

New formulation of wormhole (ワームホールの新しい定式化)

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Abstract of original part

Wormhole is one of the non-trivial spacetimes. The wormhole is often discussed in the context of general relativity. It has so-called throat structure, and exotic matter is required to keep its non-trivial structure. However, there are no definitions for the wormhole that can cover wide class of spacetimes. This is because the concept of wormhole is quasi-local. Therefore, we formulate the wormhole based on null geodesic congruences and show some general feature on wormholes. And we examine some examples based on our formulation and then we confirm that it can work for wide class of spacetimes. We also present a new wormhole spacetime without exotic matter in a certain high-dimensional model.

Chapter 1

Introduction

Wormhole is one of the non-trivial spacetimes that is often discussed in the context of general relativity [1, 2, 3, 4, 5]. Wormhole is naively supposed to have a non-trivial space structure like Fig. 1.1.

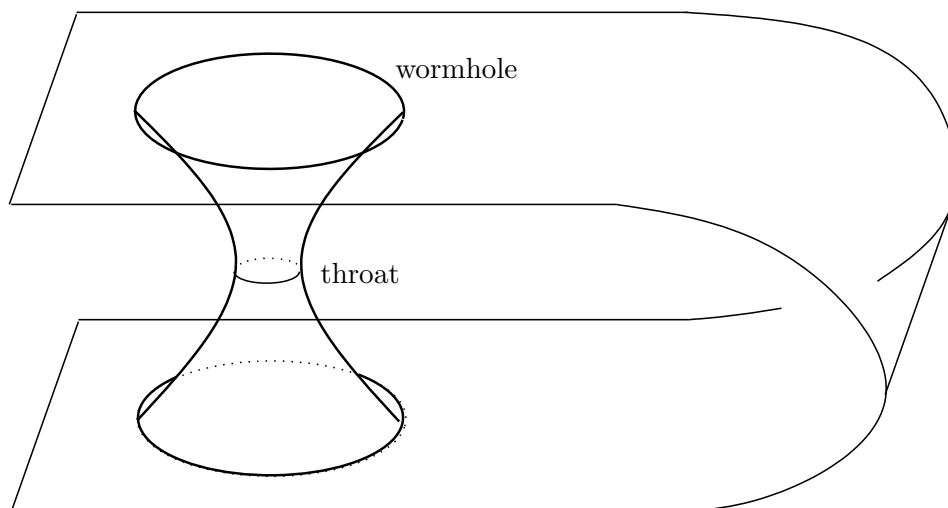


Figure 1.1: An image of wormhole. Wormhole has a throat and flares out from the throat.

Moreover, if we can traverse the wormhole, it is possible to travel with a high speed near or over the velocity of light seen from a distant observer [2]. However, there is a problem, that is, exotic matter is usually required to keep the wormhole structure [4].

The first study about wormhole is by Flamm in 1916 [6]. Just before that, the first exact solution of the vacuum Einstein equation, the Schwarzschild solution, had been discovered [7]. Flamm considered its wormhole-like structure in the Schwarzschild solution [6]. The word of *wormhole* was intro-

duced by Misner and Wheeler in 1957 [8]. In 1988, Morris and Thorne proposed a definition of wormhole as the structure satisfying the following conditions; (i) it has no event horizon and (ii) it has a *throat* which connects two regions and satisfies *flare-out* condition [2]. The wormhole considered by Morris and Thorne is static and spherically symmetric one. It is called the Morris-Thorne wormhole. Thereafter, various wormholes were considered; for example, with rotation [9, 10], cosmological constant [11], plane symmetric case [12], cylindrical symmetric cases [13, 14], dynamical cases [15, 16, 17, 18, 19, 20, 21] and in lower/higher dimensional cases [22, 23, 24, 25]. However, the issue on the introduction of exotic matters has been solved neither in static wormhole [2, 3, 26, 27, 28, 29, 30, 31] nor in dynamical case [18, 19, 20, 32]. There are some exceptional cases which have initial singularity (cosmological wormhole) [21].

There are three typical definitions of wormhole throat, Morris and Thorne [2], Hochberg and Visser [18, 19], and Maeda, Harada and Carr [21]. Morris and Thorne considered the static and spherically symmetric spacetime. They embedded the static slices to the Euclid space and defined the throat as the minimal surface on the hypersurface. Under this definition, we can see that some exotic matters are needed at least at the throat [2, 3, 28, 29, 30]. In order to consider dynamical and non-spherically symmetric case, Hochberg and Visser proposed a new definition of the throat by considering the null hypersurface [18, 19]. They defined the throat as the surface whose area is minimal on the null hypersurfaces. We can also see that some exotic matters are needed for existence of wormhole [18, 19, 26].

It was thought that Hochberg-Visser's definition can work for a wide class of wormhole spacetimes. But, an exception, called cosmological wormhole, was found by Maeda, Harada and Carr [21]. This wormhole in Maeda, Harada and Carr's sense is constructed without any exotic matters, but there is an initial singularity. They considered the spherically symmetric case and defined the throat as the surface whose area radius is minimal on spacelike hypersurfaces [21]. Since the above cosmological wormhole does not have a minimal surface on null hypersurfaces, it is not wormhole in Hochberg and Visser's sense.

However, it is hard to show mathematical properties of wormhole throat if one adopts these definitions. And, in Maeda-Harada-Carr's throat definition, spacetimes known to be non-wormhole may be categorized into wormhole. Then, we proposed a new definition of throat and wormhole by using two expansion rates of outgoing/ingoing null geodesic congruence [33]. Therein, the throat is defined as the surface where the difference between above two expansion rates is zero. Here, we note that our formulation does not depend on the choice of time or null slices unlike former three definitions.

In higher-dimensional theory, wormhole may be constructed without any exotic matters. Actually, we found such wormhole in the Dvali-Gabadadze-

Porrati (DGP) braneworld model [34]. Here, the DGP braneworld model is one of higher-dimensional cosmological models to explain accelerating expansion of the Universe without cosmological constant [35, 36]. This wormhole is realized on a vacuum brane in 5-dimensional vacuum spacetime and the 4-dimensional Einstein equation does not hold on the brane [34].

The remaining part of this thesis is as follows. In the next chapter, we will provide short review of the general relativity and wormhole. Therein we will also see that there are no definitions which can apply for a wide class of wormhole spacetimes. In Chap. 3, we will introduce a new definition of the wormhole throat. Then we will show some features of wormhole and confirm that our proposal is applicable at least for spherically symmetric cases. This part is based on my original work [33]. In Chap. 4, we will discuss the spacetime structure on the DGP brane constructed in Ref. [36] and confirm that it is a wormhole. This part is based on my original works [33, 34]. In Chap. 5, we will give a summary.

If we do not specify, we use the geometrized unit system, where the speed of light is $c = 1$ and the gravitational constant is $G = 1$ [37].

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Chapter 2

General Relativity and Wormhole

2.1 General relativity

General relativity is based on two principles, general covariance and equivalence principle. The general covariance means that physical phenomenon does not depend on coordinate systems. The second one requires that there is a local coordinate system where gravitational force vanishes. Mathematically, the general relativity is formulated in terms of the Riemann geometry.

We basically follow Ref. [38] for the notations.

2.1.1 Einstein equation, Geodesic congruence and Trapped surface

In this section, we introduce the Einstein equation and some geometrical basics for the definition of wormhole. We consider n -dimensional Lorentzian manifold. The signature of metric is $(-, +, +, +, \dots)$. In general relativity, the metric is determined by the Einstein equation

$$G_{\mu\nu} = 8\pi T_{\mu\nu}, \quad (2.1)$$

where $G_{\mu\nu} := R_{\mu\nu} - (1/2)g_{\mu\nu}R$ is the Einstein tensor, $g_{\mu\nu}$ is the metric tensor, $R_{\mu\nu} := R^\rho_{\mu\rho\nu}$ is the Ricci tensor and $R := R^\mu_\mu$ is the Ricci scalar. $T_{\mu\nu}$ is the stress-energy tensor for matter. $R_{\mu\nu\rho}{}^\sigma := \partial_\nu \Gamma^\sigma_{\mu\rho} - \partial_\mu \Gamma^\sigma_{\nu\rho} + \Gamma^\lambda_{\mu\rho} \Gamma^\sigma_{\lambda\nu} - \Gamma^\lambda_{\nu\rho} \Gamma^\sigma_{\lambda\mu}$ is the Riemann curvature tensor and $\Gamma^\mu_{\nu\rho} := (1/2)g^{\mu\sigma}(\partial_\nu g_{\sigma\rho} + \partial_\rho g_{\nu\sigma} - \partial_\sigma g_{\nu\rho})$ is the Christoffel symbols. Here, we employed the *Einstein convention*, that is, we take the summation if the same tensor indices appear.

Since we will use the geodesic congruence for the definition of worm-hole throat, we introduce the basics for that. Let us define the tensor $B_{ab} := \nabla_b \xi_a$, where ξ^a is the tangent vector of geodesic. Here Latin indices $\{a, b, c, \dots\}$ stand for Wald's abstract index [38]. They indicate the type of tensor, not the component.

When ξ^a is *timelike*, we can derive the Raychaudhuri equation as

$$\xi^c \nabla_c \theta = -\frac{1}{3}\theta^2 - \sigma_{ab}\sigma^{ab} + \omega_{ab}\omega^{ab} - R_{ab}\xi^a\xi^b, \quad (2.2)$$

where $\theta := B_{ab}q^{ab}$ is the expansion rate, $\sigma_{ab} := B_{(ab)} - (1/3)\theta q_{ab}$ is the shear and $\omega_{ab} := B_{[ab]}$ is the twist. And, $q_{ab} := g_{ab} + \xi_a\xi_b$, $B_{(ab)} := (1/2)(B_{ab} + B_{ba})$ and $B_{[ab]} := (1/2)(B_{ab} - B_{ba})$. Note that, if the vector ξ^a is orthogonal to hypersurface, we can take $\omega_{ab} = 0$ from Frobenius's theorem [38].

When ξ^a is *null*, we have

$$\xi^c \nabla_c \theta = -\frac{1}{2}\theta^2 - \sigma_{ab}\sigma^{ab} + \omega_{ab}\omega^{ab} - R_{ab}\xi^a\xi^b, \quad (2.3)$$

where $\theta := B_{ab}h^{ab}$, $\sigma_{ab} := B_{(cd)}h_a^c h_b^d - (1/2)\theta h_{ab}$, $\omega_{ab} := B_{[cd]}h_a^c h_b^d$, $h_{ab} := g_{ab} + \xi_a\eta_b + \eta_a\xi_b$ and η^a is a null normal vector to ξ^a satisfying $\xi_a\eta^a = -1$. Note that, if the vector ξ^a is orthogonal to surface, we can take $\omega_{ab} = 0$ again. Note that the Einstein equation is not used for the derivation of the Raychaudhuri equation.

Next, we introduce the trapped surface [21, 39]. Let us consider two future directed outgoing/ingoing null vectors ξ_\pm^a normal to a compact spacelike surface S . Then, the surface S satisfying

$$\theta_+\theta_-|_S > 0 \quad (2.4)$$

is called the trapped surface, where θ_\pm are null expansion rates corresponding to ξ_\pm^a , respectively. In particular, it is called the future trapped surface when $\theta_\pm|_S < 0$, the past trapped surface when $\theta_\pm|_S > 0$.

And if S satisfies

$$\theta_+\theta_-|_S < 0, \quad (2.5)$$

it is called the untrapped surface. If

$$\theta_+\theta_-|_S = 0 \quad (2.6)$$

is satisfied, S is called the marginal surface. The collection of the hypersurface foliated by marginal surface is called the bifurcating trapping horizon.

2.1.2 Energy condition

Usually classical matters satisfy so called energy conditions; roughly speaking, energy density is positive or the magnitude of pressure is smaller than that of energy density. Here, we give the definition of energy conditions [4, 38].

(i) The *null energy condition* requires that $T_{ab}\xi^a\xi^b \geq 0$ holds for any null vector ξ^a .

When the Einstein equation holds, the null energy condition is equivalent with $R_{ab}\xi^a\xi^b \geq 0$.

(ii) The *weak energy condition* requires that $T_{ab}\xi^a\xi^b \geq 0$ holds for any timelike vector ξ^a .

(iii) The *strong energy condition* requires that $(T_{ab} - (1/2)Tg_{ab})\xi^a\xi^b \geq 0$ holds for any timelike vector ξ^a , where $T = T^a_a$.

When the Einstein equation holds, the strong energy condition is equivalent with $R_{ab}\xi^a\xi^b \geq 0$.

(iv) The *dominant energy condition* requires that $-T^a_b\xi^b$ is a future directed timelike or null vector for any future directed timelike vector ξ^a . This means that the speed of matters is always less than the speed of light [38].

When the stress-energy tensor is

$$T^a_b = \text{diag}[-\rho, p_1, p_2, p_3], \quad (2.7)$$

the energy conditions can be written as follows:

$$\text{null energy condition : } \rho + p_i \geq 0, \quad (2.8)$$

$$\text{weak energy condition : } \rho \geq 0, \quad \rho + p_i \geq 0, \quad (2.9)$$

$$\text{strong energy condition : } \rho + \sum_{i=1}^3 p_i \geq 0, \quad \rho + p_i \geq 0, \quad (2.10)$$

$$\text{dominant energy condition : } \rho \geq |p_i|, \quad (2.11)$$

where $i = 1, 2, 3$. ρ and p_i are usually interpreted as the energy density and pressures, respectively. Note that, the null energy condition is always satisfied if the weak or strong energy condition is satisfied. But the weak energy condition is not necessarily satisfied even if strong energy condition is satisfied. And the null and weak energy condition are always satisfied if the dominant energy condition is satisfied.

2.1.3 Black hole

Black hole is a strong gravity region that even light can not escape from that. The simplest exact black hole solution of the Einstein equation is the Schwarzschild spacetime [7]. It is known that the spherically symmetric and vacuum solution of the Einstein equation is only the Schwarzschild solution (Birkhoff's theorem) [38].

To see the global structure of spacetimes, one often uses the Penrose diagram [37]. Therein, the spacetime we consider is conformally embedded into a compact region in another artificial spacetime. Under the conformal transformation, light cone is not changed and then the causal structure is preserved. As a simple example, we first consider the Minkowski spacetime.

The metric of the Minkowski spacetime is

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega_2^2, \quad (2.12)$$

where $d\Omega_2^2 := d\theta^2 + \sin^2 \theta d\varphi^2$. We first perform the coordinate transformation from (t, r) to (η, ξ) through

$$t + r = \tan \frac{\eta + \xi}{2}, \quad t - r = \tan \frac{\eta - \xi}{2}, \quad (2.13)$$

where $-\pi \leq \eta \pm \xi \leq \pi$. The metric becomes

$$ds^2 = \left\{ 4 \cos^2 \left(\frac{\eta + \xi}{2} \right) \cos^2 \left(\frac{\eta - \xi}{2} \right) \right\}^{-1} (-d\eta^2 + d\xi^2 + \sin^2 \xi d\Omega_2^2). \quad (2.14)$$

Now we consider the conformal transformation

$$d\tilde{s}^2 = \Omega^2 ds^2 = -d\eta^2 + d\xi^2 + \sin^2 \xi d\Omega_2^2, \quad (2.15)$$

where $\Omega := 2 \cos(\frac{\eta+\xi}{2}) \cos(\frac{\eta-\xi}{2})$. The conformally transformed spacetime is the Einstein static universe. Then we see that the Minkowski spacetime is conformally embedded into a compact region of the Einstein static universe because of $-\pi \leq \eta \pm \xi \leq \pi$. Focusing on the (η, ξ) part, we can have Fig. 2.1. The boundaries correspond to infinities. The boundary of $r \rightarrow \infty$ and $r - t < \infty$ is called the future null infinity \mathcal{J}^+ and that of $r \rightarrow \infty$ and $r + t < \infty$ is called the past null infinity \mathcal{J}^- .

Next, we consider the Schwarzschild spacetime. The metric of the Schwarzschild spacetime is given by

$$ds^2 = -\left(1 - \frac{r_g}{r}\right) dt^2 + \left(1 - \frac{r_g}{r}\right)^{-1} dr^2 + r^2 d\Omega_2^2, \quad (2.16)$$

where $r_g := 2M$ is the Schwarzschild radius and M is the mass. Let us introduce the Kruskal coordinate (T, X) defined by

$$X^2 - T^2 = \left(\frac{r}{r_g} - 1\right) e^{r/r_g} \quad (2.17)$$

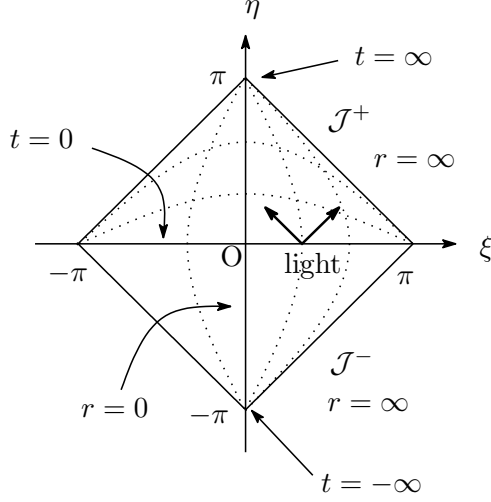


Figure 2.1: The Penrose diagram of the Minkowski spacetime.

and

$$\begin{cases} \frac{T}{X} = \tanh\left(\frac{t}{2r_g}\right) & \text{for } r > r_g, \\ \frac{X}{T} = \tanh\left(\frac{t}{2r_g}\right) & \text{for } 0 < r < r_g. \end{cases} \quad (2.18)$$

The metric (2.16) becomes

$$ds^2 = \frac{4r_g^3}{r} e^{-r/r_g} (-dT^2 + dX^2) + r^2 d\Omega_2^2. \quad (2.19)$$

Now, we consider the coordinate transformation from (T, X) to a new one (η, ξ) through

$$T + X = \tan\left(\frac{\eta + \xi}{2}\right), \quad T - X = \tan\left(\frac{\eta - \xi}{2}\right), \quad (2.20)$$

where $-\pi \leq \eta \pm \xi \leq \pi$. Then, the metric becomes

$$ds^2 = \frac{4r_g^3}{r} e^{-r/r_g} \left\{ 4 \cos^2\left(\frac{\eta + \xi}{2}\right) \cos^2\left(\frac{\eta - \xi}{2}\right) \right\}^{-1} (-d\eta^2 + d\xi^2) + r^2(\eta, \xi) d\Omega_2^2. \quad (2.21)$$

Now we consider the conformal transformation given by

$$d\tilde{s}^2 = \Omega^2 ds^2 = -d\eta^2 + d\xi^2 + R^2(\eta, \xi) d\Omega_2^2, \quad (2.22)$$

where $\Omega := (r/4r_g^3)^{1/2} e^{r/(2r_g)} \cdot 2 \cos(\frac{\eta+\xi}{2}) \cos(\frac{\eta-\xi}{2})$ and $R = \Omega r$. Then we see that the Schwarzschild spacetime is conformally embedded into a compact

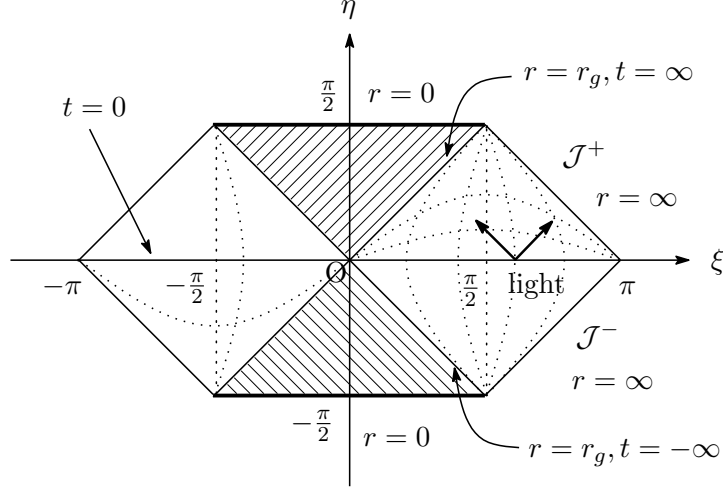


Figure 2.2: The Penrose diagram of the Schwarzschild spacetime. \mathcal{J}^+ is the future null infinity and \mathcal{J}^- is the past null infinity. The bold lines show the curvature singularity. The upper slashed region is the black hole and lower is the white hole. Their boundaries, $r = r_g$ lines, are event horizons.

region because of $-\pi \leq \eta \pm \xi \leq \pi$. Focusing on the (η, ξ) part, we can have Fig. 2.2. In particular, at $r = r_g$, we can see $\eta \pm \xi = 0$, and then $t = \pm\infty$. And at $r = 0$, we can see $\eta = \pm\pi/2$ and $-\pi/2 < \xi < \pi/2$.

The region reached by the future (past) directed timelike/null curve starting from a point p in the spacetime is called causal future (past) of the point p . This region is denoted by $J^+(p)$ ($J^-(p)$) [38]. When \mathcal{M} is asymptotically flat spacetime, the *black hole* is defined as the region of $\mathcal{M} - J^-(\mathcal{J}^+) \subset \mathcal{M}$, where \mathcal{J}^+ is the future null infinity [38]. And, its boundary is called (future) *event horizon*.

2.1.4 Cosmological solution

Since one may be interested in cosmological wormhole, we briefly review cosmological solutions as basics. Now we introduce the cosmological constant term, Λ , in the Einstein equation,

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}. \quad (2.23)$$

Let us consider the homogeneous and isotropic universe [40]. The metric is given by

$$ds^2 = -dt^2 + a^2(t) \left(\frac{1}{1 - Kr^2} dr^2 + r^2 d\Omega_2^2 \right), \quad (2.24)$$

where K is a constant which is positive/zero/negative depending on the spatial topology (closed/flat/open). We assume that the matter is perfect fluid and the stress-energy tensor is given by

$$T_{ab} = \rho u_a u_b + p(g_{ab} + u_a u_b), \quad (2.25)$$

where u^a is the 4-velocity of fluid [38]. For the metric (2.24), the Einstein equation implies two equations;

$$3 \left(\frac{\dot{a}}{a} \right)^2 + \frac{3K}{a^2} - \Lambda = 8\pi\rho \quad (2.26)$$

and

$$2 \frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a} \right)^2 + \frac{K}{a^2} - \Lambda = -8\pi p, \quad (2.27)$$

where $\dot{a} = da(t)/dt$. They are arranged as

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3}(\rho + 3p) + \frac{\Lambda}{3} \quad (2.28)$$

and

$$\left(\frac{\dot{a}}{a} \right)^2 = \frac{8\pi}{3}\rho - \frac{K}{a^2} + \frac{\Lambda}{3}. \quad (2.29)$$

Eq. (2.29) is so called the Friedmann equation.

If we define

$$\hat{\rho} := \rho + \frac{\Lambda}{8\pi}, \quad \hat{p} := p - \frac{\Lambda}{8\pi}, \quad (2.30)$$

we can regard that the cosmological constant is included in the stress-energy.

When $T_{\mu\nu} = 0$, the Einstein equation (2.23) becomes

$$R_{\mu\nu} = \Lambda g_{\mu\nu} \quad (2.31)$$

and then Eqs. (2.28) and (2.29) become

$$3 \frac{\ddot{a}}{a} = \Lambda, \quad (2.32)$$

$$\frac{\ddot{a}}{a} + 2 \left(\frac{\dot{a}}{a} \right)^2 + \frac{2K}{a^2} = \Lambda. \quad (2.33)$$

For $\Lambda > 0$, the solution is given by

$$ds^2 = \begin{cases} -dt^2 + H^{-2} \cosh^2(Ht)(d\chi^2 + \sin^2 \chi d\Omega_2^2), & (K = 1), \\ -dt^2 + H^{-2} e^{2Ht}(dr^2 + r^2 d\Omega_2^2), & (K = 0), \\ -dt^2 + H^{-2} \sinh^2(Ht)(d\bar{\chi}^2 + \sinh^2 \bar{\chi} d\Omega_2^2), & (K = -1), \end{cases} \quad (2.34)$$

$$(2.35)$$

$$(2.36)$$

where $H = \sqrt{\Lambda/3}$. These metrics represent the de Sitter spacetime. Indeed, the de Sitter spacetime is the hyperboloid of

$$-x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 = H^{-2} \quad (2.37)$$

in the 5-dimensional Minkowski spacetime [41]

$$ds^2 = -dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2. \quad (2.38)$$

When we choose appropriate coordinates, we can obtain the metrics (2.34), (2.35) and (2.36).

Furthermore, if we take the coordinate system (t, r, θ, φ) satisfying

$$x_0 = \sqrt{H^{-2} - r^2} \sinh(Ht), \quad (2.39)$$

$$x_1 = r \sin \theta \sin \varphi, \quad (2.40)$$

$$x_2 = r \sin \theta \cos \varphi, \quad (2.41)$$

$$x_3 = r \cos \theta, \quad (2.42)$$

$$x_4 = \sqrt{H^{-2} - r^2} \cosh(Ht), \quad (2.43)$$

we can write the de Sitter metric as

$$ds^2 = -(1 - H^2 r^2) dt^2 + (1 - H^2 r^2)^{-1} dr^2 + r^2 d\Omega_2^2. \quad (2.44)$$

The Killing vector ∂_t is null on $r = H^{-1}$, timelike in the region $0 < r < H^{-1}$ and spacelike in the region $r > H^{-1}$. $r = H^{-1}$ is the cosmological horizon and the spacetime is static for $0 < r < H^{-1}$. The Penrose diagram of this spacetime is drawn as Fig. 2.3.

2.2 Wormhole

There are three typical definitions of wormhole throat. In this section, we consider 4-dimensional spacetime, but it is easy to extend the arguments to higher-dimensional spacetime.

2.2.1 Throat

The wormhole is usually defined to satisfy the following; (i) no event horizon and (ii) having the *throat* which satisfies the *flare-out* condition. Here, the *throat* and the *flare-out* are characteristic features of wormholes. The former is the boundary to connect two regions, and the latter is the condition to decide the "shape" for spreading to outside [2]. Note that, because of the item (i), the Einstein-Rosen bridge and Schwarzschild wormhole [1] are not wormhole even if they have the throat satisfying the flare-out condition.

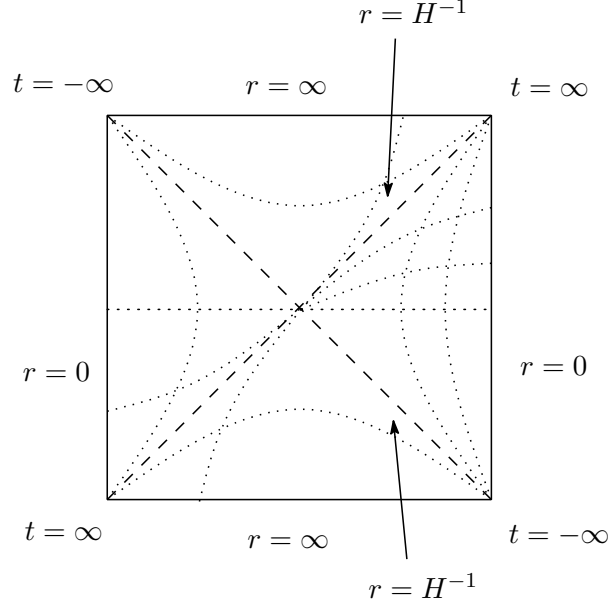


Figure 2.3: The Penrose diagram of de Sitter spacetime (2.44). The dashed lines are cosmological horizons.

There are three typical definitions for the throat (and flare-out condition) [2, 18, 21].

(i) First, we introduce Morris and Thorne's throat definition [2]. We consider the spherically symmetric and asymptotically flat spacetime with the polar coordinate system (t, r, θ, φ) .

Definition 1. (*Wormhole by Morris and Thorne*) [2]

Let us consider the embedded hypersurface of $t = \text{constant}$ into the 4-dimensional Euclid space with the cylindrical coordinate (z, r, θ, φ) . This embedded hypersurface is determined by $z = z(r)$ (Fig. 2.4).

Throat is defined as a surface $r = r_0$ satisfying

$$\left. \frac{dz}{dr} \right|_{r=r_0} \rightarrow \infty \quad (2.45)$$

and flare-out condition

$$\left. \frac{d^2r}{dz^2} \right|_{r=r_0} > 0. \quad (2.46)$$

And wormhole is defined as a spacetime having the throat and without event horizon near the throat.

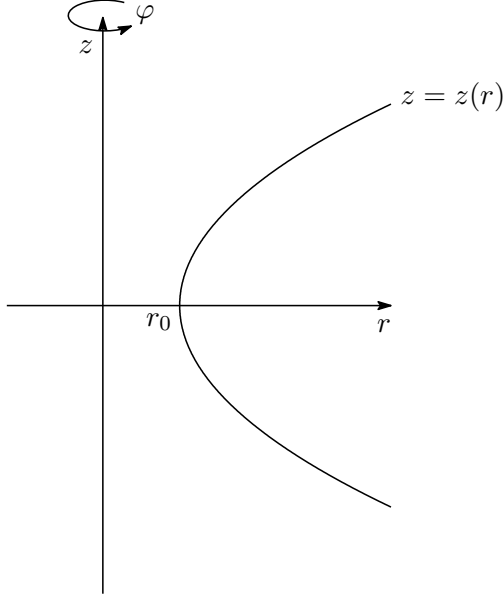


Figure 2.4: $z = z(r)$ is an embedded surface. The throat is located at $r = r_0$. Note that, in this figure, we set as $\theta = \pi/2$.

(ii) Second, we introduce Hochberg and Visser's throat definition [18, 19]. Their definition is not restricted to spherically symmetric spacetimes.

Definition 2. (*Wormhole by Hochberg and Visser*) [18, 19]

Let us consider a 2-dimensional compact spacelike surface S and two null geodesic congruences normal to S which span the two null hypersurfaces \mathcal{H}_\pm . We take u_\pm as null coordinates on \mathcal{H}_\pm and $u_\pm = 0$ on S .

Throat is defined as S satisfying

$$\theta_+|_S = 0 \quad (2.47)$$

and flare-out condition

$$\left. \frac{d\theta_+}{du_+} \right|_S \geq 0, \quad (2.48)$$

or

$$\theta_-|_S = 0 \quad (2.49)$$

and flare-out condition

$$\left. \frac{d\theta_-}{du_-} \right|_S \geq 0, \quad (2.50)$$

where θ_{\pm} is the null expansion rate for null geodesic congruence.

And wormhole is defined as a spacetime having the throat and without event horizon near the throat.

In this definition, the throat is a minimal surface on null hypersurface.

(iii) Finally, we introduce Maeda, Harada and Carr's throat definition in spherically symmetric spacetimes [21].

Definition 3. (Wormhole by Maeda, Harada and Carr) [21]

Let us consider spherically symmetric spacetime and a spherically symmetric spacelike hypersurface Σ .

Throat is defined as a minimal sphere S on Σ , that is,

$$\zeta^a \nabla_a r|_S = 0 \quad (2.51)$$

and flare-out condition

$$\zeta^a \zeta^b \nabla_a \nabla_b r|_S > 0 \quad (2.52)$$

hold, where ζ^a is any non-vanishing radial spacelike vector and r is the area radius.

And wormhole is defined as a spacetime having the throat and without event horizon near the throat.

We consider the double null coordinate (ξ_+, ξ_-) . The metric is written as

$$ds^2 = -2e^{-2f} d\xi_+ d\xi_- + r^2 d\Omega_2^2, \quad (2.53)$$

where f and r are functions of ξ_{\pm} ; $f = f(\xi_+, \xi_-)$, $r = r(\xi_+, \xi_-)$. We denote $\partial_{\pm} := \partial_{\xi_{\pm}}$ which are future-directed null normal vectors to the 2-dimensional sphere [42].

Here, let us consider the null expansion rate

$$\theta_{\pm} = 2e^f r^{-1} \partial_{\pm} r. \quad (2.54)$$

From Eq. (2.51), it holds

$$\zeta^+ \theta_+ + \zeta^- \theta_- = 0 \quad (2.55)$$

on S . Since ζ^a is the radial spacelike vector, we can set as $\zeta^+ > 0, \zeta^- < 0$. Then we see from Eq. (2.55) that

$$\theta_+ \theta_- > 0 \quad \text{or} \quad \theta_+ = \theta_- = 0 \quad (2.56)$$

is satisfied at the throat. Then, the throat has to be in the trapped region or at the bifurcating trapping horizon.

However, Maeda-Harada-Carr's throat definition may identify non-wormhole spacetimes like de Sitter spacetime as wormhole if one chooses a spacelike hypersurface carefully [33].

2.2.2 Energy condition and Wormhole in Hochberg-Visser's definition

Here we show that violation of the null energy condition is required if wormhole is in the sense of Hochberg-Visser's definition [18, 19, 32]. In the current setup, the two null geodesic tangent vectors l_{\pm}^a are orthogonal to a throat. Since the twist ω_{ab} vanishes, the Raychaudhuri equation becomes

$$l_{\pm}^c \nabla_c \theta_{\pm} = -\frac{1}{2} \theta_{\pm}^2 - \sigma_{\pm ab} \sigma_{\pm}^{ab} - R_{ab} l_{\pm}^a l_{\pm}^b. \quad (2.57)$$

Now, $l_{\pm}^c \nabla_c \theta_{\pm} \geq 0$ from the flare-out condition (2.48) or (2.50). Then Eq. (2.57) tells us that $R_{ab} l_{\pm}^a l_{\pm}^b$ must be negative at the throat. When we use the Einstein equation, this means the violation of null energy condition.

2.2.3 Morris-Thorne wormhole

In Ref. [2], Morris and Thorne presented a static and spherically symmetric wormhole whose metric is given by

$$ds^2 = -e^{2\Phi(r)} dt^2 + \left(1 - \frac{b(r)}{r}\right)^{-1} dr^2 + r^2 d\Omega_2^2, \quad (2.58)$$

where $\Phi(r)$ and $b(r)$ are regular functions. Note that, especially, the case of $\Phi(r) = 0$ and $b(r) = r_0^2/r$ is called the Ellis wormhole, where r_0 is a positive constant [43]. Note that this metric (2.58) is not a solution of the Einstein equation. Rather say, the stress-energy is determined from given metric through the Einstein equation.

Let us look at the geometry of the Morris-Thorne wormhole. Here we use Morris-Thorne's definition for the wormhole throat. We consider the embedded hypersurface $z = z(r)$ of fixed t into the Euclid space (z, r, θ, φ) . From the metric (2.58), we can see

$$\frac{dz}{dr} = \pm \left(\frac{r}{b} - 1\right)^{-\frac{1}{2}} \quad (2.59)$$

and

$$\frac{d^2 r}{dz^2} = \frac{b - rb'}{2b^2}, \quad (2.60)$$

where $b' = db(r)/dr$. If there is the throat at $r = r_0$,

$$b(r_0) = r_0 \quad (2.61)$$

and

$$b'(r_0) < 1 \quad (2.62)$$

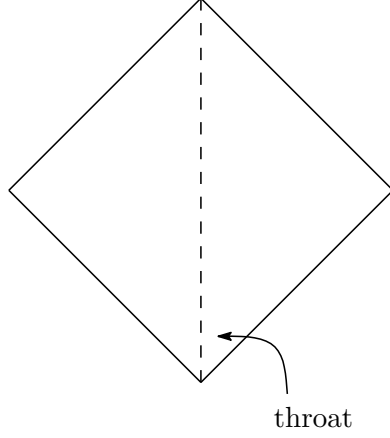


Figure 2.5: The Penrose diagram of the Morris-Thorne wormhole.

must be satisfied from Eqs. (2.45) and (2.46).

Note that we can have the same result even if we adopt Hochberg-Visser's and Maeda-Harada-Carr's definition.

Next, we consider what kind matters is required to construct the Morris-Thorne wormhole through the Einstein equation. Then we discuss whether it violates the null energy condition. From the metric (2.58), we obtain the following non-zero Einstein tensors,

$$G_t^t = -\frac{b'}{r^2}, \quad (2.63)$$

$$G_r^r = -\frac{b}{r^3} + \frac{2\Phi'}{r} \left(1 - \frac{b}{r}\right), \quad (2.64)$$

$$G_B^A = \left\{ \left(1 - \frac{b}{r}\right) \left(\Phi'' + \Phi'^2 + \frac{\Phi'}{r}\right) - \frac{b'r - b}{2r^2} \left(\Phi' + \frac{1}{r}\right) \right\} \delta_B^A, \quad (2.65)$$

where indices A, B run over the angular coordinate θ, φ . If we write the stress-energy tensor as $T_b^a = \text{diag}[-\rho(r), p_r(r), p(r), p(r)]$, from the Einstein equation (2.1), we can see

$$8\pi\rho = \frac{b'}{r^2}, \quad (2.66)$$

$$8\pi p_r = -\frac{b}{r^3} + \frac{2\Phi'}{r} \left(1 - \frac{b}{r}\right), \quad (2.67)$$

$$8\pi p = \left(1 - \frac{b}{r}\right) \left(\Phi'' + \Phi'^2 + \frac{\Phi'}{r}\right) - \frac{b'r - b}{2r^2} \left(\Phi' + \frac{1}{r}\right). \quad (2.68)$$

So we have

$$8\pi(\rho + p_r) = -\frac{1}{r^3} \{b - rb' - 2r(r - b)\Phi'\}. \quad (2.69)$$

At the throat, because of Eq. (2.62), it is easy to see that

$$8\pi(\rho + p_r) = -\frac{1}{r_0^2}(1 - b'(r_0)) < 0. \quad (2.70)$$

This tells us that the null energy condition is violated.

2.2.4 Cosmological wormhole

Any known cosmological wormholes are dynamical with initial singularity. The most attracting feature of this wormhole may be absence of exotic matters.

In Ref. [21], the dynamical Ellis wormhole was introduced;

$$ds^2 = -dt^2 + a^2(t)\{dr^2 + (r^2 + b^2)d\Omega_2^2\}, \quad (2.71)$$

where b is a positive constant. If we adopt the Maeda-Harada-Carr definition, we can see that it has throat at $r = 0$ as follows. In the double null coordinate (ξ_+, ξ_-) , the metric (2.71) can be rewritten as

$$ds^2 = -a^2 d\xi_+ d\xi_- + R^2 d\Omega_2^2, \quad (2.72)$$

where $R = a\sqrt{r^2 + b^2}$, $\xi_{\pm} = \eta \pm r$ and $d\eta = a^{-1}dt$. Because of

$$\partial_{\pm} R = \frac{a}{2\sqrt{r^2 + b^2}}\{\dot{a}(r^2 + b^2) \pm r\}, \quad (2.73)$$

from $\zeta^a \nabla_a R = 0$, we see

$$\zeta^+ = -\frac{\dot{a}(r^2 + b^2) - r}{\dot{a}(r^2 + b^2) + r} \zeta^- \quad (2.74)$$

is satisfied at the throat. Here, $\dot{a} = da/dt$ and ζ^a is non-zero radial spacelike vector satisfying Eq. (2.74). Now, we can suppose it as satisfying $\zeta^+ = -\zeta^-$. In this case, the throat candidate is at $r = 0$. After a few calculations, we see that the flare-out condition $\zeta^a \zeta^b \nabla_a \nabla_b R > 0$ is also satisfied when

$$\dot{a}^2 b^2 < 1. \quad (2.75)$$

Next, we consider the null energy condition. Using the Einstein equation with the dynamical Ellis metric (2.71), we compute the stress-energy tensor $T_{\mu\nu}$. We assume that the null energy condition is satisfied,

$$-T_t^t + T_r^r = \frac{2}{a^2} \left(-a\ddot{a} + \dot{a}^2 - \frac{b^2}{(r^2 + b^2)^2} \right) \geq 0, \quad (2.76)$$

$$-T_t^t + T_{\theta}^{\theta} = \frac{2}{a^2} (-a\ddot{a} + \dot{a}^2) \geq 0, \quad (2.77)$$

where θ is the angular coordinate appeared as $d\Omega_2^2 = d\theta^2 + \sin^2\theta d\varphi^2$. Let us suppose

$$a(t) \propto t^{\frac{2}{3(1+w)}}, \quad (2.78)$$

where $w \neq -1$ is a constant. Then, by substituting Eq. (2.78) into Eqs. (2.75), (2.76) and (2.77), we can see that the dynamical Ellis wormhole satisfying the null energy condition exists when

$$w > -\frac{1}{3}. \quad (2.79)$$

This result does not contradict with the feature introduced in Sec. 2.2.2. This is because the throat is defined as a minimal surface on the spacelike hypersurface in Maeda-Harada-Carr's definition in contrast with it on the null hypersurface in Hochberg-Visser's one.

Chapter 3

New definition of wormhole throat

In the previous chapter, we gave a brief review on the three definitions; Morris and Thorne [2], Hochberg and Visser [18, 19], and Maeda, Harada and Carr [21]. Morris-Thorne's throat definition is restricted to spherically symmetric spacetimes. So Hochberg and Visser improved it so that their definition can treat non-spherical spacetimes. And they proved that exotic matter is necessarily required to maintain wormhole structure. But, Maeda, Harada and Carr proposed a third definition of throat and found non-exotic wormhole in cosmological case [21]. In above three definitions, however, the throat depends on the choice of spacelike or null hypersurface. And the Maeda-Harada-Carr definition identifies spacetimes, which are naively regarded as non-wormhole spacetime, as wormhole spacetime. Therefore, we would suggest a new formulation of wormhole to fix the above problems. Especially, we emphasize that our new definition does not depend on the choice of hypersurface.

This chapter is based on my original work [33].

3.1 New definition

Let us consider a codimension-2 compact spacelike surface S and the future directed outgoing and ingoing null geodesic congruences with the affine parameter λ_{\pm} emanating from the surface S . The null expansion rates is given by $\theta_{\pm} := h^{ab}\nabla_b\xi_{\pm a}$, where ξ_{\pm}^a is the tangent vector of future directed outgoing/ingoing null geodesics which are orthogonal to S , h_{ab} is the induced metric of S . Then we introduce new quantities

$$k := \theta_+ - \theta_-, \quad (3.1)$$

$$\bar{k} := \theta_+ + \theta_-. \quad (3.2)$$

Defining

$$r^a := (\partial_+ - \partial_-)^a, \quad (3.3)$$

$$t^a := (\partial_+ + \partial_-)^a, \quad (3.4)$$

we rewrite k and \bar{k} as

$$k = r^a \nabla_a \log \sqrt{h}, \quad (3.5)$$

$$\bar{k} = t^a \nabla_a \log \sqrt{h}, \quad (3.6)$$

where $\partial_{\pm} := \partial_{\lambda_{\pm}}$ and h is the determinant of h_{ab} .

Definition 4. (*Wormhole by Tomikawa, Izumi and Shiromizu*) [33]

Throat is defined as a codimension-2 surfaces S satisfying

$$k = 0 \quad (3.7)$$

and flare-out condition

$$r^a \nabla_a k > 0. \quad (3.8)$$

Wormhole is defined as a spacetime having the throat S and without event horizon near the throat.

And traversable wormhole is defined as one satisfying the traversability that the tangent vector z^a of the time sequence of the throat S , which is normal to S , is timelike¹.

We can write $z^a = \alpha(\partial_+)^a + \beta(\partial_-)^a$, where α, β are positive constants. And

$$z^a \nabla_a k|_S = 0 \quad (3.9)$$

holds along the time sequence of the throat.

In the definition by Morris and Thorne [2], they defined the throat as the minimal surface that embedded static slices to the Euclid space (Def. 1). In the Hochberg-Visser definition [18, 19], the throat is defined as the minimal surface on the null hypersurface (Def. 2). And in Maeda-Harada-Carr's definition [21], they defined the throat as the minimal surface on the spacelike hypersurface (Def. 3). In their definitions, the throat depends on time or null slices. However, in our definition, we do not take hypersurfaces for the throat definition.

Our definition seems to be a hybrid of Hochberg-Visser's and Maeda-Harada-Carr's one. This is because we consider the expansion rates of null geodesic congruences (like Hochberg-Visser's definition) and a quantity k (Eq. (3.1)) roughly corresponds to the trace of extrinsic curvature for the spatial normal vector to surface on time slices (like Maeda-Harada-Carr's definition). We emphasize that time slice is not used in our formulation.

¹Note that the similar definition of the traversability was given by Hayward [20].

3.2 General features of throat/wormhole

If we adopt this new definition for wormhole throat, we can obtain general features of the throat more simply than adopting other definitions. In this section, we see two propositions for wormhole throat.

First, we consider a consequence for traversable wormhole satisfying the null energy condition.

Proposition 5. [33]

Let us consider traversable wormhole satisfying the null energy condition. Then

$$\partial_+ \theta_-|_S < 0 \quad (3.10)$$

must hold on the throat S .

Proof. The Raychaudhuri equation for null geodesics vector is

$$\partial_\pm \theta_\pm = -\frac{1}{2} \theta_\pm^2 - \sigma_{\pm ab} \sigma_\pm^{ab} - R_{ab} n_\pm^a n_\pm^b, \quad (3.11)$$

where n_\pm^a are the tangent vectors of null geodesics which are orthogonal to the throat S . Then the null energy condition implies

$$\partial_\pm \theta_\pm \leq 0. \quad (3.12)$$

Meanwhile, the traversability gives us

$$\partial_- k|_S = -\frac{\alpha}{\beta} \partial_+ k|_S. \quad (3.13)$$

Then, the flare-out condition becomes

$$r^a \nabla_a k|_S = \left(1 + \frac{\alpha}{\beta}\right) \partial_+ k|_S \quad (3.14)$$

$$= \left(1 + \frac{\alpha}{\beta}\right) (\partial_+ \theta_+ - \partial_+ \theta_-)|_S > 0. \quad (3.15)$$

Therefore, we see

$$\partial_+ \theta_-|_S < 0 \quad (3.16)$$

for the presence of traversable wormhole satisfying the null energy condition. \square

Around a normal region, we expect that $\partial_+ \theta_- > 0$ holds. So, the above opposite property is rather irregular.

Note that, at the throat S , we have

$$\theta_+|_S = \theta_-|_S \quad (3.17)$$

from Eq. (3.7). When $\theta_+|_S = \theta_-|_S < 0$ (> 0), it means the existence of the future (past) trapped surface. Then if the null energy condition holds, the singularity theorem implies the presence of singularity in the future (past) [38]. In addition, we assume the cosmic censorship conjecture to be held, the future trapped region is always inside the event horizon [38]. And then, it is not identified as the wormhole throat. Meanwhile, with $\theta_+|_S = \theta_-|_S > 0$, the singularity theorem predicts the existence of the past singularity. For $\theta_-|_S > 0$, the past singularity may be unified to the initial one. The realization of $\theta_-|_S > 0$ will be easy in expanding universe, then the dynamical wormhole satisfying the null energy condition can exist in the cosmological context. In this case, the wormhole throat exists in the past trapped region (see the latter half of Sec. 2.1.1 for the definition).

Next, we can show the violation of the null energy condition for static wormhole.

Proposition 6. [33]

The static wormhole must violate the null energy condition.

Proof. Since we have

$$r^a \nabla_a k + t^a \nabla_a \bar{k} = 2(\partial_+ \theta_+ + \partial_- \theta_-) \quad (3.18)$$

and the Raychaudhuri equation with the null energy condition shows us

$$r^a \nabla_a k + t^a \nabla_a \bar{k} \leq 0, \quad (3.19)$$

the flare-out condition is not satisfied when $t^a \nabla_a \bar{k} = 0$ holds.

If we choose the affine parameter λ_{\pm} appropriately, $t^a \nabla_a \bar{k} = 0$ at least locally holds in the static case. Then, this result presents that the static wormhole needs the violation of null energy condition at least at the throat. \square

Although this is a well-known fact, our argument is simpler than that in Ref. [26].

3.3 Spherically symmetric spacetime

To examine our throat definition for spherically symmetric wormholes, we will write down some useful formulae for spherically symmetric cases.

In the double null coordinate, the metric of spherically symmetric spacetimes is written as

$$ds^2 = -a^2(u, v) du dv + R^2(u, v) d\Omega_2^2. \quad (3.20)$$

The throat is supposed to be a 2-dimensional surface located at $u = u_0, v = v_0$.

We consider the radial null geodesic on $v = v_0$ that satisfies the geodesic equation

$$\frac{d^2 u}{d\lambda_u^2} + 2 \frac{\partial_u a}{a} \left(\frac{du}{d\lambda_u} \right)^2 = 0, \quad (3.21)$$

where λ_u is the affine parameter. In a formal way, we can solve it as

$$\lambda_u = C_u^{-1} \int a^2(u) du =: U, \quad (3.22)$$

where $a(u) := a(u, v_0)$ and C_u is a positive integration constant. And we choose λ_u such that $du/d\lambda_u > 0$. In the same way, for the geodesic on $u = u_0$, we have

$$\lambda_v = C_v^{-1} \int a^2(v) dv =: V, \quad (3.23)$$

where $a(v) := a(u_0, v)$ and C_v is a positive integration constant. We choose λ_v such that $dv/d\lambda_v > 0$.

Using U and V , we can rewrite the metric (3.20) as

$$ds^2 = -C_u C_v \frac{a^2(u, v)}{a^2(u) a^2(v)} dU dV + R^2(u, v) d\Omega_2^2. \quad (3.24)$$

In this coordinate, the null expansion rates become

$$\theta_U = \theta_- = \frac{2}{R} \partial_U R = \frac{2}{R} \frac{C_u}{a^2(u)} \partial_u R, \quad (3.25)$$

$$\theta_V = \theta_+ = \frac{2}{R} \partial_V R = \frac{2}{R} \frac{C_v}{a^2(v)} \partial_v R. \quad (3.26)$$

Then, because $k = 0$ holds on the throat S , we see

$$C_u \partial_u R|_S = C_v \partial_v R|_S. \quad (3.27)$$

Here, we defined $a_0 := a(u_0) = a(v_0)$ and used the fact $a(u)|_S = a(u_0)$, $a(v)|_S = a(v_0)$.

In Eqs. (3.8) and (3.9), we can compute $r^a \nabla_a k|_S$ and $z^a \nabla_a k|_S$ as

$$\begin{aligned} r^a \nabla_a k|_S &= (\partial_V - \partial_U) k|_S \\ &= \frac{2}{a_0^4 R} \left\{ -2C_v^2 \partial_v \log a(v) \partial_v R - 2C_u^2 \partial_u \log a(u) \partial_u R \right. \\ &\quad \left. + C_v^2 \partial_v^2 R - 2C_u C_v \partial_u \partial_v R + C_u^2 \partial_u^2 R \right\} |_S, \end{aligned} \quad (3.28)$$

$$\begin{aligned} z^a \nabla_a k|_S &= (\alpha \partial_V + \beta \partial_U) k|_S \\ &= \frac{2}{a_0^4 R} \left\{ -2\alpha C_v^2 \partial_v \log a(v) \partial_v R + 2\beta C_u^2 \partial_u \log a(u) \partial_u R \right. \\ &\quad \left. + \alpha C_v^2 \partial_v^2 R - (\alpha - \beta) C_u C_v \partial_u \partial_v R - \beta C_u^2 \partial_u^2 R \right\} |_S, \end{aligned} \quad (3.29)$$

respectively.

3.4 Specific examples

In this section, we confirm the validity of our new definition of wormhole throat by considering some specific examples.

3.4.1 Schwarzschild metric

The throat of the Schwarzschild spacetime is not that of the wormhole because the Schwarzschild spacetime has the event horizon. However it is nice to see the features with respect to our definition.

We adopt the Kruskal coordinate (2.19). If we choose u, v as $u = T - X, v = T + X$, Eq. (3.27) become

$$C_u(T + X)|_S = C_v(T - X)|_S, \quad (3.30)$$

where S is a surface of the throat candidate and C_u, C_v are positive constants. We can see from Eq. (2.17) that the throat candidate S is in the region $0 < r \leq r_g$.

Eqs. (3.28) and (3.29) become

$$r^a \nabla_a k|_S = \frac{4r_g^4}{a_0^4 r^4} e^{-r/r_g} C_u C_v|_S, \quad (3.31)$$

$$z^a \nabla_a k|_S = \frac{2r_g^4}{a_0^4 r^4} e^{-r/r_g} (\alpha - \beta) C_u C_v|_S, \quad (3.32)$$

where $a_0^2 = 4(r_g^3/r)e^{-r/r_g}$. Since C_u and C_v are positive, the flare-out condition is satisfied as expected. The condition of traversability holds if one sets $\alpha = \beta$.

However the Schwarzschild spacetime has the event horizon at $r = r_g$. Then it is not regarded as a wormhole even if it has the throat.

3.4.2 de Sitter metric

We examine the de Sitter spacetime

$$ds^2 = -dt^2 + e^{2Ht}(dr^2 + r^2 d\Omega_2^2). \quad (3.33)$$

This spacetime is known as non-wormhole. However, if one elaborates the selection of a spacelike hypersurface and follows Maeda-Harada-Carr's definition, there is a case where wormhole is.

If we transform the coordinate r to r/H for the de Sitter metric (3.33), we see

$$ds^2 = a^2(\eta)(-d\eta^2 + dr^2 + r^2 d\Omega_2^2), \quad (3.34)$$

where $e^{-Ht}dt = a(\eta)d\eta$ and $a(\eta) = -1/(H\eta)$. When we choose $u = \eta - r, v = \eta + r$, the above metric becomes

$$ds^2 = a^2(\eta)(-dudv + r^2d\Omega_2^2). \quad (3.35)$$

Now Eq. (3.27) is written as

$$C_u(Har - 1)|_S = C_v(Har + 1)|_S. \quad (3.36)$$

Then, if we choose $C_u > C_v(> 0)$, Eq. (3.36) can be satisfied. Since

$$Har|_S = \frac{C_u + C_v}{C_u - C_v} \Big|_S > 1, \quad (3.37)$$

the throat candidate is in the outside of the cosmological horizon.

However, in this metric, the flare-out condition is not satisfied because Eq. (3.28) becomes

$$r^a \nabla_a k|_S = -\frac{2H^2}{a^2} C_u C_v|_S < 0. \quad (3.38)$$

Then, there is no wormhole throat in the de Sitter spacetime as expected.

3.4.3 Friedmann-Lemaître-Robertson-Walker metric

We consider the Friedmann-Lemaître-Robertson-Walker (FLRW) spacetime in order to check the possibility for the existence of wormhole.

The FLRW metric (2.24) becomes

$$\begin{aligned} ds^2 &= a^2(\eta)(-d\eta^2 + d\zeta^2 + r^2d\Omega_2^2) \\ &= a^2(\eta)(-dudv + r^2d\Omega_2^2), \end{aligned} \quad (3.39)$$

where $a^{-1}(t)dt = d\eta, (1 - Kr^2)^{-1/2}dr = d\zeta$ and $u = \eta - \zeta, v = \eta + \zeta$. Eq. (3.27) gives us

$$C_u \left(\dot{a}r - \sqrt{1 - Kr^2} \right) \Big|_S = C_v \left(\dot{a}r + \sqrt{1 - Kr^2} \right) \Big|_S. \quad (3.40)$$

If we choose $C_u > C_v(> 0)$, the above equation has the solution as

$$H(t) \frac{a(t)r}{\sqrt{1 - Kr^2}} \Big|_S = \frac{C_u + C_v}{C_u - C_v} \Big|_S > 1, \quad (3.41)$$

where $H(t) := \dot{a}(t)/a(t)$ is called the Hubble parameter. Roughly speaking, as in the de Sitter spacetime, this means that the throat candidate is in the outside of the cosmological horizon.

When we use Eq. (3.28), the flare-out condition becomes

$$r^a \nabla_a k|_S = 2C_u C_v \frac{a\ddot{a}(1 - Kr^2) - \dot{a}^2 r^2(\ddot{a} + K)}{a^4 \{\dot{a}^2 r^2 - (1 - Kr^2)\}} \Big|_S > 0. \quad (3.42)$$

Then, from Eq. (3.41),

$$a\ddot{a}(1 - Kr^2)|_S > \dot{a}^2 r^2(\dot{a}^2 + K)|_S \quad (3.43)$$

is satisfied for the existence of the throat. Together with Eq. (3.41), Eq. (3.43) implies

$$\dot{a}^2 r^2(\dot{a}^2 + K)|_S < a\ddot{a}(1 - Kr^2)|_S < a\ddot{a}\dot{a}^2 r^2|_S, \quad (3.44)$$

and then

$$(\dot{a}^2 + K)|_S < a\ddot{a}|_S. \quad (3.45)$$

If we use the Einstein equation (2.28) and (2.29), the above inequality (3.45) is equivalent with the violation of the null energy condition,

$$(\rho + p)|_S < 0, \quad (3.46)$$

where ρ and p are the energy density and the pressure of the perfect fluid, respectively. It is consistent with common sense.

Note that there is room for the existence of wormhole if the null energy condition is violated.

3.4.4 Morris-Thorne metric

We consider the Morris-Thorne wormhole (2.58). The metric is

$$\begin{aligned} ds^2 &= e^{2\Phi}(-dt^2 + d\zeta^2) + r^2 d\Omega_2^2 \\ &= -e^{2\Phi} du dv + r^2 d\Omega_2^2, \end{aligned} \quad (3.47)$$

where $d\zeta = e^{-\Phi(r)} dr / \sqrt{1 - b(r)/r}$ and $u = t - \zeta, v = t + \zeta$. Eq. (3.27) becomes

$$-C_u \sqrt{1 - \frac{b(r)}{r}} e^{\Phi(r)} \Big|_S = C_v \sqrt{1 - \frac{b(r)}{r}} e^{\Phi(r)} \Big|_S. \quad (3.48)$$

Because of $C_u > 0, C_v > 0$, the throat candidate S is the surface satisfying

$$b(r) = r. \quad (3.49)$$

Eq. (3.28) becomes

$$r^a \nabla_a k|_S = \frac{(C_u + C_v)^2 (1 - b'(r))}{4r^2} e^{-2\Phi(r)} \Big|_S, \quad (3.50)$$

where $b'(r) = db(r)/dr$. Then, the flare-out condition $r^a \nabla_a k|_S > 0$ implies

$$b'(r)|_S < 1. \quad (3.51)$$

Finally, we consider the traversability. Eq. (3.29) becomes

$$z^a \nabla_a k|_S = \frac{(\alpha C_v - \beta C_u)(C_v + C_u)(1 - b'(r))}{4r^2} e^{-2\Phi(r)} \Big|_S. \quad (3.52)$$

Then, if we take

$$\alpha C_v = \beta C_u, \quad (3.53)$$

z^a is timelike and the traversability is satisfied.

3.4.5 Dynamical Ellis metric

We consider the dynamical Ellis wormhole with the metric of Eq. (2.71). As said in Sec. 2.2.4, this wormhole is the cosmological wormhole in the sense of Maeda-Harada-Carr's definition.

The metric (2.71) is

$$\begin{aligned} ds^2 &= a^2(\eta) \{-d\eta^2 + dr^2 + (r^2 + b^2)d\Omega_2^2\} \\ &= a^2(\eta) \{-dudv + (r^2 + b^2)d\Omega_2^2\}, \end{aligned} \quad (3.54)$$

where $a^{-1}(t)dt = d\eta$ and $u = \eta - r, v = \eta + r$. $a(\eta)$ is a function of the conformal time η . Eq. (3.27) gives us

$$C_u \{\dot{a}(r^2 + b^2) - r\}|_S = C_v \{\dot{a}(r^2 + b^2) + r\}|_S. \quad (3.55)$$

Then, we obtain the condition

$$r^2|_S < \dot{a}^2(r^2 + b^2)^2|_S \quad (3.56)$$

because of $C_u > 0$ and $C_v > 0$.

With Eq. (3.55), the flare-out condition becomes

$$r^a \nabla_a k|_S = \frac{2C_u C_v \{a\ddot{a}r^2 - \dot{a}^4(r^2 + b^2)^2 + \dot{a}^2 b^2\}}{a^4 \{\dot{a}^2(r^2 + b^2)^2 - r^2\}} \Big|_S > 0, \quad (3.57)$$

and then, from Eq. (3.56), it has to satisfy

$$f(r) := a\ddot{a}r^2 - \dot{a}^4(r^2 + b^2)^2 + \dot{a}^2 b^2 > 0 \quad (3.58)$$

at the throat candidate S .

Now, using the Einstein equation (2.1) with the dynamical Ellis metric (2.71), we compute the stress-energy tensor $T_{\mu\nu}$. Because we are interested in physically natural wormhole, we assume that the dominant energy condition is satisfied. Note that the null energy condition is also satisfied (see

Sec. 2.1.2). Then, the following inequality are required,

$$-T_t^t - T_r^r = \frac{2}{a^2}(a\ddot{a} + 2\dot{a}^2) \geq 0, \quad (3.59)$$

$$-T_t^t + T_\theta^\theta = \frac{2}{a^2}(-a\ddot{a} + \dot{a}^2) \geq 0, \quad (3.60)$$

$$-T_t^t + T_r^r = \frac{2}{a^2}\left(-a\ddot{a} + \dot{a}^2 - \frac{b^2}{(r^2 + b^2)^2}\right) \geq 0, \quad (3.61)$$

$$-T_t^t - T_\theta^\theta = \frac{2}{a^2}\left(a\ddot{a} + 2\dot{a}^2 - \frac{b^2}{(r^2 + b^2)^2}\right) \geq 0, \quad (3.62)$$

where θ is the angular coordinate appeared as $d\Omega_2^2 = d\theta^2 + \sin^2\theta d\varphi^2$. Eqs. (3.61) and (3.62) are stronger than Eqs. (3.60) and (3.59), respectively. At $r = 0$, both Eqs. (3.61) and (3.62) become to be the hardest. These inequality are

$$\dot{a}^2 - b^{-2} \geq a\ddot{a}, \quad (3.63)$$

$$a\ddot{a} \geq -2\dot{a}^2 + b^{-2}, \quad (3.64)$$

respectively. Therefore, the condition

$$-2\dot{a}^2 + b^{-2} \leq a\ddot{a} \leq \dot{a}^2 - b^{-2} \quad (3.65)$$

is satisfied, and then we can see

$$\dot{a}^2 b^2 \geq \frac{2}{3}. \quad (3.66)$$

Note that $f(r)$ is rewritten as

$$(0 <) f(r) = -\dot{a}^4 \left\{ \left(r^2 + b^2 - \frac{a\ddot{a}}{2\dot{a}^4} \right)^2 - \left(b^2 - \frac{a\ddot{a}}{2\dot{a}^4} \right)^2 + b^4 - \frac{b^2}{\dot{a}^2} \right\}. \quad (3.67)$$

Because of Eq. (3.63), we see that there is a maximum value of $f(r)$ at $r = 0$. Then, from $f(0) > 0$ and Eq. (3.66),

$$\frac{2}{3} \leq \dot{a}^2 b^2 < 1 \quad (3.68)$$

has to hold. Together with this, Eq. (3.65) tells us

$$-\dot{a}^2 < -2\dot{a}^2 + b^{-2} \leq a\ddot{a} \leq \dot{a}^2 - b^{-2} < 0. \quad (3.69)$$

Here let us suppose

$$a(t) \propto t^{\frac{2}{3(1+w)}}, \quad (3.70)$$

where $w \neq -1$ is a constant. In this case, the loosest inequality in Eq. (3.69) implies that w satisfies

$$-\frac{1}{3} \leq w < \frac{1}{3}. \quad (3.71)$$

From Eq. (3.68), we can see that this wormhole satisfying the dominant energy condition must have a size of about the Hubble radius.

Note that we did not give the equation of state like $p = w\rho$. In the current case, w is determined through the Einstein equation. Moreover, we also did not give the isotropic pressure. Therefore, Eq. (3.71) is not related to the equation of state, directly.

Finally, we consider the traversability. Eq. (3.29) becomes

$$\begin{aligned} z^a \nabla_a k|_S = & \frac{1}{2a^4} \left[(\alpha C_v^2 - \beta C_u^2) \left\{ (a\ddot{a} - \dot{a}^2) + \frac{b^2}{(r^2 + b^2)^2} \right\} \right. \\ & \left. - (\alpha - \beta) C_u C_v \left\{ (a\ddot{a} + \dot{a}^2) - \frac{b^2}{(r^2 + b^2)^2} \right\} \right] \Big|_S. \end{aligned} \quad (3.72)$$

For simplicity, we set $C_u = C_v$. Then we can see that the throat is located at $r = 0$ from Eq. (3.55), and the tangent vector z^a with $\alpha = \beta$ is timelike and satisfies the traversability $z^a \nabla_a k|_S = 0$.

3.4.6 Dynamical Schwarzschild metric

We consider other dynamical wormhole that the metric is written by

$$ds^2 = -e^{2\Phi(r)} dt^2 + a^2(t) \left\{ \left(1 - \frac{r_g}{r}\right)^{-1} dr^2 + r^2 d\Omega_2^2 \right\}, \quad (3.73)$$

where $\Phi(r)$ is the regular function and r_g is the Schwarzschild radius. This is one of the special case of dynamical wormholes studied in Ref. [17]. The metric is rewritten as

$$\begin{aligned} ds^2 &= a^2 e^{2\Phi} (-d\eta^2 + d\zeta^2) + a^2 r^2 d\Omega_2^2 \\ &= -a^2 e^{2\Phi} du dv + a^2 r^2 d\Omega_2^2, \end{aligned} \quad (3.74)$$

where $d\eta = a^{-1} dt$, $d\zeta = e^{-\Phi} dr / \sqrt{1 - r_g/r}$ and $u = \eta - \zeta$, $v = \eta + \zeta$.

We consider a case of $\Phi(r) = 0$ to compare with the dynamical Ellis wormhole. In this case, Eq. (3.27) gives us

$$C_u a \left(\dot{a}r - \sqrt{1 - \frac{r_g}{r}} \right) \Big|_S = C_v a \left(\dot{a}r + \sqrt{1 - \frac{r_g}{r}} \right) \Big|_S. \quad (3.75)$$

Then, at the throat candidate S , it holds

$$1 - \frac{r_g}{r} < \dot{a}^2 r^2. \quad (3.76)$$

With Eq. (3.75), the flare-out condition becomes

$$r^a \nabla_a k|_S = \frac{2C_u C_v \{a\ddot{a} - \dot{a}^4 r^2 + \frac{r_g}{2r}(\dot{a}^2 - 2a\ddot{a})\}}{a^4(\dot{a}^2 r^2 - 1 + \frac{r_g}{r})} \Big|_S > 0. \quad (3.77)$$

Then,

$$f(r) := a\ddot{a}r - \dot{a}^4 r^3 + \frac{r_g}{2}(\dot{a}^2 - 2a\ddot{a}) > 0 \quad (3.78)$$

has to hold.

Now, using the Einstein equation (2.1) with the dynamical Schwarzschild metric (3.73), we compute the stress-energy tensor $T_{\mu\nu}$. If we assume that the dominant energy condition is satisfied, the following inequality are required;

$$-T_t^t - T_r^r = \frac{1}{a^2} \left(2a\ddot{a} + 4\dot{a}^2 + \frac{r_g}{r^3} \right) \geq 0, \quad (3.79)$$

$$-T_t^t + T_\theta^\theta = \frac{1}{a^2} \left(-2a\ddot{a} + 2\dot{a}^2 + \frac{r_g}{2r^3} \right) \geq 0, \quad (3.80)$$

$$-T_t^t + T_r^r = \frac{1}{a^2} \left(-2a\ddot{a} + 2\dot{a}^2 - \frac{r_g}{r^3} \right) \geq 0, \quad (3.81)$$

$$-T_t^t - T_\theta^\theta = \frac{1}{a^2} \left(2a\ddot{a} + 4\dot{a}^2 - \frac{r_g}{2r^3} \right) \geq 0. \quad (3.82)$$

Eqs. (3.81) and (3.82) are stronger than others. From these, the condition

$$-2\dot{a}^2 + \frac{r_g}{4r^3} \leq a\ddot{a} \leq \dot{a}^2 - \frac{r_g}{2r^3} \quad (3.83)$$

is satisfied, and then we can see

$$\dot{a}^2 \geq \frac{1}{4r_g^2}. \quad (3.84)$$

From Eq. (3.78), $f(r)$ has a maximum value at $r = r_g$ and then we obtain

$$\dot{a}^2 < \frac{1}{2r_g^2}. \quad (3.85)$$

With the inequality (3.85), Eq. (3.83) becomes

$$-\frac{3}{2}\dot{a}^2 < -2\dot{a}^2 + \frac{1}{4r_g^2} \leq a\ddot{a} \leq \dot{a}^2 - \frac{1}{2r_g^2} < 0. \quad (3.86)$$

On the other hand, from Eqs. (3.84) and (3.85), we obtain the condition of \dot{a} ;

$$\frac{1}{4r_g^2} \leq \dot{a}^2 < \frac{1}{2r_g^2}. \quad (3.87)$$

As the dynamical Ellis wormhole, we suppose $a(t) \propto t^{\frac{2}{3(1+w)}}$. Then, from both the outermost sides of Eq. (3.86), we can obtain the condition of w ;

$$-\frac{1}{3} \leq w < \frac{2}{3}. \quad (3.88)$$

This is a similar result with the dynamical Ellis case that is Eq. (3.71).

Finally, we consider the traversability. Eq. (3.29) becomes

$$z^a \nabla_a k|_S = \frac{1}{2a^4} \left\{ (\alpha C_v^2 - \beta C_u^2) \left(-\dot{a}^2 + a\ddot{a} + \frac{r_g}{2r^3} \right) - (\alpha - \beta) C_u C_v \left(\dot{a}^2 + a\ddot{a} - \frac{r_g}{2r^3} \right) \right\} \Big|_S. \quad (3.89)$$

For simplicity, we set $C_u = C_v$. Then we can see that the throat is located at $r = r_g$ from Eq. (3.75), and the tangent vector z^a with $\alpha = \beta$ is timelike and satisfies the traversability $z^a \nabla_a k|_S = 0$.

Chapter 4

Dvali-Gabadadze-Porrati wormhole

This chapter is based on my original work [34] (and a part of Ref. [33]). The Dvali-Gabadadze-Porrati (DGP) braneworld model is one of higher-dimensional cosmological models to explain the current acceleration of the Universe without introducing the cosmological constant [35, 36]. In Ref. [36], a new vacuum solution in the DGP model was obtained. And it seems to have a wormhole structure on the brane without introducing any exotic matters because the brane has two asymptotically flat regions without event horizon [36]. Therefore, we examine the details of the spacetime structure on the brane.

4.1 Dvali-Gabadadze-Porrati brane

The new solution by Izumi and Shiromizu [36] is in the vacuum 5-dimensional spacetime. The induced metric on the brane is written by

$$ds^2 = \gamma^{-2}(r)dr^2 + r^2(-d\tau^2 + \cosh^2 \tau d\Omega_2^2), \quad (4.1)$$

where

$$\gamma^2(r) = \frac{-(r^2 - 2r_c^2) + \sqrt{r^4 - 4r_0^2 r_c^2}}{2r_c^2} \quad (4.2)$$

and r_0, r_c are positive constants satisfying $r_0 > r_c$. The range of r is

$$r \geq r_* := \sqrt{r_0^2 + r_c^2}, \quad (4.3)$$

and then $0 \leq \gamma^2(r) < 1$.

It is known that this spacetime has no event horizon as follows [36]. We introduce the double null coordinate u_{\pm} . The metric (4.1) becomes

$$ds^2 = -r^2 du_+ du_- + \mathcal{R}^2 d\Omega_2^2, \quad (4.4)$$

where $du_{\pm} = d\tau \pm dr/(r\gamma)$ and $\mathcal{R} = r \cosh \tau$. The null expansion rates θ_{\pm} can be obtained as

$$\theta_{\pm} = \frac{2}{r} \mathcal{R}^{-1} \partial_{u_{\pm}} \mathcal{R} = \frac{1}{r} (\tanh \tau \pm \gamma). \quad (4.5)$$

From Eq. (4.5), we see that the timelike hypersurface \mathcal{H}_+ with $\theta_+ > 0$ and $\theta_- = 0$ exists for $\tau > 0$ and the timelike hypersurface \mathcal{H}_- with $\theta_+ = 0$ and $\theta_- < 0$ exists for $\tau < 0$. When $\tau = 0$ and $r = r_*$, $\theta_+ = \theta_- = 0$ holds. In the region of $\tau > 0$, $\theta_+ > 0$ holds and the spacetime is regular everywhere. Then, there is no event horizon in this region. But in the region of $\tau < 0$, there are future trapped surfaces because $\theta_{\pm} < 0$ is possible. Then, if the null energy condition is satisfied, the singularity theorem holds. And if the cosmic censorship conjecture holds, the event horizon exists [38] (see also Sec. 3.2). However, in the current case, the null energy condition for the effective stress-energy tensor on the brane is not satisfied as seen in the next section. Then the singularity theorem does not hold on the brane.

Next, we consider the throat of wormhole. In fact, we can see that the throat exists even if we adopt any definitions. We confirm it adopting Hochberg-Visser's, Maeda-Harada-Carr's and our new throat definition in this order.

First, we adopt Hochberg-Visser's definition. We see that $\theta_{\pm} = 0$ implies

$$\gamma = \mp \tanh \tau \quad (4.6)$$

at the throat candidate. Note that we can see that the throat candidate exists on the hypersurface \mathcal{H}_{\mp} .

To check the flare-out condition, we look at

$$\frac{d\theta_{\pm}}{du_{\pm}} = \frac{1}{2r} \{ \mp \gamma \tanh \tau - \tanh^2 \tau + (1 - \gamma^2 + r\gamma\gamma') \}, \quad (4.7)$$

where $\gamma' := d\gamma(r)/dr$. Then, at the throat candidate S , we see that

$$\begin{aligned} \left. \frac{d\theta_{\pm}}{du_{\pm}} \right|_S &= \frac{1 - \gamma^2 + r\gamma\gamma'}{2r} \\ &= \frac{1 - \gamma^2}{2r} \left(1 + \frac{r^2}{\sqrt{r^4 - 4r_0^2 r_c^2}} \right) > 0. \end{aligned} \quad (4.8)$$

Here, we used the fact that $0 \leq \gamma < 1$. Therefore, the throat exists.

Second, we adopt Maeda-Harada-Carr's definition. We already saw in the above that there is a point satisfying $\theta_+ = \theta_- = 0$ or the region satisfying

$\theta_+\theta_- > 0$ which are the inside surrounded by \mathcal{H}_\pm . In this region, it has the minimal radius at $r = r_*$ on $\tau = \text{constant}$ slices. And we know that the surface of $r = r_*$ is minimal. Then, we can see that the throat exists on the DGP brane and locates at $r = r_*$.

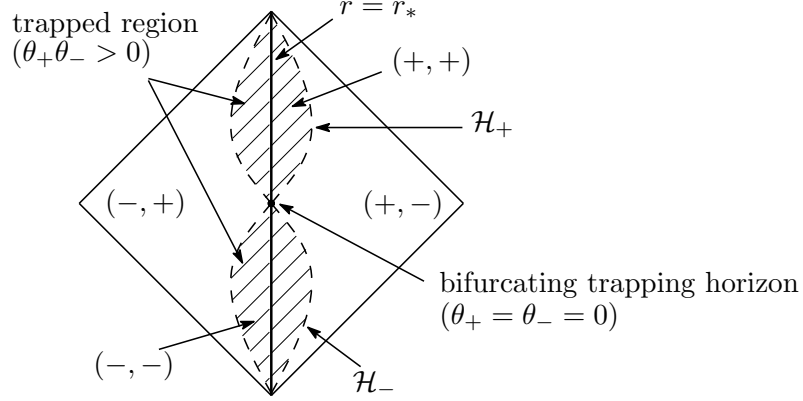


Figure 4.1: The Penrose diagram of the DGP wormhole. The slashed regions are the trapped regions $\theta_+\theta_- > 0$ and the dot at the center is the bifurcating trapping horizon $\theta_+ = \theta_- = 0$. The dashed lines are the time-like hypersurfaces \mathcal{H}_\pm . The bold line is $r = r_*$. The symbol (\cdot, \cdot) means the signature of (θ_+, θ_-) .

By the way, the trajectory of $r = \text{constant}$ is not "static" even at $r = \infty$. In the Maeda-Harada-Carr definition, the throat depends on slices. Then, it is better to change the coordinate to the asymptotically flat one. As an example, we introduce a new coordinate (T, R) defined as

$$T = rh(r) \sinh \tau, \quad R = rh(r) \cosh \tau, \quad (4.9)$$

where

$$\log(h(r)) = \int \frac{1-\gamma}{\gamma r} dr \quad (4.10)$$

and

$$\lim_{r \rightarrow \infty} h(r) = 1. \quad (4.11)$$

In this new coordinate, the metric (4.1) becomes

$$ds^2 = h^{-2}(-dT^2 + dR^2 + R^2 d\Omega_2^2). \quad (4.12)$$

We look for $R = R_*$ that minimizes the area $4\pi R^2 h^{-2}(R, T)$ on $T = \text{constant}$

slices. On the minimal surface,

$$\begin{aligned}
0 &= \partial_R(Rh^{-1}) \\
&= h^{-1} \left\{ 1 - (1 - \gamma) \frac{R^2}{h^2 r^2} \right\} \\
&= h^{-1} \left\{ 1 - (1 - \gamma) \frac{R^2}{R^2 - T^2} \right\}
\end{aligned} \tag{4.13}$$

must hold. From this, we can see

$$(1 - \gamma(R_*, T)) R_*^2 = R_*^2 - T^2. \tag{4.14}$$

It is difficult to solve Eq. (4.14) because we can not have the explicit expression for $\gamma(R, T)$. However, in the (r, τ) coordinate, we can consider the trajectory $r = r_{\min}(\tau)$ satisfying Eq. (4.14) as

$$r_{\min}^2(\tau) = r_c^2(1 - \tanh^4 \tau) + r_0^2(1 - \tanh^4 \tau)^{-1}. \tag{4.15}$$

If we adopt Maeda-Harada-Carr's throat definition, we regard this as the location of the throat on $T = \text{constant}$ slices.

We explain that the throat $r = r_{\min}(\tau)$ on the induced metric (4.1) is in the interval $r_* \leq r_{\min}(\tau) \leq r_{\mathcal{H}_{\pm}}(\tau)$, where the equality holds if and only if $\tau = 0$. Here, $r = r_{\mathcal{H}_{\pm}}(\tau)$ are the trajectories of the timelike hypersurface \mathcal{H}_{\pm} [34].

For $r_0 > r_c$, we can see

$$\begin{aligned}
r_{\min}^2(\tau) - r_*^2 &= -r_c^2 \tanh^4 \tau + r_0^2 \frac{\tanh^4 \tau}{1 - \tanh^4 \tau} \\
&\geq -r_0^2 \tanh^4 \tau + r_0^2 \frac{\tanh^4 \tau}{1 - \tanh^4 \tau} \\
&= r_0^2 \frac{\tanh^8 \tau}{1 - \tanh^4 \tau} \geq 0,
\end{aligned} \tag{4.16}$$

where the equality holds if and only if $\tau = 0$.

On the hypersurface \mathcal{H}_{\pm} ,

$$\gamma^2|_{\mathcal{H}_{\pm}} = \tanh^2 \tau \tag{4.17}$$

holds. Then, the trajectory of \mathcal{H}_{\pm} satisfies

$$r_{\mathcal{H}_{\pm}}^2(\tau) = \frac{r_c^2}{\cosh^2 \tau} + r_0^2 \cosh^2 \tau. \tag{4.18}$$

Therefore,

$$r_{\mathcal{H}_{\pm}}^2(\tau) - r_{\min}^2(\tau) = \frac{\sinh^2 \tau}{1 + \tanh^2 \tau} \left(r_0^2 - r_c^2 \frac{1 + \tanh^2 \tau}{\cosh^4 \tau} \right) \geq 0, \tag{4.19}$$

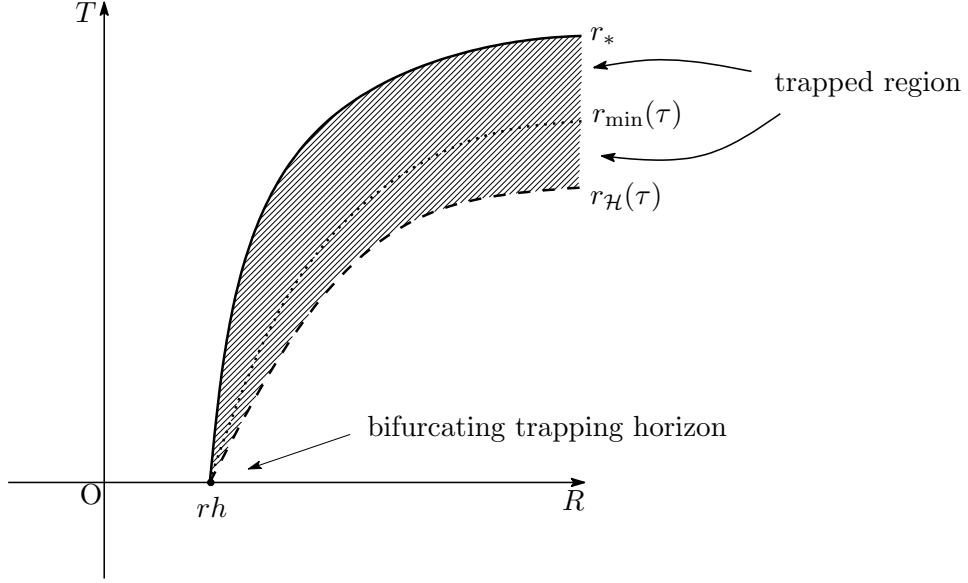


Figure 4.2: The throat locates in the trapped region or at the bifurcating trapping horizon. The surface $r_{\min}(\tau)$ does not cross the surfaces r_* and $r_{\mathcal{H}}(\tau)$.

where the equality holds if and only if $\tau = 0$.

We can see that the minimal surface r_{\min} on $T = \text{constant} (\neq 0)$ slices are in the trapped region (while that on $T = 0$ slice is at the bifurcating trapping horizon). Although the location of the minimal surface depends on time slice, it is not contrary to the expectation. To see it, we define \mathcal{K} as the trace of the extrinsic curvature of 2-dimensional surface S . We also define \tilde{t}^a as the timelike unit normal vector to S and \tilde{r}^a as the spacelike unit normal vector to S satisfying $\tilde{t}_a \tilde{r}^a = 0$. When we choose a different time slice, the spacelike normal vector is shifted to $\bar{r}^a = C\tilde{r}^a \pm \sqrt{C^2 - 1}\tilde{t}^a$, where C is a constant. Then,

$$\mathcal{K} = h^{ab} \nabla_a \tilde{r}_b = \frac{1}{C} \bar{\mathcal{K}} \mp \frac{\sqrt{C^2 - 1}}{C} h^{ab} \tilde{K}_{ab}, \quad (4.20)$$

where h_{ab} is the induced metric of S , $\bar{\mathcal{K}}$ is the trace of the extrinsic curvature of the 2-dimensional surface S on the shifted hypersurface and \tilde{K}_{ab} is the extrinsic curvature of the spacelike hypersurface orthogonal to \tilde{t}^a . Because $\mathcal{K} = 0$ has to be satisfied on the minimal surface, it depends on slices except for the time-symmetric slice of $\tilde{K}_{ab} = 0$.

Finally, we adopt Tomikawa-Izumi-Shiromizu's throat definition. We choose u, v as $u = T - R, v = T + R$ and $a(u, v) = h^{-1}(r)$. Then, Eq. (3.27)

gives us

$$C_u \{(1 - \gamma)e^\tau \cosh \tau - 1\}|_S = -C_v \{(1 - \gamma)e^{-\tau} \cosh \tau - 1\}|_S, \quad (4.21)$$

where C_u, C_v are positive constants and S is the throat candidate. From this,

$$\gamma^2 < \tanh^2 \tau \quad (4.22)$$

holds. Note that $\gamma^2 < \gamma^2|_{\mathcal{H}_\pm}$ is satisfied from Eq. (4.17).

We consider the flare-out condition. Eq. (3.28) becomes

$$r^a \nabla_a k|_S = \frac{2C_v^2 h^2 \{\gamma^2(1 - \gamma^2) \cosh^2 \tau + r\gamma\gamma' \sinh^2 \tau\}}{r^2 \{(1 - \gamma)e^\tau \cosh \tau - 1\}^2} \Big|_S. \quad (4.23)$$

We see that the flare-out condition $r^a \nabla_a k|_S > 0$ is satisfied everywhere because $0 \leq \gamma^2(r) < 1$ and $r\gamma\gamma' > 0$ hold.

We consider the traversability. From Eq. (3.29), the traversability requires

$$\begin{aligned} & \alpha [C_v^2 e^{-2\tau} \{r\gamma\gamma' + (1 - \gamma^2)\} + C_u C_v \{r\gamma\gamma' - (1 - \gamma^2)\}]|_S \\ & = \beta [C_u^2 e^{2\tau} \{r\gamma\gamma' + (1 - \gamma^2)\} + C_u C_v \{r\gamma\gamma' - (1 - \gamma^2)\}]|_S, \end{aligned} \quad (4.24)$$

where α, β are positive constants. Note that

$$r\gamma\gamma' - (1 - \gamma^2) = (1 - \gamma^2) \left(\frac{r^2}{\sqrt{r^4 - 4r_0^2 r_c^2}} - 1 \right) > 0 \quad (4.25)$$

holds. Then, α and β exist such that they satisfy Eq. (4.24). Therefore, there is the throat r_S in the region satisfying Eq. (4.22) and it depends on τ . In particular, if we take $C_u = C_v = 1$, the throat $r_S(\tau)$ is

$$r_S^2(\tau) = r_c^2(1 - \tanh^4 \tau) + r_0^2(1 - \tanh^4 \tau)^{-1} \quad (4.26)$$

from Eq. (4.21). This result coincides with the throat adopting Maeda-Harada-Carr's definition (Eq. (4.15)).

To summarize this section, the DGP brane whose induced metric is written by Eq. (4.1) has a wormhole structure even if we adopt any definitions of the throat. Moreover, this wormhole is an example without any exotic matters and initial singularity.

4.2 Non-exoticity

We did not introduce any exotic matters in this model. However, the null energy condition for the effective stress-energy tensor on the brane is not satisfied. To see this, we introduce the effective stress-energy tensor $T_{\mu\nu}^{(\text{eff})}$ defined as

$$T_{\mu\nu}^{(\text{eff})} := {}^{(4)}G_{\mu\nu}, \quad (4.27)$$

where ${}^{(4)}G_{\mu\nu}$ is the 4-dimensional Einstein tensor. For the DGP wormhole, ${}^{(4)}G_{\mu\nu}$ is computed as follows;

$${}^{(4)}G_{rr} = -\frac{3(1-\gamma^2)}{\gamma^2 r^2}, \quad (4.28)$$

$${}^{(4)}G_{ab} = -(1-\gamma^2 - 2r\gamma\gamma')\gamma_{ab}, \quad (4.29)$$

where $\gamma' = d\gamma/dr$, $\gamma_{ab} = -d\tau^2 + \cosh^2 \tau d\Omega_2^2$ and the indices a, b run over τ, θ, φ . If we write $T_{\hat{\mu}\hat{\nu}}^{(\text{eff})} = \text{diag}[\rho^{(\text{eff})}(r), p_r^{(\text{eff})}(r), p^{(\text{eff})}(r), p^{(\text{eff})}(r)]$ in the orthonormal coordinate, we have

$$\rho^{(\text{eff})}(r) = -p^{(\text{eff})}(r) = \frac{1}{r^2}(1-\gamma^2 - 2r\gamma\gamma'), \quad (4.30)$$

$$p_r^{(\text{eff})}(r) = -\frac{3}{r^2}(1-\gamma^2). \quad (4.31)$$

Because of $0 \leq \gamma^2 < 1$ and $1-\gamma^2 - 2r\gamma\gamma' < 0$, we can see that $\rho^{(\text{eff})} = -p^{(\text{eff})} < 0$ and $p_r^{(\text{eff})} < 0$. Then, from $\rho^{(\text{eff})} + p_r^{(\text{eff})} < 0$, we can see that the null energy condition is not satisfied.

Because of this, we see that the non-exoticity of DGP wormhole is consistent with the feature introduced in Sec. 2.2.2.

Chapter 5

Summary

For the wormhole throat, we gave a brief review of three typical definitions by Morris and Thorne, by Hochberg and Visser, and by Maeda, Harada and Carr. However there are no definitions which can cover a wide class of spacetimes. And the wormhole throat in their definitions depends on time or null slices. Furthermore, it is difficult to show mathematical properties of the throat. To solve these problems, we proposed a new definition of wormhole throat [33]. Then we can confirm that our definition can work for a wide class of spacetimes.

On the other hand, except for the cosmological wormhole, wormholes need some exotic matters violating the null energy condition at least at the throat. However, in the Dvali-Gabadadze-Porrati (DGP) braneworld model, we found a wormhole structure on the brane without exotic matters and initial singularity regardless of the throat definition [33, 34].

For the wormhole throat, the investigation on non-spherically symmetric spacetimes is left for future study. When we considered the DGP wormhole, we used the single-brane solution. However, the multi-branes solution also exists [36]. Then, we want to consider the wormhole in multi-branes case. It is also future study.

Bibliography

- [1] A. Einstein and N. Rosen, “The Particle Problem in the General Theory of Relativity,” *Phys. Rev.* **48**, 73 (1935).
- [2] M. S. Morris and K. S. Thorne, “Wormholes in spacetime and their use for interstellar travel: A tool for teaching general relativity,” *Am. J. Phys.* **56**, 395 (1988).
- [3] M. S. Morris, K. S. Thorne and U. Yurtsever, “Wormholes, Time Machines, and the Weak Energy Condition,” *Phys. Rev. Lett.* **61**, 1446 (1988).
- [4] M. Visser, “Lorentzian Wormholes: from Einstein to Hawking”, (New York, Springer-Verlag 1996).
- [5] F. S. N. Lobo, “Exotic solutions in General Relativity: Traversable wormholes and ‘warp drive’ spacetimes,” *Classical and Quantum Gravity Research*, 1-78, (Nova Sci. Pub., 2008).
- [6] L. Flamm, “Beiträge zur Einsteinschen Gravitationstheorie,” *Phys. Z.* **17**, 448 (1916).
- [7] K. Schwarzschild, “Über das Gravitationsfeld eines Massenpunktes nach der Einsteinschen Theorie,” *Sitzungsber. Preuss. Akad. Wiss. (Berlin)* **1916**, 189 (1916).
- [8] C. W. Misner and J. A. Wheeler, “Classical Physics as Geometry: Gravitation, Electromagnetism, Unquantized Charge, and Mass as Properties of Curved Empty Space,” *Annals Phys.* **2**, 525 (1957).
- [9] E. Teo, “Rotating traversable wormholes,” *Phys. Rev. D* **58**, 024014 (1998).
- [10] P. K. F. Kuhfittig, “Axially symmetric rotating traversable wormholes,” *Phys. Rev. D* **67**, 064015 (2003).
- [11] J. P. S. Lemos, F. S. N. Lobo and S. Quinet de Oliveira, “Morris-Thorne wormholes with a cosmological constant,” *Phys. Rev. D* **68**, 064004 (2003).

- [12] J. P. S. Lemos and F. S. N. Lobo, “Plane symmetric traversable wormholes in an anti-de Sitter background,” *Phys. Rev. D* **69**, 104007 (2004).
- [13] P. K. F. Kuhfittig, “Cylindrically symmetric wormholes,” *Phys. Rev. D* **71**, 104007 (2005).
- [14] K. A. Bronnikov and J. P. S. Lemos, “Cylindrical wormholes,” *Phys. Rev. D* **79**, 104019 (2009).
- [15] T. A. Roman, “Inflating Lorentzian wormholes,” *Phys. Rev. D* **47**, 1370 (1993).
- [16] S. Kar and D. Sahdev, “Evolving Lorentzian wormholes,” *Phys. Rev. D* **53**, 722 (1996).
- [17] S. -W. Kim, “Cosmological model with a traversable wormhole,” *Phys. Rev. D* **53**, 6889 (1996).
- [18] D. Hochberg and M. Visser, “Null Energy Condition in Dynamic Wormholes,” *Phys. Rev. Lett.* **81**, 746 (1998).
- [19] D. Hochberg and M. Visser, “Dynamic wormholes, antitrapped surfaces, and energy conditions,” *Phys. Rev. D* **58**, 044021 (1998).
- [20] S. A. Hayward, “Dynamic wormholes,” *Int. J. Mod. Phys. D* **08**, 373 (1999).
- [21] H. Maeda, T. Harada and B. J. Carr, “Cosmological wormholes,” *Phys. Rev. D* **79**, 044034 (2009).
- [22] M. S. R. Delgaty and R. B. Mann, “Traversable wormholes in (2+1) and (3+1) dimensions with a cosmological constant,” *Int. J. Mod. Phys. D* **04**, 231 (1995).
- [23] K. A. Bronnikov and S. W. Kim, “Possible wormholes in a brane world,” *Phys. Rev. D* **67**, 064027 (2003).
- [24] F. S. N. Lobo, “General class of braneworld wormholes,” *Phys. Rev. D* **75**, 064027 (2007).
- [25] J. P. de Leon, “Static wormholes on the brane inspired by Kaluza-Klein gravity,” *JCAP* **0911**, 013 (2009).
- [26] D. Hochberg and M. Visser, “Geometric structure of the generic static traversable wormhole throat,” *Phys. Rev. D* **56**, 4745 (1997).
- [27] M. Visser and D. Hochberg, “Generic wormhole throats,” In **Haifa 1997, Internal structure of black holes and spacetime singularities** 249-295 [gr-qc/9710001].

- [28] M. Visser, S. Kar and N. Dadhich, “Traversable Wormholes with Arbitrarily Small Energy Condition Violations,” *Phys. Rev. Lett.* **90**, 201102 (2003).
- [29] P. K. F. Kuhfittig, “Can a wormhole supported by only small amounts of exotic matter really be traversable?,” *Phys. Rev. D* **68**, 067502 (2003).
- [30] C. J. Fewster and T. A. Roman, “On wormholes with arbitrarily small quantities of exotic matter,” *Phys. Rev. D* **72**, 044023 (2005).
- [31] S. A. Hayward, “Wormhole dynamics in spherical symmetry,” *Phys. Rev. D* **79**, 124001 (2009).
- [32] D. Hochberg and M. Visser, “General Dynamic Wormholes and Violation of the Null Energy Condition,” *gr-qc/9901020*.
- [33] Y. Tomikawa, K. Izumi and T. Shiromizu, “New definition of a wormhole throat,” *Phys. Rev. D* **91**, 104008 (2015).
- [34] Y. Tomikawa, T. Shiromizu and K. Izumi, “Wormhole on DGP brane,” *Phys. Rev. D* **90**, 126001 (2014).
- [35] G. R. Dvali, G. Gabadadze and M. Porrati, “4D Gravity on a Brane in 5D Minkowski Space,” *Phys. Lett. B* **485**, 208 (2000).
- [36] K. Izumi and T. Shiromizu, “DGP braneworld with a bubble of nothing,” *Phys. Rev. D* **90**, 046005 (2014).
- [37] C. W. Misner, K. S. Thorne and J. A. Wheeler, “Gravitation,” (San Francisco, W.H.Freeman 1973).
- [38] R. M. Wald, “General Relativity,” (Chicago, Chicago Univ. Press 1984).
- [39] S. A. Hayward, “General laws of black-hole dynamics,” *Phys. Rev. D* **49**, 6467 (1994).
- [40] For example, S. Weinberg, “Cosmology,” (Oxford, Oxford Univ. Press 2008).
- [41] For example, J. J. Callahan, “The Geometry of Spacetime: An Introduction to Special and General Relativity,” (New York, Springer-Verlag 2000).
- [42] S. A. Hayward, “Gravitational energy in spherical symmetry,” *Phys. Rev. D* **53**, 1938 (1996).
- [43] H. G. Ellis, “Ether flow through a drainhole: A particle model in general relativity,” *J. Math. Phys.* **14**, 104 (1973).