > -3$ by Corollary 2.5.1. Similarly, we have

$$\begin{aligned} I_2 = &\frac{1}{1-s_1}\left(\frac{N^{2-s_1-s_2}}{s_1+s_2-2} - \frac{1}{2}N^{1-s_1-s_2} + \frac{s_1+s_2-1}{12}N^{-s_1-s_2} + \right. \\ &\left. + O\big(|t|^3 N^{1-\sigma_1-\sigma_2-3}\big)\right) \\ &\ll t^{1-\sigma_1-\sigma_2} \end{aligned}$$

for $\sigma_1+\sigma_2 > -2$. Since $\sigma_1+\sigma_2 > 1$, we have

$$I_j \ll t^{1-\sigma_1-\sigma_2} \qquad (j=3,4).$$

On the other hand, $R_{M,n}(s_1) = O(t^M n^{-\sigma_1-M})$ for $\sigma_1 > -M$ by Lemma 2.5.2. Hence we have

$$I_5 \ll t^M \sum_{n>N} \frac{1}{n^{\sigma_1+\sigma_2+M}} \ll t^{1-\sigma_1-\sigma_2}.$$

This implies the lemma. □

We prove Theorem 2.1.3 for $\sigma_1 > 1$, $1/2 < \sigma_2 \le 1$ or $\sigma_1 \le 1$, $3/2 < \sigma_1+\sigma_2 \le 2$. First we consider the case $\sigma_1 > \sigma_2$. In particular, this condition is satisfied when $\sigma_1 > 1$ and $1/2 < \sigma_2 \le 1$.

Proof of Theorem 2.1.3 for $\sigma_1 > \sigma_2$. If we set

$$A(s_1,s_2) = \sum_{n\le t} n^{-s_2} \sum_{m=1}^{n-1} m^{-s_1}$$

then we have

$$
\begin{aligned}
\int_2^T |A(s_1,s_2)|^2 dt &= \int_2^T \left(\sum_{n_1 \le t} n_1^{-s_2} \sum_{m_1=1}^{n_1-1} m_1^{-s_1} \sum_{n_2 \le t} n_2^{-\overline{s_2}} \sum_{m_2=1}^{n_2-1} m_2^{-\overline{s_1}} \right) dt \\
&= \sum_{2 \le n_1 \le T} \sum_{m_1=1}^{n_1-1} \sum_{2 \le n_2 \le T} \sum_{m_2=1}^{n_2-1} n_1^{-\sigma_2} m_1^{-\sigma_1} n_2^{-\sigma_2} m_2^{-\sigma_1} \int_{M(n_1,n_2)}^T \left(\frac{m_2 n_2}{m_1 n_1} \right)^{it} dt \\
&= \sum_{m_1 n_1 = m_2 n_2} \sum_{\substack{1 \le m_1 \le n_1 - 1 \\ 2 \le n_1 \le T}} \sum_{\substack{1 \le m_2 \le n_2 - 1 \\ 2 \le n_2 \le T}} n_1^{-\sigma_2} m_1^{-\sigma_1} n_2^{-\sigma_2} m_2^{-\sigma_1} \Big(T - M(n_1,n_2) \Big) \\
&\quad + \sum_{m_1 n_1 \ne m_2 n_2} \sum_{\substack{1 \le m_1 \le n_1 - 1 \\ 2 \le n_1 \le T}} \sum_{\substack{1 \le m_2 \le n_2 - 1 \\ 2 \le n_2 \le T}} n_1^{-\sigma_2} m_1^{-\sigma_1} n_2^{-\sigma_2} m_2^{-\sigma_1} \\
&\qquad \times \frac{\exp\left(iT \log\left(\frac{m_2 n_2}{m_1 n_1} \right) \right) - \exp\left(iM(n_1,n_2) \log\left(\frac{m_2 n_2}{m_1 n_1} \right) \right)}{i \log\left(\frac{m_2 n_2}{m_1 n_1} \right)} \\
&= S_1 T - S_2 + S_3,
\end{aligned}
$$

say, where $M(n_1,n_2) = \max(n_1,n_2)$. First, we rewrite

$$
\begin{aligned}
S_1 &= \sum_{2 \le k \le T} \left(\sum_{\substack{mn=k \\ m<n \le T}} m^{-\sigma_1} n^{-\sigma_2} \right)^2 + \sum_{T < k < T^2} \left(\sum_{\substack{mn=k \\ m<n \le T}} m^{-\sigma_1} n^{-\sigma_2} \right)^2 \\
&= \sum_{k=2}^{\infty} \left(\sum_{\substack{mn=k \\ m<n}} m^{-\sigma_1} n^{-\sigma_2} \right)^2 - \sum_{k>T} \left(\sum_{\substack{mn=k \\ m<n}} m^{-\sigma_1} n^{-\sigma_2} \right)^2 + \sum_{T < k < T^2} \left(\sum_{\substack{mn=k \\ m<n \le T}} m^{-\sigma_1} n^{-\sigma_2} \right)^2 .
\end{aligned}
$$

Since

$$
\begin{aligned}
\sum_{k>T} \left(\sum_{mn=k} m^{-\sigma_1} n^{-\sigma_2} \right)^2 &= \sum_{k>T} \left(\sum_{m|k} m^{-\sigma_1} m^{\sigma_2} k^{-\sigma_2} \right)^2 \\
&\ll \sum_{k>T} k^{-2\sigma_2+\epsilon} \ll T^{1-2\sigma_2+\epsilon},
\end{aligned}
$$

we have

$$
S_1 = \zeta_2^{\square}(\sigma_1,\sigma_2) + O(T^{1-2\sigma_2+\epsilon}).
$$

Next, we rewrite

$$
\begin{aligned}
S_2 &\ll \sum_{m_1n_1=m_2n_2} \sum_{\substack{1\le m_1\le n_1-1\\ 2\le n_1\le T}} \sum_{\substack{1\le m_2\le n_2-1\\ 2\le n_2\le T}} n_1^{-\sigma_2}m_1^{-\sigma_1}n_2^{-\sigma_2}m_2^{-\sigma_1}O(n_1+n_2)\\
&\ll \sum_{2\le k<T^2}\Bigg(\sum_{\substack{mn=k\\ 1\le m<n\le T}} m^{-\sigma_1}n^{1-\sigma_2}\Bigg)\Bigg(\sum_{\substack{mn=k\\ 1\le m<n\le T}} m^{-\sigma_1}n^{-\sigma_2}\Bigg)\\
&\ll \sum_{2\le k\le T}\Bigg(\sum_{\substack{mn=k\\ 1\le m<n\le T}} m^{-\sigma_1}n^{1-\sigma_2}\Bigg)\Bigg(\sum_{\substack{mn=k\\ 1\le m<n\le T}} m^{-\sigma_1}n^{-\sigma_2}\Bigg)\\
&\quad + \sum_{T<k<T^2}\Bigg(\sum_{\substack{mn=k\\ 1\le m<n\le T}} m^{-\sigma_1}n^{1-\sigma_2}\Bigg)\Bigg(\sum_{\substack{mn=k\\ 1\le m<n\le T}} m^{-\sigma_1}n^{-\sigma_2}\Bigg)\\
&= A_1+A_2,
\end{aligned}
$$

say. Since we have

$$
\begin{aligned}
A_1 &= \sum_{2\le k\le T} k^{1-2\sigma_2}\Bigg(\sum_{\substack{m|k\\ m<\sqrt{k}}} m^{\sigma_2-\sigma_1-1}\Bigg)\Bigg(\sum_{\substack{m|k\\ m<\sqrt{k}}} m^{\sigma_2-\sigma_1}\Bigg)\\
&\ll \sum_{2\le k\le T} k^{1-2\sigma_2+\epsilon} \ll T^{2-2\sigma_2+\epsilon}
\end{aligned}
$$

and

$$
\begin{aligned}
A_2 &\ll \sum_{T<k<T^2} k^{-2\sigma_1}\Bigg(\sum_{\substack{n|k\\ n\le T}} n^{1+\sigma_1-\sigma_2}\Bigg)\Bigg(\sum_{\substack{n|k\\ n\le T}} n^{\sigma_1-\sigma_2}\Bigg)\\
&\ll T^{1+2\sigma_1-2\sigma_2+\epsilon}\sum_{T<k<T^2} k^{-2\sigma_1} \ll T^{2-2\sigma_2+\epsilon},
\end{aligned}
$$

we have $S_2 \ll T^{2-2\sigma_2+\epsilon}$. Next, we have

$$\begin{aligned}
S_3 = & \sum_{m_1n_1 \neq m_2n_2} \sum_{\substack{1\le m_1\le n_1-1\\2\le n_1\le T}} \sum_{\substack{1\le m_2\le n_2-1\\2\le n_2\le T}} n_1^{-\sigma_2} m_1^{-\sigma_1} n_2^{-\sigma_2} m_2^{-\sigma_1} \\
& \times \frac{\exp\left(iT\log\left(\frac{m_2n_2}{m_1n_1}\right)\right) - \exp\left(iM(n_1,n_2)\log\left(\frac{m_2n_2}{m_1n_1}\right)\right)}{i\log\left(\frac{m_2n_2}{m_1n_1}\right)} \\
\ll & \sum_{m_1n_1 < m_2n_2} \sum_{\substack{1\le m_1\le n_1-1\\2\le n_1\le T}} \sum_{\substack{1\le m_2\le n_2-1\\2\le n_2\le T}} n_1^{-\sigma_2} m_1^{-\sigma_1} n_2^{-\sigma_2} m_2^{-\sigma_1} \frac{1}{\log\left(\frac{m_2n_2}{m_1n_1}\right)} \\
= & \sum_{m_1n_1 < m_2n_2 < 2m_1n_1} \sum_{\substack{1\le m_1\le n_1-1\\2\le n_1\le T}} \sum_{\substack{1\le m_2\le n_2-1\\2\le n_2\le T}} n_1^{-\sigma_2} m_1^{-\sigma_1} n_2^{-\sigma_2} m_2^{-\sigma_1} \frac{1}{\log\left(\frac{m_2n_2}{m_1n_1}\right)} \\
+ & \sum_{m_2n_2 \ge 2m_1n_1} \sum_{\substack{1\le m_1\le n_1-1\\2\le n_1\le T}} \sum_{\substack{1\le m_2\le n_2-1\\2\le n_2\le T}} n_1^{-\sigma_2} m_1^{-\sigma_1} n_2^{-\sigma_2} m_2^{-\sigma_1} \frac{1}{\log\left(\frac{m_2n_2}{m_1n_1}\right)} \\
= & B_1 + B_2,
\end{aligned}$$

say. We have $B_2 \ll T^{2-2\sigma_2}$ in the case $\sigma_1 > 1$. In the case $\sigma_1 \le 1$ we have

$$B_2 \ll \sum_{\substack{1\le m_1\le n_1-1\\2\le n_1\le T}} \sum_{\substack{1\le m_2\le n_2-1\\2\le n_2\le T}} n_1^{-\sigma_2} m_1^{-\sigma_1} n_2^{-\sigma_2} m_2^{-\sigma_1} \ll T^{4-2\sigma_1-2\sigma_2+\epsilon}. \tag{2.5.1}$$

Hence, we have

$$B_2 \ll \begin{cases} T^{2-2\sigma_2+\epsilon} & (\sigma_1 > 1), \\ T^{4-2\sigma_1-2\sigma_2+\epsilon} & (\sigma_1 \le 1). \end{cases}$$

Next we evaluate B_1. In the case $\sigma_1 > 1$ we have

$$\begin{aligned}
B_1 \ll & \sum_{r\le T^2} \sum_{\substack{m_2n_2 - r = m_1n_1\\ r< m_2n_2}} \sum_{1\le m_1<n_1\le T} \sum_{1\le m_2<n_2\le T} n_1^{-\sigma_2} m_1^{-\sigma_1} n_2^{-\sigma_2} m_2^{-\sigma_1} \frac{m_1n_1}{r} \\
\ll & \sum_{r\le T^2} \sum_{2\le n_1\le T} \sum_{1\le m_2<n_2\le T} \sum_{\substack{n_1|(m_2n_2-r)\\ r<m_2n_2}} n_1^{1-\sigma_2} n_2^{-\sigma_2} m_2^{-\sigma_1} \frac{1}{r} \\
\ll & \sum_{r\le T^2} \sum_{2\le n_1\le T} \sum_{1\le m_2<n_2\le T} T^{1-\sigma_2+\epsilon} n_2^{-\sigma_2} m_2^{-\sigma_1} \frac{1}{r} \ll T^{2-2\sigma_2+\epsilon}.
\end{aligned}$$

If $\sigma_1 \leq 1$ we have

$$\begin{aligned} B_1 &\ll \sum_{r \leq T^2} \sum_{1 \leq m_1 < n_1 \leq T} \sum_{1 \leq m_2 < n_2 \leq T} \sum_{\substack{m_2 n_2 = m_1 n_1 + r \\ r < m_2 n_2}} n_1^{-\sigma_2} m_1^{-\sigma_1} n_2^{-\sigma_2} m_2^{-\sigma_1} \frac{m_1 n_1}{r} \\ &\ll \sum_{r \leq T^2} \sum_{1 \leq m_1 < n_1 \leq T} \sum_{1 \leq m_2 < n_2 \leq T} \sum_{m_2 n_2 = m_1 n_1 + r} n_1^{-\sigma_2} m_1^{-\sigma_1} n_2^{-\sigma_2} m_2^{-\sigma_1} \frac{m_1 n_1}{r} \frac{m_2 n_2}{m_1 n_1} \\ &\ll \sum_{r \leq T^2} \sum_{1 \leq m_1 < n_1 \leq T} n_1^{-\sigma_2} m_1^{-\sigma_1} \frac{T^{2-\sigma_1-\sigma_2+\epsilon}}{r} \ll T^{4-2\sigma_1-2\sigma_2+\epsilon}, \end{aligned} \tag{2.5.2}$$

since we are in the case $\sigma_2 \leq \sigma_1 \leq 1$. Hence, we have

$$B_1 \ll \begin{cases} T^{2-2\sigma_2+\epsilon} & (\sigma_1 > 1), \\ T^{4-2\sigma_1-2\sigma_2+\epsilon} & (\sigma_1 \leq 1). \end{cases}$$

This implies

$$S_3 \ll \begin{cases} T^{2-2\sigma_2+\epsilon} & (\sigma_1 > 1), \\ T^{4-2\sigma_1-2\sigma_2+\epsilon} & (\sigma_1 \leq 1). \end{cases}$$

Therefore we have

$$\int_2^T |A(s_1, s_2)|^2 dt = \zeta_2^{\square}(\sigma_1, \sigma_2) T + \begin{cases} O(T^{2-2\sigma_2+\epsilon}) & (\sigma_1 > 1), \\ O(T^{4-2\sigma_1-2\sigma_2+\epsilon}) & (\sigma_1 \leq 1). \end{cases}$$

Now, if we set

$$\lambda = \begin{cases} -\sigma_2 & (\sigma_1 > 1), \\ -\sigma_2 + \epsilon & (\sigma_1 = 1), \\ 1 - \sigma_1 - \sigma_2 & (\sigma_1 < 1) \end{cases}$$

then we have

$$\begin{aligned} \int_2^T |\zeta(s_1, s_2)|^2 dt &= \int_2^T |A(s_1, s_2) + O(t^{\lambda})|^2 dt \\ &= \int_2^T |A(s_1, s_2)|^2 dt + O\left(\int_2^T |A(s_1, s_2) t^{\lambda}| dt \right) + O(1). \end{aligned}$$

By the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\int_2^T |A(s_1,s_2)t^\lambda|dt &\ll \left(\int_2^T |A(s_1,s_2)|^2dt\right)^{\frac{1}{2}}\left(\int_2^T t^{2\lambda}dt\right)^{\frac{1}{2}}\\ &\ll T^{\frac{1}{2}}.\end{aligned}$$

This implies the theorems. □

Next, we consider the case $\sigma_1 \le \sigma_2$.

Proof of Theorem 2.1.3 for $\sigma_1 \le \sigma_2$. Hereafter we use the same notations as in the previous proof. First we evaluate S_1. Since

$$\begin{aligned}\sum_{k>T}\Bigg(\sum_{\substack{mn=k\\ m<n}} m^{-\sigma_1}n^{-\sigma_2}\Bigg)^2 &= \sum_{k>T}\Bigg(\sum_{\substack{m|k\\ m<\sqrt{k}}} m^{-\sigma_1}m^{\sigma_2}k^{-\sigma_2}\Bigg)^2\\ &\ll \sum_{k>T} k^{-2\sigma_2}\left(k^{\frac{1}{2}(\sigma_2-\sigma_1)+\epsilon}\right)^2\\ &\ll \sum_{k>T} k^{-\sigma_1-\sigma_2+\epsilon} \ll T^{1-\sigma_1-\sigma_2+\epsilon},\end{aligned}$$

we have

$$S_1 = \zeta_2^{\square}(\sigma_1,\sigma_2) + O(T^{1-\sigma_1-\sigma_2+\epsilon}).$$

Next we evaluate S_2. Since

$$\sum_{\substack{mn=k\\ m<n}} m^{-\sigma_1}n^{-\sigma_2} = \sum_{\substack{m|k\\ m<\sqrt{k}}} m^{\sigma_2-\sigma_1}k^{-\sigma_2} \ll k^{-\frac{1}{2}(\sigma_1+\sigma_2)+\epsilon}, \tag{2.5.3}$$

$$\sum_{\substack{mn=k\\ m<n}} m^{-\sigma_1}n^{1-\sigma_2} = \sum_{\substack{m|k\\ m<\sqrt{k}}} k^{1-\sigma_2}m^{\sigma_2-\sigma_1-1} \ll \begin{cases} k^{1-\sigma_2+\epsilon} & (\sigma_2-\sigma_1-1\le 0)\\ k^{\frac{1}{2}(1-\sigma_1-\sigma_2)+\epsilon} & (\sigma_2-\sigma_1-1>0)\end{cases}$$

hold, we have

$$A_1 \ll \begin{cases} \displaystyle\sum_{2\le k\le T} k^{-\frac{1}{2}(\sigma_1+\sigma_2)+\epsilon} k^{1-\sigma_2+\epsilon} & (\sigma_2-\sigma_1-1\le 0) \\ \displaystyle\sum_{2\le k\le T} k^{-\frac{1}{2}(\sigma_1+\sigma_2)+\epsilon} k^{\frac{1}{2}(1-\sigma_1-\sigma_2)+\epsilon} & (\sigma_2-\sigma_1-1> 0) \end{cases}$$
$$= \begin{cases} \displaystyle\sum_{2\le k\le T} k^{1-\frac{1}{2}\sigma_1-\frac{3}{2}\sigma_2+\epsilon} & (\sigma_2-\sigma_1-1\le 0) \\ \displaystyle\sum_{2\le k\le T} k^{\frac{1}{2}-\sigma_1-\sigma_2+\epsilon} & (\sigma_2-\sigma_1-1> 0). \end{cases}$$

We note that $1-\frac{1}{2}\sigma_1-\frac{3}{2}\sigma_2<-1$ is equivalent to $\sigma_2>-\frac{1}{3}\sigma_1+\frac{4}{3}$. Hence we have

$$A_1 \ll \begin{cases} T^{2-\frac{1}{2}\sigma_1-\frac{3}{2}\sigma_2+\epsilon} & (\sigma_2-\sigma_1-1\le 0 \text{ and } \sigma_2\le -\frac{1}{3}\sigma_1+\frac{4}{3}), \\ 1 & (\text{otherwise}) \end{cases}$$

because $\sigma_1+\sigma_2>3/2$. Similarly, we have

$$A_2 \ll \begin{cases} \displaystyle\sum_{T<k<T^2} k^{1-\frac{1}{2}\sigma_1-\frac{3}{2}\sigma_2+\epsilon} & (\sigma_2-\sigma_1-1\le 0) \\ \displaystyle\sum_{T<k<T^2} k^{\frac{1}{2}-\sigma_1-\sigma_2+\epsilon} & (\sigma_2-\sigma_1-1> 0) \end{cases}$$
$$\ll \begin{cases} T^{4-\sigma_1-3\sigma_2+\epsilon} & (\sigma_2-\sigma_1-1\le 0 \text{ and } \sigma_2\le -\frac{1}{3}\sigma_1+\frac{4}{3}) \\ 1 & (\text{otherwise}). \end{cases}$$

Therefore we have

$$S_2 \ll \begin{cases} T^{4-\sigma_1-3\sigma_2+\epsilon} & (\sigma_2-\sigma_1-1\le 0 \text{ and } \sigma_2\le -\frac{1}{3}\sigma_1+\frac{4}{3}) \\ 1 & (\text{otherwise}). \end{cases}$$

Next we evaluate S_3. Since estimation (2.5.1) remains also valid in this case, we have $B_2 \ll T^{4-2\sigma_1-2\sigma_2+\epsilon}$. Since we have

$$\sum_{1\le m_2<n_2\le T}\ \sum_{m_2n_2=m_1n_1+r} n_2^{1-\sigma_2} m_2^{1-\sigma_1} \ll \left(\frac{m_1n_1+r}{T}\right)^{1-\sigma_2} T^{1-\sigma_1+\epsilon} \ll T^{2-\sigma_1-\sigma_2+\epsilon}$$

for $\sigma_1\le 1$, $\sigma_2>1$, $0\ne m_1n_1+r\ll T^2$, estimation (2.5.2) remains also valid in this case. Therefore we have $S_3 \ll T^{4-2\sigma_1-2\sigma_2+\epsilon}$. Since $4-2\sigma_1-2\sigma_2-$

$(4-\sigma_1-3\sigma_2)=\sigma_2-\sigma_1 \geq 0$, we have

$$\int_2^T |A(s_1,s_2)|^2 dt = \zeta_2^{\square}(\sigma_1,\sigma_2)T + O(T^{4-2\sigma_1-2\sigma_2+\epsilon}).$$

By the same argument as in the case $\sigma_1 > \sigma_2$, we obtain the theorem. $\square$

Proof of Theorem 2.1.3 for $\sigma_1 > 1$ *and* $\sigma_2 = 1/2$. By Theorem 2.2 in [28] we have

$$\int_2^T |\zeta(1/2+it)|^2 |\zeta(\sigma_1+it)|^2 dt \sim \frac{\zeta(2\sigma_1)\zeta(\sigma_1+1/2)^2}{\zeta(2\sigma_1+1)} T\log T.$$

By (1.1.1) and the Cauchy-Schwarz inequality we have

$$I^{\square}(T) \sim \int_2^T |\zeta(1/2+it)|^2 |\zeta(\sigma_1+it)|^2 dt.$$

This completes the proof. $\square$

Chapter 3

Certain mean values of the multiple zeta function

In this chapter we study certain mean values of the k-ple zeta function. This is a joint work with Soichi Ikeda.

3.1 Introduction

In the theory of the Riemann zeta function $\zeta(s)$, estimation of the bounds for $\zeta(s)$ in the critical strip is important and difficult. In [13] and [14], Kiuchi and Tanigawa studied the bounds for $\zeta_2(s_1, s_2)$ and $\zeta_3(s_1, s_2, s_3)$, respectively, but it is difficult to determine the correct order of $\zeta_2(s_1, s_2)$ and $\zeta_3(s_1, s_2, s_3)$.

On the other hand it is well-known that

$$\int_2^T |\zeta(\sigma+it)|^2 dt \sim \begin{cases} \zeta(2\sigma)T & (\sigma > 1/2) \\ T\log T & (\sigma = 1/2) \\ \frac{(2\pi)^{2\sigma-1}}{2-2\sigma}\zeta(2-2\sigma)T^{2-2\sigma} & (\sigma < 1/2). \end{cases} \tag{3.1.1}$$

Recently, Matsumoto and Tsumura studied

$$\int_2^T |\zeta_2(s_1, s_2)|^2 dt_2 \tag{3.1.2}$$

in [22]. In [10] Ikeda, Matsuoka and Nagata improved the results of Matsumoto and Tsumura and studied

$$\int_2^T |\zeta_2(s_1, s_2)|^2 dt_1$$

and

$$\int_2^T |\zeta_2(\sigma_1+it,\sigma_2+it)|^2 dt.$$

More generally, we may consider the mean values

$$I_k^{[j]}(T;s_1,\ldots,\sigma_j,\ldots,s_k)=\int_2^T |\zeta_k(s_1,\ldots,s_k)|^2 dt_j \qquad (1\le j\le k).$$

The following are our main results.

Theorem 3.1.1. *Let $s_1=\sigma_1+it_1,\ldots,s_k=\sigma_k+it_k\in\mathbb{C}$ with $k\in\mathbb{N}$, $k\ge 2$ and $T\ge 2$. Assume that when t_1 moves from 2 to T, the point $(s_1,\ldots,s_k)\in\mathbb{C}^k$ does not encounter the singularities of $\zeta_k(s_1,\ldots,s_k)$. In the case $\sigma_1+\cdots+\sigma_k>k$, we have*

$$I_k^{[1]}(T;\sigma_1,s_2,\ldots,s_k)=\zeta_k^{[1]}(2\sigma_1,s_2,\ldots,s_k)T+O(1),$$

where the implied constant depends on $\sigma_1,s_2,\ldots,s_k$ and $\zeta_k^{[1]}(2\sigma_1,s_2,\ldots,s_k)$ is a series which converges $\sigma_1+\cdots+\sigma_k>k-1/2$ (we define $\zeta_k^{[1]}(\sigma_1,s_2,\ldots,s_k)$ in the next section). In the case $k-1/2<\sigma_1+\cdots+\sigma_k\le k$, we have

$$I_k^{[1]}(T;\sigma_1,s_2,\ldots,s_k)=\zeta_k^{[1]}(2\sigma_1,s_2,\ldots,s_k)T+ \\ +\begin{cases} O(T^{2k-2(\sigma_1+\cdots+\sigma_k)}) & (k-1/2<\sigma_1+\cdots+\sigma_k<k), \\ O((\log T)^2) & (\sigma_1+\cdots+\sigma_k=k). \end{cases}$$

In the case $\sigma_1+\cdots+\sigma_k=k-1/2$, we have

$$I_k^{[1]}(T;\sigma_1,s_2,\ldots,s_k)=|F_k(s_2,\ldots,s_k)|^2 T\log T+O(T),$$

where

$$F_k(s_2,\ldots,s_k)=\prod_{i=0}^{k-2}\Big(\sum_{j=k-i}^{k} s_j-(i+1)\Big)^{-1}.$$

In the case $\sigma_1+\cdots+\sigma_k<k-1/2$, we have

$$I_k^{[1]}(T;\sigma_1,s_2,\ldots,s_k)\sim |F_k(s_2,\ldots,s_k)|^2\frac{(2\pi)^{2(\sigma_1+\cdots+\sigma_k)-2k+1}}{2k-2(\sigma_1+\cdots+\sigma_k)}\times \\ \times\zeta(2k-2(\sigma_1+\cdots+\sigma_k))T^{2k-2(\sigma_1+\cdots+\sigma_k)}.$$

In Theorem 1.1 in [10], Ikeda, Matsuoka and Nagata studied $I_2^{[1]}(T;\sigma_1,s_2)$ for $\sigma_1+\sigma_2 \geq 3/2$. Therefore our result includes an improvement of their result.

By Theorem 3.1.1 we may conjecture that the analytic properties of $\zeta_k(s_1,\dots,s_k)$ are special in the region $\sigma_1+\cdots+\sigma_k = k-1/2$. Matsumoto and Tsumura conjectured that $\sigma_1+\sigma_2 = 3/2$ might be the double analogue of the critical line of the Riemann zeta-function (see Remark 1.6 in [22]). In fact, Theorem 1.2 in [10] and the present Theorem 3.1.1 support this conjecture.

Remark 3.1.1. Theorem 3.1.1 can be proved by the methods similar to those of [10]. By using the methods similar to those of [10], one can probably obtain the asymptotic behaviors of $I_k^{[j]}(T;s_1,\dots,\sigma_j,\dots,s_k)$ for all j and k. In the case $j,k \geq 2$, we guess that the asymptotic behaviors of $I_k^{[j]}(T;s_1,\dots,\sigma_j,\dots,s_k)$ are different from those of $I_k^{[1]}(T;\sigma_1,s_2,\dots,s_k)$ and complicated.

3.2 Lemmas for the proof of the theorem

In this section, we collect some auxiliary results and definitions.

Definition 3.2.1. *Let $k \in \mathbb{N}$ and $N \in \mathbb{N}\cup\{0\}$. We define*

$$Z_k(s_1,\dots,s_k;N) = \sum_{n_1=N+1}^{\infty} \frac{1}{n_1^{s_1}} \sum_{n_2=n_1+1}^{\infty} \frac{1}{n_2^{s_2}} \cdots \sum_{n_k=n_{k-1}+1}^{\infty} \frac{1}{n_k^{s_k}}$$

for $\sigma_k > 1, \sigma_{k-1}+\sigma_k > 2, \dots, \sigma_1+\cdots+\sigma_k > k$.

Note that we have $Z_k(s_1,\dots,s_k;0) = \zeta_k(s_1,\dots,s_k)$,

$$Z_k(s_1,\dots,s_k;N) = \zeta_k(s_1,\dots,s_k) - \sum_{1\leq n_1\leq N} \frac{1}{n_1^{s_1}} \sum_{n_2=n_1+1}^{\infty} \frac{1}{n_2^{s_2}} \cdots \sum_{n_k=n_{k-1}+1}^{\infty} \frac{1}{n_k^{s_k}} \tag{3.2.1}$$

and, for $k \geq 2$,

$$\zeta_k(s_1,\dots,s_k) = \sum_{n_1=1}^{\infty} \frac{1}{n_1^{s_1}} Z_{k-1}(s_2,\dots,s_k;n_1). \tag{3.2.2}$$

The following lemma is a generalization of (4) in [1].

Lemma 3.2.1. *Let $k, l \in \mathbb{N}$ and $N \in \mathbb{N} \cup \{0\}$. The function $Z_k(s_1, \ldots, s_k; N)$ is continued meromorphically to $\mathbb{C}^k$ and satisfies*

$$
\begin{aligned}
&Z_{k+1}(s_1, \ldots, s_{k+1}; N) \\
&= \frac{Z_k(s_1, \ldots, s_{k-1}, s_k + s_{k+1} - 1; N)}{s_{k+1} - 1} - \frac{Z_k(s_1, \ldots, s_{k-1}, s_k + s_{k+1}; N)}{2} + \\
&+ \sum_{j=1}^{2l} \frac{B_{j+1}}{(j+1)!} (s_{k+1})_j Z_k(s_1, \ldots, s_{k-1}, s_k + s_{k+1} + j; N) + \\
&- \sum_{N < n_1 < \cdots < n_k} \frac{\phi_l(n_k, s_{k+1})}{n_1^{s_1} \cdots n_k^{s_k}}
\end{aligned}
$$

for $\sigma_{k+1} + 2l > 0, \sigma_k + \sigma_{k+1} + 2l > 1, \ldots, \sigma_1 + \cdots + \sigma_{k+1} + 2l > k$.

Proof. We prove the continuation by induction. Our method is similar to that of Akiyama, Egami and Tanigawa(see p. 109 in [1]). Since we have

$$
Z_1(s_1; N) = \zeta(s_1) - \sum_{n_1 \le N} \frac{1}{n_1^{s_1}},
$$

the function $Z_1(s_1; N)$ is meromorphic in $\mathbb{C}$. We assume that $Z_k(s_1, \ldots, s_k; N)$ $(k \ge 1)$ is continued meromorphically to $\mathbb{C}^k$. By Lemma 1.1.1 we have

$$
\begin{aligned}
&Z_{k+1}(s_1, \ldots, s_{k+1}; N) \\
&= \sum_{n_1 = N+1}^{\infty} \frac{1}{n_1^{s_1}} \cdots \sum_{n_k = n_{k-1}+1}^{\infty} \frac{1}{n_k^{s_k}} \Bigg(\frac{n_k^{1 - s_{k+1}}}{s_{k+1} - 1} - \frac{n_k^{-s_{k+1}}}{2} \\
&\qquad + \sum_{j=1}^{2l} \frac{B_{j+1}}{(j+1)!} (s_{k+1})_j n_k^{-(s_{k+1} + j)} - \phi_l(n_k, s_{k+1}) \Bigg) \\
&= \frac{Z_k(s_1, \ldots, s_{k-1}, s_k + s_{k+1} - 1; N)}{s_{k+1} - 1} - \frac{Z_k(s_1, \ldots, s_{k-1}, s_k + s_{k+1}; N)}{2} + \\
&\qquad + \sum_{j=1}^{2l} \frac{B_{j+1}}{(j+1)!} (s_{k+1})_j Z_k(s_1, \ldots, s_{k-1}, s_k + s_{k+1} + j; N) + \\
&\qquad - \sum_{N < n_1 < \cdots < n_k} \frac{\phi_l(n_k, s_{k+1})}{n_1^{s_1} \cdots n_k^{s_k}} \\
&=: A_1 + A_2 + A_3 + A_4,
\end{aligned}
$$

say. By the assumption A_1, A_2, A_3 are continued meromorphically to $\mathbb{C}^{k+1}$. The sum A_4 is absolutely convergent for $\sigma_{k+1} + 2l > 0, \sigma_k + \sigma_{k+1} + 2l > 1, \ldots, \sigma_1 + \cdots + \sigma_{k+1} + 2l > k$. Therefore, by taking sufficiently large l, the function $Z_{k+1}(s_1, \ldots, s_{k+1}; N)$ is continued meromorphically to $\mathbb{C}^{k+1}$. This implies the lemma. □

The following lemma is an analogue of Theorem 1 in [1].

Lemma 3.2.2. *Let $k, l \in \mathbb{N}$ and $N \in \mathbb{N} \cup \{0\}$. Let $p = (s_1, \ldots, s_k) \in \mathbb{C}^k$. The point p is a singularity of $Z_k(s_1, \ldots, s_k; N)$ if and only if the point p is a singularity of $\zeta_k(s_1, \ldots, s_k)$.*

Proof. We prove the lemma by induction. Our method is similar to that of Akiyama, Egami and Tanigawa(see p. 110 in [1]). In the case $k = 1$, the assertion of the lemma is obviously true. We assume that the assertion of the lemma is true for $k \geq 1$. We use the notation in the proof of Lemma 3.2.1. Note that, by Lemma 3.2.1, the singular part of $Z_{k+1}(s_1, \ldots s_{k+1}; N)$ is $A_1 + A_2 + A_3$ for all $N \in \mathbb{N} \cup \{0\}$. First we show that no singularities appeared in A_1, A_2 or A_3 identically vanish. This can be shown by changing variables:

$$u_1 = s_1, \ldots, u_{k-1} = s_{k-1}, u_k = s_k + s_{k+1}, u_{k+1} = s_{k+1}.$$

In fact, we see that the singular part of $Z_{k+1}(u_1, \ldots, u_k - u_{k+1}, u_{k+1}; N)$ is given by

$$\frac{Z_k(u_1, \ldots, u_k - 1; N)}{u_{k+1} - 1} + \sum_{j=0}^{2l} \frac{B_{j+1}}{(j+1)!}(u_{k+1})_j Z_k(u_1, \ldots, u_k + j; N). \quad (3.2.3)$$

By this expression we see that the singularities of $Z_k(u_1, \ldots, u_k + j; N)$ are summed with functions of u_{k+1} of different degree. Thus the singularities, as a weighted sum by another variable u_{k+1}, will not vanish identically. This implies the lemma, because, in (3.2.3), the singularities of $Z_k(u_1, \ldots, u_k + j; N)$ coincide with $\zeta_k(u_1, \ldots, u_k + j)$ by the assumption of induction. □

Remark 3.2.1. By Lemma 3.2.2 and Theorem 1 in [1], we see that the function $Z_k(s_1, \ldots, s_k; N)$ has singularities on

$$s_k = 1, \qquad s_{k-1} + s_k = 2, 1, 0, -2, -4, \ldots,$$

and

$$\sum_{i=1}^{j} s_{k-i+1} \in \mathbb{Z}_{\le j} \quad (j = 3, 4, \ldots, k),$$

where $\mathbb{Z}_{\le j}$ is the set of integers less than or equal to j. In addition, if $Z_k(s_1, \ldots, s_k; N)$ is holomorphic at the point $(s_1, \ldots, s_k)$, then $Z_{k-1}(s_2, \ldots, s_k; N)$ is holomorphic at $(s_2, \ldots, s_k)$.

The following lemmas can be obtained by Lemma 3.2.1 and induction.

Lemma 3.2.3. *Let $k, N \in \mathbb{N}$. For fixed $s_1, \ldots, s_k \in \mathbb{C}$, we have*

$$Z_k(s_1, \ldots, s_k; N) = \prod_{i=0}^{k-1} \left(\sum_{j=k-i}^{k} s_j - (i+1) \right)^{-1} N^{k-(s_1+\cdots+s_k)} + O(N^{k-1-(\sigma_1+\cdots+\sigma_k)}),$$

where the implied constant depends on $s_1, \ldots, s_k \in \mathbb{C}$.

Proof. We prove the lemma by induction. First we consider the case $k = 1$. Let $l \in \mathbb{N}$ with $\sigma_1 > -2l$. In order to obtain the analytic continuation of $Z_1(s_1; N)$, we regard s_1 as a complex variable and assume $\sigma_1 > 1$ temporarily. Then, by Lemma 1.1.1, we obtain the analytic continuation of $Z_k(s_1; N)$ for the region $\sigma_1 > -2l$. If we regard s_1 as a constant, then, by Lemma 1.1.1, we have

$$Z_1(s_1; N) = \frac{N^{1-s_1}}{s_1 - 1} + O(N^{-\sigma_1}).$$

This implies the lemma for $k = 1$.

We assume that the assertion of the lemma is true for $k \ge 1$. Let $l \in \mathbb{N}$ with $\sigma_{k+1} + 2l > 0, \sigma_k + \sigma_{k+1} + 2l > 1, \ldots, \sigma_1 + \cdots + \sigma_{k+1} + 2l > k$. In order to obtain the analytic continuation of $Z_{k+1}(s_1, \ldots, s_{k+1}; N)$, we regard $s_1, \ldots, s_{k+1}$ as complex variables and assume $\sigma_1, \ldots, \sigma_{k+1} > 1$ temporarily. Then, by Lemma 3.2.1, we obtain the analytic continuation of $Z_{k+1}(s_1, \ldots, s_{k+1}; N)$ for the region $\sigma_{k+1} + 2l > 0, \sigma_k + \sigma_{k+1} + 2l > 1, \ldots, \sigma_1 + \cdots + \sigma_{k+1} + 2l > k$. If we regard $s_1, \ldots, s_{k+1}$ as constants, then, by Lemma 3.2.1 and the assumption of induction, we obtain the lemma. $\square$

Lemma 3.2.4. *Let $k \in \mathbb{N}$ with $k \ge 2$. Let $\sigma_1 + \cdots + \sigma_k \le k - 1/2$. For fixed $\sigma_1, s_2, \ldots, s_k$, we have*

$$\zeta_k(s_1, \ldots, s_k) = F_k(s_2, \ldots, s_k)\zeta(s_1+\cdots+s_k-k+1) + O(\zeta(s_1+\cdots+s_k-k+2)),$$

where the implied constant does not depend on t_1.

Proof. This lemma can be obtained by the method similar to that of Lemma 3.2.3. □

The following lemma can be obtained by Lemma 2.2.1, Lemma 3.2.1 and the argument similar to that of Lemma 3.2.3. This lemma is a generalization of Lemma 3.1 in [10].

Lemma 3.2.5. *Let $N \in \mathbb{N}$ and $k \in \mathbb{N}$ with $k \geq 2$. Let $C > 1$ be a given constant. Assume that the point $(s_1, \dots, s_k) \in \mathbb{C}^k$ does not encounter the singularities of $\zeta_k(s_1, \dots, s_k)$. If $1 < |t_1 + \dots + t_k| < 2\pi N/C$ and $\sigma_1 + \dots + \sigma_k > k-1$ hold for fixed $\sigma_1 \in \mathbb{R}$ and $s_2, \dots, s_k \in \mathbb{C}$, then we have*

$$\zeta_k(s_1, \dots, s_k) = \sum_{n_1 \leq N} \frac{1}{n_1^{s_1}} Z_{k-1}(s_2, \dots, s_k; n_1) + O(t_1^{-1} N^{k-(\sigma_1 + \dots + \sigma_k)}).$$

We define

$$\zeta_k^{[1]}(\sigma_1, s_2, \dots, s_k) = \sum_{n_1=1}^{\infty} \frac{1}{n_1^{\sigma_1}} |Z_{k-1}(s_2, \dots, s_k; n_1)|^2,$$

where the point $(s_2, \dots, s_k) \in \mathbb{C}^{k-1}$ is not a singularity of $Z_{k-1}(s_2, \dots, s_k; n_1)$. By Lemma 3.2.3, the series $\zeta_k^{[1]}(2\sigma_1, s_2, \dots, s_k)$ is absolutely convergent in the region $\sigma_1 + \dots + \sigma_k > k - 1/2$.

3.3 Proof of Theorem 3.1.1

In this section, we regard $\sigma_1, s_2, \dots, s_k$ as constants and we define

$$a_{n_1} = n_1^{-\sigma_1} Z_{k-1}(s_2, \dots, s_k; n_1).$$

Note that

$$\sum_{n_1=1}^{\infty} |a_{n_1}|^2 = \zeta_k^{[1]}(2\sigma_1, s_2, \dots, s_k).$$

We divide the proof into three cases.

Proof of Theorem 3.1.1 for the case $\sigma_1 + \dots + \sigma_k > k$. By (3.2.2) we have

$$\zeta_k(s_1, \dots, s_k) = \sum_{n_1=1}^{\infty} a_{n_1} n_1^{-it_1}.$$

By Lemma 3.2.3 the last series converges absolutely in $\sigma_1 + \cdots + \sigma_k > k$. Since the series

$$\sum_{n_1=1}^{\infty} n_1 |a_{n_1}|^2 = \sum_{n_1=1}^{\infty} \frac{1}{n_1^{2\sigma_1 - 1}} |Z_{k-1}(s_2, \ldots, s_k; n_1)|^2$$

converges by Lemma 3.2.3, we have

$$I_k^{[1]}(T; \sigma_1, s_2, \ldots, s_k) = \zeta_k^{[1]}(2\sigma_1, s_2, \ldots, s_k) T + O(1)$$

by Lemma 1.1.3. □

Proof of Theorem 3.1.1 for the case $k - 1/2 \le \sigma_1 + \cdots + \sigma_k \le k$. Let

$$m_0 = \max\{m \in \mathbb{N} \mid T/2^m > |t_2 + \cdots + t_k| + 1\}.$$

Note that

$$m_0 < \frac{\log T - \log(|t_2 + \cdots + t_k| + 1)}{\log 2} \le m_0 + 1$$

holds. We take $T \ge 2$ and $N \in \mathbb{N}$ with $|t_2 + \cdots + t_k| + 1 < T$ and $3T < 2\pi N / C$, where $C > 1$, and we assume $T < t_1 < 2T$. Then we have

$$1 < t_1 - |t_2 + \cdots + t_k| < |t_1| + |t_2 + \cdots + t_k| < 3T < \frac{2\pi N}{C}.$$

Therefore we can use Lemma 3.2.5. We have

$$\zeta_k(s_1, \ldots, s_k) = \sum_{n_1=1}^{N} a_{n_1} n_1^{-it_1} + O(t_1^{-1} N^{k - \sigma_1 - \cdots - \sigma_k}) =: I_1 + I_2,$$

say. Since $a_{n_1} \ll n_1^{k-1-\sigma_1-\cdots-\sigma_k}$ by Lemma 3.2.3, we obtain

$$\sum_{n_1=1}^{N} n_1 |a_{n_1}|^2 \ll \sum_{n_1=1}^{N} n_1^{2k-1-2(\sigma_1+\cdots+\sigma_k)} \ll \begin{cases} \log N & (\sigma_1 + \cdots + \sigma_k = k) \\ N^{2k-2(\sigma_1+\cdots+\sigma_k)} & (\sigma_1 + \cdots + \sigma_k < k) \end{cases}$$

and

$$I_1 \ll \sum_{n_1=1}^{N} |a_{n_1}| \ll \begin{cases} \log N & (\sigma_1 + \cdots + \sigma_k = k) \\ N^{k-(\sigma_1+\cdots+\sigma_k)} & (\sigma_1 + \cdots + \sigma_k < k). \end{cases}$$

Therefore we have

$$\int_T^{2T} |I_1|^2 dt_1 = T \sum_{n_1=1}^{N} |a_{n_1}|^2 + \begin{cases} O(\log N) & (\sigma_1 + \cdots + \sigma_k = k) \\ O(N^{2k-2(\sigma_1+\cdots+\sigma_k)}) & (\sigma_1 + \cdots + \sigma_k < k) \end{cases}$$

by Lemma 1.1.3 and

$$\int_T^{2T} |I_1 I_2| dt_1 \ll \begin{cases} \log N & (\sigma_1 + \cdots + \sigma_k = k) \\ N^{2k-2(\sigma_1+\cdots+\sigma_k)} & (\sigma_1 + \cdots + \sigma_k < k). \end{cases}$$

On the other hand, we have

$$\int_T^{2T} |I_2|^2 dt_1 \ll N^{2k-2(\sigma_1+\cdots+\sigma_k)} \int_T^{2T} \frac{dt_1}{t_1^2} \ll T^{-1} N^{2k-2(\sigma_1+\cdots+\sigma_k)}.$$

Therefore we have

$$\int_T^{2T} |\zeta_k(s_1,\ldots,s_k)|^2 dt_1 = T \sum_{n_1=1}^{N} |a_{n_1}|^2 + \begin{cases} O(\log N) & (\sigma_1 + \cdots + \sigma_k = k) \\ O(N^{2k-2(\sigma_1+\cdots+\sigma_k)}) & (\sigma_1 + \cdots + \sigma_k < k). \end{cases}$$

By setting $N = [T] + 1$, we obtain

$$\int_T^{2T} |\zeta_k(s_1,\ldots,s_k)|^2 dt_1 = T \sum_{n_1 \le T} |a_{n_1}|^2 + \begin{cases} O(\log T) & (\sigma_1 + \cdots + \sigma_k = k) \\ O(T^{2k-2(\sigma_1+\cdots+\sigma_k)}) & (\sigma_1 + \cdots + \sigma_k < k). \end{cases} \tag{3.3.1}$$

Therefore, in the case $k - 1/2 < \sigma_1 + \cdots + \sigma_k$, we have

$$\int_T^{2T} |\zeta_k(s_1,\ldots,s_k)|^2 dt_1 = \zeta_k^{[1]}(2\sigma_1, s_2, \ldots, s_k) T + \begin{cases} O(\log T) & (\sigma_1 + \cdots + \sigma_k = k) \\ O(T^{2k-2(\sigma_1+\cdots+\sigma_k)}) & (\sigma_1 + \cdots + \sigma_k < k). \end{cases}$$

Note that $1/2^{m_0} = O(1/T)$. By

$$\begin{aligned} \int_{|t_2+\cdots+t_k|+1}^{T} |\zeta_k(s_1,\ldots,s_k)|^2 dt_1 &= \int_{T/2^{m_0}}^{T} |\zeta_k(s_1,\ldots,s_k)|^2 dt_1 + O(1) \\ &= \sum_{1 \le k \le m_0} \int_{T/2^k}^{T/2^{k-1}} |\zeta_k(s_1,\ldots,s_k)|^2 dt_1 + O(1), \end{aligned}$$

and Remark 2.2.1, we obtain the theorem for $k-1/2<\sigma_1+\cdots+\sigma_k\le k$.

In the case $\sigma_1+\cdots+\sigma_k=k-1/2$, since

$$a_{n_1}=F_k(s_2,\ldots,s_k)n_1^{k-1-\sigma_1-s_2-\cdots-s_k}+O(n_1^{k-2-\sigma_1-\sigma_2-\cdots-\sigma_k})$$

by Lemma 3.2.3, we have

$$|a_{n_1}|^2=|F_k(s_2,\ldots,s_k)|^2n_1^{-1}+O(n_1^{-2}).$$

Therefore, by (3.3.1), we obtain

$$\int_T^{2T}|\zeta_k(s_1,\ldots,s_k)|^2dt_1=|F_k(s_2,\ldots,s_k)|^2T\log T+O(T).$$

This implies the theorem for $\sigma_1+\cdots+\sigma_k=k-1/2$. □

Proof of Theorem 3.1.1 for the case $\sigma_1+\cdots+\sigma_k<k-1/2$. By (3.1.1), Lemma 3.2.4 and the Cauchy-Schwarz inequality we can easily obtain the theorem. □

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