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Compositional Z: Confluence Proofs for Permutative Conversion

Abstract. This paper gives new confluence proofs for several lambda calculi with permutation-like reduction, including lambda calculi corresponding to intuitionistic and classical natural deduction with disjunction and permutative conversions, and a lambda calculus with explicit substitutions. For lambda calculi with permutative conversion, naïve parallel reduction technique does not work, and (if we consider untyped terms, and hence we do not use strong normalization) traditional notion of residuals is required as Ando pointed out. This paper shows that the difficulties can be avoided by extending the technique proposed by Dehornoy and van Oostrom, called the Z theorem: existence of a mapping on terms with the Z property concludes the confluence. Since it is still hard to directly define a mapping with the Z property for the lambda calculi with permutative conversions, this paper extends the Z theorem to compositional functions, called compositional Z, and shows that we can adopt it to the calculi.

Keywords: lambda calculus, lambda-mu calculus, confluence, permutative conversion.

1. Introduction

The permutative conversion was introduced by Prawitz [12] as one of proof normalization processes for the natural deduction with disjunctions and existential quantifiers. It permutes order of applications of elimination rules, and then normal proofs have some nice properties such as the subformula property.

The rules of the permutative conversion are quite simple, but the combination of it with the β-reduction makes confluence proofs complicated if we do not depend on strong normalization, as Ando discussed in [2]. He pointed out that an extension of Parigot's $\lambda\mu$-calculus [11] with permutative conversion brings some big troubles with confluence proofs. Baba et al. [3] also discussed similar problem. They proved confluence of some variants of the $\lambda\mu$-calculus with the so-called renaming reductions. The structural reduction (or the μ-reduction) of the $\lambda\mu$-calculus is a variant of the permutative conversion, and they pointed out that the combination of the structural reduction and the renaming reduction makes a trouble in confluence proofs

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and it requires a modification of the parallel reduction.

From their observation, we can see that the difficulties are caused by the following. First, we cannot naïvely adopt the parallel reduction technique of Tait and Martin-Löf, since a parallel reduction defined in an ordinary way does not have the diamond property. Therefore, Ando generalized the parallel reduction with the notion of the segment trees. Secondly, it is also difficult to adopt Takahashi's technique with complete development [13], and Ando used traditional notion of the residuals [4] to define the complete development.

This paper shows that we can avoid these troubles by adapting another proof technique for confluence proposed by Dehornoy and van Oostrom [5], called the *Z theorem*: if there is a mapping which satisfies the *Z property*, then the reduction system is confluent. A major candidate for the mapping with the Z property is the complete development used in Takahashi's proof, and hence defining such a mapping is still hard. In this paper, we extend the Z theorem to compositional functions, called the *compositional Z*, and show that a mapping satisfying the Z property can be easily defined as a composition of two complete developments for the β-reduction and the permutative conversion, respectively. The compositional Z can be adopted to several λ-calculi with permutative conversion such as the λ-calculus extended with disjunctions, the $\lambda\mu$-calculus with disjunctions, and a λ-calculus with explicit substitutions.

2. Compositional Z

First, we summarize Dehornoy and van Oostrom's Z theorem, and then extend it for compositional functions, called the *compositional Z*. It gives a sufficient condition for that a compositional function satisfies the Z property, and enables us to consider a reduction system by dividing into two parts to prove confluence.

Definition 2.1 ((Weak) Z property)**.** Let $(A, \to)$ be an abstract rewriting system, and $\twoheadrightarrow$ be the reflexive transitive closure of $\to$. Let $\to_{\mathsf{x}}$ be another relation on A, and $\twoheadrightarrow_{\mathsf{x}}$ be its reflexive transitive closure.

1. A mapping f satisfies the *weak Z property for* $\to$ *by* $\to_{\mathsf{x}}$ if $a \to b$ implies $b \twoheadrightarrow_{\mathsf{x}} f(a) \twoheadrightarrow_{\mathsf{x}} f(b)$ for any $a, b \in A$.

2. A mapping f satisfies the *Z property for* $\to$ if it satisfies the weak Z property by $\to$ itself.

When f satisfies the (weak) Z property, we also say that f *is (weakly) Z*.

$$
\begin{tikzcd}
a \arrow[r, "1"] & b \arrow[dl, dotted, twoheadrightarrow, "1"'] \\
f_1(a) \arrow[r, dotted, twoheadrightarrow, "1"] \arrow[d, dotted, twoheadrightarrow] & f_1(b) \\
f_2(f_1(a)) \arrow[r, dotted, twoheadrightarrow] & f_2(f_1(b))
\end{tikzcd}
\qquad
\begin{tikzcd}
a \arrow[r, "2"] & b \arrow[dl, dotted, twoheadrightarrow] \\
f_2(f_1(a)) \arrow[r, dotted, twoheadrightarrow] & f_2(f_1(b))
\end{tikzcd}
$$

Figure 1. Proof of Theorem 2.3

It becomes clear why we call it the Z property when we draw the condition as the following diagram.

$$
\begin{tikzcd}
a \arrow[r] & b \arrow[dl, dotted, twoheadrightarrow] \\
f(a) \arrow[r, dotted, twoheadrightarrow] & f(b)
\end{tikzcd}
$$

Theorem 2.2 (Z theorem [5]). *If there exists a mapping satisfying the Z property for an abstract rewriting system, then it is confluent.*

This theorem has been applied to confluence proofs for some variants of λ-calculus in [5, 9, 1, 10]. In fact, we can often prove that the usual complete developments have the Z property.

The compositional Z is the following, which is easily proved from Theorem 2.2 with the diagrams in Figure 1.

Theorem 2.3 (Compositional Z). *Let* $(A, \to)$ *be an abstract rewriting system, and* $\to$ *be* $\to_1 \cup \to_2$. *If there exist mappings* $f_1, f_2 : A \to A$ *such that*

(a) f_1 *is Z for* $\to_1$

(b) $a \to_1 b$ *implies* $f_2(a) \twoheadrightarrow f_2(b)$

(c) $a \twoheadrightarrow f_2(a)$ *holds for any* $a \in \mathrm{Im}(f_1)$

(d) $f_2 \circ f_1$ *is weakly Z for* $\to_2$ *by* $\to$,

then $f_2 \circ f_1$ *is Z for* $(A, \to)$, *and hence* $(A, \to)$ *is confluent.*

One easy example of the compositional Z is a confluence proof for the $\beta\eta$-reduction on the untyped λ-calculus (although it can be directly proved by the Z theorem as in [9]). Let $\to_1 = \to_\eta$, $\to_2 = \to_\beta$, and f_1 and f_2 be the usual complete developments of η and β, respectively. Then, it is easy to see the conditions of the compositional Z hold. The point is that we can forget the other reduction in the definition of each complete development.

Furthermore, we have another sufficient condition for the Z property of compositional functions as follows. It is a special case of the compositional Z where $f_1(a) = f_1(b)$ holds for any $a \to_1 b$. All of the examples (except for $\beta\eta$ above) of the application of compositional Z in the following sections are in this case.

Corollary 2.4. *Let $(A, \to)$ be an abstract rewriting system, and $\to$ be $\to_1 \cup \to_2$. Suppose that there exist mappings $f_1, f_2 : A \to A$ such that*

(a) $a \to_1 b$ implies $f_1(a) = f_1(b)$

(b) $a \twoheadrightarrow_1 f_1(a)$ for any a

(c) $a \twoheadrightarrow f_2(a)$ holds for any $a \in \mathrm{Im}(f_1)$

(d) $f_2 \circ f_1$ is weakly Z for $\to_2$ by $\to$.

Then, $f_2 \circ f_1$ is Z for $(A, \to)$, and hence $(A, \to)$ is confluent.

Proof. It is easily proved from Theorem 2.3. The condition (a) in Theorem 2.3 comes from the new conditions (a) and (b), and (b) in Theorem 2.3 is not necessary since we have $f_2(f_1(a)) = f_2(f_1(b))$ for any $a \to_1 b$. □

Corollary 2.4 can be seen as generalization of the *Z property modulo*, proposed by Accattoli and Kesner [1]. For an abstract rewriting system $(A, \to)$ and an equivalence relation $\sim$ on A, the reduction modulo $\sim$, denoted $a \to_{\sim} b$, is defined as $a \sim c \to c' \sim b$ for some c and c'. The Z property modulo says that it is a sufficient condition for the confluence of $\to_{\sim}$ that there exists a mapping which is well-defined on $\sim$ and weakly Z for $\to$ by $\to_{\sim}$. If we consider $\sim$ as the first reduction relation $\to_1$, and define $f_1(a)$ as a fixed representative of the equivalence class including a, then the conditions of the Z property modulo implies the conditions of the compositional Z, since the reflexive transitive closure of $\to \cup \sim$ is $\twoheadrightarrow_{\sim}$.

3. Intuitionistic natural deduction with disjunction

3.1. Calculus

The following is the definition of the (untyped) terms (denoted by M, N,...), eliminators (denoted by e,...), and the reduction rules for the first-order natural deduction, where a ranges over the first-order variables, and t over the first-order terms. We call the system λ_{NJ}.

$$M ::= x \mid \lambda x.M \mid \langle M, M\rangle \mid \iota_1 M \mid \iota_2 M \mid \lambda\mathsf{a}.M \mid \langle M, \mathsf{t}\rangle \mid Me$$
$$e ::= M \mid \pi_1 \mid \pi_2 \mid [x.M, x.M] \mid \mathsf{t} \mid [x\mathsf{a}.M]$$

$$
\begin{array}{rcll}
(\lambda x.M)N & \to & M[x := N] & (\beta_{\mathsf{I}}) \\
\langle M_1, M_2\rangle \pi_i & \to & M_i & (\beta_{\mathsf{C}}) \\
(\iota_i M)[x_1.N_1, x_2.N_2] & \to & N_i[x_i := M] & (\beta_{\mathsf{D}}) \\
(\lambda \mathsf{a}.M)\mathsf{t} & \to & M[\mathsf{a} := \mathsf{t}] & (\beta_{\mathsf{A}}) \\
\langle M, \mathsf{t}\rangle[x\mathsf{a}.N] & \to & N[x, \mathsf{a} := M, \mathsf{t}] & (\beta_{\mathsf{E}}) \\
M[x_1.N_1, x_2.N_2]e & \to & M[x_1.N_1e, x_2.N_2e] & (\pi_{\mathsf{D}}) \\
M[x\mathsf{a}.N]e & \to & M[x\mathsf{a}.Ne] & (\pi_{\mathsf{E}})
\end{array}
$$

The last two rules (π_{D}) and (π_{E}) are called *permutative conversion*, or just *permutation*. We adopt the notion of the eliminators to write the permutative conversion for several destructors in a uniform way. As usual, the juxtaposition notation Me is supposed to be left associative, and hence Me_1e_2 denotes $(Me_1)e_2$. Bound variables are defined as usual. In particular, variable occurrences of x_1 (and x_2) in N_1 (and N_2, resp.) are supposed to be bound in the term $M[x_1.N_1, x_2.N_2]$. We can freely rename the bound variables. In the π-rules, we assume capture-avoiding conditions, that is, e must not contain either x_1 or x_2 freely in the rule (π_{D}) and e must not contain either x or a freely in the rule (π_{D}). In this paper, we consider untyped terms including ill-typed ones such as $(\iota_1 M)N$ and $(\lambda x.M)[x_1.N_1, x_2.N_2]$. They do not affect anything since there is no applicable reduction rule.

This is the whole calculus of λ_{NJ}, but we can discuss the essence of our idea in the following simple subcalculus λ^-_{NJ}, which has terms only for implications and "unary" disjunctions. The discussion in this paper can be extended to λ_{NJ} in a straightforward way.

Definition 3.1 (λ^-_{NJ})**.** The terms of λ^-_{NJ} are defined as follows.

$$
\text{(terms)} \quad M ::= x \mid \lambda x.M \mid \iota M \mid Me \qquad \text{(eliminators)} \quad e ::= M \mid [x.M]
$$

The reduction rules for λ^-_{NJ} are the following.

$$
\begin{array}{rcll}
(\lambda x.M)N & \to & M[x := N] & (\beta_{\mathsf{I}}) \\
(\iota M)[x.N] & \to & N[x := M] & (\beta_{\mathsf{D}}) \\
M[x.N]e & \to & M[x.Ne] & (\pi)
\end{array}
$$

The relation $\to_\beta$ is the compatible closure of (β_{I}) and (β_{D}), and $\to_\pi$ is similarly defined from (π). The relation $\to$ is the union of $\to_\beta$ and $\to_\pi$, and $\twoheadrightarrow$ is its reflexive transitive closure.

$$
\begin{tikzcd}
 & M_0 = M[x.N][y.L]e \arrow[dl, "\pi"'] \arrow[dr, "\pi"] & \\
M_1 = M[x.N][y.Le] \arrow[dr, two heads, "\pi"'] & & M[x.N[y.L]]e = M_2 \arrow[dl, two heads, "\pi"] \\
 & M_3 = M[x.N[y.Le]] &
\end{tikzcd}
$$

Figure 2. Critical pair induced by the π-reduction

3.2. Problems on confluence proof

In confluence proofs for $\lambda^{-}_{\mathsf{NJ}}$, no matter whether we adopt the traditional parallel reduction or the original Z theorem (Theorem 2.2), the permutation raises some difficulties. A common reason can be explained by the example in Figure 2. The term M_3 is the join point from M_1 and M_2. Hence, if we define a parallel reduction satisfying the diamond property, it has to contain $M_2 \twoheadrightarrow_\pi M_3$ as one-step, whereas in the reduction sequence

$$M_2 = M[x.N[y.L]]e \rightarrow_\pi M[x.N[y.L]e] \rightarrow_\pi M[x.N[y.Le]] = M_3$$

the π-redex $N[y.L]e$ of the second step does not occur in M_2, and it is not a simple extension of the usual parallel reduction. This example also shows that, if we want to find a mapping f satisfying the Z property, we have to define $f(M_0)$ as $M_1 \twoheadrightarrow f(M_0)$ and $M_2 \twoheadrightarrow f(M_0)$ hold, and then $M_3 \twoheadrightarrow f(M_0)$. It means that we have to do the permutation completely in f. This observation leads the following definition.

Definition 3.2. The *complete permutation* $M@e$ is defined as follows.

$$
\begin{aligned}
M[x.N]@e &= M[x.N@e] \\
M@e &= Me & \qquad (M \neq M'[x.N'])
\end{aligned}
$$

Then, we expect that a complete development with the complete permutation can be defined as follows and it is Z:

$$
\begin{aligned}
x^\bullet &= x & \qquad M^\bullet_{\mathsf{E}} &= M^\bullet \\
(\lambda x.M)^\bullet &= \lambda x.M^\bullet & \qquad [x.N]^\bullet_{\mathsf{E}} &= [x.N^\bullet]. \\
(\iota M)^\bullet &= \iota M^\bullet \\
(\lambda x.M)N^\bullet &= M^\bullet[x := N^\bullet] \\
(\iota M)[x.N]^\bullet &= N^\bullet[x := M^\bullet] \\
Me^\bullet &= M^\bullet @ e^\bullet_{\mathsf{E}} \quad \text{(o.w.)}
\end{aligned}
$$

However, this naïve definition does not work. Let $N_1 = (\iota(x[y.y]))[z.z]w$, and $N_2 = (\iota(x[y.y]))[z.zw]$. Then, we have $N_1 \to_\pi N_2$ and

$$N_1{}^\bullet = (x[y.y])@w = x[y.yw] \qquad N_2{}^\bullet = (zw)^\bullet[z := x[y.y]] = x[y.y]w.$$

Hence this mapping is not Z since $N_1^\bullet \twoheadrightarrow N_2{}^\bullet$ does not hold. The reason of the failure is that the π-redex $(x[y.y])w$ produced by the β-reduction is also reduced in $N_1{}^\bullet$.

3.3. Confluence by compositional Z

The compositional Z can be used to solve the problem.

Definition 3.3. The mappings M^P and e^P_E are inductively defined as follows.

$$\begin{aligned} x^\mathsf{P} &= x & M^\mathsf{P}_\mathsf{E} &= M^\mathsf{P} \\ (\lambda x.M)^\mathsf{P} &= \lambda x.M^\mathsf{P} & [x.N]^\mathsf{P}_\mathsf{E} &= [x.N^\mathsf{P}] \\ (\iota M)^\mathsf{P} &= \iota M^\mathsf{P} \\ (Me)^\mathsf{P} &= M^\mathsf{P}@e^\mathsf{P}_\mathsf{E} \end{aligned}$$

The mappings M^B and e^B_E are defined as follows.

$$\begin{aligned} x^\mathsf{B} &= x & M^\mathsf{B}_\mathsf{E} &= M^\mathsf{B} \\ (\lambda x.M)^\mathsf{B} &= \lambda x.M^\mathsf{B} & [x.N]^\mathsf{B}_\mathsf{E} &= [x.N^\mathsf{B}] \\ (\iota M)^\mathsf{B} &= \iota M^\mathsf{B} \\ ((\lambda x.M)N)^\mathsf{B} &= M^\mathsf{B}[x := N^\mathsf{B}] \\ ((\iota M)[x.N])^\mathsf{B} &= N^\mathsf{B}[x := M^\mathsf{B}] \\ (Me)^\mathsf{B} &= M^\mathsf{B}e^\mathsf{B}_\mathsf{E} \quad \text{(o.w.)} \end{aligned}$$

We define $M^\mathsf{PB} = (M^\mathsf{P})^\mathsf{B}$.

Note that, in the definition of $(\cdot)^\mathsf{P}$, we consider only π, and not β. Similarly, We do not need to consider π in the definition of $(\cdot)^\mathsf{B}$.

Then, we can use Corollary 2.4 to show the confluence of λ^-_NJ with the help of the following lemmas.

Lemma 3.4. *1.* $Me \twoheadrightarrow_\pi M@e$.
2. $M@[x.N]@e = M@[x.N@e]$.
3. $(M@e)[x := N] \twoheadrightarrow_\pi M[x := N]@e[x := N]$.
4. $M \to M'$ *implies* $M@e \twoheadrightarrow M'@e$.
5. $e \to e'$ *implies* $M@e \twoheadrightarrow M@e'$.

Proof. 1. By induction on M. The only nontrivial case is the following, where $M = P[x.Q]$.

$$\begin{aligned} P[x.Q]e &\to_\pi P[x.Qe] \\ &\twoheadrightarrow_\pi P[x.Q@e] && \text{(I.H.).} \end{aligned}$$

2. By induction on M. The only nontrivial case is the following, where $M = P[y.Q]$.

$$\begin{aligned} (P[y.Q])@[x.N]@e &= P[y.Q@[x.N]@e] \\ &= P[y.Q@[x.N@e]] && \text{(I.H.)} \\ &= (P[y.Q])@[x.N@e]. \end{aligned}$$

3. By induction on M. We use θ to denote the substitution $[x := N]$. Interesting cases are the following.

(Case $M = P[y.Q]$) We have the following.

$$\begin{aligned} (M@e)\theta &= (P\theta)[y.(Q@e)\theta] \\ &\twoheadrightarrow_\pi (P\theta)[y.Q\theta@e\theta] && \text{(I.H.)} \\ &= (P\theta[y.Q\theta])@e\theta \\ &= M\theta@e\theta. \end{aligned}$$

(Case $M = x$ and $N = P[y.Q]$) We have the following.

$$\begin{aligned} (M@e)\theta &= (xe)\theta \\ &= P[y.Q]e\theta \\ &\to_\pi P[y.Q(e\theta)] \\ &\twoheadrightarrow_\pi P[y.Q@e\theta] && (1) \\ &= x\theta@e\theta. \end{aligned}$$

4. By induction on $M \to M'$. The only nontrivial cases are the following.

(Case $(\iota P)[x.Q] \to_\beta Q[x := P]$) We suppose that x does not occur in e.

$$\begin{aligned} (\iota P)[x.Q]@e &= (\iota P)[x.Q@e] \\ &\to (Q@e)[x := P] \\ &\twoheadrightarrow_\pi Q[x := P]@e && (3). \end{aligned}$$

(Case $P[x.Q][y.R] \to_\pi P[x.Q[y.R]]$)

$$\begin{aligned} P[x.Q][y.R]@e &= P[x.Q][y.R@e] \\ &\to_\pi P[x.Q[y.R@e]] \\ &= P[x.Q[y.R]]@e. \end{aligned}$$

5. By induction on $e \to e'$. $\square$

Lemma 3.5. *$M \to_\pi N$ implies $M^{\mathsf{P}} = N^{\mathsf{P}}$.*

Proof. By induction on $\to_\pi$. In the case of π-redex, we have the following.

$$\begin{aligned}
(P[x.Q]e)^{\mathsf{P}} &= P^{\mathsf{P}}@[x.Q^{\mathsf{P}}]@e^{\mathsf{P}}_{\mathsf{E}} \\
&= P^{\mathsf{P}}@[x.Q^{\mathsf{P}}@e^{\mathsf{P}}_{\mathsf{E}}] && (3.4.2) \\
&= (P[x.Qe])^{\mathsf{P}}.
\end{aligned}$$

In the case of $M = Pe$ and $N = P'e'$, we have the following.

$$\begin{aligned}
(Pe)^{\mathsf{P}} &= P^{\mathsf{P}}@e^{\mathsf{P}}_{\mathsf{E}} \\
&= P'^{\mathsf{P}}@e'^{\mathsf{P}}_{\mathsf{E}} && \text{(I.H.)} \\
&= P'@e'^{\mathsf{P}}.
\end{aligned}$$

□

Lemma 3.6. *The following hold for $\langle \mathsf{X}, \xi \rangle \in \{\langle \mathsf{P}, \pi \rangle, \langle \mathsf{B}, \beta \rangle\}$.*
1. $M^{\mathsf{X}}[x := N^{\mathsf{X}}] \twoheadrightarrow_\xi (M[x := N])^{\mathsf{X}}$.
2. $M^{\mathsf{X}} e^{\mathsf{X}}_{\mathsf{E}} \twoheadrightarrow_\xi (Me)^{\mathsf{X}}$.

Proof. 1. By induction and case analysis on M. The only nontrivial cases are those where some redexes are created by substitutions.

($\mathsf{X} = \mathsf{P}$) The case where $M = xe$ and $N = P[y.Q]$ is proved as follows.

$$\begin{aligned}
M^{\mathsf{P}}[x := N^{\mathsf{P}}] &= (P^{\mathsf{P}}@[y.Q^{\mathsf{P}}])e^{\mathsf{P}}_{\mathsf{E}}[x := N^{\mathsf{P}}] \\
&\twoheadrightarrow_\pi (P^{\mathsf{P}}@[y.Q^{\mathsf{P}}])(e[x := N])^{\mathsf{P}}_{\mathsf{E}} && \text{(I.H.)} \\
&\twoheadrightarrow_\pi P^{\mathsf{P}}@[y.Q^{\mathsf{P}}]@(e[x := N])^{\mathsf{P}}_{\mathsf{E}} && (3.4.1) \\
&= (P[y.Q]e[x := N])^{\mathsf{P}}.
\end{aligned}$$

($\mathsf{X} = \mathsf{B}$) The case where $M = xP$ and $N = \lambda y.Q$ is proved as follows.

$$\begin{aligned}
M^{\mathsf{B}}[x := N^{\mathsf{B}}] &= (\lambda y.Q^{\mathsf{B}})P^{\mathsf{B}}[x := N^{\mathsf{B}}] \\
&\twoheadrightarrow_\beta (\lambda y.Q^{\mathsf{B}})(P[x := N])^{\mathsf{B}} && \text{(I.H.)} \\
&\to_\beta Q^{\mathsf{B}}[y := (P[x := N])^{\mathsf{B}}] \\
&= (\lambda y.Q)P[x := N]^{\mathsf{B}}.
\end{aligned}$$

The case where $M = x[y.P]$ and $N = \iota Q$ is similar.

2. We have $M^{\mathsf{X}} e^{\mathsf{X}}_{\mathsf{E}} = (xe^{\mathsf{X}}_{\mathsf{E}})[x := M^{\mathsf{X}}] = (xe)^{\mathsf{X}}[x := M^{\mathsf{X}}] \twoheadrightarrow_\xi (Me)^{\mathsf{X}}$ by 1. □

Lemma 3.7. *For any* $\mathsf{X} \in \{\mathsf{P}, \mathsf{B}\}$*, if* $M \to N$ *holds, then we have* $M^{\mathsf{X}} \twoheadrightarrow N^{\mathsf{X}}$*.*

Proof. $(\mathsf{X} = \mathsf{P})$ The case where $M \to_\pi N$ immediately follows from Lemma 3.5. The case where M is a β-redex is proved as follows.

$$\begin{aligned}((\lambda x.P)Q)^{\mathsf{P}} &= (\lambda x.P^{\mathsf{P}})Q^{\mathsf{P}} \\ &= P^{\mathsf{P}}[x := Q^{\mathsf{P}}] \\ &\twoheadrightarrow_\pi (P[x := Q])^{\mathsf{P}} && (3.6.1).\end{aligned}$$

The case of $Me \to M'e'$ is proved by 4 and 5 of Lemma 3.4.

$(\mathsf{X} = \mathsf{B})$ The only nontrivial case is the following: $M = (\iota P)[x.Q][y.R]$, $N = (\iota P)[x.Q[y.R]]$, and $M \to_\pi N$. In this case, we have the following.

$$\begin{aligned}M^{\mathsf{B}} &= Q^{\mathsf{B}}[x := P^{\mathsf{B}}][y.R^{\mathsf{B}}] \\ &= (Q^{\mathsf{B}}[y.R^{\mathsf{B}}])[x := P^{\mathsf{B}}] && x \notin FV([y.R^{\mathsf{B}}]) \\ &\twoheadrightarrow_\beta (Q[y.R])^{\mathsf{B}}[x := P^{\mathsf{B}}] && (3.6.2) \\ &= N^{\mathsf{B}}.\end{aligned}$$

□

Theorem 3.8 (Confluence of λ^-_{NJ})**.** λ^-_{NJ} *is confluent.*

Proof. By Corollary 2.4, it is sufficient to prove the following.

(a) $M \to_\pi N$ implies $M^{\mathsf{P}} = N^{\mathsf{P}}$.

(b) $M \twoheadrightarrow_\pi M^{\mathsf{P}}$ holds for any M.

(c) $M \twoheadrightarrow M^{\mathsf{B}}$ holds for any M.

(d) $M \to_\beta N$ implies $N \twoheadrightarrow M^{\mathsf{PB}} \twoheadrightarrow N^{\mathsf{PB}}$.

(a) is Lemma 3.5. (b) and (c) are straightforward by induction on M. (d) $M^{\mathsf{PB}} \twoheadrightarrow N^{\mathsf{PB}}$ follows from Lemma 3.7. For $N \twoheadrightarrow M^{\mathsf{PB}}$, it is proved by induction on $M \to_\beta N$. The cases of β-redexes are easy by the fact $M \twoheadrightarrow M^{\mathsf{PB}}$ for any M, which follows from (b) and (c). The case where $M = Pe$ and $N = P'e'$ is proved as follows.

$$\begin{aligned}P'e' &\twoheadrightarrow P'^{\mathsf{PB}} e'^{\mathsf{PB}}_{\mathsf{E}} && \text{(I.H.)} \\ &\twoheadrightarrow (P'^{\mathsf{P}} e'^{\mathsf{P}}_{\mathsf{E}})^{\mathsf{B}} && (3.6.2) \\ &\twoheadrightarrow (P'^{\mathsf{P}} @ e'^{\mathsf{P}}_{\mathsf{E}})^{\mathsf{B}} && (3.4.1,\ 3.7).\end{aligned}$$

□

$$\begin{tikzcd}
 & (\mu\alpha.M)[x.N]e \arrow[dl, "\pi"'] \arrow[dr, "\mu"] & \\
(\mu\alpha.M)[x.Ne] \arrow[dr, twoheadrightarrow, "\mu"'] & & (\mu\alpha.M[\alpha \Leftarrow [x.N]])e \arrow[dl, twoheadrightarrow, "\pi\mu"] \\
 & \mu\alpha.M[\alpha \Leftarrow [x.Ne]] &
\end{tikzcd}$$

Figure 3. Critical pair induced by the $\pi\mu$-reduction

4. Classical natural deduction with disjunction

The idea in the previous section can be extended to the Parigot's $\lambda\mu$-calculus [11]. As Ando's proof in [2], the proof of its confluence requires some complicated notions such as generalized parallel reduction, which is realized by means of the notion of segment trees, and residuals of redexes to define the complete development. The compositional Z makes the proof much simpler.

Definition 4.1 ($\lambda\mu^-_{\mathsf{NK}}$)**.** The terms of $\lambda\mu^-_{\mathsf{NK}}$ are the extension of those of λ^-_{NJ} as follows.

$$\text{(terms)} \quad M ::= \ \cdots \mid \mu\alpha.M \mid [\alpha]M$$

The following is the additional reduction rule.

$$(\mu\alpha.M)e \rightarrow \mu\alpha.M[\alpha \Leftarrow e] \qquad\qquad (\mu),$$

where the structural substitution $M[\alpha \Leftarrow e]$ is obtained by recursively replacing subterms of the form $[\alpha]N$ by $[\alpha]N[\alpha \Leftarrow e]e$. The relation $\rightarrow_{\pi\mu}$ is defined in a similar way to $\rightarrow_\beta$ from (π) and (μ).

In the following, we use the notation $M[\alpha \Leftarrow e_1, e_2, \cdots, e_n]$ to denote $M[\alpha \Leftarrow e_1][\alpha \Leftarrow e_2]\cdots[\alpha \Leftarrow e_n]$. This term is obtained by replacing $[\alpha]N$ by $[\alpha]N[\alpha \Leftarrow e_1, e_2, \cdots, e_n]e_1 e_2 \cdots e_n$ recursively.

Extending the complete permutation to $\lambda\mu^-_{\mathsf{NK}}$ is not straightforward. First, we have to do μ-reduction simultaneously in the complete permutation because of the example in Figure 3. Here, the right bottom arrow needs one μ-step followed by some π-steps as

$$(\mu\alpha.M[\alpha \Leftarrow [x.N]])e \rightarrow_\mu \mu\alpha.M[\alpha \Leftarrow [x.N], e] \twoheadrightarrow_\pi \mu\alpha.M[\alpha \Leftarrow [x.Ne]].$$

$$
\begin{aligned}
x[\overline{\alpha} \Leftarrow @\overline{\varepsilon}]@\varepsilon &= x\varepsilon \\
(\lambda x.M)[\overline{\alpha} \Leftarrow @\overline{\varepsilon}]@\varepsilon &= (\lambda x.M[\overline{\alpha} \Leftarrow @\overline{\varepsilon}]@\circ)\varepsilon \\
(\iota M)[\overline{\alpha} \Leftarrow @\overline{\varepsilon}]@\varepsilon &= (\iota M[\overline{\alpha} \Leftarrow @\overline{\varepsilon}]@\circ)\varepsilon \\
(MN)[\overline{\alpha} \Leftarrow @\overline{\varepsilon}]@\varepsilon &= (M[\overline{\alpha} \Leftarrow @\overline{\varepsilon}]@\circ)(N[\overline{\alpha} \Leftarrow @\overline{\varepsilon}]@\circ)\varepsilon \\
(M[x.N])[\overline{\alpha} \Leftarrow @\overline{\varepsilon}]@\varepsilon &= (M[\overline{\alpha} \Leftarrow @\overline{\varepsilon}]@\circ)[x.N[\overline{\alpha} \Leftarrow @\overline{\varepsilon}]@\varepsilon] \\
(\mu\beta.M)[\overline{\alpha} \Leftarrow @\overline{\varepsilon}]@\varepsilon &= \mu\beta.M[\overline{\alpha}, \beta \Leftarrow @\overline{\varepsilon}, \varepsilon]@\circ \\
([\alpha_i]M)[\overline{\alpha} \Leftarrow @\overline{\varepsilon}]@\varepsilon &= ([\alpha_i]M[\overline{\alpha} \Leftarrow @\overline{\varepsilon}]@\varepsilon_i)\varepsilon && (\alpha_i \in \overline{\alpha}) \\
([\beta]M)[\overline{\alpha} \Leftarrow @\overline{\varepsilon}]@\varepsilon &= ([\beta]M[\overline{\alpha} \Leftarrow @\overline{\varepsilon}]@\circ)\varepsilon && (\beta \notin \overline{\alpha})
\end{aligned}
$$

Figure 4. Definition of $M[\overline{\alpha} \Leftarrow @\overline{\varepsilon}]@\varepsilon$

Note that the latter π-reduction holds by π-reducing subterms of the form $[\alpha]P[x.N]e$ in $M[\alpha \Leftarrow [x.N], e]$ to $[\alpha]P[x.Ne]$. Secondly, the following naïve definition is not inductive on (the size of) terms:

$$
\begin{aligned}
(\mu\alpha.M)@e &= \mu\alpha.M[\alpha \Leftarrow @e] \\
([\alpha]M)[\alpha \Leftarrow @e] &= [\alpha](M[\alpha \Leftarrow @e]@e).
\end{aligned}
$$

Hence, we need some generalization for the definition of the complete permutation with respect to both π- and μ-reduction.

Definition 4.2. We use the following notation. The metavariable ε ranges over eliminators or $\circ$ denoting "nothing", and we define

$$
M\varepsilon = \begin{cases} Me & (\varepsilon = e) \\ M & (\varepsilon = \circ). \end{cases}
$$

$\overline{\alpha}$ and $\overline{\varepsilon}$ denote finite sequences such as $\alpha_1, \alpha_2, \cdots, \alpha_n$ and $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n$, respectively, and $\bullet$ denotes the empty sequence.

We define $M[\overline{\alpha} \Leftarrow @\overline{\varepsilon}]@\varepsilon$ as Figure 4, where we suppose that $\overline{\alpha}$ and $\overline{\varepsilon}$ have the same length. Then, we define $M@e = M[\bullet \Leftarrow @\bullet]@e$ and $M[\overline{\alpha} \Leftarrow @\overline{\varepsilon}] = M[\overline{\alpha} \Leftarrow @\overline{\varepsilon}]@\circ$.

Note that, the following equations hold as we expect.

$$
\begin{aligned}
(M[x.N])@e &= M[x.N@e] \\
(\mu\alpha.M)@e &= \mu\alpha.M[\alpha \Leftarrow @e] \\
M@e &= Me && \text{(o.w.)}
\end{aligned}
$$

Furthermore, we also have

$$M[\overline{\alpha} \Leftarrow @\overline{\varepsilon}]@\varepsilon = (M[\overline{\alpha} \Leftarrow @\overline{\varepsilon}])@\varepsilon.$$

Now, we can define a function with the Z property by composing two functions in a similar way to the case of $\lambda^{-}_{\mathsf{NJ}}$.

Definition 4.3. The mappings M^{P} and $e^{\mathsf{P}}_{\mathsf{E}}$ are inductively defined as follows.

$$\begin{aligned}
x^{\mathsf{P}} &= x & M^{\mathsf{P}}_{\mathsf{E}} &= M^{\mathsf{P}}\\
(\lambda x.M)^{\mathsf{P}} &= \lambda x.M^{\mathsf{P}} & [x.N]^{\mathsf{P}}_{\mathsf{E}} &= [x.N^{\mathsf{P}}]\\
(\iota M)^{\mathsf{P}} &= \iota M^{\mathsf{P}}\\
([\alpha]M)^{\mathsf{P}} &= [\alpha]M^{\mathsf{P}}\\
(\mu\alpha.M)^{\mathsf{P}} &= \mu\alpha.M^{\mathsf{P}}\\
(Me)^{\mathsf{P}} &= M^{\mathsf{P}}@e^{\mathsf{P}}_{\mathsf{E}}
\end{aligned}$$

The mappings M^{B} and $e^{\mathsf{B}}_{\mathsf{E}}$ are defined as follows.

$$\begin{aligned}
x^{\mathsf{B}} &= x & M^{\mathsf{B}}_{\mathsf{E}} &= M^{\mathsf{B}}\\
(\lambda x.M)^{\mathsf{B}} &= \lambda x.M^{\mathsf{B}} & [x.N]^{\mathsf{B}}_{\mathsf{E}} &= [x.N^{\mathsf{B}}]\\
(\iota M)^{\mathsf{B}} &= \iota M^{\mathsf{B}}\\
([\alpha]M)^{\mathsf{B}} &= [\alpha]M^{\mathsf{B}}\\
(\mu\alpha.M)^{\mathsf{B}} &= \mu\alpha.M^{\mathsf{B}}\\
((\lambda x.M)N)^{\mathsf{B}} &= M^{\mathsf{B}}[x := N^{\mathsf{B}}]\\
((\iota M)[x.N])^{\mathsf{B}} &= N^{\mathsf{B}}[x := M^{\mathsf{B}}]\\
(Me)^{\mathsf{B}} &= M^{\mathsf{B}}e^{\mathsf{B}}_{\mathsf{E}} \quad \text{(o.w.)}
\end{aligned}$$

We define $M^{\mathsf{PB}} = (M^{\mathsf{P}})^{\mathsf{B}}$.

We can use Theorem 2.3 to show the confluence of $\lambda\mu^{-}_{\mathsf{NK}}$ with the help of several lemmas. The following is the extension of Lemma 3.4.

Lemma 4.4. *1. If $\overline{\alpha} \cap FV(\overline{\varepsilon}, \varepsilon) = \emptyset$, then $M[\overline{\alpha} \Leftarrow \overline{\varepsilon}]\varepsilon \twoheadrightarrow_{\pi\mu} M[\overline{\alpha} \Leftarrow @\overline{\varepsilon}]@\varepsilon$.*

2. (a) $(M[\overline{\alpha} \Leftarrow @\overline{\varepsilon}]@[x.N])@\varepsilon = M[\overline{\alpha} \Leftarrow @\overline{\varepsilon}]@[x.N@\varepsilon]$.

(b) $M[\overline{\alpha}, \gamma \Leftarrow @\overline{\varepsilon}, [x.N]][\gamma \Leftarrow @\varepsilon] = M[\overline{\alpha}, \gamma \Leftarrow @\overline{\varepsilon}, [x.N@\varepsilon]]$.

3. If $\overline{\alpha} \cap FV(N) = \emptyset$, then $(M[\overline{\alpha} \Leftarrow @\overline{\varepsilon}]@\varepsilon)[x := N] \twoheadrightarrow_{\pi\mu} M[\overline{\alpha} \Leftarrow @\overline{\varepsilon}[x := N]]@\varepsilon[x := N]$.

4. If $x \notin FV(\overline{\varepsilon}, \varepsilon)$*, then* $(M[\overline{\alpha} \Leftarrow @\overline{\varepsilon}]@\varepsilon)[x := N[\overline{\alpha} \Leftarrow @\overline{\varepsilon}]] \twoheadrightarrow_{\pi\mu} M[x := N][\overline{\alpha} \Leftarrow @\overline{\varepsilon}]@\varepsilon$.

5. If $\gamma \notin \overline{\alpha} \cup FV(\overline{\varepsilon}, \varepsilon)$*, then* $M[\gamma \Leftarrow \varepsilon'][\overline{\alpha} \Leftarrow @\overline{\varepsilon}]@\varepsilon = (M[\overline{\alpha} \Leftarrow @\overline{\varepsilon}]@\varepsilon)[\gamma \Leftarrow \varepsilon'[\overline{\alpha} \Leftarrow @\overline{\varepsilon}]]$.

6. $M[\alpha \Leftarrow [x.N]][\alpha \Leftarrow @\varepsilon] = M[\alpha \Leftarrow [x.N@\varepsilon]]$.

7. If $\overline{\alpha} \cap FV(\varepsilon) = \emptyset$ *and* $M \to N$*, then* $M[\overline{\alpha} \Leftarrow @\overline{\varepsilon}]@\varepsilon \twoheadrightarrow N[\overline{\alpha} \Leftarrow @\overline{\varepsilon}]@\varepsilon$.

Proof. 1 is proved by induction on M. We will see the only interesting cases.

(Case $M = P[x.Q]$) We have the following.

$$\begin{aligned} M[\overline{\alpha} \Leftarrow \overline{\varepsilon}]\varepsilon &\to_\pi P[\overline{\alpha} \Leftarrow \overline{\varepsilon}][x.Q[\overline{\alpha} \Leftarrow \overline{\varepsilon}]\varepsilon] \\ &\twoheadrightarrow_{\pi\mu} P[\overline{\alpha} \Leftarrow @\overline{\varepsilon}][x.Q[\overline{\alpha} \Leftarrow @\overline{\varepsilon}]@\varepsilon] && \text{(I.H.)} \\ &= M[\overline{\alpha} \Leftarrow @\overline{\varepsilon}]@\varepsilon. \end{aligned}$$

(Case $M = \mu\beta.P$) We can suppose $\beta \notin FV(\overline{\varepsilon}, \varepsilon)$. We have the following.

$$\begin{aligned} M[\overline{\alpha} \Leftarrow \overline{\varepsilon}]\varepsilon &\to_\mu \mu\beta.P[\overline{\alpha}, \beta \Leftarrow \overline{\varepsilon}, \varepsilon] \\ &\twoheadrightarrow_{\pi\mu} \mu\beta.P[\overline{\alpha}, \beta \Leftarrow @\overline{\varepsilon}, \varepsilon] && \text{(I.H.)} \\ &= M[\overline{\alpha} \Leftarrow @\overline{\varepsilon}]@\varepsilon. \end{aligned}$$

For 2, (a) and (b) are simultaneously proved by induction on M.

3 to 6 are respectively proved by induction on M.

7 is proved by induction on $M \to N$. We will see some interesting cases. Let θ be $[\overline{\alpha} \Leftarrow @\overline{\varepsilon}]@\varepsilon$.

Consider the case of $M = (\mu\beta.M')e$, $N = \mu\beta.M'[\beta \Leftarrow e]$, and $M \to_\mu N$. We have two subcases. When $e = [x.P]$, we have the following.

$$\begin{aligned} M\theta &= (\mu\beta.M'[\overline{\alpha} \Leftarrow @\overline{\varepsilon}])[x.P\theta] \\ &\to_\mu \mu\beta.M'[\overline{\alpha} \Leftarrow @\overline{\varepsilon}][\beta \Leftarrow [x.P\theta]] \\ &= \mu\beta.M'[\overline{\alpha} \Leftarrow @\overline{\varepsilon}][\beta \Leftarrow [x.P][\overline{\alpha} \Leftarrow @\overline{\varepsilon}]][\beta \Leftarrow @\varepsilon] && \text{(by 6)} \\ &= \mu\beta.M'[\beta \Leftarrow [x.P]][\overline{\alpha} \Leftarrow @\overline{\varepsilon}][\beta \Leftarrow @\varepsilon] && \text{(by 5)} \\ &= \mu\beta.M'[\beta \Leftarrow [x.P]][\overline{\alpha}, \beta \Leftarrow @\overline{\varepsilon}, \varepsilon] \\ &= N\theta. \end{aligned}$$

Otherwise, when $e = M''$, we have the following.

$$\begin{aligned} M\theta &= (\mu\beta.M'[\overline{\alpha} \Leftarrow @\overline{\varepsilon}])M''[\overline{\alpha} \Leftarrow @\overline{\varepsilon}]\varepsilon \\ &\twoheadrightarrow_\mu \mu\beta.M'[\overline{\alpha} \Leftarrow @\overline{\varepsilon}][\beta \Leftarrow M''[\overline{\alpha} \Leftarrow @\overline{\varepsilon}]][\beta \Leftarrow \varepsilon] \\ &= \mu\beta.M'[\beta \Leftarrow M''][\overline{\alpha} \Leftarrow @\overline{\varepsilon}][\beta \Leftarrow \varepsilon] && \text{(by 5)} \\ &\twoheadrightarrow_{\pi\mu} \mu\beta.M'[\beta \Leftarrow M''][\overline{\alpha}, \beta \Leftarrow @\overline{\varepsilon}, \varepsilon] && \text{(by 1)} \\ &= N\theta. \end{aligned}$$

The other base cases are similarly proved by the previous lemmas, and the induction steps are proved by the induction hypotheses. □

The lemmas corresponding to Lemma 3.5, 3.6, and 3.7 are similarly proved by Lemma 4.4.

Theorem 4.5 (Confluence of $\lambda\mu^{-}_{\mathsf{NK}}$)**.** $\lambda\mu^{-}_{\mathsf{NK}}$ *is confluent.*

Proof. By Corollary 2.4, it is sufficient to prove the following.
(a) $M \to_{\pi\mu} N$ implies $M^{\mathsf{P}} = N^{\mathsf{P}}$
(b) $M \twoheadrightarrow_{\pi\mu} M^{\mathsf{P}}$ holds for any M.
(c) $M \twoheadrightarrow M^{\mathsf{B}}$ holds for any M.
(d) $M \to_{\beta} N$ implies $N \twoheadrightarrow M^{\mathsf{PB}} \twoheadrightarrow N^{\mathsf{PB}}$.
Proofs are similar to those for Theorem 3.8 by the previous lemmas.
□

5. Explicit substitutions

As another example of an application of the compositional Z, we show confluence of a calculus with explicit substitutions, in which the propagation rules look like the permutation rules.

Definition 5.1 (λ_{x})**.** Terms of λ_{x} are defined as follows.

$$M ::= x \mid \lambda x.M \mid MM \mid M\langle x := M\rangle$$

The expression $\langle x := M\rangle$ is called an explicit substitution. In the term $M\langle x := N\rangle$, the variable occurrences of x in M are bound, and it is supposed that we can freely rename bound variables as usual. We call a term *pure* if it contains no explicit substitution.

Reduction rules of λ_{x} are the following, where x and y are distinct, and, in the rule (π_{abs}), x does not occur freely in N.

$$\begin{aligned}
(\lambda x.M)N &\to M\langle x := N\rangle && (\beta_{\mathsf{x}})\\
y\langle y := N\rangle &\to N && (\pi_{\mathsf{hit}})\\
x\langle y := N\rangle &\to N && (\pi_{\mathsf{gc}})\\
(\lambda x.P)\langle y := N\rangle &\to \lambda x.P\langle y := N\rangle && (\pi_{\mathsf{abs}})\\
(PQ)\langle y := N\rangle &\to P\langle y := N\rangle Q\langle y := N\rangle && (\pi_{\mathsf{app}})
\end{aligned}$$

5.1. Confluence of λ_{x} by Compositional Z

The outline of the following proof with the compositional Z is almost the same as the case of $\lambda^{-}_{\mathsf{NJ}}$ and $\lambda\mu^{-}_{\mathsf{NK}}$. In this case, what corresponds to complete permutation $(\cdot)^{\mathsf{P}}$ is replacing explicit substitutions $\langle x := M\rangle$ to meta substitutions $[x := M]$.

Definition 5.2. The mappings M^{P} and M^{B} are defined as follows.

$$\begin{aligned} x^{\mathsf{P}} &= x \\ (\lambda x.M)^{\mathsf{P}} &= \lambda x.M^{\mathsf{P}} \\ (MN)^{\mathsf{P}} &= M^{\mathsf{P}}N^{\mathsf{P}} \\ (M\langle x := N\rangle)^{\mathsf{P}} &= M^{\mathsf{P}}[x := N^{\mathsf{P}}] \end{aligned} \qquad \begin{aligned} x^{\mathsf{B}} &= x \\ (\lambda x.M)^{\mathsf{B}} &= \lambda x.M^{\mathsf{B}} \\ ((\lambda x.M)N)^{\mathsf{B}} &= M^{\mathsf{B}}[x := N^{\mathsf{B}}] \\ (MN)^{\mathsf{B}} &= M^{\mathsf{B}}N^{\mathsf{B}} \quad \text{(o.w.)} \\ (M\langle x := N\rangle)^{\mathsf{B}} &= M^{\mathsf{B}}\langle x := N^{\mathsf{B}}\rangle \end{aligned}$$

Then, we define $M^{\mathsf{PB}} = (M^{\mathsf{P}})^{\mathsf{B}}$.

In fact, the last equation of the definition of $(\cdot)^{\mathsf{B}}$ is not used, because it is applied only to pure terms in the following.

It is easy to see the following auxiliary lemmas.

Lemma 5.3. *1. $M \to_\pi N$ implies $M^{\mathsf{P}} = N^{\mathsf{P}}$.*

2. M^{P} is pure.

3. If M is pure, then we have $M^{\mathsf{P}} = M$.

4. If M is pure, then we have $M\langle x := N\rangle \twoheadrightarrow_\pi M[x := N]$.

Proof. 1 is proved by induction on $M \to_\pi N$, and 2, 3, and 4 are by induction on M. □

Lemma 5.4. *1. If $M \to N$ holds in λ_{x}, then we have $M^{\mathsf{P}} \twoheadrightarrow_\beta N^{\mathsf{P}}$ in the ordinary λ-calculus without explicit substitutions.*

2. For M and N are pure, if $M \to_\beta N$ holds in the ordinary λ-calculus, then we have $M \twoheadrightarrow N$ in λ_{x}.

Proof. 1 is proved by induction on $\to$, and 2 is by induction on $\to_\beta$. □

Note that, on pure terms, the mapping $(\cdot)^{\mathsf{B}}$ is the ordinary complete development, and it has the Z property for the β-reduction in the λ-calculus [9].

Now we can prove confluence of λ_{x} by the compositional Z.

Theorem 5.5 (Confluence of λ_{x})**.** *λ_{x} is confluent.*

Proof. By Corollary 2.4, it is sufficient to prove the following.
(a) $M \to_\pi N$ implies $M^{\mathsf{P}} = N^{\mathsf{P}}$
(b) $M \twoheadrightarrow_\pi M^{\mathsf{P}}$ holds for any M.
(c) $M \twoheadrightarrow M^{\mathsf{B}}$ holds for any pure M.
(d) $M \to_{\beta_{\mathsf{x}}} N$ implies $N \twoheadrightarrow M^{\mathsf{PB}} \twoheadrightarrow N^{\mathsf{PB}}$.
(a) is Lemma 5.3.1. (b) is easy by Lemma 5.3.4. (c) is also easy since we have

$$(\lambda x.P)Q \twoheadrightarrow_{\beta_{\mathsf{x}}} P\langle x := Q\rangle \twoheadrightarrow_\pi P[x := Q]$$

by Lemma 5.3.4.

(d) is proved by induction on $M \to_{\beta_{\mathsf{x}}} N$. For $N \twoheadrightarrow M^{\mathsf{PB}}$, the only non-trivial case where $P\langle x := Q\rangle \to_{\beta_{\mathsf{x}}} P'\langle x := Q'\rangle$ is proved as follows.

$$\begin{aligned} P'\langle x := Q'\rangle &\twoheadrightarrow P^{\mathsf{PB}}\langle x := Q^{\mathsf{PB}}\rangle && \text{(I.H., (b), (c))} \\ &\twoheadrightarrow P^{\mathsf{PB}}[x := Q^{\mathsf{PB}}] && (5.3.4) \\ &\twoheadrightarrow (P^{\mathsf{P}}[x := Q^{\mathsf{P}}])^{\mathsf{B}}, \end{aligned}$$

where, for the last line, we can prove $M^{\mathsf{B}}[x := N^{\mathsf{B}}] \twoheadrightarrow_\beta (M[x := N])^{\mathsf{B}}$ in the λ-calculus in a similar way to Lemma 3.6.1, and hence we have $M^{\mathsf{B}}[x := N^{\mathsf{B}}] \twoheadrightarrow (M[x := N])^{\mathsf{B}}$ in λ_{x} by Lemma 5.4.2.

The rest part of (d), $M^{\mathsf{PB}} \twoheadrightarrow N^{\mathsf{PB}}$, is proved as follows. Suppose that $M \to_{\beta_{\mathsf{x}}} N$ holds, and we have $M^{\mathsf{P}} \twoheadrightarrow_\beta N^{\mathsf{P}}$ in the λ-calculus by Lemma 5.4.1. Then, $M^{\mathsf{PB}} \twoheadrightarrow_\beta N^{\mathsf{PB}}$ since $(\cdot)^{\mathsf{B}}$ is Z for β, and hence $M^{\mathsf{PB}} \twoheadrightarrow N^{\mathsf{PB}}$ in λ_{x} by Lemma 5.4.2. □

5.2. Confluence of Refined Rewriting Systems

Confluence of calculi with explicit substitutions is often proved by the interpretation method [6, 7, 8]. This idea can be explained as follows: For $A \subset B$, a rewriting system $(B, \to_B)$ is a refinement of $(A, \to_A)$ if $a \to_A a'$ implies $a \twoheadrightarrow_B b'$. Then, the confluence of a refinement $\to_B$ of $\to_A$ can be reduced to the confluence of $\to_A$ if there exists an interpretation mapping $f : B \to A$ such that $b \twoheadrightarrow_B f(b)$ and $b \to_B b' \Rightarrow f(b) \twoheadrightarrow_A f(b')$ holds. The confluence of λ_{x} can be proved by this idea with $A = \lambda_\beta$, $B = \lambda_{\mathsf{x}}$, and $f = (\cdot)^{\mathsf{P}}$.

The confluence proof for λ_{x} in this section is not a simple application of this interpretation method to the Z property, because it cannot be directly used to obtain the Z property for refinements. In Figure 5, f is an interpretation mapping of B in A. In general, we do not have $f(b_2) \twoheadrightarrow_A f'(f(b_1))$ under the assumption that f' is Z for $\to_A$, since $f(b_1) \twoheadrightarrow_A f(b_2)$ is not necessarily

$$
\begin{tikzcd}
b_1 \arrow[r, "B"] & b_2 \arrow[d, mapsto, "f"] \\
f(b_1) \arrow[r, dotted, two heads, "A"] & f(b_2) \arrow[dl, dotted, two heads, "A?"'] \\
f'(f(b_1)) \arrow[r, dotted, two heads, "A"'] & f'(f(b_2))
\end{tikzcd}
$$

Figure 5. Z and interpretation method

one step. In the proof of Theorem 5.5, we directly prove $b_2 \twoheadrightarrow f'(f(b_1))$ as (d).

6. Concluding remark

We have proposed an extension of Dehornoy and van Oostrom's Z theorem, called the compositional Z. This idea can be widely applied to lambda calculi with permutative conversions, including the λ- and the $\lambda\mu$-calculi with disjunction and permutative conversion, and a lambda calculus with explicit substitutions, where the propagation of the explicit substitutions is similar to the permutation rules. In particular, the combination of the β-reduction and the permutative conversions makes the confluence proofs much difficult to define the parallel reduction or a mapping with the Z property. We have seen that the latter is easily defined as a compositional function, and hence the compositional Z gives simple confluence proofs for these calculi.

The compositional Z also gives a new possibility toward modular (or gradual) proofs of confluence. In general, it is hard to prove confluence by dividing a reduction system into some parts because of the non-modular character of confluence. The compositional Z enables us to reuse the Z property for a subsystem, that is, for $\rightarrow_1$, a subrelation of $\rightarrow$, the Z property for $\rightarrow_1$ can be used to prove the Z property for $\rightarrow$ by the compositional Z.

As we stated in the introduction, the renaming reduction of the $\lambda\mu$-calculus also poses a problem. Baba's solution in [3] suggests that we can apply the compositional Z to those variants of $\lambda\mu$-calculus, but it does not work well for naïve definition of the complete developments. We would like to solve this in our framework of the compositional Z.

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