Modified Gravity Theories on a Nutshell: Inflation, Bounce and Late-time Evolution

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We systematically review some standard issues and also the latest developments of modified gravity in cosmology, emphasizing on inflation, bouncing cosmology and late-time acceleration era. Particularly, we present the formalism of standard modified gravity theory representatives, like $F(R)$, $F(\mathcal{G})$ and $F(T)$ gravity theories, but also several alternative theoretical proposals which appeared in the literature during the last decade. We emphasize on the formalism developed for these theories and we explain how these theories can be considered as viable descriptions for our Universe. Using these theories, we present how a viable inflationary era can be produced in the context of these theories, with the viability being justified if compatibility with the latest observational data is achieved. Also we demonstrate how bouncing cosmologies can actually be described by these theories. Moreover, we systematically discuss several qualitative features of the dark energy era by using the modified gravity formalism, and also we critically discuss how a unified description of inflation with dark energy era can be described by solely using the modified gravity framework. Finally, we also discuss some astrophysical solutions in the context of modified gravity, and several qualitative features of these solutions. The aim of this review is to gather the different modified gravity techniques and form a virtual modified gravity "toolbox", which will contain all the necessary information on inflation, dark energy and bouncing cosmologies in the context of the various forms of modified gravity.

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I. INTRODUCTION

With this work we shall try to provide a concise pedagogical review on the latest developments in modified gravity aspects of inflation, dark energy and bouncing cosmology. The attempt is challenging and it is conceivable that it is impossible to cover all the different approaches in the field, since the number of these is vast. Hence we focus on modified gravity aspects of early, late-time acceleration and bounce dynamics and we shall try to provide a pedagogical text accessible by non-experts but also useful to experts.

One of the most fundamental questions in modern theoretical cosmology is, whether the genesis of the Universe was singular or non-singular. This question is equivalent in asking if the Big Bang theory or the Big Bounce theory actually describes the evolution of our Universe. Naturally thinking, the initial singularity described by the Big Bang theory, is a mentally more convenient description, since we can easily imagine a zero sized Universe, with infinite temperature and energy density, and also in which all fundamental interactions are unified under the yet unknown same theoretical framework. However, no one can actually exclude a cyclic cosmological evolution, in which the Universe never shrinks to zero. Actually the latter perspective seems to be supported by quantum cosmologies, as we discuss later on. One of the main purposes of this review article is to present the tools to study these two types of cosmological scenarios, in the context of modified gravity.

During the last 25 years, the cosmologists community has experienced great surprises, since the observations indicated in the late 90's [1] that the Universe is expanding, but in an accelerating way. This observation utterly changed the cosmologists way of thinking and also it revived theories containing a cosmological constant. It was Einstein that had firstly proposed a theory of cosmological evolution with a cosmological constant, and later on he admitted that this was the greatest scientific mistake of his life. However, it seems that nature can actually incorporate such a cosmological constant, so it seems eventually that Einstein was not wrong to some extent. We need to note that finding the correct model for the late-time acceleration is not an easy task at all. The late-time acceleration era is known as the dark energy era [2–15] (see also [16–19]), and up to date there are many proposals that try to model this era, with other using a scalar field, known as quintessence models [20–38], while other models use modified gravity in its various forms. With regards to quintessence, there exist various alternative proposals on this topic, see for example [39–43], and also approaches with non-minimal coupling, see [44–47]. This field of research is still developing rapidly up to date.

Apart from the late-time acceleration era, the Universe experienced another acceleration era during its first stages of evolution, and the latest observational data have actually constrained this era [48, 49], which is known as inflation [50–70]. During the inflationary era, the Universe increased its size at an exponential rate, so it expanded quite quickly and gained a large size in a relatively small amount of time. From a theoretical point of view, the inflationary description of the early time Universe, solved quite many theoretical problems that the original Big-Bang theory had, for example the horizon problem etc. The inflationary evolution had originally two different forms, which now are known as the old inflationary scenario [71], and the new inflationary scenario [53, 55]. From the time that the new inflationary scenario was introduced, many models where proposed that could actually describe in a successful way the inflationary era, with many of the models using a scalar field or multiple scalar fields, see [70] for details. Also for supergravity models of inflation, see the review [69], and for string cosmology consult Refs. [59, 60]. There also exist alternative scenarios for inflation, involving the axion field, see for example [68] for a review. In addition for inflationary theories in the context of modified gravity in its various forms, see [62].

However, a successful model at present time (2017), has to be confronted with the observational data coming from the Planck satellite, which severely constraints the inflationary era [48]. The Planck data have excluded many inflationary models from the viable inflationary models list, and nowadays these data are a benchmark with regard to the viability of an inflationary model. Thus, it is compelling that a model has to be compatible with the Planck data, before it can be considered as a candidate for inflation. Apart from the scalar-tensor models of inflation, there are alternative proposals in the literature that use modified gravity in its various forms [62, 72–78], in order to describe this early-time era. We need to note that in this review we shall adopt the metric modified gravity formalism, but there also exists the Palatini formalism, for which we refer the reader to Refs. [79–89].

Most of the metric modified gravity descriptions provide a consistent framework for the description of the early-time acceleration, and also the compatibility of the model with the data can be achieved in many of these models. Moreover, each modification of Einstein's theory of general relativity eventually is confronted with the successes of general relativity. Hence all the constraints on modified gravity imposed by local astrophysical data but also from global constraints, have to be satisfied to an adequate level. Therefore it is conceivable that a modified gravity has many challenges and obstacles to overcome in order to be considered as a viable cosmological theory. Apart from the constraints, the ultimate goal in modified gravity is to offer a self-consistent theoretical framework in the context of which the early-time and late-time acceleration will be described by the same theory. In this direction there are many works using various theoretical contexts, that provide such a framework [90–108]. Also dark energy [109–118] and other aspects of cosmological evolution and cosmological implications [119–131] are also explained in an appealing way by modified gravity. In Refs. [95–98, 132, 133] the unification of Λ-Cold Dark Matter model (ΛCDM hereafter) with inflation was developed, in the context of $F(R)$ gravity, while in Refs. [90, 91], the same issue was addressed in the context of phantom cosmology. In principle the possibility of phantom inflation always exists in various cosmological contexts [134–139].

The first unified description of the inflation with dark energy in modified gravity was proposed in Ref. [140], while in Ref. [93] the unified description of dark energy and inflation was developed in the context of string theory, and in [91, 94] a holographic approach to the same issue was performed. Also in Refs. [102] the problem under discussion was addressed in the context of a dynamical involving dark energy component, and in [105] the problem was addressed in the context of modified $F(R)$ Horava-Lifshitz gravity. The most successful theory will be consistent with observations, with the local and global constraints and will describe simultaneously all or at least most of the evolution eras of our Universe. Finally for some different scenarios see [100–103, 141–147]. Also modified gravity usually brings along various exotic features, for example the equivalence principle can be different in comparison to the ordinary Einstein gravity [148], or the energy conditions may differ [149] and in addition the thermodynamic interpretation is different [150], see also [151]. But inevitably modified gravity has eventually to confront the successes of the Einstein-Hilbert gravity, so stringent rules are imposed from solar system tests. Some relevant studies with regards to solar system implications and constraints of modified gravity can be found in Refs. [95, 152–160], while the Newtonian limit of $F(R)$ gravity was studied in Ref. [161]. Also some studies on the stability of $F(R)$ can be found for example in [162–165] and a recent study on the observational constraints on $F(R)$ gravity from cosmic chronometers can be found in [166]. Finally, we need to note that the phase space of modified gravity theories is quite richer in geometric structures, in comparison to the Einstein gravity phase space, see for example [167, 168].

An alternative description to the inflationary paradigm and to the Big Bang cosmology, is offered by bouncing cosmologies, see Ref. [169] for the original idea of a bounce cosmology, and for an important stream of reviews see Refs. [170–179]. Particularly, in Ref. [170], the study was focused on the matter bounce scenario and various ways that may realize this scenario were discussed. Also several observational signatures that may distinguish the matter bounce from the standard inflationary paradigm were also discussed. A more focused review on bouncing cosmologies is Ref. [171], were the origin of primordial perturbations was discussed in the context of bouncing cosmologies, and also several examples that realize bounce cosmologies, including pre-big-bang scenarios, ekpyrotic scenarios, string gas cosmology, bouncing cosmologies from modified gravity and string theory, were also discussed. Moreover, the observational signatures that may distinguish a bounce from the inflationary paradigm were examined too, like for example the existence of non-Gaussianities. In Ref. [172], the bouncing cosmologies were discussed and in addition certain potentially fatal effects that undermine non-singular bouncing models were pointed out. Also, the unstable growth of curvature fluctuations and the growth of the quantum induced anisotropy, in conjunction with the study of various gravitational instabilities, were discussed too. In Ref. [173], the study performed covered the topics of higher-order gravitational theories, theories with a scalar field, bounces in the braneworld scenarios and several quantum cosmology scenarios. Also the cyclic Universes were discussed and the issue of perturbations in the context of bouncing Universes were addressed too. In Ref. [174], the matter-ekpyrotic bounce scenario was extensively studied, and also various realizations of this scenario were presented, like for example the two scalar field realization. Also the observational constraints were thoroughly discussed and also several mechanisms for generating a red tilt for primordial perturbations were presented too. In Ref. [175], the matter bounce scenario was also presented, and its realization was achieved by using a single scalar field with a nearly exponential potential. The main result was that the rolling of the scalar field leads to a running of the spectral index, and specifically a negative running is obtained. Also possible theories that realize such a scenario are discussed, such as holonomy corrected loop quantum cosmology theories and also teleparallel $F(T)$ gravity. In addition, and insightful study on the reheating process is discussed too. In Ref. [177], several issues concerning cyclic cosmologies were discussed, including, the ekpyrotic phase of a bounce, how to avoid chaos in such models, the Milne Universe and finally several ekpyrotic models were presented. Also, the scalar and tensor perturbations were addressed, and in addition the link of these theories to a more fundamental theory, like heterotic M-theory was discussed too. Also in Ref. [179], matter bounce scenarios in which the matter content consists of dark energy and dark matter were reviewed. Specifically, the ΛCDM bounce scenario was discussed, and also theories with interacting dark matter and dark energy were addressed too. Moreover, the observational signatures that may distinguish bounces from the inflationary paradigm were discussed, and also several theories that may realize a bounce were also presented, including, loop quantum cosmology, string Cosmology, $F(R)$ gravity, kinetic gravity braiding theories and finally the Fermi bounce mechanism. Furthermore, some interesting information for the occurrence of bounces can be found in [180] and for a pioneer version of the non-singular bounce in the context of modified gravity see [181]. The Big Bounce cosmology [182–185] is an appealing alternative to inflation, since the initial singularity which haunts the Big Bang cosmology is absent, hence these cosmologies are essentially non-singular [186–188]. However, other types of singular bounces appeared in the literature, in which case the singularity which occurs at the bouncing point is a soft type singularity [189–192]. In the context of bouncing cosmologies there are various scenarios in the literature, and bouncing cosmologies are often studied in ekpyrotic scenarios of some sort [193–195] (see Refs. [196, 197] for the ekpyrotic scenario per se). In the case of a bounce, the Universe is described by a repeating cycle of evolution, in which initially the Universe contracts, until a minimal radius is reached, which is called the bouncing point. After that point, the Universe starts to expand again and this cycle is continuously repeated. Hence bouncing cosmologies are essentially cyclic cosmologies or equivalently oscillating cosmologies [198–201]. Cosmological perturbations in bouncing cosmologies are generated usually during the contracting phase [171], however this is not always true, see for example the singular bounce of Ref. [192], where the primordial perturbations are actually generated during the expanding phase after the Type IV singular bouncing point. For some very relevant studies of perturbations in bouncing cosmologies, see Refs. [202–205]. In principle, a scale invariant or nearly scale invariant power spectrum can be generated by a bounce cosmology [171, 192], and also the recent observational data can be consistent with cyclic cosmologies [206–208]. There are many bouncing cosmologies in the literature and some of these scenarios naturally occur in Loop Quantum Cosmology [209–224]. One quite well known bounce cosmology is the matter bounce scenario [175, 184, 185, 214, 224–235], which is known to provide a scale invariant spectrum during the contracting phase, see for example [171]. Also for some alternative scenarios in the context of cosmological bounces, see the informative Refs. [236–241]. An interesting bouncing cosmology scenario appeared in [236], called quintom scenario, see Ref. [242] for a comprehensive review on quintom cosmology. The quintom scenario is highly motivated by the current observations which indicate that the dark energy equation of state crosses the phantom divide line. In order that the quintom scenario is realized two scalar fields are required, since a no-go theorem forbids the single scalar field realization of the quintom scenario [242]. In the two scalar field realization of the quintom scenario, one scalar field is quintessential and the other scalar field is a phantom one, and the drawback of these theories is the existence of ghost degrees of freedom. Also the quintom scenario may be obtained from string theory motivated higher derivative scalar field theories and from braneworld scenarios [242]. The quintom bounce can be realized if the null energy condition is violated, and as we already mentioned, the no-go theorem in quintom cosmology makes compelling to use two scalar fields. One important feature of the two scalar field realization of the quintom bounce is the fact that for each scalar component, the effective equation of state needs not to cross the phantom divide line, and thus the classical perturbations remain stable.

Some bouncing cosmologies scenarios have been proposed to describe the pre-inflationary era [243–245], and thus in these scenarios the inflationary paradigm is combined with cosmological bounces. So bouncing cosmologies can produce an exact scale invariant power spectrum of primordial curvature perturbations, for example the matter bounce scenario [171] during the contraction era. However, in such cases, during the expansion era, entropy is produced and the perturbation modes grow with the cosmic time [225]. Such a continuous cycle of cosmological bounces can be stopped if a crushing singularity occurs at the end of the expanding era, as for example in the deformed matter bounce scenario studied in Ref. [246], where it was shown that the infinite repeating evolution of the Universe stops at the final attractor of the theory, which is a Big Rip singularity [30, 247–277]. Modified gravity in general offers a consistent theoretical framework in the context of which bouncing cosmologies can be realized, without the need to satisfy specific constraints which are compelling in the case of the standard general relativity approach. Hence the study of bouncing cosmologies in the context of modified gravity is important, and these cosmologies need to be critically examined with regards to their observational consequences. For some recent studies on bouncing cosmologies in the context of modified gravity, see for example [278–281], see also Refs. [282–284] for bouncing cosmologies in the context of Palatini gravity.

The modified gravity description of our Universe cosmological evolution is one physically appealing theoretical framework, which can potentially explain the various evolution eras of the Universe, for the simple reason that it can provide a unified and theoretically consistent description. There exist a plethora of modified gravity models, that can potentially describe our Universe evolution and the most important criterion for the viability of a modified gravity theory is the compatibility of the theory with present time observations. The observations related to our

Universe, mainly consist of large scale observations and astrophysical observations, related to compact gravitational objects or gravitationally bound objects. In both cases, a successful modified gravity theory, should pass all the tests related to observations. But there is also another feature that may render a modified gravity theoretical description as viable, namely the fact that the theory will be able to predict new, yet undiscovered phenomena. We believe that by studying alternative modified gravity models, even if we do not succeed in finding the ultimate modified gravity theory, at least we will pave the way towards finding the most successful theory. In the context of modified gravity, the new Lagrangian terms, introduce new degrees of freedom beyond the standard General Relativity, and the Standard Model of particle physics. When these new terms are applied at a cosmological level, the extra degrees of freedom alter the evolution of the Universe, and may have as an effect the desired behavior of the Universe. These theories may however introduce certain pathologies or extra instabilities, as it happens for example in the case of $F(R,\mathcal{G})$ gravity, where superluminal extra modes appear, which are absent in the $F(R)$ gravity and the $f(\mathcal{G})$ gravity, or modified Gauss-Bonnet gravity cases, and of course these are absent in General Relativity.

Also with regards to astrophysical solutions, like black holes, neutron stars, and wormholes, modified gravity utterly changes the conditions that are needed to be satisfied in order for the solutions to be consistent. For example, in the case of wormholes in the ordinary Einstein gravity, an exotic matter fluid needs to be present, in order for the wormhole solution to be self-consistent. On the contrary, in the modified gravity case, the modified gravity part can offer a theoretically appealing and simple remedy to this problem, see for example [285].

Our aim with this review is to present the latest developments in the description of the inflationary era, dark energy and also in bouncing cosmology, in the context of modified gravity. Our motivation is the fact that up to date the ΛCDM model is successful but does not provide a complete description of the Universe. Also we shall provide all the necessary information for the models we shall study, that make these models consistent both at astrophysical and at cosmological level. Our presentation will have a pedagogical and introductive character, in order to make this review a pedagogical tool available to experts and non-experts. It is conceivable that our work does not cover all the modified gravity applications on inflation, dark energy and bouncing cosmologies, this task would be very hard to be materialized, since the subject is vast. However we shall provide the most important tools that will enable one to study in some detail inflation and bounces with modified gravity. Also for completeness and in order to render this review an autonomous study on modified gravity, we shall discuss some astrophysical applications of modified gravity and specifically of $F(R)$ gravity, which is the most sound representative theory of modified gravity. In most cosmological applications, the background space-time geometry will be that of the flat Friedmann-Robertson-Walker (FRW) geometry. We will start our presentation with chapter II, where we shall present the most important modified gravity descriptions. Specifically we shall provide the theoretical framework of each modified gravity version, and also we also present some illustrative examples in each case. Specifically, in section II-A, we discuss the most important representative theory of modified gravity, namely $F(R)$ gravity. We present in detail the equations of motion of the FRW Universe and also we discuss the criteria that render an $F(R)$ gravity theory a astrophysical and cosmologically viable theory. We also discuss the stability, in terms of the scalaron mass, and we present the corresponding scalar-tensor description in the Einstein frame. Several viable $F(R)$ gravities are presented, and the conditions of their viability are also quoted. Also we discuss in brief the case that a non-minimal coupling between the $F(R)$ gravity sector and the matter Lagrangian exists. In section II-B, we present the Gauss-Bonnet modified gravities. We present the basic equations of motion, which can be used as a reconstruction method in order to realize various cosmologies. Also we calculate the corrections to the Newton law due to the Gauss-Bonnet gravity corrections. The string inspired gravity follows in section II-C, along with several characteristic examples. In section II-D, we discuss an important modified gravity theory, namely that of an $F(T)$ gravity, with T being the torsion scalar. Both the $f(\mathcal{G})$ and $F(T)$ gravity will be thoroughly discussed in this review article, since these theories have appealing characteristics and are important alternative theories to $F(R)$ gravity. In section II-E we discuss massive gravity and bigravity theories, and in sections II-F and II-G, we discuss mimetic $F(R)$ and mimetic $f(\mathcal{G})$ gravity. The mimetic framework has offered a quite interesting appealing theoretical framework, so we present various versions of the mimetic framework. In section II-H we present another $F(R)$ gravity extension, namely that of unimodular $F(R)$ gravity, and in sections II-I, II-J, we discuss some combinations of unimodular and mimetic $F(R)$ gravity, which are accompanied by illustrative examples.

Chapter III is devoted on the study of inflationary dynamics in various theoretical contexts. Traditionally, inflation was firstly studied in the context of scalar-tensor theory in its simplest form, namely that of a canonical scalar field, the inflaton, but in this chapter we also present some modified gravity descriptions of the inflationary era. For completeness in III-A we first study the canonical scalar field inflationary paradigm, and in order to illustrate the methods, we calculate the spectral index of primordial curvature perturbations and also the scalar-to-tensor ratio. In the same section we study the non-canonical scalar field case and we also discuss in brief the form of the inflationary dynamics in multi-scalar theories of gravity. In section III-B we present the inflationary dynamics formalism for general $f(R,\phi)$ theories of gravity. We provide detailed calculations for the slow-roll indices and also for the observational indices. Also we focus on interesting subcases of $f(R,\phi)$ gravity, and particularly $\phi f(R)$, mimetic $F(R)$ gravity and

$F(R)$ gravity, and we use various illustrative examples in order to better support the theoretical formalism. In the same section we present a very useful reconstruction technique which offers the possibility of obtaining the $F(R)$ gravity from the Einstein frame theory with a given potential. In section III-C we present the study of inflationary dynamics in the context of the modified Gauss-Bonnet gravity of the form $f(\mathcal{G})$, where $\mathcal{G}$ is the Gauss-Bonnet scalar. We use a different approach in comparison to the previous sections of this chapter, so we calculate directly the power spectrum of primordial curvature perturbations, and from this we calculate the spectral index of the primordial curvature perturbations. As a peripheral study we discuss how the perturbations evolve after the horizon crossing, by exploiting a specifically chosen example. We also briefly present the formalism of inflationary dynamics in the context of $F(T)$ and one loop quantum gravity. Particularly, in the $F(T)$ case we realize the intermediate inflation scenario in section III-D, while section III-E is devoted to the singular inflation scenario and various phenomenological implications of this cosmological evolution. Finally,in section III-F we present a useful theoretical approach for the graceful exit issue and in III-G the reheating era in the context of modified gravity is studied. Usually in modified gravity the graceful exit instance is identified with the moment that the first slow-roll index becomes of the order one. However, the growing perturbations that are caused by the unstable de Sitter points can also provide a mechanism for graceful exit from inflation. So we present in brief all the essential information for this alternative approach to the graceful exit issue.

In chapter IV we address the dark energy issue in the context of modified gravity. Particularly, we discuss how the dark energy era can be realized by $F(R)$ and $f(\mathcal{G})$ gravity, and also we discuss various phenomena related to the late-time era realization with modified gravity. But firstly, we will show in sections IV-A and IV-B, how the successful ΛCDM model can be realized by $F(R)$ and $f(\mathcal{G})$ gravity respectively. In sections IV-C and IV-D we will demonstrate how it is possible to provide a unified description of early and late-time acceleration with $F(R)$ and $f(\mathcal{G})$ gravity. In section IV-E we discuss how a phantom cosmological evolution can be realized in the context of $F(R)$ gravity. Finally, in section IV-F we will discuss an important feature of $F(R)$ gravity when the late-time era is studied. Particularly, we present the dark energy oscillations issue in $F(R)$ gravity, and we discuss how this may affect the late-time era. We shall use various matter fluids present, from perfect matter fluids to collisional, and we critically discuss how the fluid viscosity may affect the dark energy oscillations issue.

In chapter V we discuss some astrophysical applications of modified gravity, emphasizing in $F(R)$ gravity applications. We shall present some neutron stars and quark stars solutions in $F(R)$ gravity, and we briefly examine the astrophysical consequences of these solutions, for general equations of state for the neutron star. Also we discuss the possibility of anti-evaporation from Reissner-Nordström black holes in $F(R)$ gravity and finally we present some wormhole solutions from $F(R)$ gravity.

In chapter VI we present the general reconstruction techniques which can be used in order to realize bouncing cosmologies with $F(R)$, $f(\mathcal{G})$ and $F(T)$ gravity. We shall present how to realize bounces by vacuum modified gravity, but also in some cases we shall present how the results are modified in the presence of perfect matter fluids. In the beginning of the chapter we provide an informative overview of bouncing cosmologies which is necessary in order to understand the fundamental characteristics of a cosmological bounce. In section VI-A we discuss how a specific bounce ca be realized with vacuum $F(R)$ gravity, but we also present the case that perfect matter fluids are present. In section VI-B we discuss the $f(\mathcal{G})$ realization of the same bounce cosmology as before, and finally in section VI-C we discuss the $F(T)$ gravity realization. In all three cases, we examine the stability of the modified gravity solutions we found, at the level of the equations of motion. Specifically, we study the stability of the equations of motion, if these are viewed as a dynamical system, and we demonstrate in which cases stability can be ensured.

Finally the conclusions follow at the end of this review, were we summarize the successes of modified gravity and we outline its shortcomings as a theory. Also we discuss in brief various challenging problems in cosmology, which still need to be incorporated successfully to a future theory.

II. MODIFIED GRAVITIES AND COSMOLOGY

A. $F(R)$ Gravity

The theory of $F(R)$ gravity could be considered as the most popular among modified gravity theories. In this section, a general review of the $F(R)$ gravity theory is given. In the literature there are various reviews also discussing this topic, see [62, 72–75, 77, 78].

1. General properties

The action of the $F(R)$ gravity [73] is given by replacing the scalar curvature R in the Einstein-Hilbert action which is,[1]

$$S_{\rm EH} = \int d^4 x \sqrt{-g} \left(\frac{R}{2\kappa^2} + \mathcal{L}_{\rm matter} \right) \,, \tag{1}$$

by using some appropriate function of the scalar curvature, as follows,

$$S_{F(R)} = \int d^4 x \sqrt{-g} \left(\frac{F(R)}{2\kappa^2} + \mathcal{L}_{\rm matter} \right) \,. \tag{2}$$

ndequation In Eqs. (1) and (2), $\mathcal{L}_{\rm matter}$ is the matter Lagrangian density. We now review in brief the general properties of $F(R)$ gravity.

For later convenience, we define the effective equation of state (EoS) parameter for the $F(R)$ gravity theory. The expression can be used in any other modified gravity context. We start with the FRW equations, which in the Einstein gravity coupled with perfect fluid are:

$$\rho_{\rm matter} = \frac{3}{\kappa^2} H^2 \,, \quad p_{\rm matter} = -\frac{1}{\kappa^2} \left(3H^2 + 2\dot H \right) \,. \tag{3}$$

Then, the EoS parameter can be given by using the Hubble rate H, in the following way,

$$w_{\rm eff} = -1 - \frac{2\dot H}{3H^2} \,. \tag{4}$$

In principle we can use the expression given in Eq. (4) even for modified gravity theories, since it is useful for a generalized fluid description of modified gravity.

By varying the action (2) with respect to the metric, we obtain the equation of motion for the $F(R)$ gravity theory as follows,

$$\frac{1}{2} g_{\mu\nu} F(R) - R_{\mu\nu} F'(R) - g_{\mu\nu} \Box F'(R) + \nabla_\mu \nabla_\nu F'(R) = - \frac{\kappa^2}{2} T_{{\rm matter}\,\mu\nu} \,. \tag{5}$$

For the spatially flat FRW Universe, in which case the metric is given by,

$$ds^2 = -dt^2 + a(t)^2 \sum_{i=1,2,3} \left(dx^i \right)^2 \,, \tag{6}$$

Eq. (5) gives the FRW equations,

$$\begin{aligned} 0 =& -\frac{F(R)}{2} + 3\left(H^2 + \dot H \right) F'(R) - 18 \left(4H^2 \dot H + H \ddot H \right) F''(R) + \kappa^2 \rho_{\rm matter} \,, & (7) \\ 0 =& \frac{F(R)}{2} - \left(\dot H + 3H^2 \right) F'(R) + 6 \left(8H^2 \dot H + 4 \dot H^2 + 6 H \ddot H + \dddot H \right) F''(R) + 36 \left(4H\dot H + \ddot H \right)^2 F'''(R) \\ & + \kappa^2 p_{\rm matter} \,, & (8) \end{aligned}$$

where the Hubble rate H is equal to $H = \dot a / a$. In terms of the Hubble rate H, the scalar curvature R is equal to $R = 12H^2 + 6\dot H$.

We can find several (in many cases exact) solutions of Eq. (7). Without the presence of matter, a simple solution is given by assuming that the Ricci tensor is covariantly constant, that is, $R_{\mu\nu} \propto g_{\mu\nu}$. Then Eq. (5) is simplified to the following algebraic equation [140] (see also [286]):

$$0 = 2F(R) - RF'(R) \,. \tag{9}$$

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[1] We use the following convention for the curvatures and connections:

$$R = g^{\mu\nu} R_{\mu\nu} \,, \quad R_{\mu\nu} = R^\lambda_{\ \mu\lambda\nu} \,, \quad R^\lambda_{\ \mu\rho\nu} = -\Gamma^\lambda_{\mu\rho,\nu} + \Gamma^\lambda_{\mu\nu,\rho} - \Gamma^\eta_{\mu\rho}\Gamma^\lambda_{\nu\eta} + \Gamma^\eta_{\mu\nu}\Gamma^\lambda_{\rho\eta} \,, \quad \Gamma^\eta_{\mu\lambda} = \frac{1}{2} g^{\eta\nu} \left(g_{\mu\nu,\lambda} + g_{\lambda\nu,\mu} - g_{\mu\lambda,\nu} \right) \,.$$

If Eq. (9) has a solution the (anti-)de Sitter and/or Schwarzschild- (anti-)de Sitter space

$$ds^2 = -\left(1 - \frac{2MG}{r} \mp \frac{r^2}{L^2}\right) dt^2 + \left(1 - \frac{2MG}{r} \mp \frac{r^2}{L^2}\right)^{-1} dr^2 + r^2 d\Omega^2 \,, \tag{10}$$

or the Kerr - (anti-)de Sitter space is an exact solution in vacuum. In Eq. (10), the minus and plus signs in $\pm$ correspond to the de Sitter and anti-de Sitter space, respectively. In Eq. (10), M is the mass of the black hole, $G = \frac{\kappa^2}{8\pi}$, and L is the length parameter of (anti-)de Sitter space, which is related to the curvature as follows $R = \pm\frac{12}{L^2}$ (the plus sign corresponds to the de Sitter space and the minus sign to the anti-de Sitter space).

We now consider the perfect fluid representation and scalar-tensor representation of the $F(R)$ gravity. For convenience, we write $F(R)$ as the sum of the scalar curvature R and the part which expresses the difference from the Einstein gravity case,

$$F(R) = R + f(R) \,. \tag{11}$$

Eqs. (7) and (8) indicate that we can express the effective energy density $\rho_{\rm eff}$ and also the effective pressure $p_{\rm eff}$ including the contribution from $f(R)$ gravity as follows, (see, for instance, [287])

$$\rho_{\rm eff} = \frac{1}{\kappa^2}\left(-\frac{1}{2}f(R) + 3\left(H^2 + \dot H\right) f'(R) - 18\left(4H^2\dot H + H\ddot H\right) f''(R)\right) + \rho_{\rm matter} \,, \tag{12}$$

$$p_{\rm eff} = \frac{1}{\kappa^2}\left(\frac{1}{2}f(R) - \left(3H^2 + \dot H\right) f'(R) + 6\left(8H^2\dot H + 4\dot H^2 + 6H\ddot H + \dddot H\right) f''(R) + 36\left(4H\dot H + \ddot H\right)^2 f'''(R)\right) + p_{\rm matter} \,, \tag{13}$$

which enables us to rewrite the equations (7) and (8) as in the Einstein gravity case (3),

$$\rho_{\rm eff} = \frac{3}{\kappa^2}H^2 \,, \quad p_{\rm eff} = -\frac{1}{\kappa^2}\left(3H^2 + 2\dot H\right) \,. \tag{14}$$

The fluid representation for the FRW equations, however, often leads to an unjustified treatment. For example, it is often ignored that the generalized gravitational fluid contains higher-derivative curvature invariants. The viable dark energy models of $F(R)$ gravity are discussed in Refs. [288–292], while the unified description of inflation with dark energy are discussed in Refs. [140, 293], for a review see [77].

2. *Scalar-tensor description*

We should note that we can also rewrite $F(R)$ gravity in a scalar-tensor form. We introduce an auxiliary field A and rewrite the action (2) of the $F(R)$ gravity in the following form:

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\left\{F'(A)\left(R - A\right) + F(A)\right\} \,. \tag{15}$$

We obtain $A = R$ by the variation of the action with respect to A and by substituting the obtained equation $A = R$ into the action (15), we find that the action in (2) is reproduced. If we rescale the metric by a kind of a scale transformation (canonical transformation),

$$g_{\mu\nu} \to \mathrm{e}^{\sigma} g_{\mu\nu} \,, \quad \sigma = -\ln F'(A) \,, \tag{16}$$

we obtain the action in the Einstein frame [2],

$$S_E = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\left(R - \frac{3}{2}g^{\rho\sigma}\partial_\rho\sigma\partial_\sigma\sigma - V(\sigma)\right) \,,$$

[2] Note that the difference between the $F(R)$ (Jordan) frame and the scalar-tensor (Einstein) frame description, may lead to a number of issues, for example the Universe in one frame may be accelerating, while decelerating in the other frame [294, 295], or the singularity types changes from frame to frame [277, 296].

$$V(\sigma) = \mathrm{e}^{\sigma} g\left(\mathrm{e}^{-\sigma}\right) - \mathrm{e}^{2\sigma} f\left(g\left(\mathrm{e}^{-\sigma}\right)\right) = \frac{A}{F'(A)} - \frac{F(A)}{F'(A)^2}\,. \tag{17}$$

Here $g\left(\mathrm{e}^{-\sigma}\right)$ is given by solving the equation $\sigma = -\ln\left(1 + f'(A)\right) = -\ln F'(A)$ as $A = g\left(\mathrm{e}^{-\sigma}\right)$. Due to the scale transformation (16), a coupling of the scalar field σ with usual matter is introduced. The mass of the scalar field σ is given by

$$m_\sigma^2 \equiv \frac{3}{2}\frac{d^2V(\sigma)}{d\sigma^2} = \frac{3}{2}\left\{\frac{A}{F'(A)} - \frac{4F(A)}{\left(F'(A)\right)^2} + \frac{1}{F''(A)}\right\}, \tag{18}$$

and if the mass m_σ is not very large, there appears a large correction to the Newton law. Since we would like to explain the accelerating expansion of the current Universe by using the $F(R)$ gravity, we may naively expect that the order of the mass m_σ should be that of the Hubble rate, that is, $m_\sigma \sim H \sim 10^{-33}\,\mathrm{eV}$. Since the mass is very small, very large corrections to the Newton law could appear. In order to avoid the above problem in the Newton law, a so-called "realistic" model was proposed in [297]. In the model, the mass m_σ becomes large enough in the regions where the curvature $R = A$ is large or under the presence of matter fluids, as in the Solar System, or in the Earth. Therefore the force mediated by the scalar field becomes short-ranged, which also introduces a screening effect in which only the surface of the massive objects like the planets can contribute to the correction to the Newton law even in the vacuum. This is called the Chameleon mechanism [298], which prevents the large correction to the Newton law.

For example, we may consider the following model, [98] (see also [299, 300]),

$$f(R) = 2\Lambda_{\mathrm{eff}}\left(\mathrm{e}^{-bR} - 1\right)\,. \tag{19}$$

In the Solar System, where $R \sim 10^{-61}\,\mathrm{eV}^2$, if we choose $1/b \sim R_0 \sim \left(10^{-33}\,\mathrm{eV}\right)^2$, we obtain $m_\sigma^2 \sim 10^{1,000}\,\mathrm{eV}^2$, which is ultimately heavy. In the atmosphere of the Earth, where $R \sim 10^{-50}\,\mathrm{eV}^2$, and if we choose $1/b \sim R_0 \sim \left(10^{-33}\,\mathrm{eV}\right)^2$, again, we obtain $m_\sigma^2 \sim 10^{10,000,000,000}\,\mathrm{eV}^2$. Then, the correction to the Newton law looks to be extremely small in this kind of model. The corresponding Compton wavelength, however, is very small and much shorter than the distance between the atoms. Since the region between the atoms can be regarded as a vacuum except several quantum corrections and we cannot use the approximation to regard the matter fluids as continuous fluid, the Chameleon mechanism does not apply. Therefore, the Compton length cannot be shorter than the distance between the atoms and the scalar field cannot become so large, although this is not in conflict with any observation nor experiment.

We now also need to mention the problem of antigravity. Eq. (15) tells that the effective gravitational coupling is given by $\kappa_{\mathrm{eff}}^2 = \frac{\kappa^2}{F'(R)}$. Therefore when $F'(R)$ is negative, it is possible to have antigravity regions [140]. Then, we need to require

$$F'(R) > 0\,. \tag{20}$$

We should note that from the viewpoint of the field theory, the graviton becomes ghost in the antigravity region.

It should be noted that the de Sitter or anti-de Sitter space solution in (9) corresponds to the extremum of the potential $V(\sigma)$. In fact, we find,

$$\frac{dV(\sigma)}{dA} = \frac{F''(A)}{F'(A)^3}\left(-AF'(A) + 2F(A)\right)\,. \tag{21}$$

Therefore, if Eq. (9) is satisfied, the scalar field σ should be on the local maximum or local minimum of the potential and σ can be a constant. When the condition (9) is satisfied, the mass given by (18) has the following form:

$$m_\sigma^2 = \frac{3}{2F'(A)}\left(-A + \frac{F'(A)}{F''(A)}\right)\,. \tag{22}$$

Therefore, in the case that the condition (20) for avoiding the antigravity holds true, the mass squared m_σ^2 is positive, showing that the scalar field is on the local minimum if,

$$-A + \frac{F'(A)}{F''(A)} > 0\,. \tag{23}$$

On the other hand, the scalar field is on the local maximum of the potential if,

$$-A + \frac{F'(A)}{F''(A)} < 0\,. \tag{24}$$

In this case the mass squared m_σ^2 is negative. The condition (23) is nothing but the stability condition of the de Sitter space.

Note that in the Einstein frame, the Universe is always in the non-phantom phase, where the effective EoS w_{eff} in (4) is larger than -1 although in the Jordan frame, the Universe can be, in general, in a phantom phase. This is because the scale transformation in (16) changes the time coordinate. We should note that we observe the expansion of the Universe via matter. In the Einstein frame, the metric for the matter is not given by $g_{\mu\nu}$ but by $\mathrm{e}^{\sigma} g_{\mu\nu}$, which is nothing but the metric in the Jordan frame. Therefore the observations in the Einstein frame is not changed from the observation in the Jordan frame. This indicates that in the Einstein frame, the metric $g_{\mu\nu}$ is not physical but $\mathrm{e}^{\sigma} g_{\mu\nu}$ is the physical metric [294].

We now shortly mention on the matter instability issue pointed out in [301]. The instability may appear when the energy density or the scalar curvature is large compared with that in the Universe, as for example on the Earth. Let R_b the background scalar curvature and separate the scalar curvature R as a sum of R_b and the perturbed part R_p as $R = R_b + R_p$ ($|R_p| \ll |R_b|$). Then we obtain the following perturbed equation:

$$\begin{aligned} 0 =& \Box R_b + \frac{F^{(3)}(R_b)}{F^{(2)}(R_b)} \nabla_\rho R_b \nabla^\rho R_b + \frac{F'(R_b) R_b}{3F^{(2)}(R_b)} \\ & - \frac{2F(R_b)}{3F^{(2)}(R_b)} - \frac{R_b}{3F^{(2)}(R_b)} + \Box R_p + 2\frac{F^{(3)}(R_b)}{F^{(2)}(R_b)} \nabla_\rho R_b \nabla^\rho R_p + U(R_b) R_p \,, \end{aligned} \tag{25}$$

$$\begin{aligned} U(R_b) \equiv & \left(\frac{F^{(4)}(R_b)}{F^{(2)}(R_b)} - \frac{F^{(3)}(R_b)^2}{F^{(2)}(R_b)^2} \right) \nabla_\rho R_b \nabla^\rho R_b + \frac{R_b}{3} \\ & - \frac{F^{(1)}(R_b) F^{(3)}(R_b) R_b}{3F^{(2)}(R_b)^2} - \frac{F^{(1)}(R_b)}{3F^{(2)}(R_b)} + \frac{2F(R_b) F^{(3)}(R_b)}{3F^{(2)}(R_b)^2} - \frac{F^{(3)}(R_b) R_b}{3F^{(2)}(R_b)^2} \,. \end{aligned} \tag{26}$$

If we assume R_b and R_p are uniform, we can replace the d'Alembertian the second derivative with respect to the time coordinate and therefore Eq. (25) has the following structure:

$$0 = -\partial_t^2 R_p + U(R_b) R_p + \text{const.} \,. \tag{27}$$

Then, if $U(R_b) > 0$, R_p becomes exponentially large with time t, we have $R_p \sim \mathrm{e}^{\sqrt{U(R_b)}t}$ and the system is rendered unstable. We should note, however, that the scalar field in (17) is nothing but the scalar curvature and therefore the above matter instability occurs when the mass of the scalar field is negative, $m_\sigma^2 < 0$. Conversely, if we choose $F(R)$ for large R in such a way, so that the mass becomes positive, $m_\sigma^2 > 0$, then the instability does not occur.

3. *Viable $F(R)$ gravities*

Using the previous arguments, we summarize the conditions that need to hold true in order for an $F(R)$ gravity to be a viable cosmological model, which unifies the accelerating expansion of the present Universe and the early-time acceleration of the Universe. The first unified inflation dark energy $F(R)$ gravity model appeared in Ref. [140], and in Refs. [95, 97–99, 302–305], various viable models of $F(R)$ gravity unifying both of the late-time and the early-time acceleration were proposed by requiring several conditions, which we list here:

1. A condition to generate the inflationary era is given by,

$$\lim_{R\to\infty} f(R) = -\Lambda_i \,. \tag{28}$$

   Here, Λ_i is an effective cosmological constant characterizing the early Universe.

2. In order to generate the accelerating expansion of the Universe at present time, the current value of $f(R)$ should be a small constant,

$$f(R_0) = -2\tilde{R}_0 \,, \quad f'(R_0) \sim 0 \,. \tag{29}$$

   Here, R_0 expresses the present time curvature $R_0 \sim \left(10^{-33}\mathrm{eV}\right)^2$. Note that $R_0 > \tilde{R}_0$ because we need to take into account the contribution from matter, since the trace part of Eq. (5) indicates that $R_0 \sim \tilde{R}_0 - \kappa^2 T_{\text{matter}}$. Here, T_{matter} is the trace part of the energy-momentum tensor of all matter fluids. We should note that the quantity $f'(R_0)$ needs not to vanish completely but instead it should satisfy $|f'(R_0)| \ll \left(10^{-33}\,\mathrm{eV}\right)^4$. This is due to the fact that we consider the time scale to be 10^{12-13} years.

3. The last condition is given by

$$\lim_{R\to 0} f(R) = 0\,, \tag{30}$$

that is, a flat space-time solution (Minkowski space-time) should exist.

Let us consider the power-law model where $F(R)$ behaves as

$$f(R) \sim F_0 + F_1 R^\epsilon\,, \tag{31}$$

when R is large. Here F_0 and F_1 are arbitrary constants. The constant F_0 may vanish but F_1 should not, $F_1 \neq 0$. Then, the trace equation, which is the trace part of Eq. (5),

$$3\Box f'(R) = R + 2f(R) - Rf'(R) - \kappa^2 T_{\text{matter}}\,, \tag{32}$$

indicates that,

$$3F_1\Box R^{\epsilon-1} = \begin{cases} R & \text{when } \epsilon < 0 \text{ or } \epsilon = 2 \\ (2-\epsilon)\,F_1 R^\epsilon & \text{when } \epsilon > 1 \text{ or } \epsilon \neq 2 \end{cases}\,. \tag{33}$$

We now assume that the Hubble rate has a structural singularity as follows,

$$H \sim \frac{h_0}{(t_s - t)^\beta}\,, \tag{34}$$

where h_0 and β are arbitrary constants suitably chosen so that the Hubble rate is real. Then the scalar curvature $R = 6\dot{H} + 12H^2$ behaves as follows,

$$R \sim \begin{cases} \frac{12h_0^2}{(t_s-t)^{2\beta}} & \text{when } \beta > 1 \\ \frac{6h_0 + 12h_0^2}{(t_s-t)^2} & \text{when } \beta = 1 \\ \frac{6\beta h_0}{(t_s-t)^{\beta+1}} & \text{when } \beta < 1 \end{cases}\,. \tag{35}$$

In Eqs. (34) and (35), the case $\beta \geq 1$ corresponds to a Type I (Big Rip) singularity, see Refs. [30, 248, 249, 253–260, 262–271, 306–312]. The case $1 > \beta > 0$ corresponds to a Type III singularity, the case $0 > \beta > -1$ corresponds to a Type II, and finally the case $\beta < -1$ (but $\beta \neq$ integer) corresponds to a Type IV singularity.

The above classification of the finite-time future singularities was proposed in Ref. [313]. Particularly, the finite-time singularity classification is the following:

- Type I ("Big Rip") : This type of singularity occurs for $t \to t_s$, $a \to \infty$, $\rho_{\text{eff}} \to \infty$ and $|p_{\text{eff}}| \to \infty$. Its manifestations in various models and theoretical contexts, have been studied in Ref. [313].
- Type II ("sudden") [275, 276, 314–322]: This type of singularity occurs for $t \to t_s$, $a \to a_s$, $\rho_{\text{eff}} \to \rho_s$ and $|p_{\text{eff}}| \to \infty$.
- Type III : This type of singularity occurs for $t \to t_s$, $a \to a_s$, $\rho_{\text{eff}} \to \infty$ and $|p_{\text{eff}}| \to \infty$.
- Type IV : This type of singularity occurs for $t \to t_s$, $a \to a_s$, $\rho_{\text{eff}} \to 0$, $|p_{\text{eff}}| \to 0$ and higher derivatives of H diverge. This also includes the case in which p_{eff} (ρ_{eff}) or both p_{eff} and ρ_{eff} tend to some finite values, whereas higher derivatives of the Hubble rate H diverge. This type of singularity was proposed in [313].

Here, ρ_{eff} and p_{eff} are defined in Eq. (3).

Substituting Eq. (35) into Eq. (33), one finds that there are two classes of consistent solutions. One solution is given by $\beta = 1$ and $\epsilon > 1$ (but $\epsilon \neq 2$), which corresponds to the Big Rip ($h_0 > 0$ and $t < t_s$) or Big Bang ($h_0 < 0$ and $t > t_s$) singularity at $t = t_s$. Another solution is $\epsilon < 1$, and $\beta = -\epsilon/(\epsilon - 2)$ $(-1 < \beta < 1)$, which corresponds to the Type II future singularity. We should note that when $\epsilon = 2$, that is, $f(R) \sim R^2$, there is no singular solution.

Therefore, if we add the above term $R^2\tilde{f}(R)$, where $\lim_{R\to 0}\tilde{f}(R) = c_1$, $\lim_{R\to\infty}\tilde{f}(R) = c_2$ to the action of $f(R)$ and if the added term dominates for large R, the modified gravity part of the $F(R)$ gravity, namely $f(R)$, behaves as $f(R) \sim R^2$ and the future singularity could disappear. On the other hand, if we add the term it dominates as it behaves as an R^n-term with $n = 3, 4, 5, \cdots$ for large R. Then the singularity appears because this case corresponds to $\epsilon = n > 1$, that is, the case of the Big Rip singularity. In case of $0 < \epsilon < 2$, the future singularity does not appear [323].

In order to avoid the occurrence of a finite-time future singularity, an additional R^2-term is needed to be added in the $F(R)$ gravity Lagrangian. The addition of this term was first proposed first in Ref. [324] in order for the Big Rip singularity to disappear. Furthermore, the R^2-term, which effectively eliminates the future singularity, generates the early-time acceleration simultaneously. In other words, adding the R^2 term to a gravitational dark energy model, may lead to the emergence of an inflationary phase in the model as first observed in Ref. [140] and also the singularity is removed. In Refs. [323, 325–327], it has been investigated that the R^2-term could cancel all types of future singularities. The R^2-term often solves other phenomenological problems [14, 328–331] of $F(R)$ dark energy. Even for the models of dark energy besides the $F(R)$ gravity, e.g., the models using the perfect fluid approach or in a scalar field context, there could often appear a singularity in the finite future Universe, but by the addition of the R^2-term, the singularity can be eliminated [332].

Summarizing the above analysis, in order to obtain a realistic and viable $F(R)$ gravity model:

1. In the limit of $R \to 0$, the Einstein gravity should be recovered,

$$F(R) \to R \quad \text{that is,} \quad \frac{F(R)}{R^2} \to \frac{1}{R} . \tag{36}$$

   If this condition is satisfied, a flat space (Minkowski) is also an solution as in (30).

2. As we will discuss later, there should appear the stable de Sitter solution, which corresponds to the late-time acceleration, where the curvature is small $R \sim R_L \sim \left(10^{-33}\,\mathrm{eV}\right)^2$. This requires that, when $R \sim R_L$,

$$\frac{F(R)}{R^2} = f_{0L} - f_{1L}\left(R - R_L\right)^{2n+2} + o\left(\left(R - R_L\right)^{2n+2}\right) . \tag{37}$$

   Here, f_{0L} and f_{1L} are positive constants and n is a positive integer. In some cases this condition may not always be necessary.

3. As we will also discuss later on, the quasi-stable de Sitter solution that corresponds to the inflationary era, in the early Universe should appear. In this case, the curvature is large $R \sim R_I \sim \left(10^{16\sim 19}\,\mathrm{GeV}\right)^2$. The de Sitter space should not be exactly stable so that the curvature decreases very slowly. This requires the following condition,

$$\frac{F(R)}{R^2} = f_{0I} - f_{1I}\left(R - R_I\right)^{2m+1} + o\left(\left(R - R_I\right)^{2m+1}\right) . \tag{38}$$

   Here, f_{0I} and f_{1I} are positive constants and m is a positive integer.

4. In order to avoid the curvature singularity when $R \to \infty$, we may assume that,

$$F(R) \to f_\infty R^2 \quad \text{that is} \quad \frac{F(R)}{R^2} \to f_\infty . \tag{39}$$

   Here, f_∞ is a positive and sufficiently small constant. Instead of (39), we may choose

$$F(R) \to f_{\tilde\infty} R^{2-\epsilon} \quad \text{that is} \quad \frac{F(R)}{R^2} \to \frac{f_{\tilde\infty}}{R^\epsilon} . \tag{40}$$

   Here, $f_{\tilde\infty}$ is a positive constant and the parameter ϵ satisfies $0 < \epsilon < 1$. With the condition (39) or (40), the future singularity can be eliminated.

5. As shown in (20), in order to avoid the antigravity regime, we require the following condition,

$$F'(R) > 0 , \tag{41}$$

   which can be rewritten as follows,

$$\frac{d}{dR}\left(\ln\left(\frac{F(R)}{R^2}\right)\right) > -\frac{2}{R} . \tag{42}$$

6. By combining conditions (36) and (41), we find

$$F(R) > 0 . \tag{43}$$

The conditions (1) and (2) indicate that an extra, unstable solution describing the de Sitter space-time always appears at $R = R_e$ ($0 < R_e < R_L$). Due to the fact that the de Sitter solution $R = R_L$ is stable, the evolution of the Universe will stop at $R = R_L$ and therefore the curvature will not become smaller than R_L, which indicates that the extra de Sitter solution is not realized. An example of such $F(R)$ gravity is given in [333] (the models in [95, 97, 98] partially satisfy the above conditions). Also for a study on the constraints of $F(R)$ gravity coming from large scale structures, see [334, 335].

We may consider the non-standard non-minimal coupling of modified gravity with the matter Lagrangian [336–338] (see also [339]),

$$S = \int d^4x\sqrt{-g}\left\{\frac{R}{2\kappa^2} + f(R)\,L_d\right\}. \tag{44}$$

Here, L_d is the Lagrangian density similar to that of a standard matter fluid. In a more generalized theoretical context, we may extend the model of Eq. (44) in the form of an $F(R)$ gravity,

$$S = \int d^4x\sqrt{-g}\left\{\frac{F(R)}{2\kappa^2} + f(R)\,L_d\right\}. \tag{45}$$

Then by varying of the action (45) with respect to the metric we obtain,

$$0 = \frac{1}{2}g_{\mu\nu}F(R) - R_{\mu\nu}\frac{\partial F(R)}{\partial R} - g_{\mu\nu}\Box\frac{\partial F(R)}{\partial R} + \nabla_\mu\nabla_\nu\frac{\partial F(R)}{\partial R} + \frac{\kappa^2}{2}\tilde{T}_{\mathrm{matter}\,\mu\nu}\,, \tag{46}$$

where $\tilde{T}_{\mu\nu}$ is the effective energy momentum tensor defined as follows,

$$\begin{aligned}\tilde{T}^{\mu\nu} \equiv& - f'(R)R^{\mu\nu}L_d + \left(\nabla^\mu\nabla^\nu - g^{\mu\nu}\nabla^2\right)\left(f'(R)L_d\right) + f'(R)T^{\mu\nu}\,,\\ T^{\mu\nu} \equiv& \frac{1}{\sqrt{-g}}\frac{\delta}{\delta g_{\mu\nu}}\left(\int d^4x\sqrt{-g}L_d\right).\end{aligned} \tag{47}$$

We should note that although $\tilde{T}^{\mu\nu}$ is conserved, that is, $\nabla_\mu\tilde{T}^{\mu\nu} = 0$, in contrast the tensor $T^{\mu\nu}$ is not conserved in general. We can define this model by specifying the Lagrangian density L_d to be that of a free massless scalar,

$$L_d = -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\,. \tag{48}$$

In the FRW Universe (6), the (t,t) component and (i,j) component in Eq. (46) give the following equations,

$$0 = -\frac{F(R)}{2} + 3\left(H^2 + \dot{H}\right)F'(R) - 18\left(4H^2\dot{H} + H\ddot{H}\right)F''(R) + \kappa^2\tilde{\rho}\,, \tag{49}$$

$$0 = \frac{F(R)}{2} - \left(\dot{H} + 3H^2\right)F'(R) + 6\left(8H^2\dot{H} + 4\dot{H}^2 + 6H\ddot{H} + \dddot{H}\right)F''(R) + 36\left(4H\dot{H} + \ddot{H}\right)^2 F'''(R) + \kappa^2\tilde{p}\,. \tag{50}$$

Here $\tilde{\rho}$ and $\tilde{p}$ have the following expressions,

$$\tilde{\rho} \equiv 3\left(H^2 + \dot{H}\right)f'(R)L_d - 3H\frac{d\left(f'(R)L_d\right)}{dt} + f(R)\rho\,, \tag{51}$$

$$\tilde{p} \equiv -\left(\dot{H} + 3H^2\right)f'(R)L_d + \frac{d^2\left(f'(R)L_d\right)}{dt^2} + 4H\frac{d\left(f'(R)L_d\right)}{dt} + f(R)p\,, \tag{52}$$

where ρ and p are the energy density and the pressure given by $T_{\mu\nu}$. This model may generate the accelerating expansion of the Universe, due to the non-trivial coupling of the curvature, although there are problems with geodesics [340].

As a variation of the $F(R)$ gravity, we may also consider $F(R,T)$ gravity [341], whose action is given by,

$$S_{F(R,T)} = \int d^4x\sqrt{-g}\left(\frac{F(R,T)}{2\kappa^2} + \mathcal{L}_{\mathrm{matter}}\right). \tag{53}$$

Here T stands for the trace of some "energy-momentum" tensor $T_{\mu\nu}$ in some sense. Then, by varying the action with respect to the metric, we obtain,

$$\frac{1}{2}g_{\mu\nu}F(R,T) - R_{\mu\nu}\frac{\partial F(R,T)}{\partial R} - g_{\mu\nu}\Box\frac{\partial F(R,T)}{\partial R} + \nabla_\mu\nabla_\nu\frac{\partial F(R,T)}{\partial R} + \frac{\partial T}{\partial g_{\mu\nu}}\frac{\partial F(R,T)}{\partial T} = -\frac{\kappa^2}{2}T_{\mathrm{matter}\,\mu\nu}\,. \tag{54}$$

Due to the presence of the term including $\frac{\partial T}{\partial g_{\mu\nu}}$, the theory cannot be correctly defined without specifying the metric dependence of T. Usually, the energy momentum tensor $T_{\mu\nu}$ is given by the variation of some action S_T with respect to the metric. If the action S_T is given in terms of the fields, the action (53) can be expressed in terms of the fields directly. We should also note that due to the existence of the $F(R,T)$ term in the action of Eq. (53), the conservation of $T_{\mu\nu}$ is violated in general. For the FRW Universe (6), the (t,t) component and (i,j) component in Eq. (54) give the following equations,

$$
\begin{aligned}
0 =& -\frac{F(R,T)}{2} + 3\left(H^2+\dot{H}\right)\frac{\partial F(R,T)}{\partial R} - 18\left(4H^2\dot{H} + H\ddot{H}\right)\frac{\partial^2 F(R,T)}{\partial R^2} + \frac{\partial T}{\partial g_{tt}}\frac{\partial F(R,T)}{\partial T} + \kappa^2 \rho_{\mathrm{matter}}\,, && (55) \\
0 =& \frac{F(R,T)}{2} - \left(\dot{H}+3H^2\right)\frac{\partial F(R,T)}{\partial R} + 6\left(8H^2\dot{H} + 4\dot{H}^2 + 6H\ddot{H} + \dddot{H}\right)\frac{\partial^2 F(R,T)}{\partial R^2} \\
& + 36\left(4H\dot{H} + \ddot{H}\right)^2 \frac{\partial^3 F(R,T)}{\partial R^3} + \frac{1}{3}g^{ij}\frac{\partial T}{\partial g_{ij}}\frac{\partial F(R,T)}{\partial T} + \kappa^2 p_{\mathrm{matter}}\,. && (56)
\end{aligned}
$$

If we require the conservation laws for $T_{\mu\nu}$, the additional constraints must be imposed on the model. Furthermore by imposing additional assumptions just for the solvability of the theory, several solutions have been studied [342–348].

B. Modified Gauss-Bonnet Gravity

We consider another class of models in modified gravity in which an arbitrary function of the topological Gauss-Bonnet invariant is added to the action of General Relativity. We call this class of modified gravity theory "modified Gauss-Bonnet gravity". This class of modified gravity could be closely related with (super)string theory.

1. General properties

The starting action is given by [115, 349–352]:

$$
S = \int d^4x\sqrt{-g}\left(\frac{1}{2\kappa^2}R + f(\mathcal{G}) + \mathcal{L}_{\mathrm{matter}}\right)\,. \tag{57}
$$

By varying the action with respect to the metric $g_{\mu\nu}$, we obtain the following equations of motion,

$$
\begin{aligned}
0 =& \frac{1}{2\kappa^2}\left(-R^{\mu\nu} + \frac{1}{2}g^{\mu\nu}R\right) + T^{\mu\nu}_{\mathrm{matter}} + \frac{1}{2}g^{\mu\nu}f(\mathcal{G}) - 2f'(\mathcal{G})RR^{\mu\nu} \\
& + 4f'(\mathcal{G})R^{\mu}_{\ \rho}R^{\nu\rho} - 2f'(\mathcal{G})R^{\mu\rho\sigma\tau}R^{\nu}_{\ \rho\sigma\tau} - 4f'(\mathcal{G})R^{\mu\rho\sigma\nu}R_{\rho\sigma} + 2\left(\nabla^\mu\nabla^\nu f'(\mathcal{G})\right)R \\
& - 2g^{\mu\nu}\left(\nabla^2 f'(\mathcal{G})\right)R - 4\left(\nabla_\rho\nabla^\mu f'(\mathcal{G})\right)R^{\nu\rho} - 4\left(\nabla_\rho\nabla^\nu f'(\mathcal{G})\right)R^{\mu\rho} \\
& + 4\left(\nabla^2 f'(\mathcal{G})\right)R^{\mu\nu} + 4g^{\mu\nu}\left(\nabla_\rho\nabla_\sigma f'(\mathcal{G})\right)R^{\rho\sigma} - 4\left(\nabla_\rho\nabla_\sigma f'(\mathcal{G})\right)R^{\mu\rho\nu\sigma}\,.
\end{aligned} \tag{58}
$$

We should note that Eq. (58) does not contain higher-derivative terms, that is, the terms including third or higher derivatives.

For the FRW metric with spatially flat part (6), we obtain the equation corresponding to the first FRW equation as follows,

$$
0 = -\frac{3}{\kappa^2}H^2 - f(\mathcal{G}) + \mathcal{G}f'(\mathcal{G}) - 24\dot{\mathcal{G}}f''(\mathcal{G})H^3 + \rho_{\mathrm{matter}}\,. \tag{59}
$$

In the FRW Universe (6), the Gauss-Bonnet invariant $\mathcal{G}$ has the following form: $\mathcal{G} = 24\left(H^2\dot{H} + H^4\right)$.

If the EoS parameter $w \equiv p_{\mathrm{matter}}/\rho_{\mathrm{matter}}$ is a constant, by using the conservation of energy equation:

$$
\dot{\rho}_{\mathrm{matter}} + 3H\left(\rho_{\mathrm{matter}} + p_{\mathrm{matter}}\right) = 0\,, \tag{60}
$$

we find that $\rho = \rho_0 a^{-3(1+w)}$. If the function $f(\mathcal{G})$ is given by,

$$
f(\mathcal{G}) = f_0\left|\mathcal{G}\right|^\beta\,. \tag{61}
$$

in case $\beta < 1/2$ and in the small curvature regime, the $f(\mathcal{G})$ term becomes dominant and we can neglect the Einstein term. Then Eq. (59) has the following solution,

$$
\begin{aligned}
& a = a_0 t^{h_0} \quad \text{when } h_0 > 0 \,, \quad a = a_0 \left(t_s - t\right)^{h_0} \quad \text{when } h_0 < 0 \,, \\
& h_0 = \frac{4\beta}{3(1+w)} \,, \quad a_0 = \left[-\frac{f_0(\beta - 1)}{(h_0 - 1)\,\rho_0} \left\{ 24 \left| h_0^3 \left(-1 + h_0\right) \right| \right\}^\beta \left(h_0 - 1 + 4\beta\right) \right]^{-\frac{1}{3(1+w)}} \,.
\end{aligned} \tag{62}
$$

Then we find that the EoS parameter w_{eff} in (4) is smaller than -1 if $\beta < 0$, and for $w > -1$, w_{eff} is given by,

$$
w_{\mathrm{eff}} = -1 + \frac{2}{3h_0} = -1 + \frac{1+w}{2\beta} \,, \tag{63}
$$

which is again smaller than -1 if $\beta < 0$. Thus, in the case $\beta < 0$, we obtain an effective phantom evolution with negative h_0, even if $w > -1$. One could presume that in the phantom phase, a Big Rip singularity could occur at $t = t_s$. Near this Big Rip, however, the curvature becomes large and the Einstein term dominates, hence the $f(\mathcal{G})$ term can be neglected. This indicates that the Universe behaves as $a = a_0 t^{2/3(w+1)}$ and the Big Rip singularity does not appear. Therefore the phantom era is transient regime.

We may consider the case that $0 < \beta < 1/2$. In this case, the phantom phase does not appear because β is positive. When the curvature is large, since the $f(\mathcal{G})$ term in the action (57) can be neglected, the theory can be approximated by the Einstein gravity solely. Then, if w is positive, the matter energy density ρ_{matter} should behave as $\rho_{\mathrm{matter}} \sim t^{-2}$, but $f(\mathcal{G})$ goes as $f(\mathcal{G}) \sim t^{-4\beta}$. At late times, so at large t values, the $f(\mathcal{G})$ term may become dominant compared with the Lagrangian density of matter. Then by neglecting the contribution from matter, Eq. (59) has the de Sitter Universe solution. Even if we start from the deceleration phase with $w > -1/3$, the asymptotically de Sitter Universe emerges. Correspondingly, a transition from deceleration to acceleration of the Universe occurs. We may also construct a model which can describe both the inflation and a late-time accelerating expansion of the Universe in a unified way.

We now consider the correction to the Newton law. Let $g_{(0)}$ be a solution of (58) and we express the perturbation of the metric as $g_{\mu\nu} = g_{(0)\mu\nu} + h_{\mu\nu}$. First, we consider the perturbation around the de Sitter background, whose metric is taken as $g_{(0)\mu\nu}$. Then, the Riemann tensor is equal to

$$
R_{(0)\mu\nu\rho\sigma} = H_0^2 \left(g_{(0)\mu\rho} g_{(0)\nu\sigma} - g_{(0)\mu\sigma} g_{(0)\nu\rho} \right) \,. \tag{64}
$$

We may impose the following gauge condition: $g_{(0)}^{\mu\nu} h_{\mu\nu} = \nabla_{(0)}^{\mu} h_{\mu\nu} = 0$. Then by using Eq. (58), we find

$$
0 = \frac{1}{4\kappa^2} \left(\nabla^2 h_{\mu\nu} - 2H_0^2 h_{\mu\nu} \right) + T_{\mathrm{matter}\,\mu\nu} \,. \tag{65}
$$

Then we find that the contribution from the Gauss-Bonnet term does not appear except in the length parameter $1/H_0$ of the de Sitter space, which is determined by taking into account the Gauss-Bonnet term. This could be due to the topological structure of the Gauss-Bonnet invariant. From Eq. (65), we find that there is no correction to the Newton law in the de Sitter space and also in the flat background, which corresponds to the limit of $H_0 \to 0$. This property does not depend on the form of f. In the case of $F(R)$ gravity, significant corrections appear in the Newton law in $1/R$ gravity, as was shown in Refs. [353], which conflicts with the observations coming from the solar system.

We now consider the relation between the modified Gauss-Bonnet gravity and the scalar-Einstein-Gauss-Bonnet gravity. By introducing two auxiliary scalar fields, A and B, we can rewrite the action (57) as follows,

$$
S = \int d^4x \sqrt{-g} \left(\frac{1}{2\kappa^2} R + B\left(\mathcal{G} - A\right) + f(A) + \mathcal{L}_{\mathrm{matter}} \right) \,. \tag{66}
$$

By the variation of the action (66) with respect to B, we obtain $A = \mathcal{G}$. By substituting the expression into (66), the action (57) is reobtained. On the other hand, the variation of the action (66) with respect to A in (66), we obtain $B = f'(A)$. Then we rewrite the action (57) as follows,

$$
S = \int d^4x \sqrt{-g} \left(\frac{1}{2\kappa^2} R + f'(A)\mathcal{G} - A f'(A) + f(A) \right) \,. \tag{67}
$$

By varying the action (67) with respect to A, we reobtain the relation $A = \mathcal{G}$. The scalar field A in (67) is not dynamic because there is no kinetic term for A. We may add, however, a kinetic term for A to the action by hand as follows,

$$
S = \int d^4x \sqrt{-g} \left(\frac{1}{2\kappa^2} R - \frac{\epsilon}{2} \partial_\mu A \partial^\mu A + f'(A)\mathcal{G} - A f'(A) + f(A) \right) \,. \tag{68}
$$

Then the resulting theory is a dynamical scalar theory coupled with the Gauss-Bonnet invariant and with a potential, which is the scalar-Einstein-Gauss-Bonnet gravity. In general, the scalar-Einstein-Gauss-Bonnet gravity has no ghosts and is stable. This model is related to string-inspired dilaton gravity, which was also proposed as an alternative model for the dark energy problem [354–356]. Then, since we can obtain the limit of $\epsilon \to 0$ smoothly, the corresponding $f(\mathcal{G})$ theory could not have any ghost and could be stable.

C. String-inspired Gravity

Superstring theory is a ten dimensional space-time theory. Since the observed space-time looks four dimensional, superstring theory was equipped with a scenario which describes the compactification from ten dimensions to four. The compactification technique makes use of many kinds of scalar fields, which are known under the names "moduli" or "dilaton", and these scalar fields are coupled with the curvature invariants. We now neglect the moduli fields associated with the radii of the internal space and we only consider the following action of the low-energy effective dilaton string theories. The general action of the theory is given by,

$$S = \int d^4x\sqrt{-g}\left[\frac{R}{2} + \mathcal{L}_\phi + \mathcal{L}_c + \dots\right] . \tag{69}$$

In the above equation, ϕ is the dilaton field, which is related to the string coupling, $\mathcal{L}_\phi$ is the Lagrangian of ϕ, and $\mathcal{L}_c$ expresses the string curvature correction terms to the Einstein-Hilbert action,

$$\mathcal{L}_\phi = -\partial_\mu\phi\partial^\mu\phi - V(\phi) , \quad \mathcal{L}_c = c_1\alpha' \mathrm{e}^{2\frac{\phi}{\phi_0}}\mathcal{L}_c^{(1)} + c_{\alpha'}^2 \mathrm{e}^{4\frac{\phi}{\phi_0}}\mathcal{L}_c^{(2)} + c_3{\alpha'}^3\mathrm{e}^{6\frac{\phi}{\phi_0}}\mathcal{L}_c^{(3)} . \tag{70}$$

In addition, $1/\alpha'$ is the string tension, $\mathcal{L}_c^{(1)}$, $\mathcal{L}_c^{(2)}$, and $\mathcal{L}_c^{(3)}$ express the leading-order (the Gauss-Bonnet term $\mathcal{G}$), the second-order, and the third-order curvature corrections, respectively. The Gauss-Bonnet invariant $\mathcal{G}$, which is defined by

$$\mathcal{G} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\xi\sigma}R^{\mu\nu\xi\sigma} . \tag{71}$$

The explicit forms of the terms $\mathcal{L}_c^{(1)}$, $\mathcal{L}_c^{(2)}$ and $\mathcal{L}_c^{(3)}$ are given by,

$$\mathcal{L}_c^{(1)} = \Omega_2 , \quad \mathcal{L}_c^{(2)} = 2\Omega_3 + R^{\mu\nu}_{\alpha\beta}R^{\alpha\beta}_{\lambda\rho}R^{\lambda\rho}_{\mu\nu} , \quad \mathcal{L}_c^{(3)} = \mathcal{L}_{31} - \delta_{\mathrm{H}}\mathcal{L}_{32} - \frac{\delta_B}{2}\mathcal{L}_{33} . \tag{72}$$

Here, δ_B and δ_H equal to 0 or 1 and

$$\begin{aligned}
\Omega_2 =& \mathcal{G} , \\
\Omega_3 \propto& \epsilon^{\mu\nu\rho\sigma\tau\eta}\epsilon_{\mu'\nu'\rho'\sigma'\tau'\eta'}R_{\mu\nu}{}^{\mu'\nu'}R_{\rho\sigma}{}^{\rho'\sigma'}R_{\tau\eta}{}^{\tau'\eta'} , \\
\mathcal{L}_{31} =& \zeta(3)R_{\mu\nu\rho\sigma}R^{\alpha\nu\rho\beta}\left(R^{\mu\gamma}{}_{\delta\beta}R_{\alpha\gamma}{}^{\delta\sigma} - 2R^{\mu\gamma}{}_{\delta\alpha}R_{\beta\gamma}{}^{\delta\sigma}\right) , \\
\mathcal{L}_{32} =& \frac{1}{8}\left(R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}\right)^2 + \frac{1}{4}R_{\mu\nu}{}^{\gamma\delta}R_{\gamma\delta}{}^{\rho\sigma}R_{\rho\sigma}{}^{\alpha\beta}R_{\alpha\beta}{}^{\mu\nu} - \frac{1}{2}R_{\mu\nu}{}^{\alpha\beta}R_{\alpha\beta}{}^{\rho\sigma}R^{\mu}{}_{\sigma\gamma\delta}R_{\rho}{}^{\nu\gamma\delta} - R_{\mu\nu}{}^{\alpha\beta}R_{\alpha\beta}{}^{\rho\nu}R_{\rho\sigma}{}^{\gamma\delta}R_{\gamma\delta}{}^{\mu\sigma} , \\
\mathcal{L}_{33} =& \left(R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}\right)^2 - 10R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\sigma}R_{\sigma\gamma\delta\rho}R^{\beta\gamma\delta\rho} - R_{\mu\nu\alpha\beta}R^{\mu\nu\rho}{}_{\sigma}R^{\beta\sigma\gamma\delta}R_{\delta\gamma\rho}{}^{\alpha} .
\end{aligned} \tag{73}$$

The values of δ_B and δ_H depend on the type of string theory. The dependence is encoded in the curvature invariants, in the coefficients (c_1, c_2, c_3) and finally in the parameters δ_H, δ_B, as follows,

- For the Type II superstring theory: $(c_1, c_2, c_3) = (0, 0, 1/8)$ and $\delta_H = \delta_B = 0$.
- For the heterotic superstring theory: $(c_1, c_2, c_3) = (1/8, 0, 1/8)$ and $\delta_H = 1, \delta_B = 0$.
- For the bosonic superstring theory: $(c_1, c_2, c_3) = (1/4, 1/48, 1/8)$ and $\delta_H = 0, \delta_B = 1$.

Motivated by the string considerations, we now consider the theories of the scalar-Einstein-Gauss-Bonnet gravity[3] based on [354, 358]. It was first proposed in Ref. [354] to consider the scalar-Einstein-Gauss-Bonnet gravity as a gravitational model of the dark energy.

[3] For pioneering work on the scalar-Einstein-Gauss-Bonnet gravity, see [357].

The starting action is given by,

$$S=\int d^4x\sqrt{-g}\left[\frac{R}{2\kappa^2}-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi-V(\phi)-\xi(\phi)\mathcal{G}\right] . \tag{74}$$

Here, we do not specify the forms of the potential $V(\phi)$ and the Gauss-Bonnet coupling $\xi(\phi)$, which could be determined by the non-perturbative effects in string theory (70). We should note that the action (74) is given by adding the kinetic term for the scalar field ϕ to the action of the $f(\mathcal{G})$ gravity in the scalar-tensor form (67) or by putting $A=\phi$ and $\epsilon=1$ in (68).

By varying the action (74) with respect to the metric $g_{\mu\nu}$, we obtain the following equation,

$$\begin{aligned}0=&\frac{1}{\kappa^2}\left(-R^{\mu\nu}+\frac{1}{2}g^{\mu\nu}R\right)+\frac{1}{2}\partial^\mu\phi\partial^\nu\phi-\frac{1}{4}g^{\mu\nu}\partial_\rho\phi\partial^\rho\phi-\frac{1}{2}g^{\mu\nu}V(\phi)\\&+2\left(\nabla^\mu\nabla^\nu\xi(\phi)\right)R-2g^{\mu\nu}\left(\nabla^2\xi(\phi)\right)R-4\left(\nabla_\rho\nabla^\mu\xi(\phi)\right)R^{\nu\rho}-4\left(\nabla_\rho\nabla^\nu\xi(\phi)\right)R^{\mu\rho}\\&+4\left(\nabla^2\xi(\phi)\right)R^{\mu\nu}+4g^{\mu\nu}\left(\nabla_\rho\nabla_\sigma\xi(\phi)\right)R^{\rho\sigma}+4\left(\nabla_\rho\nabla_\sigma\xi(\phi)\right)R^{\mu\rho\nu\sigma} ,\end{aligned} \tag{75}$$

where we have used the following identity which holds true in four dimensions: $0=\frac{1}{2}g^{\mu\nu}\mathcal{G}-2RR^{\mu\nu}-4R^\mu{}_\rho R^{\nu\rho}-2R^{\mu\rho\sigma\tau}R^\nu{}_{\rho\sigma\tau}+4R^{\mu\rho\nu\sigma}R_{\rho\sigma}$. We should note that in Eq. (75), the derivatives of curvature such as ∇R, do not appear, which tells that the derivatives higher than two do not appear.

This situation is contrasted with a general $\alpha R^2+\beta R_{\mu\nu}R^{\mu\nu}+\gamma R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ gravity, where fourth derivatives of $g_{\mu\nu}$ appear in general. This tells that for the classical theory, if we specify the values of $g_{\mu\nu}$ and $\dot{g}_{\mu\nu}$ on a spatial surface as an initial condition, the time dependence is uniquely determined. This situation is similar to the case in classical mechanics, where we only need to specify the values of the position and of the velocity of particle as initial conditions. On the other hand, in general $\alpha R^2+\beta R_{\mu\nu}R^{\mu\nu}+\gamma R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ gravity, as initial conditions we need to specify the values of $\ddot{g}_{\mu\nu}$ and $\dddot{g}_{\mu\nu}$ in addition to $g_{\mu\nu}$, $\dot{g}_{\mu\nu}$ in order to obtain a unique time evolution. In the Einstein gravity, we only need to specify the values of $g_{\mu\nu}$ and $\dot{g}_{\mu\nu}$. This may tell that the scalar-Gauss-Bonnet gravity could be a natural extension of the Einstein gravity.

In the FRW Universe (6), Eqs. (75) have the following form,

$$0=-\frac{3}{\kappa^2}H^2+\frac{1}{2}\dot{\phi}^2+V(\phi)+24H^3\frac{d\xi(\phi(t))}{dt} , \tag{76}$$

$$0=\frac{1}{\kappa^2}\left(2\dot{H}+3H^2\right)+\frac{1}{2}\dot{\phi}^2-V(\phi)-8H^2\frac{d^2\xi(\phi(t))}{dt^2}-16H\dot{H}\frac{d\xi(\phi(t))}{dt}-16H^3\frac{d\xi(\phi(t))}{dt} . \tag{77}$$

In particular when we consider the following model inspired by string theory [354],

$$V=V_0\mathrm{e}^{-\frac{2\phi}{\phi_0}} ,\quad \xi(\phi)=\xi_0\mathrm{e}^{\frac{2\phi}{\phi_0}} , \tag{78}$$

we find that the solution which describes the de Sitter space is,

$$H^2=H_0^2\equiv-\frac{\mathrm{e}^{-\frac{2\varphi_0}{\phi_0}}}{8\xi_0\kappa^2} ,\quad \phi=\varphi_0 , \tag{79}$$

where φ_0 is an arbitrary constant. When φ_0 is larger, the Hubble rate $H=H_0$ is smaller. Then in the case $\xi_0\sim\mathcal{O}(1)$, if we choose $\varphi_0/\phi_0\sim 140$, the value of the Hubble rate $H=H_0$ becomes consistent with the observations of the current Universe. There is also another exact solution for the model (78) which is,

$$\begin{aligned}H=&\frac{h_0}{t} ,\quad \phi=\phi_0\ln\frac{t}{t_1}\ \text{when}\ h_0>0 ,\\ H=&-\frac{h_0}{t_s-t} ,\quad \phi=\phi_0\ln\frac{t_s-t}{t_1}\ \text{when}\ h_0<0 .\end{aligned} \tag{80}$$

Here the constant h_0 is determined by solving the following algebraic equations:

$$0=-\frac{3h_0^2}{\kappa^2}+\frac{\phi_0^2}{2}+V_0t_1^2-\frac{48\xi_0h_0^3}{t_1^2} ,\quad 0=(1-3h_0)\phi_0^2+2V_0t_1^2+\frac{48\xi_0h_0^3}{t_1^2}(h_0-1) . \tag{81}$$

We can realize an arbitrary value of h_0 by properly choosing the values of V_0 and ξ_0. By appropriately choosing V_0 and ξ_0, it is possible to obtain a negative h_0, which corresponds to an effective phantom regime, where the EoS parameter (4) is less than -1, $w_{\text{eff}}<-1$. We should note that the phantom era cannot be realized by a canonical scalar in the usual scalar-tensor theory without the Gauss-Bonnet term.

D. $F(T)$ Gravity

As a gravitational theory besides Einstein's general relativity, we may consider "teleparallelism" with the Weitzenböck connection, where the torsion T is used as a fundamental ingredient instead of the curvature R, defined by the Levi-Civita connection [359–365]. In order to explain both of the inflationary era and the accelerating expansion of the current Universe, we may consider a model whose Lagrangian density is extended to be a function of T as $F(T)$ [146, 363, 366–392], from the teleparallel Lagrangian density given by the torsion scalar T instead of R in the Einstein gravity.

In contrast to other modified gravity theories studied in this review, the $F(T)$ gravity theory does not have the local Lorentz symmetry in general. Particularly, the torsion is not invariant under the local Lorentz transformation. This is due to the fact that we start with globally defined vierbein fields and in effect, the theory depends on the choice of the vierbein even if the metric induced from the vierbein is identical. In the case of general relativity for example, we cannot define the vierbein field globally in general, because there is a curvature for the local Lorentz symmetry. The exception to this rule is the case that the action is linear to the torsion T. The difference from the Einstein-Hilbert action, when we write the action in terms of the vierbein, is a total derivative and therefore the symmetry is enhanced in this case, and the action becomes invariant under the local Lorentz transformation. Although the torsion is not invariant under the local Lorentz transformation, the torsion is invariant under the global Lorentz symmetry. The global Lorentz symmetry is the symmetry of the tangent space of the space-time manifold. Therefore, if we choose the vierbein to be diagonal, the space-time has the Lorentz symmetry, which is inherited from the global Lorentz symmetry. The lack of the local Lorentz symmetry, however, generates some difficulties. For example, we cannot consistently couple the spinor fields with $F(T)$ gravity, and therefore the model cannot be realistic.

We need to note that the lack of the local Lorentz invariance is a crucial difference between $F(R)$ gravity and $F(T)$ gravity, as it was explained in [361]. Particularly, when the standard formulation of $F(T)$ gravity is used, the Lorentz invariance is absent or strongly restricted, since the spin connection vanishes. The latter assumption makes the theory simpler for the derivation of solutions, at the expense of having local Lorentz invariance violation and therefore the theory is frame dependent. The Lorentz violation issue can cause serious complications in the theory, which may become apparent when spherically symmetric solutions are discussed. The only way to construct a Lorentz invariant $F(T)$ theory is to impose covariance in the formulation, where both the vierbein and the spin connection are used, in which case the resulting connection makes the theory covariant, see for example [393]. However, for the FRW space-time, the absence of local Lorentz invariance is not an important issue, so for the sake of simplicity we shall use the standard formulation of $F(T)$ gravity.

In the context of $F(T)$ gravity, it is possible to explain the late-time acceleration of the Universe [146, 366, 367, 370, 380, 384, 394–401], but there exist also local astrophysical solutions, and various metric solutions, see for example [373, 375, 379, 385, 390, 402]. Finally for inflationary, bouncing cosmology and perturbation evolution studies, see [368, 369, 374, 381, 388, 391, 403–405].

We shall use the orthonormal tetrad components $e_A(x^\mu)$ for the teleparallelism theory analysis. We assume that the index A runs over 0, 1, 2, 3 for the tangent space at each point x^μ of the manifold and e_A^μ form the tangent vector of the manifold. The relation with the metric $g^{\mu\nu}$ is given by $g_{\mu\nu} = \eta_{AB} \mathrm{e}_\mu^A \mathrm{e}_\nu^B$. The torsion $T^\rho{}_{\mu\nu}$ and contorsion $K^{\mu\nu}_\rho$ tensors are defined as,

$$T^\rho{}_{\mu\nu} \equiv \mathrm{e}_A^\rho \left(\partial_\mu \mathrm{e}_\nu^A - \partial_\nu \mathrm{e}_\mu^A \right) , \quad K^{\mu\nu}{}_\rho \equiv -\frac{1}{2} \left(T^{\mu\nu}{}_\rho - T^{\nu\mu}{}_\rho - T_\rho{}^{\mu\nu} \right) . \tag{82}$$

Furthermore the torsion scalar T is defined by [360, 363],

$$T \equiv S_\rho{}^{\mu\nu} T^\rho{}_{\mu\nu} , \quad S_\rho{}^{\mu\nu} \equiv \frac{1}{2} \left(K^{\mu\nu}{}_\rho + \delta^\mu_\rho T^{\alpha\nu}{}_\alpha - \delta^\nu_\rho T^{\alpha\mu}{}_\alpha \right) . \tag{83}$$

Then the action of the modified teleparallel gravity describing $F(T)$ gravity [367] is given by,

$$S = \int d^4x |e| \left[\frac{F(T)}{2\kappa^2} + \mathcal{L}_{\mathrm{matter}} \right] . \tag{84}$$

Here $|e| = \det \left(\mathrm{e}_\mu^A \right) = \sqrt{-g}$. Upon variation of the action Eq. (84) with respect to the vierbein vector field e_A^μ we obtain, [366]

$$\frac{1}{e} \partial_\mu \left(e S_A{}^{\mu\nu} \right) f' - e_A^\lambda T^\rho{}_{\mu\lambda} S_\rho{}^{\nu\mu} f' + S_A{}^{\mu\nu} \partial_\mu T f'' + \frac{1}{4} e_A^\nu f = \frac{\kappa^2}{2} e_A^\rho T_{\mathrm{matter}\,\rho}{}^\nu . \tag{85}$$

We consider the FRW metric with flat spacial part (6) and then we find the tetrad components $\mathrm{e}_\mu^A = \mathrm{diag}\,(1, a, a, a)$, which yields $g_{\mu\nu} = \mathrm{diag}\left(1, -a^2, -a^2, -a^2\right)$. Then we find that the torsion scalar T is given by $T = -6H^2$. In the flat

FRW background (6) , Eq. (85) takes the following form,

$$\frac{3}{\kappa^2}H^2 = \rho_{\rm matter} + \rho_{\rm DE} \, , \quad \frac{1}{\kappa^2}\left(H^2 + \dot H\right) = p_{\rm matter} + p_{\rm DE} \, , \tag{86}$$

where,

$$\rho_{\rm DE} = \frac{1}{2\kappa^2}\left(-T - f + 2Tf'\right) \, , \quad p_{\rm DE} = -\frac{1}{2\kappa^2}\left(\left(4 - 4f' - 8Tf''\right)\dot H - T - f + 2Tf'\right) \, . \tag{87}$$

We should note that $\rho_{\rm DE}$ and $p_{\rm DE}$ in Eq. (87) satisfy the standard continuity equation

$$0 = \dot\rho_{\rm DE} + 3H\left(\rho_{\rm DE} + p_{\rm DE}\right) \, . \tag{88}$$

In the vacuum case, that is when $\rho_{\rm matter} = p_{\rm matter} = 0$, since $T = -6H^2$, the first Eq. (86) gives

$$0 = -f + 2Tf' \, , \quad 0 = f' + 2Tf'' \, , \tag{89}$$

which can be integrated and we obtain,

$$f(T) = f_0 \left(T\right)^{\frac{1}{2}} \, , \tag{90}$$

where f_0 is a constant. Therefore, the only consistent $F(T)$ is uniquely given by (89). On the other hand, if $F(T)$ is given by (90), Eqs. (86) can be satisfied regardless of what the choice of H is, which indicates that an arbitrary cosmological evolution can be realized by (90). There are many works related with $F(T)$ gravity and several solutions have been found, see for example, [361, 378, 379, 395, 402, 406–411].

E. Massive $F(R)$ Gravity and Massive Bigravity

In the gravity models we presented in the previous sections, the accelerating expansion of the Universe is achieved by the condensation of scalar quantities. If there could occur the condensation of a vector field, and especially of the spatial components of the vector field, the isotropy of space could be broken and therefore the theory would be rendered inconsistent. Besides the case of a scalar quantity condensation, the condensation of a rank-two symmetric tensor field, does not always break the isotropy of space, if the spatial components of the tensor field are proportional to the spatial components of the metric tensor. Usually the rank-two symmetric tensor field corresponds to spin-two particles. Since the massless mode cannot condense, we need to consider a theory of massive spin-two particles, which could be regarded as massive gravitons. Since the typical scale of the accelerating expansion of the present Universe is about 10^{-33} eV, it is natural to expect that the mass of the massive graviton would be almost 10^{-33} eV. Since the mass is so small, we may naively expect that the interaction of the massive graviton could not be changed from that of the standard massless graviton, although the situation is not so simple because the degrees of freedom of the massless spin-two particle are two, which corresponds to the helicity, but the degrees of freedom of the massive spin-two field are five. There is a very long history in the study of the massive spin-two particles. The free theory was found by Fierz and Pauli in 1939 [412]. It was soon found that if the massive spin-two field couples with the matter fields as in the Einstein gravity, the extra three degrees of freedom do not decouple, because their coupling with matter diverges in the massless limit, which is called as vDVZ discontinuity [413, 414]. It was found, however, that the extra three degrees of freedom can be screened by the Vainshtein mechanism [415] if we introduce a non-linearity as in the Einstein gravity case. The non-linearity, however, generates an extra degree of freedom, which is a ghost called the Boulware-Deser ghost [416]. The symmetric tensor field has ten components in four dimensions. In case of the Einstein gravity, the re-parametrization invariance removes four degrees of freedom and one lapse function and three shift functions give in total four constraints and therefore only two degrees of freedom, corresponding to the helicity remain. In the case of the massive graviton model by Fierz and Pauli, the three components corresponding to the shift functions can be solved with respect to other components and the component corresponding to the lapse function becomes the Lagrange multiplier field, which is not dynamical and gives one constraint, and therefore there remain five degrees of freedom. If we introduce the non-linearity in the theory, however, the shift function does not become the Lagrange multiplier field but the shift function is solved with respect to other components and does not give any further constraint.

Then there appears one extra degree of freedom, which corresponds to the Boulware-Deser ghost. After that, there had not been much progress in the study of the massive gravity for a long time until the dRGT model [417, 418] was found. In the formulation of the dRGT model, it becomes clear how we can introduce the non-linearity to keep the shift function to be the Lagrange multiplier field and therefore there does not appear any ghost. It is possible if the massive spin-two particle coupled with the standard gravity, which is called as bigravity and even in this model, the ghost does not appear [419]. For the general review on the massive gravity, see [420] and for a more recent review, see [421].

In the massive gravity, the same cosmological constant is replaced by the small mass of the massive graviton. Therefore, the problem of the cosmological constant is never solved, but we need to explain the unnaturally small mass of the massive graviton. The massive gravity model could be, however, a low energy effective theory, because the unitarity is broken in high energy regime, even for the dRGT model. Therefore, the small mass might be explained by a more fundamental theory. Another problem could be that it is difficult to generate the accelerating expansion of the Universe only by the massive gravity theory or the bigravity theory. Then we need to extend the models to the $F(R)$ gravity form in the way we demonstrate in the present section. By such an extension, however, we need to introduce more small parameters beside the graviton mass and therefore the situation of the cosmological constant problem becomes much more complicated than in the standard ΛCDM model. In spite of such difficulties, we may expect that a more fundamental theory could solve the problem and the study of the $F(R)$ extension we present in this section may provide new hints for the more fundamental theory.

Now we consider a model of massive gravity which is an extension of $F(R)$ gravity, whose action is given by,

$$S_{\rm mg} = M_g^2 \int d^4x \sqrt{-\det g}\, F\left(R^{(g)} + 2m^2 \int d^4x \sqrt{-\det g} \sum_{n=0}^{4} \beta_n\, e_n \left(\sqrt{g^{-1}f} \right) \right) + S_{\rm matter}\,. \tag{91}$$

Here $R^{(g)}$ is the scalar curvature for $g_{\mu\nu}$ and $f_{\mu\nu}$ is a non-dynamical reference metric. We define the tensor $\sqrt{g^{-1}f}$ by the square root of $g^{\mu\rho} f_{\rho\nu}$, $\left(\sqrt{g^{-1}f}\right)^{\mu}_{\ \rho} \left(\sqrt{g^{-1}f}\right)^{\rho}_{\ \nu} = g^{\mu\rho} f_{\rho\nu}$. For a general tensor $X^\mu_{\ \nu}$, we define the quantities $e_n(X)$'s as follows,

$$\begin{aligned}
& e_0(X) = 1\,, \quad e_1(X) = [X]\,, \quad e_2(X) = \tfrac{1}{2}([X]^2 - [X^2])\,, \\
& e_3(X) = \tfrac{1}{6}([X]^3 - 3[X][X^2] + 2[X^3])\,, \\
& e_4(X) = \tfrac{1}{24}([X]^4 - 6[X]^2[X^2] + 3[X^2]^2 + 8[X][X^3] - 6[X^4])\,, \\
& e_k(X) = 0 \quad \text{for } k > 4\,.
\end{aligned} \tag{92}$$

Here $[X]$ expresses the trace of arbitrary tensor $X^\mu_{\ \nu}$, $[X] \equiv X^\mu_{\ \mu}$.

In the following, just for simplicity, we only consider the minimal case, where,

$$2m^2 \sum_{n=0}^{4} \beta_n\, e_n \left(\sqrt{g^{-1}f} \right) = 2m^2 \left(3 - \mathrm{tr}\, \sqrt{g^{-1}f} + \det \sqrt{g^{-1}f} \right)\,. \tag{93}$$

Then the equation motion which are obtained from the action when it is varied with respect to the metric, have the following form:

$$\begin{aligned}
0 =& M_g^2 \left(\frac{1}{2} g_{\mu\nu} F\left(\tilde R^{(g)}\right) - R^{(g)}_{\mu\nu} F'\left(\tilde R^{(g)}\right) + \nabla_\nu \nabla_\mu F'\left(\tilde R^{(g)}\right) - g_{\mu\nu} \nabla^2 F'\left(\tilde R^{(g)}\right) \right) \\
& + m^2 M_g^2 F'\left(\tilde R^{(g)}\right) \left\{ \frac{1}{2} f_{\mu\rho} \left(\sqrt{g^{-1}f}\right)^{-1\,\rho}_{\quad\ \nu} + \frac{1}{2} f_{\nu\rho} \left(\sqrt{g^{-1}f}\right)^{-1\,\rho}_{\quad\ \mu} - g_{\mu\nu} \det \sqrt{g^{-1}f} \right\} + \frac{1}{2} T_{\mathrm{matter}\,\mu\nu}\,.
\end{aligned} \tag{94}$$

Here,

$$\tilde R^{(g)} \equiv R^{(g)} + 2m^2 \left(3 - \mathrm{tr}\, \sqrt{g^{-1}f} + \det \sqrt{g^{-1}f} \right)\,. \tag{95}$$

Now the covariant derivative ∇_μ is defined, as usual, in terms of the Levi-Civita connection defined by the metric $g_{\mu\nu}$. In this section, we do not use the covariant derivative with respect to the metric $f_{\mu\nu}$. First we observe that,

$$\begin{aligned}
& \nabla^\mu \left(\frac{1}{2} g_{\mu\nu} F\left(\tilde R^{(g)}\right) - R^{(g)}_{\mu\nu} F'\left(\tilde R^{(g)}\right) + \nabla_\nu \nabla_\mu F'\left(\tilde R^{(g)}\right) - g_{\mu\nu} \nabla^2 F'\left(\tilde R^{(g)}\right) \right) \\
=& m^2 F'\left(\tilde R^{(g)}\right) \partial_\nu \left(- \mathrm{tr}\, \sqrt{g^{-1}f} + \det \sqrt{g^{-1}f} \right)\,,
\end{aligned} \tag{96}$$

which can be obtained by using the Bianchi identity $0 = \nabla^\mu \left(\frac{1}{2}g_{\mu\nu}R^{(g)} - R^{(g)}_{\mu\nu}\right)$. By multiplying the covariant derivative ∇^μ with respect to the metric g with Eq. (94) and using the conservation law $0 = \nabla^\mu T_{\mathrm{matter}\,\mu\nu}$, we find the following equation,

$$\begin{aligned}
0 =& F'\left(\tilde{R}^{(g)}\right)\partial_\nu\left(-\mathrm{tr}\sqrt{g^{-1}f} + \det\sqrt{g^{-1}f}\right) \\
&+ \left(\partial^\mu F'\left(\tilde{R}^{(g)}\right)\right)\left\{\frac{1}{2}f_{\mu\rho}\left(\sqrt{g^{-1}f}\right)^{-1\,\rho}_{\quad\ \nu} + \frac{1}{2}f_{\nu\rho}\left(\sqrt{g^{-1}f}\right)^{-1\,\rho}_{\quad\ \mu} - g_{\mu\nu}\det\sqrt{g^{-1}f}\right\} \\
&+ F'\left(\tilde{R}^{(g)}\right)\nabla^\mu\left\{\frac{1}{2}f_{\mu\rho}\left(\sqrt{g^{-1}f}\right)^{-1\,\rho}_{\quad\ \nu} + \frac{1}{2}f_{\nu\rho}\left(\sqrt{g^{-1}f}\right)^{-1\,\rho}_{\quad\ \mu} - g_{\mu\nu}\det\sqrt{g^{-1}f}\right\}.
\end{aligned} \tag{97}$$

We consider that the metric $g_{\mu\nu}$ is the flat FRW Universe and flat Minkowski space-time for $f_{\mu\nu}$. We also use the conformal time t with metric $g_{\mu\nu}$ as follows,

$$ds_g^2 = \sum_{\mu,\nu=0}^{3} g_{\mu\nu}dx^\mu dx^\nu = a(t)^2\left(-dt^2 + \sum_{i=1}^{3}\left(dx^i\right)^2\right), \quad ds_f^2 = \sum_{\mu,\nu=0}^{3} f_{\mu\nu}dx^\mu dx^\nu = -dt^2 + \sum_{i=1}^{3}\left(dx^i\right)^2. \tag{98}$$

Then, although the $\nu = i$ component in (97) is trivially satisfied, the $\nu = t$ component gives the following equation,

$$0 = \partial_t\left(-4a^{-1} + a^{-4}\right)F'\left(\tilde{R}^{(g)}\right) + \left(a^{-1} - a^{-4}\right)\partial_t F'\left(\tilde{R}^{(g)}\right) = a^{-4}\partial_t\left\{F'\left(\tilde{R}^{(g)}\right)\left(a^3 - 1\right)\right\}, \tag{99}$$

which further gives a constraint,

$$F'\left(\tilde{R}^{(g)}\right)\left(a^3 - 1\right) = C. \tag{100}$$

Here C is a constant. By using Eq. (100), we can determine the form of $F'\left(\tilde{R}^{(g)}\right)$. For a given evolution of the scale factor $a = a(t)$ with respect to the time, we can find the t-dependence of $\tilde{R}^{(g)}$, $\tilde{R}^{(g)} = \tilde{R}^{(g)}(t)$. This equation can be solved with respect to t as a function of $\tilde{R}^{(g)}$, $t = t\left(\tilde{R}^{(g)}\right)$. By using the obtained expression and Eq. (100), we find the following form of $F'\left(\tilde{R}^{(g)}\right)$,

$$F'\left(\tilde{R}^{(g)}\right) = \frac{C}{a\left(t\left(\tilde{R}^{(g)}\right)\right)^3 - 1}, \tag{101}$$

We should note that as we show shortly, the time evolution of the scale factor $a = a(t)$ cannot be arbitrary. Furthermore $F'\left(\tilde{R}^{(g)}\right)$ diverges when the scale factor a goes to unity.

In the FRW metric with conformal time in (98), the $(\mu,\nu) = (t,t)$ component in (94) has the following form:

$$0 = -\frac{1}{2}a^{-2}F\left(\tilde{R}^{(g)}\right) + 3\dot{H}F'\left(\tilde{R}^{(g)}\right) - 3H\partial_t F'\left(\tilde{R}^{(g)}\right) + \left(-a + a^{-3}\right)F'\left(\tilde{R}^{(g)}\right) + \frac{1}{2M_g^2}\rho_{\mathrm{matter}}, \tag{102}$$

and the $(\mu,\nu) = (i,j)$ component gives,

$$0 = \frac{1}{2}a^{-2}F\left(\tilde{R}^{(g)}\right) - \left(\dot{H} + 2H^2\right)F'\left(\tilde{R}^{(g)}\right) + \left(\partial_t^2 + H\partial_t\right)F'\left(\tilde{R}^{(g)}\right) - \left(-a + a^{-3}\right)F'\left(\tilde{R}^{(g)}\right) + \frac{1}{2M_g^2}p_{\mathrm{matter}}. \tag{103}$$

By combining Eqs. (102) and (103), we obtain,

$$0 = 2\left(\dot{H} - H^2\right)F'\left(\tilde{R}^{(g)}\right) + \left(\partial_t^2 - 2H\partial_t\right)F'\left(\tilde{R}^{(g)}\right) + \frac{1}{2M_g^2}\left(\rho_{\mathrm{matter}} + p_{\mathrm{matter}}\right). \tag{104}$$

In contrast to the Einstein gravity case, Eqs. (102), (103), and the conservation law (60) are independent equations. Note that the form of the conservation law in terms of the conformal time is not changed from that in terms of the cosmological time. We now treat Eqs. (100), (104), and the conservation law (60) as independent equations. By using Eq. (100), we can rewrite Eq. (104) as,

$$0 = \frac{\left\{\dot{H}\left(-a^6 - a^3 + 2\right) + H^2\left(13a^6 + 7a^3 - 2\right)\right\}C}{\left(a^3 - 1\right)^3} + \frac{1}{2M_g^2}\left(\rho_{\mathrm{matter}} + p_{\mathrm{matter}}\right). \tag{105}$$

This equation (105) describes the dynamics of the Universe, which does not depend on the form of $F\left(\tilde{R}^{(g)}\right)$.

Since it is difficult to solve (105) explicitly, we consider the following three cases, that is, a) The $a \to 1$ case, b) The $a \gg 1$ case, c) The $a \ll 1$ case. In the following, for simplicity, we consider the case that the matter has a constant EoS parameter w.

a) The $a \to 1$ case. By putting $a = 1 + \delta a$ and by using (105), we find,

$$0 \sim -9\dot{H} + 18H^2 \sim -9\delta\ddot{a}\delta a + 18\left(\delta\dot{a}\right)^2 = -9\left(\delta a\right)^3 \frac{d}{dt}\left(\frac{\delta\dot{a}}{\left(\delta a\right)^2}\right) , \tag{106}$$

which can be solved as,

$$\delta a = \frac{C_1}{t + C_2} . \tag{107}$$

Here we denote the constants of integration by C_1 and C_2. By using Eq. (107), we find that the limit $a \to 1$ ($\delta a \to 0$) is realized in the infinite past or future in conformal time, $t \to \pm\infty$.

b) The $a \gg 1$ case. In this case, we can approximate Eq. (105) as,

$$0 \sim Ca^{-3}\left(-\dot{H} + 13H^2\right) + \frac{1+w}{2}\rho_0 a^{-3(w+1)} . \tag{108}$$

1. The $\rho_0 = 0$ case. In this case, we find that the solution of (108) is,

$$H = \frac{1}{13\left(t_0 - t\right)} . \tag{109}$$

   This solution describes a phantom Universe which has a Big Rip singularity at $t = t_0$ because we assume $a \gg 1$.

2. The $\rho_0 \neq 0$ case. In case $w \neq 0$, we find the following power law solution,

$$a = a_0 t^{\frac{2}{3w}} . \tag{110}$$

   Here a_0 is given by solving the following equation,

$$0 = \frac{2C}{3w}\left(1 + \frac{26}{3w}\right) + \frac{w+1}{2} a_0^{-3w}\rho_0 . \tag{111}$$

   On the other hand, for the case $w = 0$, we obtain a solution describing the de Sitter Universe:

$$H^2 = \frac{\rho_0}{26C} . \tag{112}$$

   This could be interesting because the accelerating expansion of the present Universe can be realized by dust, which could be identified with cold dark matter. Then we find

$$\frac{1}{26C}\rho_0 \sim \left(10^{-33}\,\mathrm{eV}\right)^2 , \quad \rho_0 a^3 \sim \left(10^{-3}\,\mathrm{eV}\right)^4 . \tag{113}$$

c) The $a \ll 1$ case. We find the approximated form of Eq. (105) as follows,

$$0 = -2C\left(\dot{H} - H^2\right) + \frac{1+w}{2}\rho_0 a^{-3(w+1)} . \tag{114}$$

1. The $\rho_0 = 0$ case. The solution is given by,

$$H = \frac{1}{t_0 - t} , \tag{115}$$

   with t_0 being a constant of integration . The expression (115) is valid when $t \to \pm\infty$ because we are assuming $a \ll 1$. Then, regardless of the form (115), there does not always occur a Big Rip singularity.

2. The $\rho_0 \neq 0$ case. The solution of Eq. (115) is given by,

$$a = a_0 t^{\frac{2}{3(w+1)}} . \tag{116}$$

The constant a_0 is given by solving the following equation,

$$0 = -\frac{4C}{3(w+1)}\left(1 - \frac{2}{3(w+1)}\right) + \frac{w+1}{2} a_0^{-3w} \rho_0 . \tag{117}$$

Then we find that the qualitative behavior is similar to the Einstein gravity coupled with matter having a constant EoS parameter w.

Just for comparison reasons, instead of the model in Eq. (91), we may consider an $F(R)$ gravity extension of massive gravity proposed in [422], whose action is given by

$$S_{\rm mg} = M_g^2 \int d^4x \sqrt{-\det g}\, F\left(R^{(g)}\right) + 2m^2 M_g^2 \int d^4x \sqrt{-\det g} \sum_{n=0}^{4} \beta_n\, e_n \left(\sqrt{g^{-1}f}\right) + S_{\rm matter} . \tag{118}$$

For simplicity, we only consider the minimal case in the following, so as in (93) we have,

$$S_{\rm mg} = M_g^2 \int d^4x \sqrt{-\det g}\, F\left(R^{(g)}\right) + 2m^2 M_g^2 \int d^4x \sqrt{-\det g} \left(3 - {\rm tr}\, \sqrt{g^{-1}f} + \det \sqrt{g^{-1}f}\right) + S_{\rm matter} , \tag{119}$$

Then upon variation with respect to the metric $g_{\mu\nu}$, we find,

$$\begin{aligned} 0 =& M_g^2 \left(\frac{1}{2} g_{\mu\nu} F\left(R^{(g)}\right) - R^{(g)}_{\mu\nu} F'\left(R^{(g)}\right) + \nabla_\nu \nabla_\mu F'\left(R^{(g)}\right) - g_{\mu\nu}\nabla^2 F'\left(R^{(g)}\right)\right) \\ & + m^2 M_g^2 \left\{ g_{\mu\nu}\left(3 - {\rm tr}\,\sqrt{g^{-1}f}\right) + \frac{1}{2} f_{\mu\rho}\left(\sqrt{g^{-1}f}\right)^{-1\,\rho}{}_{\nu} + \frac{1}{2} f_{\nu\rho}\left(\sqrt{g^{-1}f}\right)^{-1\,\rho}{}_{\mu}\right\} + \frac{1}{2} T_{{\rm matter}\,\mu\nu} . \end{aligned} \tag{120}$$

Then, instead of Eq. (96), we have the following equations,

$$0 = \nabla^\mu \left(\frac{1}{2} g_{\mu\nu} F\left(R^{(g)}\right) - R^{(g)}_{\mu\nu} F'\left(R^{(g)}\right) + \nabla_\nu \nabla_\mu F'\left(R^{(g)}\right) - g_{\mu\nu}\nabla^2 F'\left(R^{(g)}\right)\right) , \tag{121}$$

and

$$0 = -g_{\mu\nu}\nabla^\mu \left({\rm tr}\,\sqrt{g^{-1}f}\right) + \frac{1}{2}\nabla^\mu \left\{ f_{\mu\rho}\left(\sqrt{g^{-1}f}\right)^{-1\,\rho}{}_{\nu} + f_{\nu\rho}\left(\sqrt{g^{-1}f}\right)^{-1\,\rho}{}_{\mu}\right\} , \tag{122}$$

which corresponds to Eq. (97). We choose the $g_{\mu\nu}$ metric to describe the FRW Universe and the metric $f_{\mu\nu}$ describing a flat Minkowski space-time and also we use the conformal time t as in Eq. (98). Then the (t,t) component of (120) has the following form,

$$0 = -3M_g^2 H^2 - 3m^2 M_g^2 \left(a^2 - a\right) + \rho_{\rm matter} , \tag{123}$$

and (i,j) components are given by,

$$0 = M_g^2 \left(2\dot{H} + H^2\right) + 3m^2 M_g^2 \left(a^2 - a\right) + p_{\rm matter} . \tag{124}$$

On the other hand, Eq. (122) gives the following constraint:

$$\frac{\dot{a}}{a} = 0 . \tag{125}$$

In contrast to Eq. (100), the identity (125) shows that a should be a constant $a = a_0$. This indicates that the only consistent solution for $g_{\mu\nu}$ is the flat Minkowski space. Therefore, we cannot obtain the expanding Universe without extra fields and/or fluids.

Before we close this section, we also review in brief the construction of ghost-free $F(R)$ bigravity by following Ref. [423]. The consistent model of bimetric gravity, which includes two metric tensors $g_{\mu\nu}$ and $f_{\mu\nu}$, was proposed in Ref. [419]. This model contains a massless spin-two field, corresponding to graviton, and also a massive spin-two field.

The gravity model which only contains the massive spin-two field is called massive gravity but now we consider the model including both the massless and massive spin-two field, which is called bigravity. The Boulware-Deser ghost [416] does not appear in such a theory.

We start with the following action,

$$
\begin{aligned}
S_{\mathrm{bi}} =& M_g^2 \int d^4x \sqrt{-\det g}\, R^{(g)} + M_f^2 \int d^4x \sqrt{-\det f}\, R^{(f)} \\
&+ 2m^2 M_{\mathrm{eff}}^2 \int d^4x \sqrt{-\det g} \sum_{n=0}^{4} \beta_n\, e_n \left(\sqrt{g^{-1} f} \right) .
\end{aligned} \tag{126}
$$

Here $R^{(g)}$ is the scalar curvature for $g_{\mu\nu}$ again and $R^{(f)}$ is the scalar curvature for $f_{\mu\nu}$. The massive parameter M_{eff} is defined by,

$$
\frac{1}{M_{\mathrm{eff}^2}} = \frac{1}{M_g^2} + \frac{1}{M_f^2} . \tag{127}
$$

Furthermore, the tensor $\sqrt{g^{-1} f}$ is defined by the square root of $g^{\mu\rho} f_{\rho\nu}$ again.

In order to construct a consistent $F(R)$ bigravity, we add the following terms to the action (126):

$$
S_\varphi = - M_g^2 \int d^4x \sqrt{-\det g} \left\{ \frac{3}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi + V(\varphi) \right\} + \int d^4x \mathcal{L}_{\mathrm{matter}} \left(\mathrm{e}^{\varphi} g_{\mu\nu}, \Phi_i \right) , \tag{128}
$$

$$
S_\xi = - M_f^2 \int d^4x \sqrt{-\det f} \left\{ \frac{3}{2} f^{\mu\nu} \partial_\mu \xi \partial_\nu \xi + U(\xi) \right\} . \tag{129}
$$

By using the conformal transformations $g_{\mu\nu} \to \mathrm{e}^{-\varphi} g^{\mathrm{J}}_{\mu\nu}$ and $f_{\mu\nu} \to \mathrm{e}^{-\xi} f^{\mathrm{J}}_{\mu\nu}$, the total action $S_F = S_{\mathrm{bi}} + S_\varphi + S_\xi$ is transformed as follows,

$$
\begin{aligned}
S_F =& M_f^2 \int d^4x \sqrt{-\det f^{\mathrm{J}}} \left\{ \mathrm{e}^{-\xi} R^{\mathrm{J}(f)} - \mathrm{e}^{-2\xi} U(\xi) \right\} \\
&+ 2m^2 M_{\mathrm{eff}}^2 \int d^4x \sqrt{-\det g^{\mathrm{J}}} \sum_{n=0}^{4} \beta_n \mathrm{e}^{\left(\frac{n}{2}-2\right)\varphi - \frac{n}{2}\xi} e_n \left(\sqrt{{g^{\mathrm{J}}}^{-1} f^{\mathrm{J}}} \right) \\
&+ M_g^2 \int d^4x \sqrt{-\det g^{\mathrm{J}}} \left\{ \mathrm{e}^{-\varphi} R^{\mathrm{J}(g)} - \mathrm{e}^{-2\varphi} V(\varphi) \right\} + \int d^4x \mathcal{L}_{\mathrm{matter}} \left(g^{\mathrm{J}}_{\mu\nu}, \Phi_i \right) .
\end{aligned} \tag{130}
$$

Then the kinetic terms for φ and ξ vanish. By the variations of the action with respect to φ and ξ, as in the case of convenient $F(R)$ gravity [140], we obtain,

$$
0 = 2m^2 M_{\mathrm{eff}}^2 \sum_{n=0}^{4} \beta_n \left(\frac{n}{2} - 2 \right) \mathrm{e}^{\left(\frac{n}{2}-2\right)\varphi - \frac{n}{2}\xi} e_n \left(\sqrt{{g^{\mathrm{J}}}^{-1} f^{\mathrm{J}}} \right) + M_g^2 \left\{ -\mathrm{e}^{-\varphi} R^{\mathrm{J}(g)} + 2\mathrm{e}^{-2\varphi} V(\varphi) + \mathrm{e}^{-2\varphi} V'(\varphi) \right\} , \tag{131}
$$

$$
0 = - 2m^2 M_{\mathrm{eff}}^2 \sum_{n=0}^{4} \frac{\beta_n n}{2} \mathrm{e}^{\left(\frac{n}{2}-2\right)\varphi - \frac{n}{2}\xi} e_n \left(\sqrt{{g^{\mathrm{J}}}^{-1} f^{\mathrm{J}}} \right) + M_f^2 \left\{ -\mathrm{e}^{-\xi} R^{\mathrm{J}(f)} + 2\mathrm{e}^{-2\xi} U(\xi) + \mathrm{e}^{-2\xi} U'(\xi) \right\} . \tag{132}
$$

Then, Eqs. (131) and (132) can be solved algebraically with respect to φ and ξ as $\varphi = \varphi \left(R^{\mathrm{J}(g)}, R^{\mathrm{J}(f)}, e_n \left(\sqrt{{g^{\mathrm{J}}}^{-1} f^{\mathrm{J}}} \right) \right)$ and $\xi = \xi \left(R^{\mathrm{J}(g)}, R^{\mathrm{J}(f)}, e_n \left(\sqrt{{g^{\mathrm{J}}}^{-1} f^{\mathrm{J}}} \right) \right)$. Substituting the φ and ξ we found above, into (130), one gets the $F(R)$ bigravity which is described as follows:

$$
\begin{aligned}
S_F =& M_f^2 \int d^4x \sqrt{-\det f^{\mathrm{J}}} F^{(f)} \left(R^{\mathrm{J}(g)}, R^{\mathrm{J}(f)}, e_n \left(\sqrt{{g^{\mathrm{J}}}^{-1} f^{\mathrm{J}}} \right) \right) \\
&+ 2m^2 M_{\mathrm{eff}}^2 \int d^4x \sqrt{-\det g} \sum_{n=0}^{4} \beta_n \mathrm{e}^{\left(\frac{n}{2}-2\right)\varphi\left(R^{\mathrm{J}(g)}, e_n \left(\sqrt{{g^{\mathrm{J}}}^{-1} f^{\mathrm{J}}} \right) \right)} e_n \left(\sqrt{{g^{\mathrm{J}}}^{-1} f^{\mathrm{J}}} \right) \\
&+ M_g^2 \int d^4x \sqrt{-\det g^{\mathrm{J}}} F^{\mathrm{J}(g)} \left(R^{\mathrm{J}(g)}, R^{\mathrm{J}(f)}, e_n \left(\sqrt{{g^{\mathrm{J}}}^{-1} f^{\mathrm{J}}} \right) \right) + \int d^4x \mathcal{L}_{\mathrm{matter}} \left(g^{\mathrm{J}}_{\mu\nu}, \Phi_i \right) ,
\end{aligned} \tag{133}
$$

$$
\begin{aligned}
F^{\mathrm{J}(g)}\left(R^{\mathrm{J}(g)},R^{\mathrm{J}(f)},e_n\left(\sqrt{g^{\mathrm{J}-1}f^{\mathrm{J}}}\right)\right)\equiv&\left\{\mathrm{e}^{-\varphi\left(R^{\mathrm{J}(g)},R^{\mathrm{J}(f)},e_n\left(\sqrt{g^{\mathrm{J}-1}f^{\mathrm{J}}}\right)\right)}R^{\mathrm{J}(g)}\right.\\
&\left.-\mathrm{e}^{-2\varphi\left(R^{\mathrm{J}(g)},R^{\mathrm{J}(f)},e_n\left(\sqrt{g^{\mathrm{J}-1}f^{\mathrm{J}}}\right)\right)}V\left(\varphi\left(R^{\mathrm{J}(g)},R^{\mathrm{J}(f)},e_n\left(\sqrt{g^{\mathrm{J}-1}f^{\mathrm{J}}}\right)\right)\right)\right\}\,,\\
F^{(f)}\left(R^{\mathrm{J}(g)},R^{\mathrm{J}(f)},e_n\left(\sqrt{g^{\mathrm{J}-1}f^{\mathrm{J}}}\right)\right)\equiv&\left\{\mathrm{e}^{-\xi\left(R^{\mathrm{J}(g)},R^{\mathrm{J}(f)},e_n\left(\sqrt{g^{\mathrm{J}-1}f^{\mathrm{J}}}\right)\right)}R^{\mathrm{J}(f)}\right.\\
&\left.-\mathrm{e}^{-2\xi\left(R^{\mathrm{J}(g)},R^{\mathrm{J}(f)},e_n\left(\sqrt{g^{\mathrm{J}-1}f^{\mathrm{J}}}\right)\right)}U\left(\xi\left(R^{\mathrm{J}(g)},R^{\mathrm{J}(f)},e_n\left(\sqrt{g^{\mathrm{J}-1}f^{\mathrm{J}}}\right)\right)\right)\right\}\,.
\end{aligned}
\tag{134}
$$

Note that it is difficult to solve Eqs. (131) and (132) with respect to φ and ξ explicitly. Therefore, it might be easier to define the model in terms of the auxiliary scalars φ and ξ as in Eq. (130) (for details see [424]).

In Ref. [425], another kind of non-linear massive $F(R)$ gravity has been proposed. The action is given by,

$$
S_{\mathrm{mg}}=M_g^2\int d^4x\sqrt{-\det g}\,F\left(R^{(g)}+2m^2\int d^4x\sqrt{-\det g}\sum_{n=0}^{4}\beta_n\,e_n\left(\sqrt{g^{-1}f}\right)\right)+S_{\mathrm{matter}}\,. \tag{135}
$$

In this action, $f_{\mu\nu}$ is a non-dynamical reference metric. By using the Hamiltonian formulation in the scalar-tensor frame, it has been shown that this kind of $F(R)$ theory is ghost-free.

F. Mimetic $F(R)$ gravity

Mimetic gravity is a modified gravity theory for which the conformal symmetry is respected as an internal degree of freedom [426] (for its generalizations, see [427–431]). The terminology mimetic gravity firstly appeared in Ref. [426], and since then, several works were devoted on mimetic gravity in various contexts, see the recent review [432]. For example in Ref. [433], see also Refs. [434–437] the generalization of mimetic gravity in the context of $F(R)$ gravity was performed and in Refs. [438–441] the mimetic gravity extensions of various modified gravity models were studied. In Refs. [442–448] several cosmological applications of mimetic gravity were studied in various theoretical contexts. In addition, in Refs. [449–453] various astrophysical solutions in mimetic gravity were presented. Finally for some late-time applications of mimetic gravity, see [454, 455] and for some more involved mimetic unimodular gravity models see [456, 457].

Although we usually regard the metric $g_{\mu\nu}$ as a fundamental variable of gravity, in the case of mimetic gravity, the metric is expressed in a different way by using new degrees of freedom. We may consider the variation of the action with respect to the new degrees of freedom and the obtained equation may admit a new or wider class of solutions compared with the equations given by the variation with respect to the metric $g_{\mu\nu}$. In the case of mimetic gravity, we use the following parametrization of the metric,

$$
g_{\mu\nu}=-\hat{g}^{\rho\sigma}\partial_\rho\phi\partial_\sigma\phi\hat{g}_{\mu\nu}\,, \tag{136}
$$

and we consider the variation with respect to $\hat{g}_{\mu\nu}$ and ϕ. Since the parametrization is invariant under the Weyl transformation $\hat{g}_{\mu\nu}\to\mathrm{e}^{\sigma(x)}\hat{g}_{\mu\nu}$, the variation over $\hat{g}_{\mu\nu}$ gives the traceless part of the equation. In fact, in case of the Einstein gravity, whose action is given by,

$$
S=\int d^4x\sqrt{-g\left(\hat{g}_{\mu\nu},\phi\right)}\left(\frac{R\left(\hat{g}_{\mu\nu},\phi\right)}{2\kappa^2}+\mathcal{L}_{\mathrm{matter}}\right)\,, \tag{137}
$$

the variation with respect to $\hat{g}_{\mu\nu}$ gives the traceless part of the Einstein equation:

$$
0=-R\left(\hat{g}_{\mu\nu},\phi\right)_{\mu\nu}+\frac{1}{2}g\left(\hat{g}_{\mu\nu},\phi\right)_{\mu\nu}R\left(\hat{g}_{\mu\nu},\phi\right)+\kappa^2T_{\mu\nu}+\partial_\mu\phi\partial_\nu\phi\left(R\left(\hat{g}_{\mu\nu},\phi\right)+\kappa^2T\right)\,. \tag{138}
$$

Here T is the trace of the matter energy-momentum tensor $T_{\mu\nu}$, $T=g\left(\hat{g}_{\mu\nu},\phi\right)^{\mu\nu}T_{\mu\nu}$. Eq. (136) shows

$$
g\left(\hat{g}_{\mu\nu},\phi\right)^{\mu\nu}\partial_\mu\phi\partial_\nu\phi=-1\,. \tag{139}
$$

Usually, the variation with respect to the Weyl factor in the metric gives the equation for the trace part, that is, in the case of the Einstein gravity, $R+\kappa^2T=0$, but due to the parametrization in Eq. (136), the variation with respect to ϕ gives,

$$
0=\nabla\left(g\left(\hat{g}_{\mu\nu},\phi\right)_{\mu\nu}\right)^{\mu}\left(\partial_\mu\phi\left(R\left(\hat{g}_{\mu\nu},\phi\right)+\kappa^2T\right)\right)\,. \tag{140}
$$

Note that ∇_μ is the covariant derivative with respect to $g_{\mu\nu}$. Eq. (140) shows that there can be a wider class of the solutions in the mimetic model, compared with the Einstein gravity. In fact, Eq. (140) effectively induces dark matter [426], which will be discussed in the following section for the case of the $F(R)$ extension of the mimetic model. We should note that in Eqs. (138) and (140), $\hat{g}_{\mu\nu}$ appears only in the combination of $g_{\mu\nu}$ in (136) and therefore $\hat{g}_{\mu\nu}$ does not appear explicitly.

Now we discuss the mimetic $F(R)$ gravity which was developed in Ref. [433]. This theory seems to be a ghost-free theory, like the conventional $F(R)$ gravity, and in addition, it is a conformal invariant theory. In the literature it was shown how early-time and late-time acceleration of the Universe, can be realized in the context of mimetic $F(R)$ gravity [434]. Also it was shown that an inflationary era which can be consistent with the current observational data, may be realized in the context of mimetic $F(R)$ gravity [435, 436]. Furthermore, it was shown in [434], that the reconstruction of ΛCDM model is also possible, as well as the unification of early-time acceleration with the late-time acceleration can be done, in the spirit of first proposal in $F(R)$ gravity [140]. Also, cosmological bounces can be realized in the context of mimetic $F(R)$ gravity. We need to note that a specified cosmological evolution which is realized by an $F(R)$ gravity, in principle it is realized by a different mimetic $F(R)$ gravity. In addition, like in the case of usual mimetic gravity, the mimetic $F(R)$ theory can be extended to include a scalar potential and a Lagrange multiplier.

We start from $F(R)$ gravity, whose action is given by,

$$S=\int d^4x\sqrt{-g}\left(F\left(R\right)+\mathcal{L}_{\mathrm{matter}}\right)\,. \tag{141}$$

The function $F(R)$ is some function of the scalar curvature R and $\mathcal{L}_{\mathrm{matter}}$ is matter Lagrangian. If we parameterize the metric as in Eq. (136), upon variation of the action with respect to the metric, we get,

$$\delta g_{\mu\nu}=\hat{g}^{\rho\tau}\delta\hat{g}_{\tau\omega}\hat{g}^{\omega\sigma}\partial_\rho\phi\partial_\sigma\phi\hat{g}_{\mu\nu}-\hat{g}^{\rho\sigma}\partial_\rho\phi\partial_\sigma\phi\delta\hat{g}_{\mu\nu}-2\hat{g}^{\rho\sigma}\partial_\rho\phi\partial_\sigma\delta\phi\hat{g}_{\mu\nu}\,. \tag{142}$$

In effect, in the case of $F(R)$ gravity, by using the parametrization of the metric as in Eq. (136), the action can be written as follows,

$$S=\int d^4x\sqrt{-g\left(\hat{g}_{\mu\nu},\phi\right)}\left(F\left(R\left(\hat{g}_{\mu\nu},\phi\right)\right)+\mathcal{L}_{\mathrm{matter}}\right)\,. \tag{143}$$

Note that in principle any solution of the standard $F(R)$ gravity, is also a solution of the mimetic $F(R)$ gravity.

Assume that the background geometry is the flat FRW space-time (6), where ϕ depends only on the cosmic time t, and due to Eq. (139), we find that the cosmic time can be identified with ϕ, that is $\phi=t$. For the FRW space-time, we obtain,

$$\begin{aligned}\frac{C_\phi}{a^3}=&2F(R)-RF'(R)-3\Box F'(R)+\frac{1}{2}T\\=&2F(R)-6\left(\dot{H}+2H^2\right)F'(R)+3\frac{d^2F'(R)}{dt^2}+9H\frac{dF'(R)}{dt}+\frac{1}{2}\left(-\rho+3p\right)\,,\end{aligned} \tag{144}$$

where C_ϕ is an arbitrary integration constant. Since the term containing C_ϕ behaves as a^{-3}, the case $C_\phi\neq 0$ describes mimetic dark matter. We also obtain the following equation,

$$0=\frac{d^2F'(R)}{dt^2}+2H\frac{dF'(R)}{dt}-\left(\dot{H}+3H^2\right)F'(R)+\frac{1}{2}F(R)+\frac{1}{2}p\,. \tag{145}$$

By combining Eqs. (144) and (145), we obtain,

$$0=\frac{d^2F'(R)}{dt^2}-H\frac{dF'(R)}{dt}+2\dot{H}F'(R)+\frac{1}{2}\left(p+\rho\right)+\frac{4C_\phi}{a^3}\,. \tag{146}$$

In the case $C_\phi=0$, the FRW equations above are identical to the ones corresponding to the standard $F(R)$ gravity, or in other words, in the case that $C_\phi\neq 0$, the FRW equations and therefore the corresponding solutions, are in principle different from those corresponding to the standard $F(R)$ gravity.

Let us briefly illustrate how it is possible to realize an arbitrary cosmological evolution, in the context of mimetic $F(R)$ gravity. Consider an arbitrary cosmological evolution with scale factor $a(t)$ and assume that the explicit ρ and p depend explicitly on the scale factor, as it happens usual perfect fluids. Consider the differential equation:

$$0=\frac{d^2f(t)}{dt^2}-H(t)\frac{df(t)}{dt}+2\dot{H}(t)f(t)\,. \tag{147}$$

We denote the two solutions of Eq. (147) as $f_1(t)$ and $f_2(t)$. Then $F'(R)$, which is the solution of Eq. (146), is equal to,

$$\begin{aligned} F'(R(t)) =& - f_1(t) \int^t dt' \frac{\gamma(t') f_2(t')}{W(f_1(t'), f_2(t'))} + f_2(t) \int^t dt' \frac{\gamma(t') f_1(t')}{W(f_1(t'), f_2(t'))} \, , \\ \gamma(t) \equiv& -\frac{1}{2}(p+\rho) - \frac{4C_\phi}{a^3} \, , \\ W(f_1, f_2) \equiv& f_1(t) f_2'(t) - f_2(t) f_1'(t) \, . \end{aligned} \tag{148}$$

By using the explicit t-dependence of the scalar curvature $R = R(t)$, we find the cosmic time t as a function of the scalar curvature R, that is $t = t(R)$. Therefore, one gets the explicit form of the expression $F'(R)$.

In Ref. [427], it has been proposed that instead of parameterizing the metric tensor as in Eq. (136), the condition of Eq. (139) can be imposed in the theory, by adding the Lagrange multiplier auxiliary field λ. In effect, instead of Eq. (143), we may consider the following gravitational action,

$$S = \int d^4x \sqrt{-g} \left(F(R(g_{\mu\nu})) + \lambda \left(g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + 1 \right) + \mathcal{L}_{\text{matter}} \right) \, . \tag{149}$$

In addition, a scalar potential $V(\phi)$ can be added as follows,

$$S = \int d^4x \sqrt{-g} \left(F(R(g_{\mu\nu})) - V(\phi) + \lambda \left(g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + 1 \right) + \mathcal{L}_{\text{matter}} \right) \, . \tag{150}$$

This gravitational action is a modified gravity with Lagrange multiplier constraints. These Lagrange multiplier techniques were developed in [458–461], see also [462]. Then by appropriately choosing the scalar potential $V(\phi)$, in principle any arbitrary cosmological evolution can be realized. Since the standard $F(R)$ gravity has no ghost issues, the explicit addition of the Lagrange multiplier constraint, in principle, does not violate the ghost-free property that the standard $F(R)$ version has. Therefore, it can be expected that the mimetic $F(R)$ gravity theory under consideration, it is also a ghost-free theory, although this should be verified explicitly by using the Hamiltonian formulation of the theory. In a similar way, it can be expected that the same applies for the Newton law, so in principle it should be the same as in standard $F(R)$ gravity.

G. Mimetic $f(\mathcal{G})$ Gravity

The mimetic framework can be extended in the context of the modified Gauss-Bonnet gravity, and in this section we present the essential features of this framework based on Ref. [440]. Consider the following $f(\mathcal{G})$ gravity action,

$$S = \int d^4x \sqrt{-g} \left[\frac{R}{2\kappa^2} + f(\mathcal{G}) \right] + S_{\text{matter}} \, . \tag{151}$$

with S_{matter} denoting the action for the matter fluids present. The Gauss-Bonnet scalar is given in (71). The signature for the Riemannian metric is $(-+++)$, and also $\kappa^2 = 8\pi G/c^4 = 1$, with G being the Newtonian constant.

In order to introduce the mimetic formalism in the Gauss-Bonnet gravity, we make the following parametrization of the metric [426, 428]:

$$g_{\mu\nu} = -\hat{g}^{\rho\sigma} \partial_\rho \phi \partial_\sigma \phi \hat{g}_{\mu\nu} \, . \tag{152}$$

Upon variation of the metric, we obtain,

$$\begin{aligned} \delta g_{\mu\nu} =& \hat{g}^{\rho\tau} \delta \hat{g}_{\tau\omega} \hat{g}^{\omega\sigma} \partial_\rho \phi \partial_\sigma \phi \hat{g}_{\mu\nu} - \hat{g}^{\rho\sigma} \partial_\rho \phi \partial_\sigma \phi \delta \hat{g}_{\mu\nu} \\ & - 2\hat{g}^{\rho\sigma} \partial_\rho \phi \partial_\sigma \delta\phi \hat{g}_{\mu\nu} \, . \end{aligned} \tag{153}$$

Moreover, upon variation of the action (151), with respect to the metric $\hat{g}_{\mu\nu}$, and also with respect to the scalar field ϕ, the gravitational equations are equal to,

$$\begin{aligned} & R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \\ & + 8 \left[R_{\mu\rho\nu\sigma} + R_{\rho\nu} g_{\sigma\mu} - R_{\rho\sigma} g_{\nu\mu} - R_{\mu\nu} g_{\sigma\rho} + R_{\mu\sigma} g_{\nu\rho} + \frac{R}{2} \left(g_{\mu\nu} g_{\sigma\rho} - g_{\mu\sigma} g_{\nu\rho} \right) \right] \nabla^\rho \nabla^\sigma f_{\mathcal{G}} + \left(f_{\mathcal{G}} \mathcal{G} - f(\mathcal{G}) \right) g_{\mu\nu} \end{aligned}$$

$$+\partial_\mu\phi\partial_\nu\phi\left(-R+8\left(-R_{\rho\sigma}+\frac{1}{2}Rg_{\rho\sigma}\right)\nabla^\rho\nabla^\sigma f_{\mathcal{G}}+4(f_{\mathcal{G}}\mathcal{G}-f(\mathcal{G}))\right)$$
$$=T_{\mu\nu}+\partial_\mu\phi\partial_\nu\phi T\,, \tag{154}$$

with $f_{\mathcal{G}}$ being equal to $f_{\mathcal{G}}=df(\mathcal{G})/d\mathcal{G}$. Also the covariant derivative ∇_μ, acts on vectors as follows,

$$\nabla_\mu V_\nu=\partial_\mu V_\nu-\Gamma^\lambda_{\mu\nu}V_\lambda\,. \tag{155}$$

Also, upon variation of the action (151), with respect to the mimetic scalar degree of freedom ϕ, we obtain,

$$\nabla^\mu\left(\partial_\mu\phi\left(-R+8\left(-R_{\rho\sigma}+\frac{1}{2}Rg_{\rho\sigma}\right)\nabla^\rho\nabla^\sigma f_{\mathcal{G}}+4(f_{\mathcal{G}}\mathcal{G}-f(\mathcal{G})\right)-T\right)\right)=0\,. \tag{156}$$

For the flat FRW metric, with line element (6), the scalar curvature and the Gauss-Bonnet invariant are given by $R=6\left(\frac{\ddot{a}}{a}+\frac{\dot{a}^2}{a^2}\right)=6(\dot{H}+2H^2)$, and $\mathcal{G}=24\frac{\ddot{a}\dot{a}^2}{a^3}=24H^2(\dot{H}+H^2)$. with $H(t)$ being the Hubble rate $H(t)=\dot{a}(t)/a(t)$. From Eq. (152) we obtain that,

$$g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi=-1\,, \tag{157}$$

and due to the fact that the mimetic scalar ϕ has an explicit dependence solely on the cosmic time t, from Eq. (157), we have $\phi=t$. In effect, the (t,t) component of Eq. (154), is equal to,

$$2\dot{H}+3H^2+16H(\dot{H}+H^2)\frac{df_{\mathcal{G}}}{dt}+8H^2\frac{d^2f_{\mathcal{G}}}{dt^2}-(f_{\mathcal{G}}\mathcal{G}-f(\mathcal{G}))=-p\,, \tag{158}$$

and as it can be checked, the same equation results if one considers the (r,r) component. Upon integration, Eq. (156) yields,

$$-R+8\left(-R_{\rho\sigma}+\frac{1}{2}Rg_{\rho\sigma}\right)\nabla^\rho\nabla^\sigma f_{\mathcal{G}}+4(f_{\mathcal{G}}\mathcal{G}-f(\mathcal{G}))+\rho-3p=-\frac{C}{a^3}\,, \tag{159}$$

and after some algebra we obtain,

$$\dot{H}+2H^2+4H^2\frac{d^2f_{\mathcal{G}}}{dt^2}+4H\left(2\dot{H}+3H^2\right)\frac{df_{\mathcal{G}}}{dt}+\frac{2}{3}(f_{\mathcal{G}}\mathcal{G}-f(\mathcal{G}))+\frac{\rho}{6}-\frac{p}{2}=-\frac{C}{a^3}\,. \tag{160}$$

with C being an arbitrary integration constant. By combining Eqs. (158) and (160), we get,

$$\dot{H}+4H^2\frac{d^2f_{\mathcal{G}}}{dt^2}+4H(2\dot{H}-H^2)\frac{df_{\mathcal{G}}}{dt}=-\frac{1}{2}(\rho+p)-\frac{C}{a^3}\,. \tag{161}$$

where the constant C is redefined for notational simplicity. We introduce the function $g(t)$, $g(t)=\frac{df_{\mathcal{G}}}{dt}$, which satisfies the following equation:

$$4H^2\frac{dg(t)}{dt}+4H(2\dot{H}-H^2)g(t)=-\dot{H}-\frac{1}{2}(\rho+p)-\frac{C}{a^3}\,. \tag{162}$$

For the value $C=0$, both Eqs. (158) and (161) are identical with the Friedmann equations of ordinary $f(\mathcal{G})$ gravity. If the right hand side of the above equation is equal to zero, that is,

$$B(t)=-\dot{H}-\frac{1}{2}(\rho+p)-\frac{C}{a^3}=0\,, \tag{163}$$

then we easily obtain the following solution,

$$g(t)=g_0\left(\frac{H_0}{H}\right)^2\exp\left(\int_0^t Hdt\right)\,, \tag{164}$$

where g_0 is an integration constant and also $H_0=H(0)$. For non-vanishing right hand side of Eq. (162), which means that $B(t)\neq 0$, then we obtain the following solution,

$$g(t)=g_0\left(\frac{H_0}{H}\right)^2\exp\left(\int_0^t Hdt\right)+\frac{1}{4H^2}\int_0^t dt_1 B(t_1)\exp\left(\int_{t_1}^t H(\tau)d\tau\right)\,. \tag{165}$$

Hence, given a cosmic evolution $H(t)$, then we can find the function $g(t)$, by applying the formulas we quoted above. Therefore one can specify the cosmic evolution of the Universe, in terms of the Hubble rate $H(t)$ and then obtain the function $g(t)$ and we may obtain the function $f(t)$, $f_{\mathcal{G}}(t) = \int g(t)dt$. By also finding the function $t(\mathcal{G})$, we may substitute it in the above equation, and then we can find the function $f_{\mathcal{G}}(\mathcal{G})$. From this a simple integration with respect to the Gauss-Bonnet scalar, yields the function $f(\mathcal{G})$.

Let us apply the method we just presented in order to realize the following cosmological evolution,

$$H(t) = \frac{H_0}{1+\alpha t}, \quad \alpha > 0\,. \tag{166}$$

The Gauss-Bonnet invariant for the evolution (166) is equal to $\mathcal{G} = 24\frac{H_0^2(\alpha - H_0)}{(1+\alpha t)^3}$. For $B = 0$, the solution to Eq. (162), is, $g(t) = g_0\,(1+\alpha t)^{H_0/\alpha + 2}$. For a Universe that does not contain any matter or mimetic dark matter, which means that $\rho = p = C = 0$, the function $g(t)$ is, $g(t) = g_0\,(1+\alpha t)^{H_0/\alpha+2} - \frac{1}{4H_0}\frac{1+\alpha t}{\alpha + H_0}$. Hence, the function $f_{\mathcal{G}}(t)$ is, $f_{\mathcal{G}}(t) = f_0\,(1+\alpha t)^{H_0/\alpha+3} - \frac{1}{8\alpha H_0}\frac{(1+\alpha t)^2}{\alpha+H_0}$. By inverting the Gauss-Bonnet scalar, we can find the function $t = t(\mathcal{G})$, and eventually the function $f_{\mathcal{G}}(G)$ is equal to, $f_{\mathcal{G}} = A\mathcal{G}^{-1-H_0/3\alpha} + D\mathcal{G}^{-2/3}$, $D = -\frac{1}{8\alpha H_0(\alpha+H_0)}\left(24H_0^2(\alpha - H_0)\right)^{2/3}$. By integrating with respect to $\mathcal{G}$, we obtain the non-mimetic $f(\mathcal{G})$ gravity,

$$f(\mathcal{G}) = F_0\mathcal{G}^{-H_0/\alpha} + 3D\mathcal{G}^{1/3}\,. \tag{167}$$

In the case that matter and mimetic dark matter is present, in which case we have $C \neq 0$, $\rho = \rho_0 a^{-3}$, $p = 0$, the function $g(t)$ is,

$$g(t) = g_0\,(1+\alpha t)^{H_0/\alpha+2} - \frac{1}{4H_0}\frac{1+\alpha t}{\alpha+H_0} + \frac{\tilde{C}}{4H_0-\alpha}\frac{1}{4H_0^2}\,(1+\alpha t)^{3(1-H_0/\alpha)}\,, \quad \tilde{C} = C + \rho_0\,, \tag{168}$$

and therefore $f_{\mathcal{G}}(t)$ is equal to,

$$f_{\mathcal{G}}(t) = A\mathcal{G}^{-1-H_0/3\alpha} + D\mathcal{G}^{-2/3} + E\mathcal{G}^{-4/3+H_0/\alpha}\,, \tag{169}$$

with E being equal to,

$$E = \frac{\tilde{C}}{4H_0^2}\frac{\alpha}{(4H_0-\alpha)(4\alpha-H_0)}\left(24H_0^2(\alpha-H_0)\right)^{4/3-H_0/\alpha}\,. \tag{170}$$

Hence, the mimetic $f(\mathcal{G})$ gravity is equal to,

$$f(\mathcal{G}) = F_0\mathcal{G}^{-H_0/\alpha} + 3D\mathcal{G}^{1/3} + \frac{3\alpha}{3H_0-\alpha}E\mathcal{G}^{-1/3+H_0/\alpha}\,. \tag{171}$$

By comparing Eqs. (171) and (167), we can easily spot the differences between mimetic and non-mimetic $f(\mathcal{G})$ gravity.

There is another formalism for mimetic $f(\mathcal{G})$ gravity, which employs a Lagrange multiplier function, in order to realize the constraint for the mimetic scalar we found in the previous formalism, namely Eq. (157). In this case, the gravitational action is,

$$S_\phi = \int d^4x\sqrt{-g}\left(-\epsilon g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right)\,, \tag{172}$$

where a mimetic potential is included too. The values of the parameter $\epsilon = \pm 1$, correspond to a canonical and a phantom scalar and also for $\epsilon = 0$, the theory is conformally invariant, and the above action is rendered equivalent to an $f(\mathcal{G})$ gravity, with $U = 1$ and $V = 0$. We consider the following generalized action with Lagrange multiplier and mimetic potential,

$$S = \int d^4x\sqrt{-g}\left(\frac{R}{2} + f(\mathcal{G}) - \epsilon g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi) + \lambda(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + U(\phi)) + \mathcal{L}_{\mathrm{matter}}\right)\,, \tag{173}$$

and upon variation with respect to $g_{\mu\nu}$, we get,

$$3H^2 + 24H^3\frac{df_{\mathcal{G}}(\mathcal{G})}{dt} + f(\mathcal{G}) - f_{\mathcal{G}}(\mathcal{G})\mathcal{G} = \rho + \epsilon\dot{\phi}^2 + V(\phi) - \lambda(\dot{\phi}^2 + U(\phi))\,, \tag{174}$$

$$-2\dot{H}-3H^2-8H^2\frac{d^2 f_{\mathcal{G}}(\mathcal{G})}{dt^2}-16H(\dot{H}+H^2)\frac{df_{\mathcal{G}}(\mathcal{G})}{dt}+f_{\mathcal{G}}(\mathcal{G})\mathcal{G}-f(\mathcal{G})=p+\epsilon\dot{\phi}^2-V(\phi)-\lambda(\dot{\phi}^2-U(\phi))\,. \tag{175}$$

for the flat FRW metric. In addition, upon variation with respect to λ , we obtain.

$$\dot{\phi}^2-U(\phi)=0\,, \tag{176}$$

which is a subcase of the constraint (157). Moreover, by varying with respect to the mimetic scalar, we get,

$$2\partial_t((\lambda-\epsilon)\dot{\phi})+6H(\lambda-\epsilon)\dot{\phi}-V'(\phi)+\lambda U'(\phi)=0\,. \tag{177}$$

In the case that non-relativistic matter is present ($p=0$), the mimetic potential $V(t)$ and the corresponding Lagrange multiplier $\lambda(t)$, are equal to,

$$V(t)=2\dot{H}+3H^2+\epsilon\dot{\phi}^2+8H^2\frac{d^2 f_{\mathcal{G}}(\mathcal{G})}{dt^2}+16H(\dot{H}+H^2)\frac{df_{\mathcal{G}}(\mathcal{G})}{dt}-f_{\mathcal{G}}(\mathcal{G})\mathcal{G}+f(\mathcal{G})\,, \tag{178}$$

$$\lambda(t)=\dot{\phi}^{-2}\left(\frac{\rho}{2}+\dot{H}+4H(2\dot{H}-H^2)\frac{df_{\mathcal{G}}(\mathcal{G})}{dt}+4H^2\frac{d^2 f_{\mathcal{G}}(\mathcal{G})}{dt^2}\right)+\epsilon\,. \tag{179}$$

Hence, given the function $f(\mathcal{G})$, we can find the cosmological evolution and the scalar potential that can realize such an evolution. The method can work in the inverse way, so for a given potential and also for an arbitrary cosmic evolution, we can find the $f(\mathcal{G})$ gravity which may realize such an evolution. We refer to Ref. [440] for some illustrative applications of this reconstruction method.

H. Unimodular $F(R)$ Gravity

In principle the cosmological constant can be regarded as the vacuum energy, but in the context of quantum field theory, the magnitude of the vacuum energy is around 60-120 orders higher in comparison to that of the observed cosmological constant. For this problem, the unimodular gravity theory [463–488] may offer an interesting and conceptually simple theoretical proposal. However, this theory cannot provide a full solution to the problem of the cosmological constant [476]. In this section, we review the extension of unimodular gravity, in the context of $F(R)$ gravity. This was proposed and studied in detail in [489].

For the standard Einstein-Hilbert unimodular gravity case, the determinant of the metric is fixed, and therefore a constraint of the metric exists, $g_{\mu\nu}\delta g^{\mu\nu}=0$. We assume, without loss of generality, that the determinant of the metric is constrained to be equal to one,

$$\sqrt{-g}=1\,. \tag{180}$$

Since the above unimodular constraint of Eq. (180) is not satisfied by the metric (6), we need to redefine the cosmic time coordinate to be $d\tau=a(t)^3dt$ and we rewrite the FRW metric (6) in the following way,

$$ds^2=-a\left(t\left(\tau\right)\right)^{-6}d\tau^2+a\left(t\left(\tau\right)\right)^2\sum_{i=1}^{3}\left(dx^i\right)^2\,. \tag{181}$$

In effect, the metric of Eq. (181) satisfies the unimodular constraint (180). By using the coordinate τ, the de Sitter Universe, for which $a(t)=\mathrm{e}^{H_0 t}$, is expressed as follows,

$$ds^2=-\left(3H_0\tau\right)^{-2}d\tau^2+\left(3H_0\tau\right)^{\frac{2}{3}}\sum_{i=1}^{3}\left(dx^i\right)^2\,. \tag{182}$$

On the other hand, for the Universe with a power-law scale factor, $a(t)=\left(\frac{t}{t_0}\right)^{h_0}$, we find that,

$$ds^2=-\left(\frac{\left(3h_0+1\right)\tau}{t_0}\right)^{\frac{-6h_0}{3h_0+1}}d\tau^2+\left(\frac{\left(3h_0+1\right)\tau}{t_0}\right)^{\frac{2h_0}{3h_0+1}}\sum_{i=1}^{3}\left(dx^i\right)^2\,. \tag{183}$$

We should note that the metric of Eq. (182) becomes identical with that of the de Sitter Universe, when $h_0, t_0 \to \infty$, by keeping $\frac{(3h_0+1)\tau}{t_0}$ to be finite, that is, $\frac{(3h_0+1)\tau}{t_0} \to 3H_0$. Hence, we can unify the form of the metric in the following way,

$$ds^2 = -\left(\frac{\tau}{\tau_0}\right)^{-6f_0} d\tau^2 + \left(\frac{\tau}{\tau_0}\right)^{2f_0} \sum_{i=1}^{3} \left(dx^i\right)^2 , \tag{184}$$

where the parameters τ_0 and f_0 are arbitrary constant real numbers. The case for which $f_0 = \frac{1}{3}$ corresponds to the de Sitter Universe. On the other hand, if the condition $\frac{1}{4} \leq f_0 < \frac{1}{3}$ holds true, which occurs when, $h_0 \geq 1$, the Universe is described by a quintessential evolution. In the case that $0 < f_0 < \frac{1}{4}$, which implies, $0 < h_0 < 1$, the Universe is expanding but in a decelerating way. Finally, when $f_0 < 0$ or if $f_0 > \frac{1}{3}$, which in turn implies, $h_0 < 0$, the Universe's evolution is a phantom evolution [247, 252, 306, 490–503].

Let us now see how the Newton law becomes in unimodular $F(R)$ gravity. First, as in the case of standard $F(R)$ gravity theory, we can rewrite the action in a scalar-tensor theory form,

$$S = \int d^4x \left\{ \sqrt{-g} \left(\frac{1}{2\kappa^2} \left(R - \frac{3}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right) - \lambda \mathrm{e}^{2\phi} \right) + \lambda \right\} + S_{\mathrm{matter}} \left(\mathrm{e}^{\phi} g_{\mu\nu}, \Psi \right) , \tag{185}$$

where the scalar potential $V(\phi)$ is given in Eq. (17). The unimodular constraint of Eq. (180) is modified in the following way $\mathrm{e}^{2\phi}\sqrt{-g} = 1$. By eliminating the scalar field ϕ, the action of Eq. (185) can be written in the following way,

$$S = \int d^4x \sqrt{-g} \left(\frac{1}{2\kappa^2} \left(R - \frac{3}{32 g^2} g^{\mu\nu} \partial_\mu g \partial_\nu g - V\left(\frac{1}{4} \ln\left(-g\right) \right) \right) \right) + S_{\mathrm{matter}} \left(\left(-g\right)^{\frac{1}{4}} g_{\mu\nu}, \Psi \right) . \tag{186}$$

Now we consider the perturbation of the metric tensor $g_{\mu\nu}$ around the background metric $g_{\mu\nu}^{(0)}$, in the following way, $g_{\mu\nu} = g_{\mu\nu}^{(0)} + h_{\mu\nu}$. In addition, we assume that the background metric is a flat metric, hence we have, $g_{\mu\nu}^{(0)} = \eta_{\mu\nu}$. In effect, we can obtain the following equations,

$$\partial_\lambda \partial^\lambda h_{\mu\nu} - \partial_\mu \partial^\lambda h_{\lambda\nu} - \partial_\nu \partial^\lambda h_{\lambda\mu} + \partial_\mu \partial_\nu h + \eta_{\mu\nu} \partial^\rho \partial^\sigma h_{\rho\sigma} - \frac{13}{16} \eta_{\mu\nu} \partial_\lambda \partial^\lambda h - m^2 \eta_{\mu\nu} h = \kappa^2 \left(T_{\mu\nu} - \frac{1}{4} \eta_{\mu\nu} T \right) , \tag{187}$$

with $T_{\mu\nu}$ being the energy-momentum tensor of the matter fluids present, and in addition T stands for the trace of $T_{\mu\nu}$, $T \equiv \eta^{\rho\sigma} T_{\rho\sigma}$. In order to study the Newton law, we may consider a point source at the origin of the coordinate system, so that the components of the energy-momentum tensor have the following form, $T_{00} = M\delta\left(\boldsymbol{r}\right)$, $T_{ij} = 0$ $(i, j = 1, 2, 3)$, so we seek a static solution of Eq. (187). In the case of the Einstein gravity, four gauge degrees of freedom exist, but in the unimodular $F(R)$ gravity case, only three gauge degrees of freedom exist, since the unimodular constraint (180) needs to be satisfied. In effects, we additionally impose three gauge conditions, $\partial^i h_{ij} = 0$. Moreover, by using appropriate boundary conditions, and by defining the Newtonian potential Φ as follows, $h_{00} = 2\Phi$, we may obtain the Poisson equation that the Newtonian potential Φ satisfies, which is, $\partial_i \partial^i \Phi = \frac{3\kappa^2}{8} M\delta\left(\boldsymbol{r}\right)$. Hence, by redefining the gravitational constant κ by $\frac{3\kappa^2}{4} \to \kappa^2 = 8\pi G$, we easily obtain the standard Poisson equation satisfied by the Newtonian potential U, which is, $\partial_i \partial^i \Phi = 4\pi G M \delta\left(\boldsymbol{r}\right)$, the solution of which is given by,

$$\Phi = -\frac{GM}{r} . \tag{188}$$

The result we found above is quite different from the solution in the context of the standard $F(R)$ gravity (for reviews, see [73–75, 77]), in which case the propagation of the scalar mode $\phi = -\ln F'(A)$ yields a non-trivial correction to the standard Newton law of gravity. In the case of unimodular $F(R)$ gravity, due to the fact that the unimodular constraint (180) is rewritten in the following way, $\mathrm{e}^{2\phi}\sqrt{-g} = 1$, the degree of the freedom corresponding to the scalar mode ϕ, is eventually eliminated from the resulting field equations, and in effect, the scalar ϕ does not propagate. In effect, no correction to the Newton law of gravity occurs.

We need to note that since the unimodular constraint (180) needs to be satisfied, the unimodular $F(R)$ gravity is not a fully covariant theory. A covariant formulation of the unimodular Einstein gravity was proposed in Ref. [465]. By using this formulation, we may start from the following action,

$$S = \int d^4x \left\{ \sqrt{-g} \left(\mathcal{L}_{\mathrm{gravity}} - \lambda \right) + \lambda \epsilon^{\mu\nu\rho\sigma} \partial_\mu a_{\nu\rho\sigma} \right\} + S_{\mathrm{matter}} \left(g_{\mu\nu}, \Psi \right) , \tag{189}$$

with $\mathcal{L}$ being the Lagrangian density of any gravitational theory, and $a_{\nu\rho\sigma}$ is a three-form field. Upon variation with respect to $a_{\nu\rho\sigma}$, we obtain the equation $0 = \partial_\mu \lambda$, so in effect, λ is a constant. On the other hand, upon variation with respect to λ, yields,

$$\sqrt{-g} = \epsilon^{\mu\nu\rho\sigma} \partial_\mu a_{\nu\rho\sigma} \,, \tag{190}$$

instead of the unimodular constraint. Since Eq. (190) can be solved with respect to $a_{\mu\nu\rho}$, there is no constraint imposed on the metric $g_{\mu\nu}$.

We should note that the quantity $a_{\mu\nu\rho}$, has four degrees of freedom. The gravitational action is invariant under the following gauge transformation $\delta a_{\mu\nu\rho} = \partial_\mu b_{\nu\rho} + \partial_\nu b_{\rho\mu} + \partial_\rho b_{\mu\nu}$, with $b_{\mu\nu}$ being an anti-symmetric tensor field, that is, $b_{\mu\nu} = -b_{\nu\mu}$, which contributes in total six degrees of freedom. We should note that the gauge transformation is actually invariant under the following gauge transformation, $\delta b_{\mu\nu} = \partial_\mu c_\nu - \partial_\nu c_\mu$. The quantity c_μ is a vector field, which contributes four degrees of freedom. Note that the gauge transformation of the original gauge transformation is actually invariant under the following gauge transformation $\delta c_\mu = \partial_\mu \varphi$. The field φ is a scalar field which contributes a degree of freedom. Therefore, the total number of degrees of freedom in the gauge transformation is $6 - 4 + 1 = 3$ and the number of degrees of freedom of the quantity $a_{\mu\nu\rho}$ is $4 - 3 = 1$. In effect, we may choose the following gauge condition,

$$a_{tij} \left(= a_{jti} = a_{ijt}\right) = 0 \,, \quad i,j = 1,2,3 \,, \tag{191}$$

and we find that the only remaining degree of freedom is given by a_{ijk} $(i,j,k = 1,2,3)$. We can rewrite the action as follows,

$$S_{\lambda\alpha} = \int d^4x \lambda \left(-\sqrt{-g} + \epsilon^{\mu\nu\rho\sigma} \partial_\mu a_{\nu\rho\sigma}\right) = \int d^4x \lambda \left(-\sqrt{-g} + \partial_t \alpha\right) \,, \tag{192}$$

where $\alpha \equiv \frac{1}{3!} a_{123}$. The system described by Eq. (192) might yield a non-trivial correction to the Newton law, or it might be a ghost and it can contribute negative norm states in the corresponding quantum field theory. We can use an analogy with the quantum mechanical system with Lagrangian,

$$L = By\dot{x} \,, \tag{193}$$

which describes the massless limit of a charged particle in a uniform magnetic field B. Then, we find the following commutation relations satisfied by the fields λ and α,

$$[\alpha, \lambda] = i\delta\left(\boldsymbol{x}\right) \,, \tag{194}$$

with $\boldsymbol{x} = \left(x^1, x^2, x^3\right)$ and the Hamiltonian H is given by,

$$H = \int_S dS \sqrt{-g} \lambda \,, \tag{195}$$

with S being an arbitrary space-like surface.

In effect, regardless that λ is a constant, the time evolution of the quantity α is given equal to,

$$\frac{d\alpha}{dt} = i\left[H, \alpha\right] = \sqrt{-g} \,, \tag{196}$$

which is compatible with the classical equation which is obtained by the variation of the action of Eq. (192) with respect to the quantity λ. The eigenstate of the Hamiltonian H, could be provided by the eigenstate of λ. In the representation of the quantum states by using α, the commutation relation of Eq. (194), yields $\lambda = i\frac{\delta}{\delta\alpha}$. In effect, the eigenstate $\Psi_{\lambda_0}(\alpha)$ of λ with the eigenvalue λ_0 can be expressed as follows,

$$\Psi_{\lambda_0}(\alpha) = \exp\left(i\lambda_0 \int_S dS \alpha\left(\boldsymbol{x}\right)\right) \,. \tag{197}$$

The eigenvalue of the Hamiltonian (195) is infinite, due to the fact that the volume of S is infinite, and in addition unbounded from below. Note however that no quantum transition between the states occurs, and in effect, the states may be considered quantum mechanically stable.

In the case that covariant unimodular $F(R)$ gravity is considered, the action is,

$$\mathcal{L}_{\text{gravity}} = \frac{F(R)}{2\kappa^2} \,, \tag{198}$$

and as in the standard $F(R)$ gravity case, we may rewrite the action in a scalar-tensor form as follows,

$$S = \int d^4x \left\{ \sqrt{-g} \left(\frac{1}{2\kappa^2} \left(R - \frac{3}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right) - \lambda \mathrm{e}^{2\phi} \right) + \lambda \epsilon^{\mu\nu\rho\sigma} \partial_\mu a_{\nu\rho\sigma} \right\} + S_{\mathrm{matter}} \left(\mathrm{e}^{\phi} g_{\mu\nu}, \Psi \right) . \tag{199}$$

In the action of Eq. (199), one obtains $0 = \partial_\mu \lambda$ and therefore λ is a constant in this case too. Thus, the scalar potential $V(\phi)$ is effectively changed as follows,

$$V(\phi) \to \tilde{V}(\phi) = \frac{A(\phi)}{F'(A(\phi))} - \frac{F(A(\phi))}{F'(A(\phi))^2} + 2\kappa^2 \lambda \mathrm{e}^{2\phi} , \tag{200}$$

Then if the mass of field ϕ, which is defined to be $m_\phi^2 = \frac{3}{2} \frac{d^2 \tilde{V}(\phi)}{d\phi^2}$ is small, a large correction to the Newton law of gravity could appear. By using the equation $\phi = -\ln F'(A)$ and Eq. (200), the explicit expression for the mass m_ϕ^2 is given by,

$$m_\phi^2 = \frac{3}{2} \left\{ \frac{A(\phi)}{F'(A(\phi))} - \frac{4F(A(\phi))}{F'(A(\phi))^2} + \frac{1}{F''(A(\phi))} + \frac{8\kappa^2 \lambda}{F'(A(\phi))^2} \right\} . \tag{201}$$

The last term is a characteristic feature of the unimodular $F(R)$ gravity theory but since λ is constant, the last term can be absorbed into the redefinition of $F(R)$, that is, $F(R) \to F(R) + 2\kappa^2 \lambda$. The expression for m_ϕ^2 obtained by the above redefinition, is identical to the expression in the standard $F(R)$ gravity. Therefore, there is no essential difference in the Newton law corrections, between the covariant unimodular $F(R)$ gravity and the standard $F(R)$ gravity [73–75, 77]. This feature is quite different from the non-covariant unimodular $F(R)$ gravity version, where the standard Newton law is recovered.

I. Unimodular Mimetic Gravity

The unimodular gravity formalism may be extended in the context of mimetic gravity, and in this section we discuss the formalism of mimetic unimodular (U-M) gravity, which was developed in [456]. The formalism we shall present constitutes a reconstruction method, and we now we shall briefly present the essential features of this reconstruction method. The standard Einstein gravity approach for unimodular gravity [470, 472, 475, 477, 479–485], is based on the basic assumption that the determinant of the metric is a fixed number, so that the metric satisfies $g_{\mu\nu}\delta g^{\mu\nu} = 0$, which in effect implies that the various components of the metric can be appropriately adjusted, so that the resulting determinant of the metric $\sqrt{-g}$ is some fixed function of space-time. Hence, it can be assumed, that the metric tensor satisfies the unimodular constraint of Eq. (180). In order to satisfy the unimodular constraint, we shall employ the Lagrange multiplier method, and in effect the unimodular constraint will appear as a part of the resulting equations of motion. Bearing this in mind, we can generalize the mimetic gravity with potential and Lagrange multiplier [427], in order to take into account the unimodular constraint of Eq. (180). Hence, the generalization is the following,

$$S = \int d^4x \left\{ \sqrt{-g} \left(\frac{R}{2\kappa^2} + f(R) - V(\phi) - \eta \left(\partial_\mu \phi \partial^\mu \phi + 1 \right) - \lambda \right) + \lambda \right\} + S_{\mathrm{matter}} , \tag{202}$$

with ϕ being the real mimetic scalar field, and R is the scalar curvature. Moreover, S_{matter} stands for the action for the matter fluids present. We need to note that the action of Eq. (202) describes the action in (11), but we shall take $f(R) = 0$. The case with $f(R) \neq 0$ will be studied in a later section. The Lagrange multipliers η and λ correspond to the mimetic gravity constraint and to the unimodular constraint respectively. We can easily see that by varying the action of Eq. (202) with respect to η, in which case we obtain,

$$\partial_\mu \phi \partial^\mu \phi = -1 , \tag{203}$$

which is the mimetic constraint we discussed earlier. Moreover, by varying the action of Eq. (202), with respect to λ, we obtain the unimodular constraint appearing in Eq. (180).

Both the functions η and λ are functions of the cosmic time variable, which can be identified with the field ϕ, as we show shortly. The last identification is owing to the mimetic constraint. Upon variation of the action (202), with respect to the metric, we obtain the equations of motion,

$$0 = \frac{1}{2} g_{\mu\nu} \left(\frac{R}{2\kappa^2} - V(\phi) - \eta \left(\partial_\mu \phi \partial^\mu \phi + 1 \right) - \lambda \right) - \frac{1}{2\kappa^2} R_{\mu\nu} + \eta \partial_\mu \phi \partial_\nu \phi + \frac{1}{2} T_{\mu\nu} , \tag{204}$$

where $T_{\mu\nu}$ denotes the energy-momentum tensor of the perfect matter fluids which are present. In addition, upon variation of the action (202), with respect to the field ϕ, we obtain,

$$0 = 2\nabla^\mu \left(\lambda \partial_\mu \phi\right) - V'(\phi) \,. \tag{205}$$

By assuming that the metric is the flat FRW metric, we can see that the metric does not satisfy the unimodular constraint (180), so the cosmic time variable needs to be redefined as follows, $d\tau = a(t)^3 dt$. In effect, the FRW metric of Eq. (6) can be rewritten as in Eq. (181). We shall refer to the metric of Eq. (181) as "the unimodular metric" hereafter. For the unimodular metric of Eq. (181), below we quote only the non-vanishing components of the Ricci tensor and of the Levi-Civita connection, which are,

$$\begin{aligned} &\Gamma^t_{tt} = -3K \,, \quad \Gamma^t_{ij} = a^8 K \delta_{ij} \,, \quad \Gamma^i_{jt} = \Gamma^i_{tj} = K \delta_j^{\ i} \,, \\ &R_{tt} = -3\frac{dK}{d\tau} - 12K^2 \,, \quad R_{ij} = a^8 \left(\frac{dK}{d\tau} + 6K^2 \right) \delta_{ij} \,. \end{aligned} \tag{206}$$

In Eq. (206), the function K is a direct generalization of the usual Hubble rate, in terms of the τ variable this time, that is, $K \equiv \frac{1}{a}\frac{da}{d\tau}$. In addition, the scalar curvature R as a function of the τ variable is given below,

$$R = a^6 \left(6\frac{dK}{d\tau} + 30K^2 \right) \,. \tag{207}$$

Due to our assumption that the auxiliary scalar field ϕ, is only dependent on the cosmic time t (or τ), we can rewrite the mimetic constraint as follows,

$$a^{-6} \left(\frac{d\phi}{d\tau} \right)^2 = 1 \,, \tag{208}$$

which can be rewritten in terms of the cosmic time t by employing $d\tau = a(t)^3 dt$, in the following way,

$$\left(\frac{d\phi}{dt} \right)^2 = 1 \,. \tag{209}$$

Therefore, we may identify the auxiliary scalar field ϕ with the cosmic time coordinate t, that is $\phi = t$. The (τ, τ) and (i, j) components of the equations appearing in Eq. (204) are equal to,

$$0 = -\frac{3a^6}{2\kappa^2} K^2 + \frac{V(\phi)}{2} + \frac{\lambda}{2} + \eta + \frac{\rho}{2} \,, \tag{210}$$

$$0 = \frac{a^6}{2\kappa^2} \left(2\frac{dK}{d\tau} + 9K^2 \right) - \frac{V(\phi)}{2} - \frac{\lambda}{2} + \frac{p}{2} \,, \tag{211}$$

with ρ and p denoting the energy density and the pressure of the matter fluids which are present. We need to note that in order to get Eqs. (210) and (211), we employed the constraint of Eq. (208). Eqs. (210) and (211) can be rewritten by making use of the cosmological time t variable, as follows,

$$0 = -\frac{3H^2}{2\kappa^2} + \frac{V(\phi)}{2} + \frac{\lambda}{2} + \eta + \frac{\rho}{2} \,, \tag{212}$$

$$0 = \frac{1}{2\kappa^2} \left(3H^2 + 2\frac{dH}{dt} \right) - \frac{V(\phi)}{2} - \frac{\lambda}{2} + \frac{p}{2} \,. \tag{213}$$

In addition, Eq. (205) may be written in the following way,

$$0 = -6H\lambda - 2\frac{d\lambda}{dt} - V'(\phi) \,. \tag{214}$$

By making use of Eqs. (213) and (214), we may eliminate λ from the equations of motion, and hence we have,

$$0 = 6HV(\phi) - 3V'(\phi) - 6Hp - 2\frac{dp}{dt} + \frac{1}{\kappa^2} \left(-18H^3 - 6H\frac{dH}{dt} + 4\frac{d^2H}{dt^2} \right) \,. \tag{215}$$

Since we identified $\phi = t$, we can integrate Eq. (216) and we obtain the scalar potential $V(\phi)$,

$$V(\phi) = \frac{a\,(t=\phi)^2}{3} \int^{\phi} dt\, a(t)^{-2} \left\{ -6H(t)p(t) - 2\frac{dp(t)}{dt} + \frac{1}{\kappa^2}\left(-18H(t)^3 - 6H(t)\frac{dH(t)}{dt} + 4\frac{d^2H(t)}{dt^2} \right) \right\} . \tag{216}$$

Therefore, by specifying the cosmological evolution in terms of its scale factor, and also by specifying the functional dependence of the energy density and pressure in terms of the scale factor, by employing Eq. (216), we can find the mimetic potential which generates the cosmology with scale factor $a(t)$. In addition, by employing Eqs. (213) and (212), we can easily find the functions $\lambda(t)$ and $\eta(t)$, and the complete unimodular mimetic gravity which realizes $a(t)$ is determined. Actually, the Eqs. (216), (213) and (212) constitute a reconstruction method for finding the unimodular-mimetic theory which can realize a specific cosmological evolution.

J. Unimodular Mimetic $F(R)$ Gravity

The unimodular mimetic $F(R)$ gravity framework [457] uses the Lagrange multiplier formalism. So we introduce two Lagrange multiplier functions η and λ, and the corresponding mimetic unimodular $F(R)$ gravity action with mimetic potential and Lagrange multipliers is,

$$S = \int dx^4 \left(\sqrt{-g}\left(F(R) - V(\phi) + \eta\left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + 1 \right) - \lambda \right) + \lambda \right) . \tag{217}$$

Upon variation of the above action (217), with respect to the metric, we obtain the gravitational equations of motion,

$$\frac{g_{\mu\nu}}{2}\left(F(R) - V(\phi) + \eta\left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + 1 \right) - \lambda \right) - R_{\mu\nu}F'(R) - \eta\partial_\mu\phi\partial_\nu\phi + \nabla_\mu\nabla_\nu F'(R) - g_{\mu\nu}\Box F'(R) = 0 . \tag{218}$$

Moreover, upon variation with respect to the mimetic scalar ϕ, we obtain,

$$-2\nabla^\mu\left(\eta\partial_\mu\phi \right) - V'(\phi) = 0 . \tag{219}$$

By varying the action (217) with respect to the function η we get,

$$g^{\mu\nu}(\hat{g}_{\mu\nu}, \phi)\partial_\mu\phi\partial_\nu\phi = -1 , \tag{220}$$

while varying the action with respect to the function $\lambda(t)$, we obtain,

$$\sqrt{-g} = 1 . \tag{221}$$

For the flat FRW metric, the (t,t) components of the expression appearing in Eq. (218), are equal to,

$$-F(R) + 6(\dot{H} + H^2)F'(R) - 6H\frac{dF'(R)}{dt} - \eta(\dot{\phi}^2 + 1) + \lambda + V(\phi) = 0 , \tag{222}$$

while the corresponding (i,j) components are equal to,

$$F(R) - 2(\dot{H} + 3H^2) + 2\frac{d^2F'(R)}{dt^2} + 4H\frac{dF'(R)}{dt} - \eta(\dot{\phi}^2 - 1) - V(\phi) - \lambda = 0 , \tag{223}$$

For the flat FRW metric, Eq. (219)can be written as follows,

$$2\frac{d(\eta\dot{\phi})}{dt} + 6H\eta\dot{\phi} - V'(\phi) = 0 . \tag{224}$$

Finally, from the mimetic constraint we have,

$$\dot{\phi}^2 - 1 = 0 . \tag{225}$$

We need to note that in Eqs. (222), (223), (224), and (225), the “prime” indicates differentiation with respect to R, the scalar curvature, and also with respect to the mimetic scalar. By taking into account Eq. (225), the equation (222), can be rewritten as follows,

$$-F(R) + 6(\dot{H} + H^2)F'(R) - 6H\frac{dF'(R)}{dt} - 2\eta + \lambda + V(\phi) = 0 , \tag{226}$$

and in addition Eq. (223) is rewritten in the following way,

$$F(R) - 2(\dot{H} + 3H^2) + 2\frac{d^2F'(R)}{dt^2} + 4H\frac{dF'(R)}{dt} - V(\phi) - \lambda = 0\,. \tag{227}$$

Furthermore, Eq. (225) becomes,

$$2\frac{d\eta}{dt} + 6H\eta - V'(t) = 0\,. \tag{228}$$

If we eliminate λ from Eqs. (226) and (227), we get,

$$6(\dot{H} + H^2)F'(R) - 2H\frac{dF'(R)}{dt} - 2\eta - 2(\dot{H} + 3H^2) + 2\frac{d^2F'(R)}{dt^2} - V(t) = 0\,. \tag{229}$$

From Eqs. (229) and (228), we get,

$$-2V'(t) - 3HV(t) + f_0(t) = 0\,, \tag{230}$$

with the function $f_0(t)$ being equal to,

$$\begin{aligned} f_0(t) = -\Bigg[& 18H(t)\left(\dot{H} + H^2\right)F'(R) - 6H^2\frac{dF'(R)}{dt} - 6H\left(\dot{H}3H^2\right) + 6H\frac{d^3F'(R)}{dt^3} \\ & 6(\ddot{H} + 2\dot{H}H)F'(R) + H\dot{H}\frac{dF'(R)}{dt} + 6H^2\frac{dF'(R)}{dt} - 2H\frac{d^2F'(R)}{dt^2} - 2\left(\ddot{H} + 6\dot{H}H\right) + 2\frac{d^2F'(R)}{dt^2}\Bigg]\,. \end{aligned} \tag{231}$$

The differential equation of Eq. (230) can be solved and we get the following general solution,

$$V(t) = \frac{a^{3/2}(t)}{2}\int a^{-3/2}(t)f_0(t)dt\,, \tag{232}$$

where $a(t)$ is the scale factor. Hence, given an arbitrary cosmological evolution, quantified in terms of its scale factor, and also the form of the $F(R)$ gravity, we can use Eq. (232) in order to obtain the mimetic potential, and with it, by using Eq. (227), we can obtain the function $\lambda(t)$. Eventually, by using the resulting function $\lambda(t)$ and also the potential $V(t)$, we can substitute these in Eq. (226), and we can obtain the function $\eta(t)$. Several illustrative examples on how this reconstruction method works, can be found in [457].

In the previous sections we provided a quick review for a number of modified gravities. Definitely, the models we presented are not all the models one can construct, different approaches and generalizations of the above can be constructed. In the next chapter we shall present cosmological applications of the above models in more detail.

III. INFLATIONARY DYNAMICS IN MODIFIED GRAVITY

The Standard Big Bang cosmology had several issues which rendered the theory incomplete. Particularly, the flatness problems, the monopole problem, the entropy problem and the horizon problem, found a successful explanation in the context of inflation. In some sense the horizon problem can represent all the above problems, since it is related to all of the above. The horizon problem refers to the question how two distinct parts of the Universe which are very far away at present time, could be in causal connection in the past. These two distinct parts of the Universe seem to have almost the same density at present time, so this could not be explained at a microphysics level, unless these two parts were in causal connection in the past. If someone adopts the Big Bang cosmology evolution, and solves the equations following the cosmic time backwards, then two parts of the Universe which are now very far away and seem to have almost the same density, cannot be causal connection in the past. The inflationary paradigm solved this issue by adding a period of nearly exponential accelerated expansion during early times.

The inflationary paradigm [50–54, 56–71] is up to date one of the most successful descriptions of the early Universe, since in most cases the outcome is a nearly scale invariant power spectrum. It is quite common in the cosmology literature to use a single canonical scalar field which slow-rolls on its potential, in order to describe the inflationary era, and actually this was the original approach of inflation [50–54, 56–70]. In the literature there are many reviews on this topic and in this section we shall provide a quick overview of the various approaches on inflation, emphasizing in the modified gravity description. We shall not expand our description going into many details, but we provide a concrete and condensed description of the quantitative features of an inflationary theory, that is, the calculation of the spectral index of the primordial curvature perturbations and of the scalar-to-tensor ratio. We start off with the single and multi-scalar field descriptions of inflation and then we proceed to the modified gravity description. For simplicity, in all the cases we shall assume that the inflationary dynamics is controlled by the vacuum modified gravity, but the same issue can be addressed in the presence of perfect matter fluids. The original papers and reviews on cosmological perturbations can be found in Refs. [51, 504], but we do not discuss perturbations in detail here.

The Hubble radius R_H, defined as $R_H = 1/(aH)$, where a is the scale factor of the Universe and H the corresponding Hubble rate, quantifies perfectly the conditions that have to be satisfied in order for inflation to occur. In terms of the Hubble radius, the primordial modes before the inflationary era were well inside the Hubble radius, and during the inflationary era these exited the Hubble radius, since the Hubble radius shrunk exponentially. In terms of the Hubble radius, the conditions for inflation are, $\dot{R}_H < 0$, which implies that $\ddot{a} > 0$. In turn the latter conditions indicate that the Universe described by the scale factor a is accelerating, and this is why the inflationary era is often referred to as early-time accelerating era.

A. Scalar Field Descriptions

1. *Canonical Scalar Field Inflation*

Consider a Universe described by the FRW metric and a canonical scalar field φ, with the following action,

$$\mathcal{S} = \sqrt{-g}\left(\frac{R}{2} - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\varphi))\right) , \tag{233}$$

where $V(\varphi)$ the canonical scalar field potential. The energy momentum tensor $T_{\mu\nu}$ corresponding to the scalar field action (233) is equal to,

$$T_{\mu\nu} = \partial_\mu\varphi\partial_\nu\varphi - g_{\mu\nu}\left(\frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - V(\varphi)\right) , \tag{234}$$

and therefore from this we obtain that the energy density is $T_0^0 = \rho$, that is,

$$\rho = \frac{1}{2}\dot{\varphi}^2 + V(\varphi) . \tag{235}$$

Also from the component $T_j^i = -P\delta_j^i$, we obtain that the total pressure in a Universe filled with a canonical scalar field is,

$$P = \frac{1}{2}\dot{\varphi}^2 - V(\varphi) . \tag{236}$$

The Friedmann equation is,

$$H^2 = \frac{\kappa^2}{3}\rho , \tag{237}$$

so by substituting the energy density (235) and taking the first derivative with respect to the cosmic time t, by also using the expression for the pressure (236) we obtain,

$$\dot{H} = -\frac{\kappa^2}{2}\dot{\varphi}^2 , \tag{238}$$

which combined with the above leads to the Klein-Gordon equation for the canonical scalar field φ,

$$\ddot{\varphi} + 3H\dot{\varphi} + V' = 0 , \tag{239}$$

where the prime denotes differentiation with respect to the scalar field φ. The slow-roll indices ϵ and η are defined in terms of the Hubble rate as follows,

$$\epsilon = -\frac{\dot{H}}{H^2} , \quad \eta = -\frac{\ddot{H}}{2H\dot{H}} . \tag{240}$$

The slow-roll conditions for the canonical scalar field require that the slow-roll indices satisfy $\epsilon, \eta \ll 1$, so in terms of the scalar field and the potential, this means $\frac{1}{2}\dot{\varphi}^2 \ll V(\varphi)$. This means that the Friedmann equation (237) is simplified as follows,

$$H^2 \simeq \kappa^2 \frac{V}{3} \,. \tag{241}$$

In terms of the scalar field φ, the slow-roll indices can easily be found by using Eqs. (237) and (238), so these are,

$$\epsilon = \frac{\kappa^2 \dot{\varphi}^2}{2H^2} \,, \quad \eta = \frac{\kappa^2 \dot{\varphi}^2}{H^2} + 2\frac{\ddot{\varphi}}{H\dot{\varphi}} \,. \tag{242}$$

The second slow-roll condition $\eta \ll 1$ implies that $\ddot{\varphi} \ll H\dot{\varphi}$, so the Klein-Gordon equation (239) is simplified as follows,

$$3H\dot{\varphi} \simeq -V' \,. \tag{243}$$

By using the expressions for the slow-roll indices (242) and combining these with the slow-roll conditions (241) and (243) we finally obtain the slow-roll indices ϵ and η in terms of the canonical scalar field potential $V(\varphi)$,

$$\epsilon \simeq \frac{1}{2\kappa^2} \left(\frac{V'(\varphi)}{V(\varphi)} \right)^2 \,, \quad \eta \simeq \frac{1}{\kappa^2} \frac{|V''(\varphi)|}{V(\varphi)} \,. \tag{244}$$

Moreover, the graceful exit from the inflationary era occurs when the first slow-roll index becomes of the order $\epsilon \sim \mathcal{O}(1)$. Having the above at hand, one can easily calculate the spectral index of primordial curvature perturbations n_s and the scalar-to-tensor ratio r, which for a slow-rolling canonical scalar can be expressed in terms of the slow-roll indices as follows,

$$n_s \simeq 1 - 6\epsilon + 2\eta \,, \quad \eta \simeq 16\epsilon \,, \tag{245}$$

where the slow-roll indices are calculated at the horizon crossing. Let us use the single scalar field inflation formalism in order to calculate the slow-roll indices and the corresponding observational indices for the Starobinsky model of inflation [505], or Higgs inflation [506],

$$V(\varphi) = \alpha \mu^2 \left(1 - \mathrm{e}^{-\sqrt{\frac{2}{3}}\varphi} \right)^2 \,, \tag{246}$$

where for simplicity we use the physical units system in which $\hbar = c = 8\pi G = \kappa^2 = 1$. So let us calculate the slow-roll indices (244) at the horizon crossing, which we assume it occurs for $\varphi = \varphi_k$, and these are,

$$\epsilon = \frac{4}{3\left(\mathrm{e}^{\sqrt{\frac{2}{3}}\varphi_k} - 1 \right)^2} \,, \quad \eta = -\frac{4\left(\mathrm{e}^{\sqrt{\frac{2}{3}}\varphi_k} - 2 \right)}{3\left(\mathrm{e}^{\sqrt{\frac{2}{3}}\varphi_k} - 1 \right)^2} \,. \tag{247}$$

Then, the observational indices n_s and r, calculated at the horizon crossing read,

$$n_s \simeq \frac{-14\mathrm{e}^{\sqrt{\frac{2}{3}}\varphi_k} + 3\mathrm{e}^{2\sqrt{\frac{2}{3}}\varphi_k} - 5}{3\left(\mathrm{e}^{\sqrt{\frac{2}{3}}\varphi_k} - 1 \right)^2} \,, \quad r \simeq \frac{64}{3\left(\mathrm{e}^{\sqrt{\frac{2}{3}}\varphi_k} - 1 \right)^2} \,. \tag{248}$$

We can express the observational indices as functions of the e-foldings number N, which for a slow-rolling canonical scalar field is,

$$N \simeq \int_{\varphi_f}^{\varphi_k} \frac{V(\varphi)}{V'(\varphi)} d\varphi \,, \tag{249}$$

where φ_k is the value of the scalar field at the horizon crossing and φ_f is the value of the scalar at the end of inflation. By using the potential (246), the e-foldings number at leading order is,

$$N \simeq \frac{-3}{4}\mathrm{e}^{\sqrt{\frac{2}{3}}\varphi_f} + \frac{3}{4}\mathrm{e}^{\sqrt{\frac{2}{3}}\varphi_k} \,, \tag{250}$$

and the expression containing the value of the scalar field at the end of the inflationary era can be determined by the condition $\epsilon(\varphi_f) \simeq 1$, which yields the condition,

$$\frac{4}{3\left(e^{\sqrt{\frac{2}{3}}\varphi_f}-1\right)^2}=1\,. \tag{251}$$

Combining Eqs. (248), (250), and (251), and also for large e-folding values, we finally obtain the observational indices of the Starobinsky model, which at leading-N order are,

$$n_s \simeq 1-\frac{2}{N}-\frac{3}{N^2}\,,\quad r\simeq\frac{12}{N^2}\,. \tag{252}$$

Before we close this section, it is worth discussing an interesting aspect with regard to the graceful exit from inflation in the context of inflationary theories with a single canonical scalar field. Actually, the slow-roll parameters ϵ and η are the lowest order terms in the Hubble slow-roll expansion [507–509], and there are many more higher order parameters in the expansion. Then it is possible that the graceful exit might occur even at higher order and thus much more earlier than in the case where $\epsilon \sim \mathcal{O}(1)$. The slow-roll expansion [507–509], quantitatively completes the slow-roll approximation, and it actually enables us to securely find the final inflationary attractor of the theory. It is worth recalling some fundamental features of the slow-roll expansion, and for details we refer to [507–509]. According to the Hubble slow-roll expansion, the physical system has an inflationary attractor, to which all the slow-roll solutions tend to asymptotically. Then, we may write the single canonical scalar field Friedmann equation as follows,

$$H^2(\varphi)=\frac{8\pi\kappa^2}{3}V(\varphi)\left(1-\frac{1}{3}\epsilon(\varphi)\right)^{-1}\,, \tag{253}$$

where we expressed the slow-roll parameter ϵ as a function of the canonical scalar field φ. By employing the binomial theorem, we can obtain the perturbative expansion of Eq. (253) in terms of the slow-roll parameters ϵ_V, ξ_V and η_V

$$H^2(\varphi)\simeq\frac{8\pi\kappa^2}{3}V(\varphi)\left(1+\epsilon-\frac{4}{3}\epsilon^2+\frac{2}{3}\epsilon\eta_V+\frac{32}{9}\epsilon^3+\frac{5}{9}\epsilon\eta_V^2-\frac{10}{3}\epsilon^2\eta_V+\frac{2}{9}\epsilon\xi_V^2+\mathcal{O}_4\right)\,, \tag{254}$$

where in the expansion above we kept terms up to fourth order in the parameters η_V, ξ_V, which in terms of the slow-roll parameters ϵ and η, are defined below,

$$\begin{aligned}\eta_V&=(3-\epsilon)^{-1}\left(3\epsilon+3\eta-\eta^2-\xi_H^2\right)\,,\\ \xi_V&=(3-\epsilon)^{-1}\left(27\epsilon\eta+9\xi_H^2-9\epsilon\eta^2-12\eta\xi_H^2-3\sigma_H^3+3\eta^2\xi_H^2+\eta\sigma_H^3\right)\,.\end{aligned} \tag{255}$$

In addition, the parameters ξ_H and σ_H appearing in Eq. (255), are also functions of the slow-roll parameters ϵ and η and are defined as follows,

$$\xi_H^2=\epsilon\eta-\sqrt{\frac{1}{4\pi\kappa^2}}\sqrt{\epsilon}\eta'\,,\quad \sigma_H^3=\xi_H^2(2\epsilon-\eta)-\sqrt{\frac{1}{\kappa^2\pi}}\sqrt{\epsilon}\xi_H\xi_H'\,. \tag{256}$$

Note that in the above equations, the prime indicates differentiation with respect to the canonical scalar field φ, and in addition we assumed the Hubble rate solution $H(t)$ is the final attractor of the theory.

Having the slow-roll expansion (254) at hand, it is easy to support our argument that the slow-roll expansion might actually break down much more earlier than the slow-roll approximation does, and this is due to the existence of the higher order terms in the expansion. In the case of the slow-roll approximation, ones takes into account only ϵ and η which are simply the lowest order terms. Moreover, as was also noted in Ref. [507–509], the Hubble slow-roll expansion can break down for large values of the slow-roll parameters or if a singularity occurs in the perturbative expansion. For example, if the canonical scalar field potential has the following form $V(\varphi)\sim\varphi^2$, then the Hubble slow-roll expansion contains inverse powers of the canonical scalar field, and the inflationary era might end before $\epsilon\sim\mathcal{O}(1)$. In fact, although definitely the end of inflation occurs at $\varphi=0$, where the absolute minimum of the potential is, the Hubble slow-roll expansion might break down earlier, due to the fact that the expansion contains terms φ^{-n}, with $n>0$ and this is clearly an indication that inflation might end earlier. For similar results on instabilities of the second slow-roll index η and its connection to the graceful exit from inflation, see [510–513].

2. Non-Canonical Scalar Field Inflation

Consider now a non-canonical scalar field, in which case the gravitational action is,

$$S = \int d^4x\sqrt{-g}\left\{\frac{1}{2\kappa^2}R - \frac{1}{2}\omega(\phi)\partial_\mu\phi\partial^\mu\phi - V(\phi) + L_{\text{matter}}\right\} . \tag{257}$$

In this case the appearance of the kinetic term $\omega(\phi)$, makes the scalar field ϕ non-canonical, so it is conceivable that the slow-roll indices in this case, are related to this kinetic function $\omega(\phi)$. In principle the kinetic function is irrelevant and can be easily absorbed by making a redefinition of the scalar field ϕ, as follows,

$$\varphi \equiv \int^\phi d\phi\sqrt{\omega(\phi)}\,, \tag{258}$$

where it is assumed that $\omega(\phi) > 0$. Then, one can obtain a canonical scalar field action, since the kinetic scalar term becomes,

$$-\omega(\phi)\partial_\mu\phi\partial^\mu\phi = -\partial_\mu\varphi\partial^\mu\varphi\,, \tag{259}$$

Then, only in the case for which the resulting expression from the integration (258), is an invertible function of ϕ, then if we find the function $\phi = \phi(\varphi)$, we may substitute ϕ in the action (257) and then we have the canonical scalar theory at hand. However, in the case that the result of the integral (258) is not an invertible function of ϕ, then it is not possible to obtain the canonical scalar theory, and therefore we need an alternative way to calculate the slow-roll indices. This case was studied in detail in [507–509], and the slow-roll indices as functions of the non-canonical kinetic term $\omega(\phi)$ and of the scalar potential $V(\phi)$ are given below [52, 507, 514],

$$\begin{aligned}
\epsilon =& \frac{1}{2\kappa^2}\left(\frac{d\phi}{d\varphi}\right)^2\left(\frac{V'(\phi)}{V(\phi)}\right)^2 = \frac{1}{2\kappa^2}\frac{1}{\omega(\phi)}\left(\frac{V'(\phi)}{V(\phi)}\right)^2 ,\\
\eta =& \frac{1}{\kappa^2V(\phi)}\left[\frac{d\phi}{d\varphi}\frac{d}{d\phi}\left(\frac{d\phi}{d\varphi}\right)V'(\phi) + \left(\frac{d\phi}{d\varphi}\right)^2 V''(\phi)\right] = \frac{1}{\kappa^2V(\phi)}\left[-\frac{\omega'(\phi)}{2\omega(\phi)^2}V'(\phi) + \frac{1}{\omega(\phi)}V''(\phi)\right] .
\end{aligned}\tag{260}$$

Then, one may study the analytic slow-roll of the non-canonical scalar field ϕ, following the research line we presented in the previous section, but we will not go into details for brevity.

Before closing this section we need to note that in the case $\omega(\phi) < 0$, the non-canonical scalar ϕ is a phantom field, and in this case instead of the integration performed in Eq. (258), one should perform the following redefinition of the scalar ϕ,

$$\varphi \equiv \int^\phi d\phi\sqrt{-\omega(\phi)}\,. \tag{261}$$

In this case then, the kinetic term reads,

$$-\omega(\phi)\partial_\mu\phi\partial^\mu\phi = \partial_\mu\varphi\partial^\mu\varphi\,, \tag{262}$$

instead of the corresponding expression appearing in Eq. (259). However, in the case of a ghost scalar field, the situation is rather physically not appealing, since phantom inflation occurs. In this case, the total energy density is unbounded from below when one deals with the classical theory at least. However, in the quantum theory, the energy becomes bounded from below, at the cost of having a negative norm in the Hilbert space states.

3. Multi-Scalar Field Inflation

The slow-roll formalism for inflation we developed in the previous sections for a single scalar field, can be generalized in the case of multiple scalar fields. In this section we briefly outline the fundamental features of this formalism and for details we refer to the original papers that address this issue, see Refs. [515–518].

The gravitational action in the presence of multiple scalar fields ϕ^I reads,

$$S = \int d^4x\sqrt{-\hat{g}}\left(\frac{\hat{R}}{2\kappa^2} - \frac{1}{2}G_{IJ}(\phi^I)\hat{g}^{\mu\nu}\partial_\mu\phi^I\partial_\nu\phi^J - V(\phi^I)\right) , \tag{263}$$

with $I, J = 1, 2, ..N$, where N is the total number of the scalar fields present. The metric $G_{IJ}(\phi^I)$ depends solely on the scalar fields and therefore it is the metric in the configuration space formed by the scalars ϕ^I.

The FRW equations corresponding to the gravitational action (263) are given below,

$$H^2 = \frac{\kappa^2}{3}\left(\frac{1}{2}G_{IJ}\dot{\varphi}^I\dot{\varphi}^J + V(\varphi^I)\right), \quad \dot{H} = -\frac{1}{\kappa^2}G_{IJ}\dot{\varphi}^I\dot{\varphi}^J, \quad \Box\phi^I + \hat{g}^{\mu\nu}\Gamma^I_{IK}\partial_\mu\phi^J\partial_\nu\phi^K - G^{IK}V_{,K} = 0, \tag{264}$$

where $V_{,K} = \partial V/\partial\phi^K$, and also the Christoffel symbols $\Gamma^I_{IK} = \Gamma^I_{IK}(\varphi, \phi)$ are the connections in the configuration space formed by the scalar fields ϕ^I, equipped with the metric Γ^I_{IK}.

We introduce the scalar field function σ and the vector field $\hat{\sigma}^I$, which are defined in terms of the scalar fields and the metric as follows,

$$\dot{\sigma} = \sqrt{G_{IJ}\dot{\phi}^I\dot{\phi}^J}, \quad \hat{\sigma}^I = \frac{\dot{\phi}^I}{\dot{\sigma}}. \tag{265}$$

The FRW equations (264) can be rewritten in terms of the variables $\dot{\sigma}$ and $\hat{\sigma}^I$, in the following way,

$$H^2 = \frac{\kappa^2}{3}\left(\frac{1}{2}\dot{\sigma}^2 + V\right), \quad \dot{H} = -\frac{\kappa^2}{2}\dot{\sigma}^2, \quad \ddot{\sigma} + 3H\dot{\sigma} + V_{,\sigma} = 0, \tag{266}$$

where $V_{,\sigma} = \hat{\sigma}^I V_{,I}$. Finally, the generalized multi-field slow-roll indices, as functions of the scalar fields, are equal to,

$$\epsilon = \frac{3\dot{\sigma}^2}{\dot{\sigma}^2 + 2V}, \quad \eta_{\sigma\sigma} = \frac{1}{\kappa^2}\frac{\hat{\sigma}_I\hat{\sigma}^J M^I_J}{V}, \quad \eta_{ss} = \frac{1}{\kappa^2}\frac{\hat{s}_I\hat{s}^J M^I_J}{V}, \tag{267}$$

where the vectors $\hat{\sigma}^I$ and $\hat{s}^I = \omega^I/\omega$ are the unit vectors of the adiabatic and iso-curvature directions respectively, in the curved configurations space of the scalar fields. In addition, the vector ω^I is the turn-rate vector given by $\omega^I = \mathcal{D}_t\hat{\sigma}^I$, with $\omega = |\omega^I|$ and $\mathcal{D}_t$ the covariant derivative along the t-direction, which on an arbitrary vector field A^I acts as follows,

$$\mathcal{D}_t A^I = \dot{A}^I + \Gamma^I_{JK}A^J\dot{\phi}^K. \tag{268}$$

For a recent work with a detailed computation with two scalar fields, we refer the reader to Ref. [513].

B. Inflation in $f(\varphi, R)$ Theories of Gravity

A quite general class of inflationary theories can be described by the following action [519–525],

$$\mathcal{S} = \int d^4x\sqrt{-g}\left(\frac{f(R,\phi)}{2\kappa^2} - \frac{1}{2}\omega(\phi)g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right), \tag{269}$$

where $f(R, \phi)$ is a smooth function of the scalar curvature and of the scalar field, which is in general a non-canonical scalar field, if the kinetic term satisfies $\omega(\phi) \neq 1$. It is conceivable that the $F(R)$ gravity theory, the canonical and non-canonical scalar field gravity, and also theories of the form $f(R, \phi) = f(\phi)R$, are subcases of the theory described by the action (269). In this section we present the essential features of the inflationary dynamics for the general action (269) and later on we specify how the dynamics are modified in each of the $F(R)$ and $f(R, \phi) = f(\phi)R$ cases, since the single scalar field case was studied in the previous sections. The slow-roll indices for the action (269) have the following general form,

$$\epsilon_1 = -\frac{\dot{H}}{H^2}, \quad \epsilon_2 = \frac{\ddot{\phi}}{H\dot{\phi}}, \quad \epsilon_3 = \frac{\dot{f}_R}{2Hf_R}, \quad \epsilon_4 = \frac{\dot{E}}{2HE}, \tag{270}$$

where $f_R = \frac{\partial f(R,\phi)}{\partial R}$. Also the function E appearing in Eq. (270) is equal to,

$$E = f_R\omega + \frac{3\dot{f}_R^2}{2\kappa^2\dot{\phi}^2}. \tag{271}$$

In addition for later convenience, we define the following function, Q_s,

$$Q_s = \dot{\phi}^2\frac{E}{f_R H^2(1+\epsilon_3)^2}, \tag{272}$$

which plays an important role for the calculation of the scalar-to-tensor ratio during the slow-roll era. Assuming the flat FRW background, then by varying the action (269) with respect to the metric and also with respect to the scalar field, we obtain the following equations of motion,

$$\begin{aligned}
\frac{3f_R(R,\phi)H^2}{\kappa^2} &= \frac{\omega(\phi)\dot{\phi}^2}{2} + V(\phi) + \frac{1}{\kappa^2}\left(Rf_R(R,\phi) - f(R,\phi)\right) - 3H\dot{f}_R(R,\phi)\,,\\
-\frac{2f(R,\phi)}{\kappa^2}\dot{H} &= \omega(\phi)\dot{\phi}^2\ddot{f}_R(R,\phi) - H\dot{f}_R(R,\phi)\,,\\
\ddot{\phi} + 3H\dot{\phi} + \frac{1}{2\omega(\phi)}&\left(\dot{\omega(\phi)}\dot{\phi} - \frac{1}{\kappa^2}\frac{\partial f(R,\phi)}{\partial\phi} + 2\frac{\partial V}{\partial\phi}\right) = 0\,.
\end{aligned} \tag{273}$$

During the slow-roll era, the following relations hold true between the Hubble rate, the scalar field and the functions $f(R,\phi)$, $\omega(\phi)$,

$$\frac{\dot{\omega}\dot{\phi}}{H^2\dot{f}_R} \ll 1\,, \quad \frac{\dot{\omega}\dot{\phi}^2}{H^2\dot{f}_R} \gg \frac{2\omega\dot{\phi}\epsilon_3}{H\dot{f}_R}\,, \tag{274}$$

and in view of the relations (274), the equations of motion (273) become,

$$\begin{aligned}
\frac{3f_R(R,\phi)H^2}{\kappa^2} &\simeq V(\phi) + \frac{Rf_R(R,\phi) - f(R,\phi)}{2}\,,\\
3H\dot{\phi} + \frac{1}{2\omega(\phi)}&\left(\dot{\omega(\phi)}\dot{\phi} - \frac{1}{\kappa^2}\frac{\partial f(R,\phi)}{\partial\phi} + 2\frac{\partial V}{\partial\phi}\right) = 0\,.
\end{aligned} \tag{275}$$

In the context of the slow-roll approximation, where the slow-roll indices satisfy $\epsilon_i \ll 1$, $i = 1,...4$, the spectral index of primordial curvature perturbations has the following form [519–524],

$$n_s \simeq 1 - 4\epsilon_1 - 2\epsilon_2 + 2\epsilon_3 - 2\epsilon_4\,. \tag{276}$$

Also, the general form of the scalar-to-tensor ratio r, as a function of the function Q_s, which we introduced in Eq. (272), is equal to,

$$r = 8\kappa^2\frac{Q_s}{f_R(R,\phi)}\,. \tag{277}$$

For example, in the case of a canonical scalar field, the function Q_s is,

$$Q_s = \frac{\dot{\phi}^2}{H^2}\,, \tag{278}$$

and the slow-roll approximation yields, $\dot{\phi}^2 = -\frac{2\dot{H}}{\kappa^2}$, so the resulting scalar-to-tensor ratio is $r \simeq 16\epsilon_1$, as we saw in the previous section. Also in the pure $F(R)$ gravity case, the parameter Q_s is,

$$Q_s = \frac{3\dot{f_R}^2}{2f_RH^2\kappa^2}\,, \tag{279}$$

and so the resulting scalar-to-tensor ratio is $r = 48\epsilon_3^2$. In the following sections we shall present several examples of inflationary theories, which belong to some types of the model (269).

1. *Inflation in $f(\phi)R$ Theories of Gravity*

As a first example of a theory belonging to the class of the model in action (269), we consider $f(\phi)R$ theories of gravity, which describe non-minimally coupled theories of gravity, in which case the action is [441, 519–523],

$$\mathcal{S} = \int d^4x\sqrt{-g}\left(\frac{f(\phi)R}{2\kappa^2} - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right)\,, \tag{280}$$

where $f(\phi)$ is an analytic function of the scalar field ϕ. By varying the action (280) with respect to the metric and with respect to the scalar field ϕ, we obtain the following equations of motion,

$$\frac{3f}{\kappa^2}H^2=\frac{\dot{\phi}^2}{2}+V(\phi)-3h\frac{\dot{f}}{\kappa^2}\,,\quad -\frac{2f}{\kappa^2}\dot{H}=\dot{\phi}^2+\frac{\ddot{f}}{\kappa^2}-H\frac{\dot{f}}{\kappa^2}\,,\quad \ddot{\phi}+3H\dot{\phi}-\frac{1}{2\kappa^2}R\frac{df}{d\phi}+\frac{dV}{d\phi}=0\,, \tag{281}$$

with the "dot" denoting as usual, differentiation with respect to the cosmic time. The slow-roll indices in the case of a non-minimally coupled scalar theory, are equal to,

$$\epsilon_1=-\frac{\dot{H}}{H^2}\,,\quad \epsilon_2=\frac{\ddot{\phi}}{H\dot{\phi}}\,,\quad \epsilon_3=\frac{\dot{f}}{2Hf}\,,\quad \epsilon_4=\frac{\dot{E}}{2HE}\,, \tag{282}$$

and in this case the function E is equal to,

$$E=f+\frac{3\dot{f}^2}{2\kappa^2\dot{\phi}^2}\,. \tag{283}$$

The spectral index of primordial curvature perturbations n_s and the scalar-to-tensor ratio in terms of the slow-roll indices, are equal to,

$$n_s\simeq 1-4\epsilon_1-2\epsilon_2+2\epsilon_3-2\epsilon_4\,,\quad r=8\kappa^2\frac{Q_s}{f}\,, \tag{284}$$

and it is assumed that the slow-roll indices satisfy the slow-roll condition $\epsilon_i \ll 1$, $i=1,..,4$. In addition, the parameter Q_s in the case at hand is equal to,

$$Q_s=\dot{\phi}^2\frac{E}{fH^2(1+\epsilon_3)^2}\,. \tag{285}$$

We can find an approximate expression for the function Q_s, by using the slow-roll approximation, so we find the slow-roll approximated gravitational equations of motion, which take the following form,

$$\frac{3fH^2}{\kappa^2}\simeq V(\phi)\,,\quad 3H\dot{\phi}-\frac{6H^2}{\kappa^2}f'+V'\simeq 0\,, \tag{286}$$

$$\dot{\phi}^2\simeq\frac{H\dot{f}}{\kappa^2}-\frac{2f\dot{H}}{\kappa^2}\,, \tag{287}$$

with the "prime" denoting this time differentiation with respect to the scalar ϕ. In view of the slow-roll approximated equations of motion, the parameter Q_s becomes,

$$Q_s=\frac{\dot{\phi}^2}{H^2}+\frac{3\dot{f}^2}{2\kappa^2fH^2}\,, \tag{288}$$

and in conjunction with Eq. (287), the parameter Q_s finally becomes,

$$Q_s\simeq\frac{H\dot{f}}{H^2\kappa^2}-\frac{2f\dot{H}}{\kappa^2H^2}\,. \tag{289}$$

Hence, by combining Eqs. (284) and (289), we can find a simplified expression for the scalar-to-tensor ratio during the slow-roll era, which is,

$$r\simeq 16(\epsilon_1+\epsilon_3)\,. \tag{290}$$

Moreover, we may express the spectral index of the primordial curvature perturbations n_s as a function of the slow-roll indices in the slow-roll approximation, which takes the following form,

$$n_s\simeq 1-2\epsilon_1\left(\frac{3H\dot{f}}{\dot{\phi}^2}+2\right)-2\epsilon_2-6\epsilon_3\left(\frac{H\dot{f}}{\dot{\phi}^2}-1\right)\,. \tag{291}$$

The formalism we just developed, enables us to calculate analytically the observational indices during the slow-roll era, for any given function $f(\phi)$. As an example we shall consider theories in which the function $f(\phi)$ has a special symmetry $\beta \to \frac{1}{\beta}$, where $\beta > 1$. These kind of potentials were considered in [526]. So suppose that the function $f(\phi)$ is equal to,

$$f(\phi) = \frac{1 + \xi \left(\mathrm{e}^{-\beta n \phi} + \mathrm{e}^{-\frac{1}{\beta} n \phi}\right)}{2}\,, \tag{292}$$

where the parameters ξ and n are positive real parameters. The scalar potential $V(\phi)$ will be chosen in such a way so that it does not affect the dynamics, in comparison to the function $f(\phi)$, for example it can contain higher powers of the exponential appearing in Eq. (292). However, in the case at hand we can simply choose $V(\phi) = \Lambda$, with Λ being a positive constant parameter. Notice that the function $f(\phi)$ of Eq. (292), is symmetric under the transformation $\beta \to \frac{1}{\beta}$. By using the formalism we developed earlier, we shall calculate analytically the slow-roll indices and the corresponding observational indices for the case at hand. For simplicity we shall choose a physical units system for which $\kappa^2 = 1$, and in addition we choose $\xi = 1$ in order to simplify the intermediate expressions. It can be shown though that the resulting expressions for n_s and r do not depend on the parameter ξ. For $\beta > 1$, the function $f(\phi)$ can be approximated as follows,

$$f(\phi) \simeq \frac{1 + \mathrm{e}^{-\frac{n}{\beta}\phi}}{2}\,, \tag{293}$$

and by using the slow-roll equations of motion, we get the following expression,

$$\dot{\phi} \simeq -H \frac{n}{\beta} \mathrm{e}^{-\frac{n}{\beta}\phi}\,. \tag{294}$$

Moreover, the function $\dot{f}$ during the slow-roll era, can be written as follows,

$$\dot{f} \simeq \frac{n^2 \mathrm{e}^{-2\frac{n}{\beta}\phi} H}{2}\,. \tag{295}$$

We can then calculate the slow-roll indices, with the first one being approximately equal to,

$$\epsilon_1 \simeq \left(\frac{\dot{\phi}}{H}\right)^2 \frac{1}{2f} - \frac{\dot{f}}{2fH}\,. \tag{296}$$

By combining Eqs. (294) and (295), the index ϵ_1 reads

$$\epsilon_1 \simeq \frac{n^2 \mathrm{e}^{-2\frac{n}{\beta}\phi}}{2\beta^2}\,. \tag{297}$$

Accordingly, the slow-roll indices ϵ_2 and ϵ_3 can be calculated, and these are equal to,

$$\epsilon_2 \simeq \frac{n^2}{\beta^2} \mathrm{e}^{-\frac{n}{\beta}\phi} + \epsilon_1\,, \quad \epsilon_3 \simeq \epsilon_1\,. \tag{298}$$

Finally, the observational indices n_s and r can be easily found by combining Eqs. (290), (291), (297), and (298), so these read,

$$n_s = 1 - 2\frac{n^2}{\beta^2} \mathrm{e}^{-\frac{n}{\beta}\phi}\,, \quad r \simeq 16 \frac{n^2}{\beta^2} \mathrm{e}^{-2\frac{n}{\beta}\phi}\,. \tag{299}$$

It is more appropriate to have the resulting expressions for n_s and r as functions of the e-foldings number, so we shall make use of the following relation-definition,

$$N = \int_{t_k}^{t_f} H(t) dt = \int_{\phi_k}^{\phi_f} \frac{H}{\dot{\phi}} d\phi \simeq \frac{\beta^2}{n^2} \mathrm{e}^{\frac{n}{\beta}\phi}\,, \tag{300}$$

with t_k being chosen as the horizon crossing time instance and in addition, ϕ_k is the value of the scalar field at the horizon crossing time instance. Finally the time instance t_f is the time that inflation ends, and also ϕ_f is the corresponding scalar field. Not that for the derivation of relation (300), we made use of the approximation $\phi_k \gg \phi_f$, which holds true during the slow-roll era. From Eq. (300) we easily obtain that $\frac{n^2}{\beta^2}e^{-\frac{n}{\beta}\phi} = \frac{1}{N}$, so the observational indices can be expressed in terms of N as follows,

$$n_s = 1 - \frac{2}{N}\,, \quad r \simeq \frac{16\beta^2}{n^2 N^2}\,. \tag{301}$$

A quite interesting choice for the parameter n is $n = \frac{2}{\sqrt{3}}$, and the observational indices become,

$$n_s = 1 - \frac{2}{N}\,, \quad r \simeq \frac{12\beta^2}{N^2}\,. \tag{302}$$

The final expressions for the observational indices (302) are identical to the ones corresponding to the α-attractor models [527–537], with the difference being that $\beta \ll 1$ in the case of the α-attractor models.

2. *Inflation from $F(R)$ Theories*

As another example of a theory belonging to the general class of models of Eq. (269), we shall consider $F(R)$ theories of gravity. As we showed in section II, the $F(R)$ gravity has a unique Einstein frame canonical scalar theory, so in many cases a viable cosmological model in the Einstein frame has a unique $F(R)$ gravity in the Jordan frame, which is also viable. In this section we shall present the theoretical study of slow-roll inflation in the pure $F(R)$ gravity case. For simplicity we shall consider only a vacuum theory, but the results can be easily extended in the case that perfect matter fluids are present.

Consider the $F(R)$ gravity. The detailed study of inflation for modified gravities was developed in Refs. [519–524, 535] (see also [538] for a study on the evolution of density perturbations in $F(R)$ gravity), and the dynamics of inflation are determined by four generalized slow-roll indices, namely ϵ_1 ,ϵ_2, ϵ_3, ϵ_4, which we shall intensively use in this section. The parameter ϵ_1 is simply equal to $\epsilon_1 = -\frac{\dot{H}}{H^2}$, and in the case of the pure $F(R)$ gravity of Eq. (2), the slow-roll parameters can be expressed in terms of the slow-roll index ϵ_1 as follows [519–524, 535],

$$\epsilon_2 = 0\,, \quad \epsilon_1 \simeq -\epsilon_3\,, \quad \epsilon_4 \simeq -3\epsilon_1 + \frac{\dot{\epsilon}_1}{H\epsilon_1}\,. \tag{303}$$

The spectral index of primordial curvature perturbations and the scalar-to-tensor ratio in the slow-roll limit, have the following form,

$$n_s \simeq 1 - 6\epsilon_1 - 2\epsilon_4, \quad r = 48\epsilon_1^2\,. \tag{304}$$

The expressions (304) are valid only in the slow-roll limit where $\epsilon_1, \epsilon_4 \ll 1$. Let us apply this methodology in order to calculate the observational indices n_s and r for some special cases.

The R^2 model of inflation [505], is widely known and also has very appealing features since it is compatible to the latest (2015) Planck data [48]. In this case, the $F(R)$ gravity is $F(R) = R + \frac{R^2}{36H_i}$, where H_i has dimensions of mass^2. During the slow-roll era of inflation, it is expected that $F'(R) \simeq R/(18H_i)$. For the FRW metric, the FRW equations read,

$$\ddot{H} - \frac{\dot{H}^2}{2H} + 3H_i H = -3H\dot{H}\,, \quad \ddot{R} + 3HR + 6H_i R = 0\,, \tag{305}$$

and during the slow-roll era, the first two terms in the first equation of Eq. (305) can be neglected, hence the evolution during the inflationary era is the quasi-de Sitter evolution of the form,

$$H(t) \simeq H_0 - H_i(t - t_k)\,. \tag{306}$$

The time instance can be arbitrarily chosen, but we assume that this is the time instance that the primordial curvature modes exit the horizon. Now let us proceed to the calculation of the indices, so at first let us note that the slow-roll approximation ends when the first slow-roll parameter becomes $\epsilon_1 \simeq \mathcal{O}(1)$, and by assuming that this occurs at the time instance $t = t_f$ at which point $H(t_f) = H_f$, the condition $\epsilon_1(t_f) \simeq 1$ yields the relation $H_f \simeq \sqrt{H_i}$. Accordingly, by substituting in Eq. (306), we obtain,

$$H_f - H_0 \simeq -H_i(t_f - t_k)\,, \tag{307}$$

and by substituting H_f we obtain,

$$t_f - t_k = \frac{H_0}{H_i} - \frac{\sqrt{H_i}}{H_i}\,. \tag{308}$$

Both the parameters H_0 and H_i are expected to have large values during the slow-roll era, so we may omit the second term in Eq. (308), and hence,

$$t_f - t_k \simeq \frac{H_0}{H_i}\,. \tag{309}$$

In terms of the Hubble rate, the e-foldings number N is equal to

$$N = \int_{t_k}^{t_f} H(t)dt\,. \tag{310}$$

By using the Hubble rate (306), and performing the integral in Eq. (310), we obtain,

$$N = H_0(t_f - t_k) - \frac{H_i(t_f - t_k)^2}{2}\,, \tag{311}$$

and by using (309), we finally get,

$$N = \frac{H_0^2}{2H_i}\,. \tag{312}$$

So at leading order we have,

$$t_f - t_k \simeq \frac{2N}{H_0}\,, \tag{313}$$

and now we can calculate the observational indices by combining Eqs. (303), (304), and (306), so we finally have,

$$n_s \simeq 1 - \frac{4H_i}{\left(H_0 - \frac{2H_iN}{H_0}\right)^2}\,, \quad r = \frac{48H_i^2}{\left(H_0 - \frac{2H_iN}{H_0}\right)^4}\,. \tag{314}$$

By taking the large N limit, the observational indices take the following form,

$$n_s \simeq 1 - \frac{H_0^2}{H_iN^2}\,, \quad r = \frac{3H_0^4}{H_i^2N^4}\,, \tag{315}$$

so by using Eq. (312) we finally obtain,

$$n_s \simeq 1 - \frac{2}{N}\,, \quad r \simeq \frac{12}{N^2}\,. \tag{316}$$

As it can be seen by comparing Eqs. (252) and (316), the Einstein frame R^2 model and the Jordan frame R^2 model yield the same observational indices at leading order, as it was possible expected. This equivalence of the indices in the Jordan and Einstein frame is not accidental, see for example [539–543]. It should be noted that the graceful exit from inflation in these theories occurs due to the R^2 term. Indeed this term induces graceful exit from inflation, since the de Sitter point becomes unstable. Particularly, as it was shown in [544], the final attractor of the theory is a de Sitter vacuum and by perturbing this solution, it is shown that the perturbations grow. Thus the final de Sitter attractor is unstable, and hence the graceful exit comes as a result of growing curvature perturbations.

3. Inflation in mimetic-$F(R)$ Theories of Gravity

Another interesting class of models that belongs to the models with action (269), is the mimetic $F(R)$ gravity models, with Lagrange multiplier and potential. These models were introduced in [433] and further studied in [142]. In this section we shall be interested mainly on the study of inflationary dynamics in the context of these theories.

Also, another issue that is related to these theories is the study of graceful exit, which in general can be a tedious study. We will address this issue at the end of this chapter.

The theoretical framework of mimetic $F(R)$ gravity in principle provides us with much freedom in realizing various cosmological scenarios, in a viable way, meaning that the theories are compatible with the recent observational data. In the literature there are various approaches for these kind of theories, with one approach being based on the perfect fluid description [434], and the other description deals with the mimetic $F(R)$ gravity as it is an $F(R)$ gravity in the presence of a non-canonical scalar field. In this section we discuss the second paradigm, in order to apply the formalism of the previous sections, since the mimetic $F(R)$ gravity can be viewed as a special case of an $F(R,\phi)$ scalar-tensor theory. The slow-roll conditions are in this case,

$$\dot{H} \ll H^2 \, , \quad \ddot{H} \ll H\dot{H} \, , \tag{317}$$

with H being the Hubble rate. The mimetic $F(R)$ gravity action describes a special class of $F(R,\phi)$ scalar-tensor theory, since the mimetic $F(R)$ action can be written in the following way,

$$S = \int \sqrt{-g} \left\{ \frac{F(R)}{2\kappa^2} + \lambda(\phi)\partial_\mu\phi\partial^\mu\phi + \lambda(\phi) - V(\phi) \right\} \, , \tag{318}$$

and therefore, the kinetic term is, $\omega(\phi) = -2\lambda(\phi)$, while the scalar potential is, $\mathcal{V}(\phi) = \lambda(\phi) - V(\phi)$. The slow-roll indices for the general $F(R,\phi)$ gravity are [545–547],

$$\epsilon_1 = -\frac{\dot{H}}{H^2} \, , \quad \epsilon_2 = \frac{\ddot{\phi}}{H\dot{\phi}} \, , \quad \epsilon_3 = \frac{\dot{F}'(R,\phi)}{2HF'(R,\phi)} \, , \quad \epsilon_4 = \frac{\dot{E}}{2HE} \, , \tag{319}$$

where the function $E(R,\phi)$ is,

$$E(R,\phi) = F(R,\phi)\omega(\phi) + \frac{3\dot{F}'(R,\phi)^2}{2\dot{\phi}^2} \, . \tag{320}$$

In the case of mimetic $F(R)$ gravity, owing to the fact that $\phi = t$ and $\omega(\phi) = -2\lambda(\phi)$, the slow-roll indices take the following form,

$$\epsilon_1 = -\frac{\dot{H}}{H^2} \, , \quad \epsilon_2 = 0 \, , \quad \epsilon_3 = \frac{\dot{F}'(R,\phi)}{2HF'(R,\phi)} \, , \quad \epsilon_4 = \frac{\dot{E}}{2HE} \, , \tag{321}$$

with the function $E(R,\phi)$ being equal to,

$$E(R,\phi) = 2F(R,\phi)\lambda(\phi) + \frac{3\dot{F}'(R,\phi)^2}{2} \, . \tag{322}$$

Then, given the form of the $F(R)$ gravity, it is possible to calculate analytically the slow-roll indices and the corresponding observational indices. However, note that without the slow-roll approximation, the calculation of the slow-roll indices can be tedious, so we assume that the slow-roll approximation holds true. Therefore, if the slow-roll indices ϵ_i, $i = 1,..,4$ satisfy $\epsilon_i \ll 1$, the observational indices can be written in terms of the slow-roll indices as follows, [545–547],

$$n_s \simeq 1 - 4\epsilon_1 - 2\epsilon_2 + 2\epsilon_3 - 2\epsilon_4 \, , \quad r = 16(\epsilon_1 + \epsilon_3) \, . \tag{323}$$

Then, by specifying the explicit form of the $F(R)$ gravity and also the Hubble rate, we can realize various cosmological scenarios which are compatible with the observational data coming from Planck [48] and the most recent BICEP2/Keck Array data [49]. For convenience we quote here the Planck constraints, for the spectral index of the primordial curvature perturbations n_s and for the scalar-to-tensor ratio r, which are constrained as follows,

$$n_s = 0.9644 \pm 0.0049 \, , \quad r < 0.10 \, . \tag{324}$$

In addition, the latest BICEP2/Keck-Array data [49] further constrain the scalar-to-tensor ratio to satisfy,

$$r < 0.07 \, , \tag{325}$$

at 95% confidence level. For the purposes of our analysis, we shall consider two quite popular $F(R)$ gravity models, first the R^2 inflation model,

$$F(R) = R + \frac{R^2}{6M^2} \, , \tag{326}$$

and secondly the power-law model,

$$F(R) = \alpha R^n \,, \tag{327}$$

with $n > 0$. With regards to the non-mimetic R^2 model, this model is compatible with the observational data, but the non-mimetic version of the model (327) is not compatible with the data. As we shall demonstrate, the mimetic version of the power-law model (327), can be compatible with observations, if the parameters of the model are suitably chosen.

Let us first consider the case that the $F(R)$ gravity is the R^2 model of Eq. (326), so by using the mimetic $F(R)$ formalism, we shall realize the cosmological evolution,

$$H(t) = H_0 - \frac{d}{6}(t - t_0) + c(t - t_0)^2 \,, \tag{328}$$

with H_0, d, c and t_0 being arbitrary free parameters of the theory. These parameters must be chosen in the following way, in order for the slow-roll condition to hold true,

$$H_0 > d, c, t_0 \,. \tag{329}$$

In addition, the parameter t_0 must be $t_0 \ll 1$, since this time instance corresponds to the beginning of the inflationary era. Moreover, we need to note that the slow-roll approximation is valid for the Hubble rate (328), due to the fact that the cosmic time take values in the interval $(10^{-35}, 10^{-15})$sec, and therefore the time-dependent terms of Eq. (328), have extremely small values. The mimetic potential and the Lagrange multiplier have to be chosen appropriately in order to realize the cosmological evolution (328). Due to the slow-roll conditions (317), the $F'(R)$ can be simplified as $F'(R) \simeq R/(3M^2)$, and also in the same way, $\dot{F'} \simeq 8H\dot{H}/M^2$. Therefore, the mimetic R^2 gravity gravitational equations are simplified as follows,

$$\ddot{H} - \frac{\dot{H}^2}{2H} + \frac{M^2 H}{2} = -3H\dot{H} + M^2 \frac{V(\phi) - 2\lambda(\phi)}{12H} \,, \tag{330}$$

and by neglecting the first two terms in Eq. (330), we obtain,

$$\frac{M^2 H}{2} = -3H\dot{H} + M^2 \frac{V(\phi) - 2\lambda(\phi)}{12H} \,. \tag{331}$$

Consequently, for the cosmological evolution (328), the Lagrange multiplier and the potential satisfy,

$$V(t) - 2\lambda(t) = \frac{\left(-d + M^2 + 12c(t - t_0)\right)\left(-6H_0 + (t - t_0)(d + 6c(-t + t_0))\right)^2}{6M^2} \,. \tag{332}$$

We can find the exact forms of the Lagrange multiplier and of the potential as follows,

$$\lambda(\phi) = 4c\phi - \frac{10cd\phi}{M^2} - \frac{120c^2 t_0 \phi}{M^2} + \frac{60c^2\phi^2}{M^2} + \mathcal{A}_1 \,, \tag{333}$$

with the constant $\mathcal{A}_1$ being,

$$\mathcal{A}_1 = -\frac{d}{3} + \frac{d^2}{3M^2} + \frac{12cH_0}{M^2} - 4ct_0 + \frac{10cdt_0}{M^2} + \frac{60c^2 t_0^2}{M^2} \,. \tag{334}$$

In the same way we can easily find the mimetic potential, but we omit this for brevity. The slow-roll indices ϵ_3 and ϵ_4 can be simplified during the slow-roll era, in the following way,

$$\epsilon_3 \simeq -2\epsilon_1 \,, \quad \epsilon_4 \simeq \frac{\dot{\lambda}}{2H\lambda} \,, \tag{335}$$

and the slow-roll index ϵ_1 remains the same. By combining Eqs. (333) and (328), we can find the observational indices, which are,

$$\epsilon_1 = \frac{\frac{d}{6} - 2c(t - t_0)}{\left(H_0 - \frac{1}{6}d(t - t_0) + c(t - t_0)^2\right)^2} \,, \quad \epsilon_3 \simeq -\frac{2\left(\frac{d}{6} - 2c(t - t_0)\right)}{\left(H_0 - \frac{1}{6}d(t - t_0) + c(t - t_0)^2\right)^2}$$

$$\epsilon_4 \simeq \frac{18c\left(-5d+2\left(M^2+30c(t-t_0)\right)\right)}{\left(d^2-d\left(M^2+30c(t-t_0)\right)+12c\left(3H_0+\left(M^2+15c(t-t_0)\right)(t-t_0)\right)\right)\left(6H_0-(t-t_0)(d+6c(-t+t_0))\right)}\,. \quad (336)$$

Accordingly, by using the e-foldings number N, we can express the spectral index n_s as a function of N, as follows,

$$n_s \simeq 1-\frac{48(d-12cz)}{(-6H_0+z(d-6cz))^2}-\frac{36c\left(-5d+2M^2+60cz\right)}{(6H_0+z(-d+6cz))\left(d^2-d\left(M^2+30cz\right)+12c\left(3H_0+z\left(M^2+15cz\right)\right)\right)}\,, \quad (337)$$

and the N-dependence of n_s is given in terms of the parameter z, the explicit form of which is,

$$\begin{aligned} z =& \frac{M^2}{12c}-\frac{144cH_0-M^4}{6\; 2^{2/3}c\left(-432cH_0M^2+2M^6+5184c^2N+\sqrt{4\left(144cH_0-M^4\right)^3+\left(-432cH_0M^2+2M^6+5184c^2N\right)^2}\right)^{1/3}} \\ &+\frac{\left(-432cH_0M^2+2M^6+5184c^2N+\sqrt{4\left(144cH_0-M^4\right)^3+\left(-432cH_0M^2+2M^6+5184c^2N\right)^2}\right)^{1/3}}{12\; 2^{1/3}c}\,. \end{aligned} \quad (338)$$

In the same way, the scalar-to-tensor ratio is equal to,

$$r \simeq -\frac{16\left(\frac{d}{6}-2cz\right)}{\left(H_0-\frac{dz}{6}+cz^2\right)^2}\,. \quad (339)$$

By assuming that the free parameters have the following values,

$$N=50\,,\quad c=0.0000009\,,\quad M=0.085\,,\quad H_0=0.26\,,\quad d=0.002\,, \quad (340)$$

the observational indices n_s and r, take the following values,

$$n_s \simeq 0.967429\,,\quad r\simeq 0.0134796\,, \quad (341)$$

which are in concordance with the Planck data (324) and also with the BICEP2/Keck Array data (325). It is conceivable that the results we just presented, strongly depend on the choice of the free parameters, and specifically on the parameter H_0, and the compatibility with the observational data is achieved at the expense of having fine-tuning. For more details on this model, see [142] for a detailed analysis.

As we just showed, the mimetic R^2 model is compatible with the observational data, or at least can be by appropriately choosing the free parameters of the theory. As we now show, the mimetic version of the power law model of Eq. (327) can also be compatible with the observations, although the ordinary power-law model is not compatible with the observational data. Particularly, in the ordinary power-law case, the slow-roll indices of Eq. (319) are equal to,

$$\epsilon_1=\frac{2-n}{(n-1)(2n-1)}\,,\quad \epsilon_2=0\,,\quad \epsilon_3=\frac{(1-n)(2-n)}{(-1+n)(-1+2n)}\,,\quad \epsilon_4=\frac{n-2}{n-1}\,, \quad (342)$$

and the resulting spectral index n_s is,

$$n_s\simeq\frac{-7+5n}{1-3n+2n^2}\,, \quad (343)$$

and in addition, scalar-to-tensor ratio r is,

$$r\simeq\frac{16(-2+n)^2}{(-1+n)(-1+2n)}\,. \quad (344)$$

In the ordinary power-law $F(R)$ gravity model, the only free parameter is n, so by choosing $n=1.8105$, the spectral index and the scalar-to-tensor ratio become,

$$n_s\simeq 0.966191\,,\quad r\simeq 0.27047\,, \quad (345)$$

so only the spectral index can be compatible with observations. As we show, the mimetic power-law $F(R)$ model can produce results compatible with observations. We first consider the quasi-de Sitter evolution of the form,

$$H(t)=H_0-\frac{M^2}{6}(t-t_0)\,, \quad (346)$$

with $H_0 > M^2$. Since $t \ll 1$, then the slow-roll conditions of Eq. (317) hold true for the evolution of Eq. (346). The gravitational equations during the slow-roll era in this case become,

$$\alpha n = 2\alpha(n-1) - \alpha n(n-1)24\dot{H} + \frac{V(t) - 2\lambda(t)}{6(12H^2)^{n-1}H^2}\,, \tag{347}$$

and therefore the Lagrange multiplier is,

$$\lambda(\phi) = -2^{-3+2n}3^{-3+n}M^2 n\left(2 - n + 4M^2(-1+n)n\right)\alpha\left(H_0 + \frac{1}{6}M^2(t_0 - \phi)\right)^{2(-1+n)}\,, \tag{348}$$

while the mimetic potential becomes,

$$V(\phi) = 3^{-3+n}4^{-1+n}\left(2 - n + 4M^2(-1+n)n\right)\alpha\left(9 - \frac{36M^2 n}{\left(6H_0 + M^2(t_0-\phi)\right)^2}\right)\left(H_0 + \frac{1}{6}M^2(t_0-\phi)\right)^{2n}\,. \tag{349}$$

We can easily find the slow-roll indices in the slow-roll approximation, which are,

$$\epsilon_3 \simeq -(n-1)\epsilon_1\,, \quad \epsilon_4 \simeq \frac{\dot{\lambda}}{2H\lambda}\,, \tag{350}$$

so by using Eqs. (346), (348), and (349), we obtain,

$$\epsilon_1 = \frac{M^2}{6\left(H_0 - \frac{1}{6}M^2(t-t_0)\right)^2}\,, \quad \epsilon_3 = \frac{M^2(1-n)}{6\left(H_0 - \frac{1}{6}M^2(t-t_0)\right)^2}\,, \quad \epsilon_4 = -\frac{6M^2(-1+n)}{\left(6H_0 + M^2(-t+t_0)\right)^2}\,. \tag{351}$$

As in the R^2 case, we shall express the observational indices as functions of the e-foldings number, so we have,

$$n_s = \frac{3H_0^2 - M^2(2+N)}{3H_0^2 - M^2 N}\,, \quad r = \frac{8M^2(-2+n)}{-3H_0^2 + M^2 N}\,. \tag{352}$$

By assuming that the free parameters and N have the following values,

$$N = 50\,, \quad H_0 = 12.04566\,, \quad M = 2\,, \quad n = 1.8\,, \tag{353}$$

the spectral index n_s and the scalar-to-tensor ration become,

$$n_s \simeq 0.966\,, \quad r \simeq 0.0272\,, \tag{354}$$

and therefore compatibility with the observational data is achieved. It can be shown that compatibility with the observations is always achieved if M and H_0 satisfy $H_0 = 6.0228 \times M$, and if n is appropriately chosen in each case. As a second example, consider the following cosmological evolution,

$$H(t) = H_0 - d/6(t-t_0) + c(t-t_0)^2\,, \tag{355}$$

and the Lagrange multiplier that generates (355), for the power-law gravity (327), is at leading order,

$$\lambda(\phi) \simeq -2^{1+2n}3^{-1+n}cH_0^{-1+2n}(-1+n)n\alpha\,, \tag{356}$$

Accordingly, the leading order mimetic potential is,

$$V(\phi) = 12^{-1+n}H_0^{-1+2n}\alpha\left(-16c(-1+n)n + H_0\left(2 + 4n^2(d + 12c(t_0-\phi)) - n(1 + 4d + 48c(t_0-\phi))\right)\right)\,. \tag{357}$$

Then, the spectral index n_s and the scalar-to-tensor ratio can be easily found, but we do not quote their explicit form here for brevity. It can be shown [142] that if we choose the parameters as follows,

$$N = 50\,, \quad c = 0.014\,, \quad d = -0.1\,, \quad n = 11\,, \quad H_0 = 11, \tag{358}$$

the observational indices take the following values,

$$n_s \simeq 0.966121\,, \quad r \simeq 0.0176679\,, \tag{359}$$

which of course are compatible with the current observational data. As we also mentioned in the R^2 model, compatibility is achieved at the expense of having to fine-tune the parameters, so we refer the reader to [142] for more details on the allowed ranges of the parameters.

Also, in the context of mimetic $F(R)$ gravity and for similar theories, it is very difficult to address the graceful exit issue by using the methods we already described in this chapter. So at the end of this chapter, we devote a whole section for this issue.

4. Reconstruction of $F(R)$ gravity from Einstein Frame Scalar Potential

In chapter II we demonstrated how to obtain the Einstein frame canonical scalar theory, from the Jordan frame $F(R)$ gravity, so in this section we shall demonstrate how it is possible to obtain the $F(R)$ gravity which corresponds to an inflationary potential. As we showed in chapter II, the Einstein frame potential of the canonical scalar field is,

$$V(\varphi)=\frac{1}{2}\left(\frac{A}{F'(A)}-\frac{F(A)}{F'(A)^2}\right)=\frac{1}{2}\left(\mathrm{e}^{-\sqrt{2/3}\varphi}R\left(\mathrm{e}^{\sqrt{2/3}\varphi}\right)-\mathrm{e}^{-2\sqrt{2/3}\varphi}F\left[R\left(\mathrm{e}^{\sqrt{2/3}\varphi}\right)\right]\right).\tag{360}$$

The Einstein and Jordan frames are connected via the following canonical transformation,

$$\varphi=\sqrt{\frac{3}{2}}\ln(F'(A)),\tag{361}$$

so by solving Eq. (361) with respect to A, we may obtain the scalar curvature R as a function of the scalar field, since $A=R$. Then, finding the $F(R)$ gravity from a given canonical scalar potential is a straightforward task, since we have only to combine Eqs. (360) and (361). Particularly, taking the first derivative of Eq. (360), with respect to R, that is with respect to the scalar curvature, and also by using $\frac{d\varphi}{dR}=\sqrt{\frac{3}{2}}\frac{F''(R)}{F'(R)}$, we get the following differential equation,

$$RF_R=2\sqrt{\frac{3}{2}}\frac{d}{d\varphi}\left(\frac{V(\varphi)}{\mathrm{e}^{-2\left(\sqrt{2/3}\right)\varphi}}\right),\tag{362}$$

where $F_R=\frac{dF(R)}{dR}$. Given the above differential equation, in conjunction with the canonical transformation Eq. (361), will give us the resulting $F(R)$ gravity, given the potential $V(\varphi)$. Let us demonstrate how the method works by using two simple examples. Firstly let us assume that the potential in the Einstein frame is the Starobinsky potential of Eq. (246), so by substituting the Starobinsky potential in Eq. (362), and also due to the fact that $F_R=\mathrm{e}^{\sqrt{\frac{2}{3}}\varphi}$, we finally get the following algebraic equation,

$$F_RR-\left(4F_R^2\mu^2-4F_R\mu^2\right)=0.\tag{363}$$

By solving the algebraic equation (363) with respect to F_R , we easily obtain,

$$F_R=\frac{4\mu^2+R}{4\mu^2},\tag{364}$$

and in turn, by integrating with respect to the scalar curvature R, we get the corresponding $F(R)$ gravity, which is $F(R)=R+\frac{R^2}{8\mu^2}$.

As another example consider the following limiting attractor potential studied in Ref. [526],

$$V(\varphi)\simeq\mu\alpha\left(1-2n\mathrm{e}^{-\frac{1}{\alpha}\sqrt{\frac{2}{3}}\varphi}\right).\tag{365}$$

The Starobinsky model tends asymptotically, that is for large field values, to the potential (365) and also the E-attractor models and T-attractor models [527–530], also tend to the potential (365). Moreover, a special class of attractors, the inverse symmetric attractors [526], also tend to the potential (365). The differential equation (362), for the potential (365), becomes,

$$F_R=\frac{R}{4\alpha\mu}-F_R^{1-\frac{1}{\alpha}}(\frac{n}{\alpha}+2n).\tag{366}$$

The algebraic equation (366) has a particularly simple solution during the slow-roll era, since $F_R\gg 1$ during this era. Therefore, for $\alpha>1$, the second term in the right hand side of Eq. (366), satisfies the inequality $1-\frac{1}{\alpha}<1$, and therefore the term $\sim F_R^{1-\frac{1}{\alpha}}$, becomes subdominant at leading order. Therefore, during the slow-roll era we have $F_R\simeq\frac{R}{4\alpha\mu}$, and in effect, upon substitution in Eq. (366), we finally obtain,

$$F(R)\simeq\frac{R^2}{8\alpha\mu}-\frac{\frac{n}{\alpha}+2n}{(2-\frac{1}{\alpha})(8\alpha\mu)}R^{2-\frac{1}{\alpha}}+\Lambda,\tag{367}$$

where the parameter Λ is an arbitrary integration constant.

An extensive analysis of various phenomenologically interesting canonical scalar field potentials was developed in Ref. [535]. In Table I we have gathered the most interesting outcomes of the research carried out in Ref. [535]. As it can be seen in Table I, we present the canonical scalar field potential in the Einstein frame and the corresponding Jordan frame $F(R)$ gravity. The parameters c_0, c_1 and n in Table I, are positive numbers.

TABLE I: Scalar Potential and the Corresponding $F(R)$ Gravity

Scalar Potential	$F(R)$ Gravity	
$V(\varphi)=c_0+c_1\mathrm{e}^{-\sqrt{\frac{2}{3}}\kappa\varphi}+c_2\mathrm{e}^{-2\sqrt{\frac{2}{3}}\kappa\varphi}$	$F(R)=-\frac{c_1}{2c_0}R+\frac{R^2}{4c_0}+\frac{c_1^2}{4c_0}-c_2$	
$V(\varphi)=c_0\mathrm{e}^{-n\sqrt{\frac{2}{3}}\kappa\varphi}$	$F(R)\simeq c_0\left(\frac{n+1}{n+2}\right)\left(\frac{1}{4(n+2)}\right)^{\frac{1}{n+1}}\left(\frac{R}{c_0}\right)^{\frac{n+2}{n+1}}$	$R\gg c_1$
$V(\varphi)=\frac{c_0}{\kappa^2}-\frac{c_1}{\kappa^2}\mathrm{e}^{-\sqrt{\frac{2}{3}}\frac{\kappa}{2}\varphi}$	$F(R)\simeq\frac{R}{2}+\frac{R^2}{6c_1}+\frac{\sqrt{3}}{36}\left(4R/c_1+3\right)^{3/2}+\frac{c_1}{4}$	$R\gg c_1$
$V(\varphi)=\frac{c_1(2-n)}{2\kappa^2}-\frac{c_1}{\kappa^2}\mathrm{e}^{-n\sqrt{\frac{2}{3}}\kappa\varphi}$	$F(R)\simeq\frac{R^2}{2c_1(2-n)}+\frac{1}{2-n}\left(\frac{1}{2c_1(2-n)}\right)^{1-n}R^{2-n}$	$R\gg c_1$

C. Inflation from Gauss-Bonnet Theories of Gravity

1. *Explicit Calculation of the Power Spectrum*

In the previous section, we calculated the spectral index of the primordial curvature perturbations by finding the explicit form of the slow-roll indices. However, there are more direct ways towards the calculation of the spectral index, for example by calculating directly the power spectrum of curvature perturbations. In this section we shall present the general calculation for a vacuum $f(\mathcal{G})$ theory of gravity [327, 548, 549], with $\mathcal{G}$ being the Gauss-Bonnet invariant.

We shall realize a quite popular inflationary scenario, namely that of intermediate inflation [550–557] in which case the scale factor is,

$$a(t)=\mathrm{e}^{f_0t^{\alpha+1}}\,, \tag{368}$$

and the corresponding Hubble rate is,

$$H(t)=f_0(\alpha+1)t^{\alpha}\,, \tag{369}$$

with $-1<\alpha<0$ and also $f_0>0$. Also for notational simplicity we set $\beta=f_0(\alpha+1)$. Note that we use a somewhat strange notation for the intermediate inflation scale factor and there is a specific reason for this (later on we shall change this "heavy" notation). The reason is that the intermediate inflation scenario is a type of singular inflation, and specifically a Type III singular inflation, as it can be seen by looking at the scale factor (369) and the singularity classification after Eq. (454). The intermediate inflation scenario is quite popular, and in Refs. [553, 554] the scalar-tensor description of this scenario was confronted with observational data. Here we shall realize the intermediate inflation with $f(\mathcal{G})$ gravity. The background metric will be assumed to be the flat FRW metric. Before we calculate the power spectrum we need to find which $f(\mathcal{G})$ gravity can realize the intermediate inflation Hubble rate (369), emphasizing on early cosmic times. To this end we shall use some very well known $f(\mathcal{G})$ gravity reconstruction schemes [327, 548, 549]. Also a detailed presentation of the issues that follow can be found in [191]. The starting point of our analysis is the Jordan frame vacuum $f(\mathcal{G})$ gravity, with the action being,

$$\mathcal{S}=\frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\left(R+f(\mathcal{G})\right)\,, \tag{370}$$

with $\kappa^2=1/M_{\mathrm{pl}}^2$, and $M_{\mathrm{pl}}=1.22\times10^{19}$GeV, and g is the trace of the background metric $g_{\mu\nu}$. If we vary the action with respect to the metric tensor $g_{\mu\nu}$, we easily obtain the following gravitational equations of motion,

$$\begin{aligned}&R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}f(\mathcal{G})-\left(-2RR_{\mu\nu}+4R_{\mu\rho}R^{\rho}_{\nu}-2R_{\mu}^{\rho\sigma\tau}R_{\nu\rho\sigma\tau}+4g^{\alpha\rho}g^{\beta\sigma}R_{\mu\alpha\nu\beta}R_{\rho\sigma}\right)F'(\mathcal{G})\\&-2\left(\nabla_{\mu}\nabla_{\nu}F'(\mathcal{G})\right)R+2g_{\mu\nu}\left(\Box F'(\mathcal{G})\right)R-4\left(\Box F'(\mathcal{G})\right)R_{\mu\nu}+4\left(\nabla_{\mu}\nabla_{\nu}F'(\mathcal{G})\right)R^{\rho}_{\nu}\\&+4\left(\nabla_{\rho}\nabla_{\nu}F'(\mathcal{G})\right)R^{\rho}_{\mu}-4g_{\mu\nu}\left(\nabla_{\rho}\nabla_{\sigma}F'(\mathcal{G})\right)R^{\rho\sigma}+4\left(\nabla_{\rho}\nabla_{\sigma}F'(\mathcal{G})\right)g^{\alpha\rho}g^{\beta\sigma}R_{\mu\alpha\nu\beta}=0\,.\end{aligned} \tag{371}$$

Recall from the previous sections that for the flat FRW metric, the Gauss-Bonnet invariant can be expressed as a function of the Hubble rate as follows,

$$\mathcal{G}=24H^2\left(\dot{H}+H^2\right)\,, \tag{372}$$

and also for the flat FRW metric, the equations of motion (371), can be written as follows,

$$6H^2+f(\mathcal{G})-\mathcal{G}F'(\mathcal{G})+24H^3\dot{\mathcal{G}}F''(\mathcal{G})=0$$

$$
\begin{aligned}
&4\dot{H}+6H^2+f(\mathcal{G})-\mathcal{G}F'(\mathcal{G})+16H\dot{\mathcal{G}}\left(\dot{H}+H^2\right)F''(\mathcal{G})\\
&+8H^2\ddot{\mathcal{G}}F''(\mathcal{G})+8H^2\dot{\mathcal{G}}^2F'''(\mathcal{G})=0\,.
\end{aligned}
\tag{373}
$$

A vital element of the reconstruction method we shall use, is the introduction of an auxiliary scalar field ϕ, which can be identified eventually with the cosmic time, as it was shown in [327, 548, 549]. Then, by using two proper functions of the cosmic time, namely $P(t)$ and $Q(t)$, the Jordan frame $f(\mathcal{G})$ gravity action (370), can be rewritten as follows,

$$
\mathcal{S}=\frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\left(R+P(t)\mathcal{G}+Q(t)\right)\,,
\tag{374}
$$

so by varying the above action with respect to the cosmic time t, we get the following differential equation,

$$
\frac{dP(t)}{dt}\mathcal{G}+\frac{dQ(t)}{dt}=0\,.
\tag{375}
$$

Then, one may solve Eq. (375) with respect to $t=t(\mathcal{G})$, and eventually by substituting the result in the following equation,

$$
f(\mathcal{G})=P(t)\mathcal{G}+Q(t)\,,
\tag{376}
$$

one may obtain the resulting $f(\mathcal{G})$ gravity. Hence, the functions $P(t)$ and $Q(t)$ are very important for the calculation of the resulting $f(\mathcal{G})$ gravity which may realize a specific cosmic evolution, so now we demonstrate how to find these, given a specific cosmological evolution. Combining Eqs. (376) and (373), we get the following differential equation,

$$
Q(t)=-6H^2(t)-24H^3(t)\frac{dP}{dt}\,,
\tag{377}
$$

and in addition, by combining (376) and (377), we get,

$$
2H^2(t)\frac{d^2P}{dt^2}+2H(t)\left(2\dot{H}(t)-H^2(t)\right)\frac{dP}{dt}+\dot{H}(t)=0\,.
\tag{378}
$$

By solving the differential equation (378), we can obtain the analytic form of the function $P(t)$, and eventually the function $Q(t)$. So by substituting these in Eq. (375), we obtain the resulting form of the function $t=t(\mathcal{G})$, and having this at hand, enables us to obtain the $f(\mathcal{G})$ gravity, by simply substituting $t=t(\mathcal{G})$ in Eq. (376). Let us apply this method for the intermediate inflation cosmology (369), at early times, in which case the differential equation (378) becomes,

$$
\frac{2\beta}{\alpha}t^{1+\alpha}\frac{d^2P}{dt^2}+4t^\alpha\beta+1=0\,,
\tag{379}
$$

and by analytically solving this we obtain,

$$
P(t)=-\frac{t^{1-2\alpha}\left(t^\alpha-2t^\alpha\alpha-2\beta C_1+2\alpha\beta C_1\right)}{2(-1+\alpha)(-1+2\alpha)\beta}+C_2\,.
\tag{380}
$$

By combining Eqs. (380) and (377), we get the function $Q(t)$, which is,

$$
\begin{aligned}
Q(t)=&-2t^{-1+2\alpha}\alpha\beta^2\\
&-72t^{-1+3\alpha}\alpha\beta^3\left(-\frac{t^{1-2\alpha}\left(t^{-1+\alpha}\alpha-2t^{-1+\alpha}\alpha^2\right)}{2(-1+\alpha)(-1+2\alpha)\beta}-\frac{t^{-2\alpha}(1-2\alpha)\left(t^\alpha-2t^\alpha\alpha-2C_1\beta+2C_1\alpha\beta\right)}{2(-1+\alpha)(-1+2\alpha)\beta}\right)\\
&+24t^{3\alpha}\beta^3\left(-\frac{t^{-2\alpha}(1-2\alpha)\left(t^{-1+\alpha}\alpha-2t^{-1+\alpha}\alpha^2\right)}{(-1+\alpha)(-1+2\alpha)\beta}-\frac{t^{1-2\alpha}\left(t^{-2+\alpha}(-1+\alpha)\alpha-2t^{-2+\alpha}(-1+\alpha)\alpha^2\right)}{2(-1+\alpha)(-1+2\alpha)\beta}\right.\\
&\left.+\frac{t^{-1-2\alpha}(1-2\alpha)\alpha\left(t^\alpha-2t^\alpha\alpha-2C_1\beta+2C_1\alpha\beta\right)}{(-1+\alpha)(-1+2\alpha)\beta}\right)\,,
\end{aligned}
\tag{381}
$$

Finally, by using the resulting $Q(t)$ and $P(t)$, we can obtain the final form of Eq. (375), which is,

$$
\frac{t^{-1-2\alpha}\left(4t^{3\alpha}\alpha\beta^3\left(11t^\alpha-12C_1\beta\right)-\mathcal{G}x\left(t^\alpha-2C_1\beta\right)\right)}{2\beta}=0\,.
\tag{382}
$$

We shall be interested for early times, so we solve the algebraic equation above for $t \to 0$, in which case, the algebraic equation (382) becomes,

$$C_1 \mathcal{G} t^{-2\alpha} - 24 C_1 t^{-1+\alpha} \alpha \beta^3 = 0\,, \tag{383}$$

which results to,

$$t = \frac{\mathcal{G}^{\frac{1}{(3\alpha-1)}}}{(24\alpha\beta^3)^{\frac{1}{(3\alpha-1)}}}\,, \tag{384}$$

Substituting (384) in $P(t)$ and $Q(t)$ and also by making use of Eq. (376), the resulting $f(\mathcal{G})$ gravity which realizes the intermediate inflation scenario at early times reads,

$$f(\mathcal{G}) = C_2 \mathcal{G} + A G^{\frac{2\alpha}{-1+3\alpha}} + B \mathcal{G}^{\frac{\alpha}{-1+3\alpha}}\,, \tag{385}$$

where the parameters A and B are equal to,

$$\begin{aligned} A =& 11\ 24^{\frac{2\alpha}{1-3\alpha}} \beta^2 \left(\alpha\beta^3\right)^{\frac{2\alpha}{1-3\alpha}} \\ B =& \frac{\left(24^{\frac{1-2\alpha}{1-3\alpha}} C_1 - 24^{\frac{1-2\alpha}{1-3\alpha}} C_1 \alpha - \frac{2^{-1+\frac{3(1-2\alpha)}{1-3\alpha}} 3^{\frac{1-2\alpha}{1-3\alpha}}}{\beta} + \frac{24^{\frac{1-2\alpha}{1-3\alpha}} \alpha}{\beta}\right) \left(\alpha\beta^3\right)^{\frac{1-2\alpha}{1-3\alpha}}}{1 - 3\alpha + 2\alpha^2} \\ & - 24^{1+\frac{\alpha}{1-3\alpha}} C_1 G^{\frac{\alpha}{-1+3\alpha}} \beta^3 \left(\alpha\beta^3\right)^{\frac{\alpha}{1-3\alpha}}\,. \end{aligned} \tag{386}$$

We can further simplify the resulting $f(\mathcal{G})$ gravity, by using the fact that we are interested in early times. Particularly, the Gauss-Bonnet invariant is equal to,

$$\mathcal{G} = 24 t^{-1+3\alpha} \alpha \beta^3 + 24 t^{4\alpha} \beta^4\,, \tag{387}$$

so for $t \to 0$, we also have $\mathcal{G} \to 0$. Hence we can simplify the $f(\mathcal{G})$ gravity by keeping only the leading order terms in the small-$\mathcal{G}$ limit, so we finally have,

$$f(\mathcal{G}) \simeq C_2 \mathcal{G} + B \mathcal{G}^{\frac{\alpha}{-1+3\alpha}}\,. \tag{388}$$

Having the resulting $f(\mathcal{G})$ gravity at hand, we can easily calculate the power spectrum of the primordial curvature perturbations, which will determine how the perturbations evolve.

To proceed to the calculation of the power spectrum, consider scalar linear perturbations of the flat FRW metric of the following form,

$$ds^2 = -(1+\psi)dt^2 - 2a(t)\partial_i \beta dt dx^i + a(t)^2 \left(\delta_{ij} + 2\phi\delta_{ij} + 2\partial_i\partial_j\gamma\right) dx^i dx^j\,, \tag{389}$$

where ψ, ϕ, γ and β are smooth scalar perturbations. The primordial perturbations are analyzed by studying a gauge invariant quantity, so we shall use the comoving curvature perturbation, which is equal to,

$$\Phi = \phi - \frac{H\delta\xi}{\dot{\xi}}\,, \tag{390}$$

where $\xi = \frac{dF}{dG}$. In the case of $f(\mathcal{G})$ gravity, there are no k^4 scalar propagating modes, where k is the wavelength, so in this case there is no superluminal propagation. Hence the propagation of the scalar perturbations contains only k^2 terms, and the evolution of perturbations is governed by the following equation,

$$\frac{1}{a(t)^3 Q(t)} \frac{d}{dt}\left(a(t)^3 Q(t) \dot{\Phi}\right) + B_1(t) \frac{k^2}{a(t)^2} \Phi = 0\,, \tag{391}$$

and it can be seen that the k^2 terms dominate the evolution. The speed of the propagating modes is determined by $B_1(t)$, which for an $f(\mathcal{G})$ theory is,

$$B_1(t) = 1 + \frac{2\dot{H}}{H^2}\,. \tag{392}$$

In addition, the term $Q(t)$ which appears in Eq. (391), for a general $f(\mathcal{G})$ gravity is,

$$Q(t)=\frac{6\left(\frac{d^2F}{d\mathcal{G}^2}\right)^2\dot{\mathcal{G}}^2\left(1+4F''(\mathcal{G})\dot{\mathcal{G}}H\right)}{\left(1+6HF''(\mathcal{G})\dot{\mathcal{G}}\right)^2}\,, \tag{393}$$

where we used $\dot{\xi}=\frac{dF^2}{d\mathcal{G}^2}\dot{\mathcal{G}}$. Note that the "prime" in Eq. (393) denotes differentiation with respect to the Gauss-Bonnet scalar, while the "dot" denotes differentiation with respect to the cosmic time. We shall try to find an approximate solution of the differential equation (393) corresponding to early times. The differential equation (391) can be recast as follows,

$$a(t)^3Q(t)\ddot{\Phi}+\left(3a(t)^2\dot{a}Q(t)+a(t)^3\dot{Q}(t)\right)\dot{\Phi}+B_1(t)Q(t)a(t)k^2\Phi=0\,. \tag{394}$$

Then after some calculations, the leading order differential equation which governs the evolution of the perturbations at early times reads,

$$t^{1+\alpha}\Omega_4\ddot{\Phi}-t^{\alpha}\Omega_2\dot{\Phi}+\Omega_1\Phi=0\,, \tag{395}$$

where the explicit form of the parameters Ω_i $(i=1,2,4)$ is,

$$\begin{aligned}
\Omega_1=&\frac{2^{-4+\frac{6\alpha}{-1+3\alpha}}3^{-1+\frac{2\alpha}{-1+3\alpha}}B^2k^2\,(1-2\alpha)^2\alpha\left(\alpha\beta^3\right)^{\frac{2\alpha}{-1+3\alpha}}}{(1-3\alpha)^4\beta^7}\,,\\
\Omega_2=&-\frac{2^{-3+\frac{6\alpha}{-1+3\alpha}}3^{-1+\frac{2\alpha}{-1+3\alpha}}B^2\,(1-2\alpha)^2\alpha\left(\alpha\beta^3\right)^{\frac{2\alpha}{-1+3\alpha}}}{(1-3\alpha)^4\beta^6}\,,\\
\Omega_3=&\frac{2^{-5+\frac{6\alpha}{-1+3\alpha}}3^{\frac{2\alpha}{-1+3\alpha}}B^2\,(1-2\alpha)^2\alpha\left(\alpha\beta^3\right)^{\frac{2\alpha}{-1+3\alpha}}}{(1-3\alpha)^4(1+\alpha)\beta^5}\,,\\
\Omega_4=&\frac{2^{-5+\frac{6\alpha}{-1+3\alpha}}3^{-1+\frac{2\alpha}{-1+3\alpha}}B^2(1-2\alpha)^2\left(\alpha\beta^3\right)^{\frac{2\alpha}{-1+3\alpha}}}{(1-3\alpha)^4\beta^6}\,.
\end{aligned} \tag{396}$$

We need to note that for deriving the resulting differential equation, we omitted a term $\sim(\Omega_3\dot{\Phi}t)^{2\alpha}$, since the term $t^{\alpha}\Omega_2\dot{\Phi}$ dominates, and the explicit form of the parameter Ω_3 can be found in Eq. (396). The differential equation (394) can be solved and the solution is,

$$\Phi(t)=\Delta_1 t^{\frac{\mu}{2(-1+\alpha)}}J_{\mu}(\zeta t^{\frac{1-\alpha}{2}})+\Delta_2 t^{\frac{\mu}{2(-1+\alpha)}}J_{-\mu}(\zeta t^{\frac{1-\alpha}{2}})\,, \tag{397}$$

where the parameters μ and ζ, are equal to,

$$\mu=\frac{\Omega_2+\Omega_4}{(-1+\alpha)\Omega_4}\,,\quad \zeta=\frac{2\sqrt{\Omega_1}}{\left(-1+\frac{1}{\alpha}\right)\alpha\sqrt{\Omega_4}}\,, \tag{398}$$

and $J_{\mu}(y)$ is the Bessel function of the first kind. Also the explicit form of the parameters Δ_i, $i=1,2$ appears below,

$$\begin{aligned}
\Delta_1=&\left(-1+\frac{1}{\alpha}\right)^{\mu}\alpha^{\mu}\Omega_1^{-\frac{\mu}{2}}\Omega_4^{\frac{\mu}{2}}\,C_3\Gamma\left[\frac{\alpha}{-1+\alpha}+\frac{\Omega_2}{(-1+\alpha)\Omega_4}\right],\\
\Delta_2=&\left(-1+\frac{1}{\alpha}\right)^{3\mu}\alpha^{\mu}\Omega_1^{-\frac{\mu}{2}}\Omega_4^{\frac{\mu}{2}}C_4\Gamma\left[\frac{\Omega_2}{\Omega_4-\alpha\Omega_4}+\frac{2\Omega_4}{\Omega_4-\alpha\Omega_4}-\frac{\alpha\Omega_4}{\Omega_4-\alpha\Omega_4}\right].
\end{aligned} \tag{399}$$

We can further simplify the solution by using the small argument limit of the Bessel function,

$$J_{\mu}(y)\simeq\frac{y^{\mu}2^{-\mu}}{\Gamma[1+\mu]}\,, \tag{400}$$

so the solution (397) is simplified as follows,

$$\Phi(t)\simeq\Delta_2\frac{2^{-\mu}\zeta^{\mu}}{\Gamma[1+\mu]}x^{\frac{\left(2-2\alpha+\alpha^2\right)\mu}{2(-1+\alpha)}}\,. \tag{401}$$

Hence, from Eq. (401) it is obvious that the evolution of the primordial curvature scalar perturbations depends linearly on the cosmic time t. Now we investigate whether the power spectrum is nearly scale invariant and also whether this can be in concordance with the observational data. The gauge invariant function Φ of Eq. (390), satisfies the following equation [558],

$$\mathcal{S}_p = \int dx^4 a(t)^3 Q_s \left(\frac{1}{2}\dot{\Phi} - \frac{1}{2}\frac{c_s^2}{a(t)^2}(\nabla\Phi)^2 \right) , \tag{402}$$

with Q_s being this time, $Q_s = \frac{4}{\kappa^2}Q(t)$ and the function $Q(t)$ was defined in Eq. (393). The power spectrum of the primordial curvature perturbations for the gauge invariant variable Φ, can be calculated by using standard approaches in cosmological perturbation theory [50–54, 56, 58–70], and it is equal to,

$$\mathcal{P}_R = \frac{4\pi k^3}{(2\pi)^3}|\Phi|^2_{k=aH} . \tag{403}$$

If we look at the analytic forms of the parameters Ω_i, Δ_2 and ζ above, one may easily conclude that the power spectrum is not scale invariant, and we demonstrate that now in some detail. The parameters Ω_1, Δ_2, ζ contain the wavenumber k implicitly via the parameter Ω_1, and in detail we have,

$$\Omega_1 \sim k^2, \quad \zeta \sim \sqrt{\Omega_1}, \quad \Delta_2 \sim \Omega_1^{-\frac{\mu}{2}} . \tag{404}$$

Due to the fact that the power spectrum depends on $\Delta\zeta^\mu$, the from Eq. (404), we get,

$$\mathcal{P}_R \sim k^3 \left| C_4 (t-t_s)^{\frac{\left(2-2\alpha+\alpha^2\right)\mu}{2(-1+\alpha)}} \right|^2_{k=aH} . \tag{405}$$

However it is not obvious if the above spectrum is scale invariant, since the constant of integration contained in the parameter Δ_2 in Eq. (399), namely C_4, depends on k, and its exact form can be found if the initial conditions on the gauge invariant quantity $\Phi(t)$ are given. Moreover, a hidden k-dependence exists in the term $t^{\frac{\left(2-2\alpha+\alpha^2\right)\mu}{2(-1+\alpha)}}$, since the power spectrum is calculated at the horizon crossing time instance, where $k = aH$. So let us calculate these in detail. Since we are working for $t \to 0$ cosmic times, the conformal time τ, which is defined as $d\tau = a^{-1}(t)dt$, satisfies $\tau \sim t$, since for $t \to 0$, the scale factor in Eq. (368), is approximately equal to $a \sim 1$. In addition, for $t \to 0$, we approximately have $k \simeq H$ at the horizon crossing time instance, and therefore the following relation holds true,

$$\beta t^\alpha \simeq k . \tag{406}$$

By solving the above equation with respect to the cosmic time t, we obtain,

$$t \simeq \left(\frac{k}{\beta}\right)^{\frac{1}{\alpha}} . \tag{407}$$

It is obvious that Eq. (407) determines the k-dependence of the cosmic time t, at the horizon crossing. Having found this, let us now proceed to find C_4 as a function of k. We introduce the canonical variable $u = z_s\Phi$, where, $z_s = Q(t)a(t)$, and owing to the fact that $a(t) \simeq 1$ for $t \to 0$, we finally conclude that $z_s \simeq Q(t)$, and this implies that,

$$u \sim \Phi Q(t) , \tag{408}$$

with $Q(t)$ being defined in Eq. (393). Then we can write the action of Eq. (402), in terms of the variable u, at early times, as follows,

$$\mathcal{S}_u \simeq \int d^3 d\tau \left[\frac{u'}{2} - \frac{1}{2}(\nabla u)^2 + \frac{z_s''}{z_s}u^2 \right] , \tag{409}$$

with the prime this time indicating differentiation with respect to $\tau \sim t$. Assuming a Bunch-Davies vacuum state for the canonical scalar field u [61], before the inflationary era starts, then we have $u \sim \frac{e^{-ik\tau}}{\sqrt{k}}$. Note that the imaginary phase plays no role, since we are interested in calculating $|\Phi(t=t_s)|^2$. Due to Eq. (408), we get,

$$\Phi(t_0) \sim C_4 \sim \frac{1}{\sqrt{k}Q(t)} , \tag{410}$$

where t_0 indicates the start of the inflationary era. In addition, the function $Q(t)$ is equal to,

$$Q(t) \simeq \frac{2^{-5+\frac{6\alpha}{-1+3\alpha}} 3^{-1+\frac{2\alpha}{-1+3\alpha}} B^2 (t-t_s)^{-4\alpha} (1-2\alpha)^2 \left(\alpha\beta^3\right)^{\frac{2\alpha}{-1+3\alpha}}}{(1-3\alpha)^4 \beta^6} \,. \tag{411}$$

Then, by using Eq. (407), we obtain,

$$Q(t(k)) \simeq \mathcal{Z}_1 k^{\frac{-4\alpha}{\alpha}} = \mathcal{Z}_1 k^{-4} \,, \tag{412}$$

with the parameter $\mathcal{Z}_1$ being equal to,

$$\mathcal{Z}_1 = \frac{2^{-5+\frac{6\alpha}{-1+3\alpha}} 3^{-1+\frac{2\alpha}{-1+3\alpha}} B^2 (1-2\alpha)^2 \left(\alpha\beta^3\right)^{\frac{2\alpha}{-1+3\alpha}}}{(1-3\alpha)^4 \beta^6 \beta^{1/\alpha}} \,. \tag{413}$$

By combining Eqs. (412) and (410), we obtain that,

$$C_4 \simeq \frac{1}{\mathcal{Z}_1} \frac{1}{\sqrt{k} k^{-4}} \sim k^{\frac{7}{2}} \,. \tag{414}$$

Finally combining Eqs. (414), (407), and (405), the power spectrum $\mathcal{P}_R$ as a function of k is,

$$\mathcal{P}_R \sim k^{\frac{7}{2}+3+\frac{\left(2-2\alpha+\alpha^2\right)\mu}{2(-1+\alpha)}} \,, \tag{415}$$

and it can be seen that the resulting power spectrum is not scale invariant. We can compute the spectral index of primordial curvature perturbations if we combine Eqs. (398) and (396) and in effect we get, $\mu = 11/(1-\alpha)$. Then, the spectral index of the primordial curvature perturbations is,

$$n_s - 1 \equiv \frac{d \ln \mathcal{P}_\mathcal{R}}{d \ln k} = \frac{7}{2} + 3 + \frac{2-2\alpha+\alpha^2}{2(\alpha-1)} \mu = 1 - \frac{11}{2(\alpha-1)^2} \,. \tag{416}$$

There can be agreement of n_s with the observational data for two values of α, namely for $\alpha = 13.9$ and for $\alpha = -12$. The latter is not allowed by the intermediate inflation scenario, and the same applies for the former, so we can see that the intermediate inflation scenario, when realized by an $f(\mathcal{G})$ gravity, does not result to a power spectrum which scale or nearly scale invariant and it is not compatible with observational data.

2. Evolution of Perturbations After the Horizon Crossing

As another task relevant to the dynamics of the inflationary era, we shall now investigate whether the cosmological perturbations remain constant or not after the horizon crossing. We are mainly presenting this issue in order to have a complete idea of all the topics related to the inflationary era.

In many scenarios, the conservation of the comoving curvature perturbations does not occur, like for example in the matter bounce scenario [61], so this issue should be properly investigated. The focus is on cosmological times for which the wavenumber satisfies $k \ll a(t)H(t)$. Due to the last relation, the differential equation (391) can be simplified since the last term of it can be safely neglected, so it becomes,

$$\frac{1}{a(t)^3 Q(t)} \frac{d}{dt} \left(a(t)^3 Q(t) \dot{\Phi} \right) = 0 \,, \tag{417}$$

and by analytically solving this we obtain,

$$\Phi(t) = \mathcal{C}_1 + \mathcal{C}_2 \int \frac{1}{a(t)^3 Q(t)} dt \,, \tag{418}$$

where $Q(t)$ is defined in Eq. (393). Clearly the integral in Eq. (418), determines the evolution of the comoving curvature perturbations after the horizon crossing. It is therefore crucial to find the function $Q(t)$ and also to determine the $f(\mathcal{G})$ gravity which governs the evolution for $k \ll a(t)H(t)$. Since the condition $t \to 0$ holds true after the horizon crossing and for $t \ll 1$sec, the $f(\mathcal{G})$ gravity which generates the evolution is still the one appearing in Eq. (388), and in effect, the function $Q(t)$ is,

$$Q(t) \simeq \mathcal{Z}_2 t^{-4\alpha} \,. \tag{419}$$

Hence, due to the fact that $a(t) \simeq e^{f_0 t^{\alpha+1}}$, the term $\frac{1}{a(t)^3 Q(t)}$, has the following behavior,

$$\frac{1}{a(t)^3 Q(t)} \sim \frac{t^{4\alpha}}{e^{f_0 t^{\alpha+1}}} \,. \tag{420}$$

Since the exponential term dominates the evolution, the integral term eventually decays, as the cosmic time increases, so in effect we have,

$$\int \frac{1}{a(t)^3 Q(t)} \to 0 \,. \tag{421}$$

Therefore, the comoving curvature perturbation can be approximated as follows,

$$\Phi(t) = \mathcal{C}_1 \,, \tag{422}$$

and therefore the comoving curvature perturbation is conserved after the horizon crossing, since $\Phi(t)$ is constant.

D. Inflation from $F(T) = T + f(T)$ Theories in the Jordan Frame

In this section we shall present in brief the formalism for obtaining the power spectrum of primordial curvature perturbations in the context of $F(T) = T + f(T)$ gravity. For detailed analysis on these issues we refer to Refs. [146, 366–371, 373, 375–390], and we adopt the notation of Ref. [368]. The analysis that follows has been presented in detail in Refs. [404, 405]. We consider only the longitudinal gauge, so only scalar-type metric fluctuations are considered, with the perturbed metric being of the following form,

$$ds^2 = (1+2\Phi)dt^2 - a(t)^2(1-2\Psi)\sum_i dx_i^2 \,, \tag{423}$$

and the scalar fluctuations in the metric are quantified in terms of the functions Φ and Ψ. The perturbation of the torsion scalar in terms of the scalar perturbations Ψ and Φ, at leading order, is equal to,

$$\delta T = 12H(\dot{\Phi} + H\Psi) \,, \tag{424}$$

with H being the Hubble rate. The $F(T) = T + f(T)$ gravity perturbed gravitational equations corresponding to the metric of Eq. (423), are equal to,

$$\begin{aligned}
&(1+F_{,T})\frac{\nabla^2}{a^2}\Psi - 3(1+f_{,T})H\dot{\Psi} - 3(1+f_{,T})H^2\Phi + 36f_{,TT}H^3(\dot{\Psi}+H\Phi) = 4\pi G\delta\rho \,, \\
&(1+f_{,T} - 12H^2 f_{,TT})(\dot{\Psi}+H\Phi) = 4\pi G\delta q \,, \\
&(1+f_{,T})(\Psi - \Phi) = 8\pi G\delta s \,, \\
&(1+f_{,T} - 12H^2 f_{,TT})\ddot{\Psi} + 3H(1+f_{,T} - 12H^2 f_{,TT} - 12\dot{H}f_{,TT} + 48H^2\dot{H}f_{,TTT})\dot{\Psi} \\
&\Big[3H^2(1+f_{,T} - 12H^2 f_{,TT}) + 2\dot{H}(1+f_{,T} - 30H^2 f_{,TT} + 72H^4 f_{,TTT})\Big]\Phi + \frac{1+f_{,T}}{2a^2}\nabla^2(\Psi - \Phi) = 4\pi G\delta p \,,
\end{aligned} \tag{425}$$

where $f_{,T}$, stands for $\partial_T f(T)$, and the rest of the derivatives, namely $f_{,TT}$ and $f_{,TTT}$ are defined correspondingly. Also, the functions $\delta\rho$, δq, δs, δp, are the fluctuations of the total energy density, of the fluid velocity, of the anisotropic stress and of the total pressure respectively. By assuming that the matter fluids present consist of a canonical scalar field with potential $V(\phi)$, we obtain the following relations,

$$\delta\rho = \dot{\phi}(\delta\dot{\phi} - \dot{\phi}\Phi) + V_{,\phi}\delta\phi \,, \quad \delta q = \dot{\phi}\delta\phi \,, \quad \delta s = 0 \,, \quad \delta p = \dot{\phi}(\delta\dot{\phi} - \dot{\phi}\Phi) - V_{,\phi}\delta\phi \,. \tag{426}$$

Due to the above equations, it can be shown [368] that we have $\Psi = \Phi$ and also that the gravitational potential Φ can be completely determined by the scalar fluctuation $\delta\phi$. Hence it is conceivable that the minimally coupled $F(T)$ gravity to a scalar field, has only one degree of freedom.

The evolution of scalar perturbations is governed by the following differential equation,

$$\ddot{\Phi}_k + \alpha\dot{\Phi}_k + \mu^2\Phi_k + c_s^2\frac{k^2}{a^2}\Phi_k = 0 \,, \tag{427}$$

where Φ_k is the scalar Fourier mode of the scalar potential Φ, and the functions α, μ^2 and c_s^2 are the frictional term, the effective mass and the speed of sound parameter for the potential Φ respectively. These functions are equal to,

$$\begin{aligned}
\alpha &= 7H + \frac{2V_{,\phi}}{\dot{\phi}} - \frac{36H\dot{H}(f_{,TT} - 4H^2 f_{,TTT})}{1 + f_{,T} - 12H^2 f_{,TT}}\,, \\
\mu^2 &= 6H^2 + 2\dot{H} + \frac{2HV_{,\phi}}{\dot{\phi}} - \frac{36H^2\dot{H}(f_{,TT} - 4H^2 f_{,TTT})}{1 + f_{,T} - 12H^2 f_{,TT}}\,, \\
c_s^2 &= \frac{1 + f_{,T}}{1 + f_{,T} - 12H^2 f_{,TT}}\,.
\end{aligned} \tag{428}$$

By taking into account that the gravitational equation of motion for the canonical scalar field is,

$$\ddot{\phi} + 3H\dot{\phi} + V_{,\phi} = 0\,, \tag{429}$$

and also by rewriting the second $F(T)$ gravity Friedmann equation as follows,

$$(a + f_{,T} - 12H^2 f_{,TT})\dot{H} = -4\pi G\dot{\phi}^2\,, \tag{430}$$

the evolution of scalar perturbations differential equation (427), becomes,

$$\ddot{\Phi}_k + \left(H - \frac{\ddot{H}}{\dot{H}}\right)\dot{\Phi}_k + \left(2\dot{H} - \frac{H\ddot{H}}{\dot{H}}\right)\Phi_k + \frac{c_s^2 k^2}{a^2}\Phi_k = 0\,. \tag{431}$$

By looking at the evolution equation (431), we can see that it is identical to the one corresponding to the Einstein-Hilbert case, except for the speed of sound parameter.

The physical quantity that quantifies perfectly the cosmological inhomogeneities is the comoving curvature fluctuation ζ, expressed in comoving coordinates, which in this case is,

$$\zeta = \Phi - \frac{H}{\dot{H}}\left(\dot{\Phi} + H\Phi\right)\,. \tag{432}$$

The parameter ζ is a gauge invariant quantity, and it can be used to simplify the calculation of the power spectrum and of the corresponding spectral index of the primordial curvature perturbation. Particularly, we can introduce the variable v which is equal to,

$$v = z\zeta\,, \tag{433}$$

with z being equal to,

$$z = a\sqrt{2\epsilon}\,, \tag{434}$$

and with ϵ being the first slow-roll index $\epsilon = -\frac{\dot{H}}{H^2}$. Eventually, the differential equation that determines the evolution of the primordial perturbations is [368],

$$v_k'' + \left(c_s^2 k^2 - \frac{z''}{z}\right)v_k = 0\,, \tag{435}$$

where the sound speed parameter was defined in Eq. (428). The prime in Eq. (435) denotes differentiation with respect to the conformal time, which is defined as follows,

$$\tau = \int dt \frac{1}{a}\,. \tag{436}$$

In order to demonstrate how the calculation of the power spectrum works in practise, we shall realize the intermediate inflation scenario [551–557], in which case, the scale factor and the Hubble rate are,

$$a(t) = \mathrm{e}^{At^n}\,, \quad H(t) = Ant^{n-1}\,, \tag{437}$$

where $0 < n < 1$ and also $A > 0$ (here we use for notational convenience, the notation n for the exponent in the scale factor). In Refs. [551–553], the intermediate inflation scenario was realized in the context of scalar-tensor theory, but here we shall realize the intermediate inflation in the context of vacuum $F(T)$ gravity, and after that we shall calculate the power spectrum in detail. For a similar presentation to ours, see [555]. Notice that the $F(T)$ gravity function appears implicitly in Eq. (435), via the sound speed parameter c_s. For the FRW background, the first Friedmann equation of $F(T)$ gravity in vacuum is,

$$H^2 = -\frac{f(T(t))}{6} - 2f_{,T}(t)H^2\,, \tag{438}$$

and also $T = -6H^2$, which in the case of the intermediate inflation Hubble rate, we get,

$$T = -6A^2n^2t^{2n-2}\,, \tag{439}$$

which can be solved with respect to t and we get the function $t(T)$,

$$t(T) = 6^{-\frac{1}{2n-2}}\left(-\frac{T}{A^2n^2}\right)^{\frac{1}{2n-2}}\,. \tag{440}$$

By inserting the intermediate inflation Hubble rate from Eq. (437) in the Friedmann equation (438), and also by substituting $t(T)$ from Eq. (440), we can easily solve it to get the approximate $F(T)$ gravity, which is,

$$f(T) = c_1T^{\frac{An}{2}} - \frac{T}{2\left(1-\frac{An}{2}\right)}\,, \tag{441}$$

where c_1 is an arbitrary integration constant. Since we are interested in the inflationary era, which corresponds to early times, when the intermediate inflation scenario is considered, the conformal time τ defined in Eq. (436) can be approximately equal with the cosmic time t, since the exponential for small values of t is approximately equal to $e^{At^n} \sim 1$. Therefore, since $t \equiv \tau$, we can calculate $z(t)$, which is,

$$z(t) = \sqrt{\frac{2(1-n)t^{-n}}{An}}\,, \tag{442}$$

and in addition, by using Eq. (441), and also the explicit form of the Hubble rate for the intermediate inflation scenario, we can calculate the sound speed parameter which is,

$$c_s^2(t) = \frac{Ac_1n6^{\frac{An}{2}}(An-2)\left(-A^2n^2t^{2n-2}\right)^{\frac{An}{2}} - 12A^2n^2(An-1)t^{2n-2}}{2(An-1)\left(c_16^{\frac{An}{2}}(An-2)\left(-A^2n^2t^{2n-2}\right)^{\frac{An}{2}} - 6A^2n^2t^{2n-2}\right)}\,, \tag{443}$$

and since the dominant term in the above expression during the inflationary era, is $\sim t^{2n-2}$, the sound speed parameter can be approximated as $c_s^2 \simeq 1$. Therefore, the evolution equation (435) becomes as follows,

$$v_k''(t) + \left(k^2 - \frac{\left(\frac{n}{2}+1\right)n}{2t^2}\right)v_k(t) = 0\,, \tag{444}$$

which can be analytically solved to yield,

$$v_k(t) = C_1\sqrt{t}J_{\frac{n+1}{2}}(kt) + C_2\sqrt{t}Y_{\frac{n+1}{2}}(kt)\,, \tag{445}$$

where $J_n(z)$ and $Y_n(z)$ are the Bessel functions of first and second kind respectively, and also C_1, C_2 are arbitrary constants. We can approximate the expression in Eq. (445) for small values of the argument of the Bessel function, so we have,

$$\begin{aligned} v_k(t) =& \frac{C_1\sqrt{t}(kt)^{n/2}\left(2^{\frac{1}{2}(-n-1)}\sqrt{kt}\right)}{\Gamma\left(\frac{n+1}{2}+1\right)} \\ &+ C_2\sqrt{t}\left(-\frac{2^{\frac{n}{2}+\frac{1}{2}}\Gamma\left(\frac{n+1}{2}\right)(kt)^{-\frac{n}{2}-\frac{1}{2}}}{\pi} - \frac{2^{-\frac{n}{2}-\frac{1}{2}}\cos\left(\frac{1}{2}\pi(n+1)\right)\Gamma\left(\frac{1}{2}(-n-1)\right)(kt)^{\frac{n}{2}+\frac{1}{2}}}{\pi}\right)\,, \end{aligned} \tag{446}$$

so by keeping the most dominant term, we finally have,

$$v_k(t) = \frac{C_2\sqrt{t}\left(2^{\frac{n}{2}+\frac{1}{2}}\Gamma\left(\frac{n+1}{2}\right)(kt)^{-\frac{n}{2}-\frac{1}{2}}\right)}{\pi}\,. \tag{447}$$

The power spectrum of the primordial curvature perturbations can be written in terms of the function v_k and z as follows,

$$\mathcal{P}_\zeta = \frac{k^3}{2\pi^2}\left|\frac{v_k}{z}\right|^2_{k=aH}\,, \tag{448}$$

which is evaluated at the horizon crossing $k = aH$. So our aim is to express the quantity $|\frac{v_k}{z}|^2$ in terms of the wavenumber k. We have already calculated the function $v_k(t)$ which appears in Eq. (447), however the integration constant C_2 has a hidden k-dependence, which can be determined if we assume that at some initial time t_0, the function $v_k(t_0)$ is described by a Bunch-Davies vacuum, that is $v_k \simeq \frac{\mathrm{e}^{-ikt}}{\sqrt{2k}}$, so in this way we obtain that the constant C_2 is equal to,

$$C_2 \simeq \frac{\pi 2^{-\frac{n}{2}-\frac{1}{2}}k^{n/2}t^{n/2}}{\Gamma\left(\frac{n+1}{2}\right)}\,. \tag{449}$$

Also the cosmic time in Eqs. (447) and (449) can be expressed in terms of the wavenumber k, by using the fact that the power spectrum is evaluated at the horizon crossing, so when $k = aH$, and since $a \sim 1$ for the intermediate inflation scenario, we obtain,

$$t \simeq \frac{k^{\frac{1}{n-1}}}{(An)^{\frac{1}{n-1}}}\,. \tag{450}$$

Therefore, by combining Eqs. (442), (447), (449), and (450), the power spectrum of Eq. (448) becomes,

$$\mathcal{P}_\zeta \simeq \frac{\left(n^{1-\frac{n}{n-1}}A^{1-\frac{n}{n-1}}\right)k^{\frac{1}{n-1}+3}}{4\pi^2(1-n)}\,, \tag{451}$$

which is not scale invariant. However, the spectral index can be compatible with the 2015 Planck data of Eq. (324), by suitably choosing the parameter n. Indeed, the spectral index of the power spectrum is,

$$n_s - 1 = \frac{d\ln\mathcal{P}_\zeta}{d\ln k}\,, \tag{452}$$

so in the case at hand, the spectral index is,

$$n_s = \frac{1}{n-1} + 4\,, \tag{453}$$

so by choosing $n = 0.670576$, the spectral index becomes $n_s = 0.9644$, which is identical with the 2015 Planck constrains on the spectral index.

E. Singular Inflation

Recently, another type of inflationary dynamics was introduced in the relevant literature, under the name "singular inflation" [559, 560], where the terminology singular refers to a soft singularity type which occurs during the inflationary era. The most interesting type of singularity is the Type IV singularity, which according to the classification of singularities we did in section II-A-3, the Type IV singularity is not of crushing type and the Universe may smoothly pass through it without having catastrophic consequences on physical observable quantities. These singularities were studied in the literature in various contexts, for example in the context of scalar-tensor gravity, see Refs. [512, 513, 559–561], while in the context of modified gravity in general, see [189, 510, 511, 562–564]. Also when inhomogeneous equation of state fluids are considered, see [565].

In this section, we shall present how a Type IV singularity can be incorporated to the R^2 model, and the details on this issue can be found in Ref. [510]. As it was shown in [510], and we now briefly discuss, the incorporation of a

Type IV singularity during the slow-roll era of the R^2 model, leads to dynamical instabilities in the second slow-roll index, which indicates that the exit from inflation might occur due to these instabilities. We can easily incorporate the Type IV singularity in the R^2 model if the quasi-de Sitter Hubble rate of the R^2 model is modified as follows,

$$H(t) \simeq H_i - \frac{M^2}{6}(t-t_i) + f_0 (t-t_s)^{\alpha} \, , \tag{454}$$

where the singularity occurs at $t=t_s$. The singularity structure which the Hubble rate (454) implies is the following,

- $\alpha < -1$ corresponds to the Type I singularity.
- $-1 < \alpha < 0$ corresponds to Type III singularity.
- $0 < \alpha < 1$ corresponds to Type II singularity.
- $\alpha > 1$ corresponds to Type IV singularity.

Therefore, in order for a Type IV singularity to occur, we must require that $\alpha > 1$. Moreover, we assume that $H_i \gg f_0$, $M \gg f_0$ and in addition that $f_0 \ll 1$. When these conditions hold true, the singularity term in Eq. (454), is much more smaller when it is compared the first two terms. In effect, near the singularity, the $F(R)$ gravity that generates the Hubble rate (454) is the following,

$$F(R) = R + \frac{1}{6M^2}R^2 \, , \tag{455}$$

with $M \gg 1$. By calculating the Hubble flow parameters [510] for the Hubble rate (454), these read,

$$\begin{aligned}
\epsilon_1 =& -\frac{-\frac{M^2}{6} + f_0(t-t_s)^{-1+\alpha}\alpha}{\left(H_i - \frac{1}{6}M^2(t-t_i) + f_0(t-t_s)^{\alpha}\right)^2} \, , \\
\epsilon_3 =& \frac{f_0(t-t_s)^{-2+\alpha}(-1+\alpha)\alpha + 4\left(H_i - \frac{1}{6}M^2(t-t_i) + f_0(t-t_s)^{\alpha}\right)\left(-\frac{M^2}{6} + f_0(t-t_s)^{-1+\alpha}\alpha\right)}{M^2\left(1 + \frac{2\left(-\frac{M^2}{6} + 2\left(H_i + \frac{1}{6}M^2(-t+t_i)\right)^2\right)}{M^2}\right)\left(H_i - \frac{1}{6}M^2(t-t_i) + f_0(t-t_s)^{\alpha}\right)} \, , \\
\epsilon_4 =& \frac{4f_0H(t)(t-t_s)^{-2+\alpha}(-1+\alpha)\alpha + f_0(t-t_s)^{-3+\alpha}(-2+\alpha)(-1+\alpha)\alpha + 4\left(-\frac{M^2}{6} + f_0(t-t_s)^{-1+\alpha}\alpha\right)^2}{H(t)\left(f_0(t-t_s)^{-2+\alpha}(-1+\alpha)\alpha + 4H(t)\left(-\frac{M^2}{6} + f_0(t-t_s)^{-1+\alpha}\alpha\right)\right)} \, .
\end{aligned} \tag{456}$$

and near the Type IV singularity at $t=t_s$, these become,

$$\begin{aligned}
\epsilon_1 =& \frac{M^2}{6\left(H_i - \frac{1}{6}M^2(t-t_i)\right)^2} \, , \\
\epsilon_3 =& \frac{f_0(t-t_s)^{-2+\alpha}(-1+\alpha)\alpha + 4\left(H_i - \frac{1}{6}M^2(t-t_i)\right)\left(-\frac{M^2}{6}\right)}{M^2\left(1 + \frac{2\left(-\frac{M^2}{6} + 2\left(H_i + \frac{1}{6}M^2(-t+t_i)\right)^2\right)}{M^2}\right)\left(H_i - \frac{1}{6}M^2(t-t_i)\right)} \, , \\
\epsilon_4 =& \frac{\frac{M^4}{9} + 4f_0\left(H_i - \frac{1}{6}M^2(t-t_i)\right)(t-t_s)^{-2+\alpha}(-1+\alpha)\alpha + f_0(t-t_s)^{-3+\alpha}(-2+\alpha)(-1+\alpha)\alpha}{\left(H_i - \frac{1}{6}M^2(t-t_i)\right)\left(-\frac{2}{3}M^2\left(H_i - \frac{1}{6}M^2(t-t_i)\right) + f_0(t-t_s)^{-2+\alpha}(-1+\alpha)\alpha\right)} \, .
\end{aligned} \tag{457}$$

As it can be seen from the equations above, the parameter ϵ_1 is identical to the one corresponding to the R^2 model, and the difference can be found when the parameters ϵ_3 and ϵ_4 are considered. In the case of ϵ_3 the difference is due to the presence of the term $\sim (t-t_s)^{-2+\alpha}$ which may render the parameter ϵ_3 singular, only in the case $1 < \alpha < 2$, at $t = t_s$. In the case of ϵ_4, the terms $\sim (t-t_s)^{-3+\alpha}$ and $(t-t_s)^{-2+\alpha}$, may become singular. So we have three possible scenarios which may occur, and now we discuss these in brief.

1. Scenario I

If the singularity occurs before inflation ends, which occurs at $t = t_f$, then we have $t_s < t_f$, and if $2 < \alpha < 3$, the slow-roll parameter ϵ_4 has a singularity at $t = t_s$, and it is simplified as follows,

$$\epsilon_4 \simeq -\frac{3\left(\frac{M^4}{9} + f_0(t-t_s)^{-3+\alpha}(-2+\alpha)(-1+\alpha)\alpha\right)}{2M^2\left(H_i - \frac{1}{6}M^2(t-t_i)\right)^2} \, , \tag{458}$$

while the rest slow-roll parameters are finite. As it was argued in Ref. [510], this singularity indicates that the slow-roll expansion breaks at a higher order, so this can be viewed as a dynamical instability of the system. Therefore, this mechanism may serve as a mechanism for generating the graceful exit from inflation. Also by calculating the spectral index of primordial curvature perturbations n_s and the scalar-to-tensor ratio r, in the case at hand, we get,

$$\epsilon_H(t_s) = \frac{M^2}{6\left(H_i - \frac{1}{6}M^2(t_s - t_i)\right)^2}\,, \quad \eta_H(t_s) = 0\,, \tag{459}$$

where N is calculated in the interval $[t_i, t_s]$

$$N = \int_{t_i}^{t_s} H(t)dt = H_i(t_s - t_i) - \frac{M^2}{12}\left(t_s - t_i\right)^2\,, \tag{460}$$

and upon solving with respect to the difference $(t_s - t_i)$, we obtain,

$$t_s - t_i = \frac{2\left(3H_i + \sqrt{3}\sqrt{3H_i^2 - M^2 N}\right)}{M^2}\,. \tag{461}$$

Finally by substituting in Eq. (459), we get,

$$\epsilon_H(t_s) = \frac{M^2}{6H_i^2 - 2M^2 N}\,, \quad \eta_H(t_s) = 0\,, \tag{462}$$

which are identical to R^2 inflation model ones.

2. Scenario II

In the case $t_s < t_f$, and if $1 < \alpha < 2$, the second Hubble slow-roll parameter η_H becomes,

$$\eta_H \simeq -\frac{f_0(t - t_s)^{-2+\alpha}(-1+\alpha)\alpha}{2\left(H_i - \frac{1}{6}M^2(t - t_i) + f_0(t - t_s)^\alpha\right)\left(-\frac{M^2}{6} + f_0(t - t_s)^{-1+\alpha}\alpha\right)}\,. \tag{463}$$

In the present case, the parameters ϵ_3, ϵ_4 and η_H diverge at $t = t_s$, which indicates a strong instability at this time instance, and therefore inflation ends abruptly at that point.

3. Scenario III

In the case that $t_s < t_f$ and $\alpha > 3$, all the parameters ϵ_3, ϵ_4 are not singular at $t = t_s$, and therefore inflation ends when ϵ_1 becomes of order one. The difference of this scenario with the ordinary R^2 model is that in the case at hand, the second slow-roll parameter is not exactly zero, but it is equal to,

$$\eta_H \simeq -\frac{f_0(t_f - t_s)^{-2+\alpha}(-1+\alpha)\alpha}{2\left(H_i - \frac{1}{6}M^2(t_f - t_i) + f_0(t_f - t_s)^\alpha\right)\left(-\frac{M^2}{6} + f_0(t_f - t_s)^{-1+\alpha}\alpha\right)}\,. \tag{464}$$

Hence, in this case, the spectral index n_s is equal to,

$$n_s = -\frac{9f_0(t_f - t_s)^{-1+\alpha}(-1+\alpha)\alpha}{\left(-\sqrt{9H_i^2 - 3M^2 N} + 3f_0(t_f - t_s)^\alpha\right)\left(M^2(-t_f + t_s) + 6f_0(t_f - t_s)^\alpha\alpha\right)}\,. \tag{465}$$

However, the observational differences between this scenario and of the Starobinsky model are minor, as was shown in [510].

4. The Scenarios with $t_s = t_f$

Finally, let us consider in brief the case that the singularity occurs at the end of inflation, so that we have $t_s = t_f$. If $2 < \alpha < 3$, the Hubble flow parameter ϵ_4 diverges at $t = t_f$. This case is similar to Scenario I, so inflation ends when $\epsilon_1 \sim 1$, but since ϵ_4 diverges at $t = t_f = t_s$, the graceful exit from inflation becomes more pronounced in comparison to Scenario I. The same situation occurs in the case $1 < \alpha < 2$, since the slow-roll indices ϵ_3, ϵ_4 and η_H diverge at $t = t_s$. In the case that $\alpha > 3$, the situation is similar to Scenario III, with the only difference being that the second Hubble slow-roll index is zero in the case at hand. Hence this case is indistinguishable for the ordinary R^2 model.

In conclusion, the Type IV singular R^2 model has the advantage that inflation ends more abruptly in comparison to the ordinary R^2 model. Also, although the singularity term is insignificant at early times, it becomes dominant at late times, so this term can be responsible for late-time acceleration, see [510] for details. Moreover we need to note that the $F(R)$ gravity of Eq. (455) generates the singular Hubble rate of Eq. (454) only at early times, so at late times this description no longer holds true, and therefore a different description applies, as was shown in detail in Ref. [510].

F. A Qualitative Study for the Graceful Exit from Inflation Issue

In the previous sections we discussed the inflationary dynamics in various modified gravities, and as we showed, in many cases, the graceful exit from inflation quantitatively occurs when the first slow-roll index becomes of the order $\mathcal{O}(1)$. In the cases of $F(R)$ gravity and also in the case of a single scalar field this is easy to address in a quantitative way. However, in more complicated cases, like for example in the mimetic $F(R)$ gravity, this problem is difficult to handle with, by using conventional approaches. In this section we shall discuss the graceful exit issue in the context of complicated theories. The formalism we shall develop is quite general and can be applied in various theories of gravity. In order to make the presentation more clear and for illustrative purposes, we shall discuss the graceful exit problem in the context of the mimetic $F(R)$ gravity we discussed earlier in this chapter. Specifically, we consider the power-law model of Eq. (327), so the Lagrange multiplier and the mimetic potential are the ones appearing in Eqs. (348) and (349). As we showed, cosmologically viable results for this specific model are obtained when $n = 1.8$, and as we show this crucially affects the graceful exit from inflation.

The formalism for the graceful exit from inflation that we now discussed, is based on the existence of the unstable de Sitter attractors in the theory. These unstable de Sitter vacua generate curvature perturbations which grow with time, so this mechanism triggers the graceful exit from inflation. More theoretical details on this issue can be found in Refs. [434, 544]. To proceed with the method, the gravitational equation for mimetic $F(R)$ gravity can be written as follows,

$$6\alpha n H(t)^2 - (\alpha n - \alpha)\left(12H(t)^2 + 6H'(t)\right) + 6\alpha n(n-1)(24H(t)H'(t) + 6H''(t)) - \frac{V(t) - 2\lambda(t)}{12H(t)^2 + 6H'(t)} = 0\,. \tag{466}$$

We investigate whether the de Sitter solutions to the above equations exist, so we seek solutions of the form $H(t) = H_d$, and by substituting this in (466), we get the following algebraic equation,

$$6H_d^2 n\alpha - 12H_d^2(-\alpha + n\alpha) - 12^{-2+n}H_d^{-2+2n}(-2(-1+n)\alpha + n\alpha) = 0\,, \tag{467}$$

which when solved, yields,

$$H_d = \left(2^{-5+2n}3^{-3+n}\right)^{\frac{1}{4-2n}}\,. \tag{468}$$

Having found the de Sitter vacuum for the theory, we need to investigate whether this is unstable or not, towards linear perturbations, so we set $H(t) = H_d + \Delta H(t)$ in Eq. (466, and we keep only linear terms in $\Delta H(t)$, $\Delta H'(t)$ and $\Delta H''(t)$. After expanding and keeping only linear terms in the aforementioned variables, we obtain,

$$\begin{aligned}
&\frac{1}{288H_d^2}\alpha\left(-4H_d\left(864H_d^2 + 12^n H_d^{2n}\right)(-2+n)\Delta H(t)\right.\\
&+\left(-12^n H_d^{2n}(-2+n) - 1728H_d^2(-1+n) + 41472H_d^3(-1+n)n + 3^{1+n}4^{2+n}H_d^{1+2n}(-1+n)n\right)\Delta H'(t)\\
&\left.+2H_d\left(\left(-864H_d^2 + 12^n H_d^{2n}\right)(-2+n) + 5184H_d(-1+n)n\Delta H''(t)\right)\right) = 0\,.
\end{aligned} \tag{469}$$

This differential equation can be solved analytically, so we obtain,

$$\Delta H(t) = -\frac{864H_d^2 - 2^{2n}3^n H_d^{2n}}{2\left(864H_d^2 + 2^{2n}3^n H_d^{2n}\right)} + c_1 e^{\mu_1 t} + c_2 e^{\mu_2 t}\,, \tag{470}$$

where the detailed form of the parameters μ_1 and μ_2 is,

$$\mu_1 = \frac{q_1 + \sqrt{q_2}}{2\left(-10368 H_d^2 n + 10368 H_d^2 n^2\right)}\,, \quad \mu_2 = \frac{q_1 - \sqrt{q_2}}{2\left(-10368 H_d^2 n + 10368 H_d^2 n^2\right)}\,, \tag{471}$$

where q_1 stands for,

$$\begin{aligned} q_1 = & -1728 H_d^2 - 2^{1+2n} 3^n H_d^{2n} + 1728 H_d^2 n + 41472 H_d^3 n \\ & + 2^{2n} 3^n H_d^{2n} n + 2^{4+2n} 3^{1+n} H_d^{1+2n} n - 41472 H_d^3 n^2 - 2^{4+2n} 3^{1+n} H_d^{1+2n} n^2\,, \end{aligned} \tag{472}$$

while q_2 stands for,

$$\begin{aligned} q_2 = & -4\left(6912 H_d^3 + 2^{3+2n} 3^n H_d^{1+2n} - 3456 H_d^3 n - 2^{2+2n} 3^n H_d^{1+2n} n\right)\left(-10368 H_d^2 n + 10368 H_d^2 n^2\right) \\ & + \left(1728 H_d^2 + 2^{1+2n} 3^n H_d^{2n} - 1728 H_d^2 n - 41472 H_d^3 n - 2^{2n} 3^n H_d^{2n} n - 2^{4+2n} 3^{1+n} H_d^{1+2n} n \right. \\ & \left. + 41472 H_d^3 n^2 + 2^{4+2n} 3^{1+n} H_d^{1+2n} n^2\right)^2 . \end{aligned} \tag{473}$$

and also c_1, c_2 are arbitrary integration constants. So by combining Eqs. (468) and (470), for $n = 1.8$, we have,

$$\Delta H(t) = -0.499989 + c_1 \mathrm{e}^{0.0775432 t} + c_2 \mathrm{e}^{0.00195451 t}\,, \tag{474}$$

which tells us that the perturbation $\Delta H(t)$ is exponentially growing with time. It is more convenient and useful to have the solution (474) in terms of the e-foldings number, so now we demonstrate how this can be done formally. Assume that the graceful exit occurs at the time instance t_f, so from the condition $\epsilon_1(t_f) \sim \mathcal{O}(1)$ and from (346), we obtain $H_f^2 \simeq \frac{M^2}{6}$. Therefore, at $t = t_f$, the quasi-de Sitter evolution of (346), yields,

$$H_f = H_0 - \frac{M^2}{6}(t_f - t_0)\,. \tag{475}$$

By solving with respect to $t_f - t_0$, and also by substituting $H_f^2 \simeq \frac{M^2}{6}$ we obtain,

$$t_f - t_0 \simeq \frac{6}{M^2}\left(H_0 - \frac{M}{\sqrt{6}}\right)\,. \tag{476}$$

The definition of the e-foldings number N is,

$$N = \int_{t_0}^{t_f} H(t) dt\,. \tag{477}$$

By choosing the initial time t_0 to be the time instance that the horizon crossing occurs, then we have from (477),

$$N = H_0(t_f - t_0) - \frac{M^2}{12}(t_f - t_0)^2\,, \tag{478}$$

and by substituting $(t_f - t_0)$ given in Eq. (476), we obtain,

$$N \simeq \frac{3H_0^2}{M^2} - \frac{1}{2}\,. \tag{479}$$

Combining Eqs. (476) and (479), we get,

$$t_f - t_i \simeq \frac{2(N + \frac{1}{2})}{H_0} - \frac{6}{M\sqrt{6}}\,, \tag{480}$$

and by substituting $(t_f - t_0)$ in Eq. (474), we get,

$$\Delta H(N) = -0.499989 + c_1 \mathrm{e}^{0.0775432\left(\frac{2(N+\frac{1}{2})}{H_0} - \frac{6}{M\sqrt{6}}\right)} + c_2 \mathrm{e}^{0.00195451\left(\frac{2(N+\frac{1}{2})}{H_0} - \frac{6}{M\sqrt{6}}\right)}\,. \tag{481}$$

Having the resulting perturbation ΔH as a function of N, enables us to decide if enough inflation is produced for at least $N \sim 50$. For the example at hand, if we choose $H_0 = 12.04566$ and $M = 2$, we obtain,

$$\Delta H(N) = -0.499989 + c_1 \exp\left\{0.0775432\left[-\sqrt{\frac{3}{2}} + 0.166035\left(\frac{1}{2} + N\right)\right]\right\}$$

$$+ c_2 \exp\left\{0.00195451\left[-\sqrt{\frac{3}{2}} + 0.166035\left(\frac{1}{2}+N\right)\right]\right\} .$$

In effect, even for small values of N, we find $\Delta H \sim -0.5 + 0.9c_1 + 1.0c_2$, and therefore unless c_1 and c_2, are chosen to be extremely small, the perturbation ΔH is of the order $\mathcal{O}(1)$ at the beginning of inflation, so not enough inflation is produced. As a conclusion, we need to note that the qualitative results of the method we presented, are quite model dependent, however, the quantitative approach is quite general, and can be applied in quite complicated modified gravity theories.

Before closing it is worth discussing an important question related to the graceful exit issue in different frames. The question is whether the Universe can exit from the inflationary era in both of the Jordan and Einstein frames, and if yes what is the physical reasoning. The answer to this question is not easy to address because this issue is strongly model dependent. It is known that conformal invariant quantities have the same description in both of the Jordan and Einstein frames, for example the observational indices of inflation [539–541], that is the spectral index n_s and the scalar-to-tensor ratio r, so only the conformal invariant quantities are the same in the Jordan and Einstein frame.

The point is that for slow-roll inflation, the Einstein and the Jordan frame give the same indexes. Then, the graceful exit occurs in both of the Einstein and Jordan frames. For example in the vacuum R^2 gravity the graceful exit occurs in the $F(R)$ frame and at the same time it occurs in the Einstein frame [506]. However, the two frames are effectively equivalent only in the vacuum case only. When we have matter fluids present, due to the fact that $F(R)$ gravity with matter is not equivalent to the Einstein frame with same matter fluids (rather it is the Einstein frame with matter coupled to a scalar field), then we expect that there is a difference between the two frames, and the impact of the difference can be seen in the graceful exit issue too. In this section we assumed that the physical theory is the $F(R)$ gravity one, so we focused on the $F(R)$ gravity frame only.

Differences between the Jordan and Einstein frames were pointed out in the literature. As it was shown in [295], the acceleration in one frame corresponds to deceleration in the other frame, so the physical interpretation in the two frames is totally different. However the graceful exit issue is more involved and we remain skeptical on answering the question with a simple yes or no, in the case that matter fluids are present. Traditionally, the exit is triggered when the first slow-roll index becomes of the order one, however this ends the slow-roll era but not inflation to our opinion. Practically, in order for the inflationary era to come to an end, the unstable de Sitter solution must exist in the theory and if this exists, it has to be the final attractor of the cosmological dynamical system. The de Sitter instabilities generate uncontrollable curvature perturbations which cease the inflationary era. According to Refs. [295], the de Sitter solution in the Jordan frame may not be the de Sitter solution in the other frame, when matter fluids are present, and in fact it may describe a totally new physical evolution. So we refrain from going into deeper details for this intriguing issue and we defer the study to a focused future research paper.

Thus we presented the various models in which inflation can be realized in the context of modified gravity. Basically speaking, all modified gravities we presented may describe the inflationary era of the sort appearing in this chapter. What now remains is to briefly discuss the reheating issue in the context of $F(R)$ gravity. This is the subject of the next section.

G. Reheating in $F(R)$ Gravity

Describing successfully the inflationary era is one step towards the understanding of early-time dynamics, however the inflationary era must be followed by the Friedmann evolution, as this is described by standard Big-Bang cosmology. The reheating period fills the gap between inflation and the radiation and matter domination era, and it is actually the most important mechanism in order to thermalize the cold Universe that results after the inflationary era. However, it is conceivable that the full understanding of how the Universe transits from a supercooled state at the end of inflation, to a thermal radiation dominated epoch, is a very complicated task. Actually, the conceptual understanding of the Universe's evolution in the era between the ending of the inflationary era and Big Bang Nucleosynthesis, is still an active research stream [566]. In some sense, during this era, the energy that drives the quasi-de Sitter inflationary evolution should be endowed to the matter fields present, which are Standard Model particles. Now in most descriptions, the inflaton directly transfers its energy to the Standard Model particles [567–569], however there are alternative ways in order for this reheating process to occur, such as the modified gravity description of reheating [570].

Although the reheating study can go quite deep in theoretical descriptions, for example considering non-perturbative resonances as preheating sources [566], in this section we shall discuss in brief this thermalization era, and we shall try to outline the basic facts about reheating, always in the context of modified gravity. Particularly, we shall focus on the $F(R)$ gravity theory description of reheating, since this is the prominent and most representative modified gravity theory. The modified gravity description offers a more refined reheating theory since the reheating occurs as a back-reaction effect on the metric, due to curvature fluctuations acting as matter sources in the field equations. This description is more refined in comparison to standard inflaton approaches, since in the modified case there is no need to fine-tune the couplings of the inflation to the Standard Model fields, in order for these to be extremely suppressed during the inflationary era.

We shall focus on the Starobinsky R^2 model we used earlier,

$$F(R) = R + \frac{R^2}{36H_i}\,, \tag{482}$$

where H_i has dimensions of mass^2, and as we will see will play an important role in determining the reheating and Friedman temperature. We shall base our analysis on Ref. [570]. Let us note here that by using the terminology "Friedman evolution" or "Friedman temperature", we refer to the era that the Big Bang description for the Universe evolution apply, so this era refers to the radiation domination era and after. During the reheating era, the first term of the second differential equation in Eq. (305), namely $\ddot{R}$ cannot be neglected anymore. Therefore, the evolution of the scalar curvature looks like a damped harmonic oscillation, as the differential equation in Eq. (305) suggests, with the restoring force being $\sim 3H_i$. The parameter H_i has a prominent role in the theory for three reasons, firstly since the observational indices of inflation strongly depend on this as we showed earlier, secondly since it affects the reheating temperature and thirdly since it affects the Friedman temperature and evolution. With regard to the reheating temperature, it has to be high enough in order for the baryogenesis to occur, but also it has to be low enough in order for the grand unified theories phase transitions not to occur, in which case the monopoles will be absent. Hence the parameter H_i is strongly constrained by the above requirements as we show later on in this section.

The evolution of the Hubble rate is given in the first differential equation of Eq. (305), which we quote here too for reading convenience,

$$\ddot{H} - \frac{\dot{H}^2}{2H} + 3H_iH = -3H\dot{H}\,. \tag{483}$$

During the slow-roll era, the first and second term in Eq. (483) can be neglected, due to the slow-roll conditions and since the following condition holds true,

$$\left|\frac{\dot{H}^2}{2H}\right| \ll \left|3H\dot{H}\right|\,, \tag{484}$$

so as we showed earlier, the evolution during the slow-roll era is the quasi-de Sitter evolution given in Eq. (306). When the condition (484) ceases to hold true, then the reheating phase begins, which is an oscillatory era, in which case the terms $\sim \ddot{H}$ and $\sim \frac{\dot{H}^2}{2H}$ dominate, but the term $\sim \dot{H}H$ becomes subdominant. Then by solving the differential equation (484), we obtain the following solution,

$$H(t) \simeq \frac{\cos^2\omega(t-t_r)}{\frac{3}{\omega} + \frac{3}{4}(t-t_r) + \frac{3}{8\omega}\sin 2\omega(t-t_r)}\,, \tag{485}$$

where $t = t_r$ is the time instance that the quasi-de Sitter phase stops and the reheating era starts, which occurs when the following equality holds true,

$$\left|\frac{\dot{H}^2}{2H}\right| = \left|3H\dot{H}\right|\,. \tag{486}$$

Also the scale factor during reheating is equal to,

$$a(t) = a_r\left(1 + \frac{\omega(t-t_r)}{4}\right)^{2/3}\,, \tag{487}$$

where a_r is equal to $a_r = a_0 e^{\frac{H_0^2}{2H_i} - \frac{1}{12}}$, and a_0 is the scale factor corresponding to the onset of inflation. The parameter ω appearing in Eq. (485) can be found by using the equation (486) and matching the quasi de-Sitter solution with the solution (485) at $t = t_r$, so the parameter ω and the reheating time $t = t_r$ are,

$$\omega = \sqrt{\frac{3H_i}{2}} \quad t_r \simeq H_iH_0\,, \tag{488}$$

where H_0 is the Hubble rate at the onset of inflation. During the reheating phase, the scalar curvature is approximately equal to $R \simeq 6\dot{H}$, and this is to be contrasted with the slow-roll era, in which case the scalar curvature is approximately equal to $R \sim 12H^2$. Hence, during the reheating era, the scalar curvature is approximately equal to,

$$R(t) \simeq -\frac{6\omega \sin 2\omega(t-t_r)}{\left(\frac{3}{\omega}+\frac{3}{4}(t-t_r)+\frac{3}{8\omega}\sin 2\omega(t-t_r)\right)} \,. \tag{489}$$

It is conceivable that the scalar curvature appearing in Eq. (489) is an approximation and not an exact solution of the differential equation (305). Also notice how the parameter H_i affects the resulting scalar curvature and Hubble rate via the parameter ω, and it affects the shape and size of the reheating phase.

During the reheating phase, the matter fields present will be excited and the Universe will be reheated. Let us consider the simplest case of a matter field by considering the effects of curvature oscillations on a scalar field ϕ, with equation of motion $g^{\mu\nu}\phi_{;\mu\nu}=0$. The energy density of this scalar field strongly depends on the averaged square of the scalar curvature as it was shown in [570], and the evolution of the energy density is determined by the following differential equation,

$$\frac{d\rho}{dt}=-4\rho H+\frac{\omega \bar{R}^2}{1152\pi} \,. \tag{490}$$

The differential equation above can determine the way that matter fields are affected by the curvature oscillations, and when the reheating era ends, the last term in (490) can be neglected, so the differential equation becomes $\frac{d\rho a^4}{dt}=0$, and the corresponding equation of state of the matter fields is $p=\frac{1}{3}$, which perfectly describes a radiation domination era. However, during the reheating era, the equation of state of the matter field becomes,

$$p=\frac{1}{3}\rho-\frac{\omega \bar{R}^2}{1152\pi H} \,. \tag{491}$$

Hence it is obvious that the curvature oscillations affect significantly the matter content during the reheating era, and this in turn has a back-reaction effect in the equations of motion. Indeed, by using (491), the curvature evolution during reheating in the presence of matter is,

$$\ddot{R}+3HR+6H_iR=\frac{GNH_i}{8H}\omega \bar{R}^2 \,, \tag{492}$$

where the factor N denotes the number of matter fields that are involved in the reheating process, excluding the massless conformal fields which are not excited during the reheating era. Also the (t,t) component of the field equations is,

$$H^2+\frac{1}{18H_i}\left(HR-\frac{R^2}{12}RH^2\right)=\frac{8\pi}{3}G\rho_{\mathrm{matter}} \,, \tag{493}$$

where the term ρ_{matter} is equal to,

$$\rho_{\mathrm{matter}}=\frac{N}{a^4}\int_{t_r}^{t}\frac{\omega}{1152\pi}\bar{R}^2a^4dt \,. \tag{494}$$

After discussing the back-reaction of the reheated matter on the equations of motion, we can have a qualitative estimate of the reheating temperature by using Eqs. (487) and (489), so for $t\simeq t_r+10\omega$, we obtain $\rho \simeq 7776\times 10^{-7}H_i^2$, and the corresponding reheating temperature is,

$$T_r \simeq 4\times 10^{17}\left(\frac{1}{36H_il_{\mathrm{pl}}^2}\right)^{-1/2}\mathrm{GeV} \,, \tag{495}$$

where l_{pl} is the Planck length. The reheating temperature T_r imposes further restrictions on the parameter H_i, which recall that is restricted from the inflationary observational indices. Particularly, the reheating temperature has to be large enough in order for the baryogenesis to take place, and at the same time it has to be smaller than the grand unified theory phase transition temperatures, so that the theory remains monopole free. Specifically, the reheating temperature imposes the following constraint on the parameter H_i,

$$36^{-1}10^{-15}-10^{-12}l_{\mathrm{pl}}^{-2}<H_i<10^{-3}l_{\mathrm{pl}}^{-2} \,. \tag{496}$$

For cosmic times $t \gg t_r + \frac{1}{\omega}$, the energy density ρ_{matter} becomes approximately,

$$\rho_{\mathrm{matter}} \simeq \frac{3}{5}\frac{32}{1152\pi}\frac{N\omega^3}{t-t_r}\,, \tag{497}$$

and the matter energy density term ρ_{matter} tends to zero, so the solution of (493) tends to the radiation domination solution. Finally, it can be shown that the Friedman temperature is equal to,

$$T_F \sim 10^{17}\left(\frac{1}{36H_i l_{\mathrm{pl}}^2}\right)^{-3/4} N^{-3/4}\mathrm{GeV}\,, \tag{498}$$

but no important constraint can imposed on this parameter.

IV. LATE-TIME DYNAMICS AND DARK ENERGY

The ΛCDM model is up-to-date quite successful in fitting the pre-2000 observational data, and specifically it predicted the location and existence of the baryon acoustic oscillations. Since this model is quite important in cosmology, in this chapter we shall demonstrate that it can be realized in the context of modified gravity. Also we shall discuss various phenomenological aspects of the late-time evolution of the Universe, which is now known as the dark energy era. Special emphasis shall be given on an especially appealing feature of modified gravity, which enables one to provide a unified theoretical description of early and late-time acceleration. Moreover, we shall discuss in brief the possibility of having a phantom dark energy era in the context of modified gravity, and we shall present in brief the implications of a phantom dark energy era in modified gravity. Finally, we shall present a vital feature of the $F(R)$ gravity description of the late-time era, namely, the dark energy oscillations era, by using various matter-fluids, not necessarily of non-interacting matter. Also the evolution of the growth index in $F(R)$ gravity is briefly addressed too.

A. ΛCDM Epoch from $F(R)$ Gravity

In this section we shall demonstrate how the ΛCDM model can be realized in the context of $F(R)$ gravity. This issue has been addressed in detail in Refs. [571, 572]. We shall consider the approach of Ref. [571], so the ΛCDM Hubble rate is equal to,

$$\frac{3}{\kappa^2}H^2 = \frac{3}{\kappa^2}H_0^2 + \rho_0 a^{-3}\,. \tag{499}$$

We shall use the reconstruction technique we developed in section IV-B, so it is vital to find the function $G(N)$ which is equal to $G(N) = H^2(N)$, which in the case at hand is,

$$G(N) = H_0^2 + \frac{\kappa^2}{3}\rho_0 a_0^{-3}\mathrm{e}^{-3N}\,, \tag{500}$$

and by using the fact that the scalar curvature in terms of $G(N)$ is given in Eq. (688), we can express the e-foldings number as a function of R, so we get,

$$N = -\frac{1}{3}\left(\frac{R-12H_0^2}{\kappa^2\rho_0 a_0^{-3}}\right)\,, \tag{501}$$

and therefore, Eq. (692) becomes in our case,

$$3(R-9H_0^2)(R-12H_0^2)F''(R) - (\frac{R}{2}-9H_0^2)F'(R) - \frac{F(R)}{2} = 0\,, \tag{502}$$

where the prime this time denotes differentiation with respect to the scalar curvature R. The differential equation (502) can solved analytically since it is the hypergeometric differential equation, which has as solution the Gauss hypergeometric function $F(\alpha,\beta,\gamma;x)$, with the parameters α, β, γ and x being equal to,

$$\alpha+\beta = -\frac{1}{6},\quad \alpha\beta = -\frac{1}{6},\quad \gamma = -\frac{1}{6}\,,\quad x = \frac{R}{3H_0^2} - 3\,. \tag{503}$$

The analysis performed in Ref. [572], has the same result for the $F(R)$ gravity that realizes the ΛCDM model, see [572] for details.

B. ΛCDM Epoch from Modified Gauss-Bonnet Gravity

The ΛCDM epoch can be realized by a modified Gauss-Bonnet gravity, and we now present the essential features of this realization. The action of Eq. (57) can be rewritten by introducing an auxiliary scalar field ϕ, as follows:

$$S = \int d^4x\sqrt{-g}\left[\frac{R}{2\kappa^2} - V(\phi) - \xi(\phi)\mathcal{G} + \mathcal{L}_{\rm matter}\right] . \tag{504}$$

In fact, upon variation of the above action with respect to the auxiliary scalar field ϕ, yields the following algebraic equation,

$$0 = V'(\phi) + \xi'(\phi)\mathcal{G} . \tag{505}$$

The equation (505) can be solved with respect to the scalar field ϕ, to yield the function $\phi = \phi(\mathcal{G})$.

By substituting the obtained expression of $\phi(\mathcal{G})$ into the action of Eq. (504), we reobtain the action of Eq. (57), where,

$$f(\mathcal{G}) \equiv -V\left(\phi(\mathcal{G})\right) + \xi\left(\phi(\mathcal{G})\right)\mathcal{G} . \tag{506}$$

We employ the formalism of Refs. [115, 349, 350], and we review the reconstruction method of scalar-$f(\mathcal{G})$ gravity. The equations of motion are obtained by the varying the action with respect to the metric, and for the FRW Universe, these have the following form,

$$\begin{aligned}
0 =& -\frac{3}{\kappa^2}H^2 + V(\phi) + 24H^3\frac{d\xi(\phi(t))}{dt} + \rho_{\rm matter} , \\
0 =& \frac{1}{\kappa^2}\left(2\dot{H} + 3H^2\right) - V(\phi) - 8H^2\frac{d^2\xi(\phi(t))}{dt^2} - 16H\dot{H}\frac{d\xi(\phi(t))}{dt} - 16H^3\frac{d\xi(\phi(t))}{dt} + p_{\rm matter} ,
\end{aligned} \tag{507}$$

which can be easily integrated in the following way,

$$\begin{aligned}
\xi(\phi(t)) =& \frac{1}{8}\int^t dt_1 \frac{a(t_1)}{H(t_1)^2}W(t_1) , \\
V(\phi(t)) =& \frac{3}{\kappa^2}H(t)^2 - 3a(t)H(t)W(t) + \rho_{\rm matter}(t) , \\
W(t) \equiv& \int^t \frac{dt_1}{a(t_1)}\left(\frac{2}{\kappa^2}\dot{H}(t_1) + \rho_{\rm matter}(t_1) + p_{\rm matter}(t_1)\right) .
\end{aligned} \tag{508}$$

We should note here, that there is no kinetic term for the auxiliary scalar field ϕ in the action (504), and therefore we can redefine the scalar field ϕ properly, and identify ϕ with the cosmological time, that is, $\phi = t$. In effect, if we consider the following potential $V(\phi)$ and ξ, given in terms of a single function g as follows,

$$\begin{aligned}
V(\phi) =& \frac{3}{\kappa^2}g'\left(\phi\right)^2 - 3g'\left(\phi\right)\mathrm{e}^{g(\phi)}U(\phi) + \rho_{\rm matter}(t) , \\
\xi(\phi) =& \frac{1}{8}\int^\phi d\phi_1 \frac{\mathrm{e}^{g(\phi_1)}}{g'(\phi_1)^2}U(\phi_1) , \\
U(\phi) \equiv& \int^\phi d\phi_1\left(\frac{2}{\kappa^2}\mathrm{e}^{-g(\phi_1)}g''\left(\phi_1\right) + \rho_{\rm matter}(t_1) + p_{\rm matter}(t_1)\right) ,
\end{aligned} \tag{509}$$

we can find the following explicit solution,

$$a = a_0\mathrm{e}^{g(t)} \ \left(H = g'(t)\right) . \tag{510}$$

The energy density $\rho_{\rm matter}$ and the pressure density $p_{\rm matter}$ of the matter fluids present, can be often expressed as the sum of the contributions from various different matter fluids, with different but constant EoS parameters w_i. Since the energy density $\rho_{{\rm matter}\,i}$ with a constant EoS parameter w_i behaves as, $\rho_{{\rm matter}\,i} = \rho_{0i}a^{-3(1+w_i)}$ and also since the pressure $p_{{\rm matter}\,i}$ with a constant EoS parameter w_i is given by $p_{{\rm matter}\,i} = w_i\rho_{{\rm matter}\,i}$, we may rewrite Eq. (509) as follows,

$$V(\phi) = \frac{3}{\kappa^2}g'\left(\phi\right)^2 - 3g'\left(\phi\right)\mathrm{e}^{g(\phi)}U(\phi) + \sum_i \rho_{0i}a_0^{-3(1+w_i)}\mathrm{e}^{-3(1+w_i)g(t)} ,$$

$$\xi(\phi) = \frac{1}{8}\int^\phi d\phi_1 \frac{\mathrm{e}^{g(\phi_1)}}{g'(\phi_1)^2} U(\phi_1)\,,$$
$$U(\phi) \equiv \int^\phi d\phi_1 \left(\frac{2}{\kappa^2} \mathrm{e}^{-g(\phi_1)} g''(\phi_1) + \sum_i (1+w_i)\,\rho_{0i} a_0^{-3(1+w_i)} \mathrm{e}^{-3(1+w_i)g(t_1)} \right)\,, \tag{511}$$

where we have used the expression in Eq. (510).

Instead of the cosmic time t, we can use the e-folding number N, and we may rewrite Eq. (509) as follows,

$$\xi(\phi(N)) = \frac{1}{8}\int^N dN_1 \frac{\mathrm{e}^{N_1}}{H(N_1)^3} \tilde{W}(N_1)\,,$$
$$V(\phi(N)) = \frac{3}{\kappa^2} H(N)^2 - 3\mathrm{e}^N H(N) \tilde{W}(N) + \sum_i \rho_{0i} a_0^{-3(1+w_i)} \mathrm{e}^{-3(1+w_i)N}\,,$$
$$\tilde{W}(N) \equiv \int^N \frac{dN_1}{\mathrm{e}^{N_1}} \left(\frac{2}{\kappa^2} \dot{H}(N_1) + \sum_i (1+w_i)\, a_0^{-3(1+w_i)} \rho_{0i} \mathrm{e}^{-3(1+w_i)N_1} \right)\,. \tag{512}$$

We can now identify the auxiliary scalar ϕ with the e-foldings number N, that is, $\phi = N$, instead of the cosmic time $\phi = t$, and therefore if $V(\phi)$ and ξ are chosen as follows,

$$V(\phi) = \frac{3}{\kappa^2} h(\phi)^2 - 3h(\phi)\, \mathrm{e}^\phi \tilde{U}(\phi) + \sum_i \rho_{0i} a_0^{-3(1+w_i)} \mathrm{e}^{-3(1+w_i)\phi}\,,$$
$$\xi(\phi) = \frac{1}{8}\int^\phi d\phi_1 \frac{\mathrm{e}^{\phi_1}}{h(\phi_1)^3} \tilde{U}(\phi_1)\,,$$
$$U(\phi) \equiv \int^\phi d\phi_1 \mathrm{e}^{-\phi_1} \left(\frac{2}{\kappa^2} h'(\phi_1) + \sum_i (1+w_i)\,\rho_{0i} a_0^{-3(1+w_i)} \mathrm{e}^{-3(1+w_i)\phi_1} \right)\,, \tag{513}$$

we obtain the solution $H = h(N)$.

We now consider the evolution in ΛCDM cosmology, without CDM (Cold Dark Matter). In case of the ΛCDM model, we have

$$\frac{3}{\kappa^2} H^2 = \Lambda + \rho_0 \mathrm{e}^{-3N}\,. \tag{514}$$

Here Λ is a constant corresponding to the cosmological constant and ρ_0 is simply a constant. The second term in Eq. (514) expresses the contribution from the CDM. Therefore, we find,

$$h(\phi) = \sqrt{\frac{\kappa^2}{3}\left(\Lambda + \rho_0 \mathrm{e}^{-3\phi}\right)}\,. \tag{515}$$

In effect, by substituting the expression from Eq. (515) into Eq. (513), we obtain the model which reproduces the ΛCDM universe.

C. Unification of Inflation with Dark Energy Era in $F(R)$ Gravity

As we already discussed in the introduction, in the context of modified gravity it is possible to construct cosmological models which describe in a unified way the early-time acceleration with the late-time acceleration of the Universe. In the literature there exist various proposals on this, see for example [95, 97, 98]. In this section we shall be interested in the unified description of the inflationary era with the dark energy era, in the context of $F(R)$ gravity. Before we start, let us summarize the conditions that need to hold true in order for the unified $F(R)$ models, to be viable:

1. The first condition is of course that the model can generate the inflationary era. This condition could be given by Eq. (28). Here we suppose that $F(R) = R + f(R)$. The viability condition indicates that $f(R)$ goes to a constant Λ_i, which plays the role of an effective cosmological constant in the early Universe. In effect, the following requirement must be satisfied $\Lambda_i \gg \left(10^{-33}\mathrm{eV}\right)^2$ and we may naturally assume that $\Lambda_i \sim 10^{20\sim 38}\,(\mathrm{eV})^2$.

   Since the inflationary era should not be eternal, but it should stop at some point, the de Sitter space-time described by Eq. (28) should not be the stable de-Sitter point, but a quasi-stable de Sitter point. We assume that the de Sitter space-time is realized in the limit of large curvature, that is $R \sim R_I \sim \left(10^{16\sim19}\,\mathrm{GeV}\right)^2$, and in effect, the quasi-stability condition requires Eq. (38). Note that, f_{0I} and f_{1I} are positive constants and also m is a positive integer. If the following condition is satisfied,

$$-R_I + \frac{F'(R_I)}{F''(R_I)} < 0\,, \tag{516}$$

the de Sitter space-time solution is rendered strongly unstable.

In order to generate the inflationary era, there are other possibilities, for example, we may assume that,

$$\lim_{R\to\infty} f(R) = \alpha R^m\,. \tag{517}$$

Here the parameter m is a positive integer greater than unity, $m > 1$ and also α is a constant. In order to avoid an antigravity era, which would imply that $f'(R) > -1$, we need to require that $\alpha > 0$, which indicates that $f(R)$ should be positive during the inflationary era.

2. The second condition is the condition that the $F(R)$ gravity model should generate the present-time cosmic acceleration. This condition could be realized by simply requiring that the $f(R)$ function in the current Universe, behaves as a small constant, as in Eq. (Uf3). We denote the curvature of the present Universe by R_0, $R_0 \sim \left(10^{-33}\mathrm{eV}\right)^2$. We should note that $R_0 > \tilde{R}_0$ due to a small extra contribution to the curvature by the matter fluids present.

   Eq. (29) indicates that $f(R_0)$ behaves as an effective cosmological constant and by using the Einstein equations, we effectively obtain $R_0 = \tilde{R}_0 - \kappa^2 T_{\mathrm{matter}}$. Note that T_{matter} expresses the trace of the matter energy-momentum tensor.

   Since we are now considering the time scale to be a few billion years, this implies that $f'(R_0)$ does not need to vanish completely but instead it should behave as, $|f'(R_0)| \ll \left(10^{-33}\,\mathrm{eV}\right)^4$.

   We should note that Eq. (29) indicates that $f(R)$ could be negative at the present Universe. Then, if we consider the model of Eq. (517) in the early Universe, $f(R)$ should cross zero in the past.

   The de Sitter space-time given by Eq. (29) should be stable. This requires that when $R \sim R_0 \sim \left(10^{-33}\,\mathrm{eV}\right)^2$, the function $F(R)$ behaves as in Eq. (37). In some cases this condition might not be strictly necessary, but we can use this condition in order to avoid a future finite-time singularity. In the case $n = 0$, the condition of Eq. (37) can be expressed as follows,

$$-R_0 + \frac{F'(R_0)}{F''(R_0)} > 0\,. \tag{518}$$

3. The third condition is given by Eq. (30), which indicates that there is a flat space-time solution. Furthermore if we require that in the small curvature limit, that is when $R \to 0$, the Einstein gravity solution should be recovered, we require Eq. (36). If the condition of Eq. (36) is satisfied, then the condition Eq. (30) is automatically satisfied.

4. In some models of $F(R)$ gravity, a curvature singularity is easily generated, when the density of the matter present is large. In order to avoid this issue, we may require that in the large curvature limit, the $F(R)$ gravity behaves as in Eq. (39). Instead of Eq. (39), $F(R)$ may behaves as in Eq. (40).

5. Since the expression $1/F'(R)$ plays the role of the effective gravitational constant, that is $1/F'(R) \sim \kappa^2$, in order to avoid the antigravity era, we should require (41) or (42). If the antigravity era occurs, the graviton becomes a ghost and in effect it generates negative norm states in the corresponding quantum theory.

6. By combining conditions (36) and (41), we find the following condition, that is, (43), and we need to note that this is not an independent one.

7. As pointed in Ref. [301], an instability could occur due to matter itself, that is a matter instability. When one considers matter perturbations of the scalar curvature, in the Einstein gravity, the perturbations often increase rapidly. In order to avoid the matter instabilities, we need to require the following,

$$U(R_b) \equiv \frac{R_b}{3} - \frac{F^{(1)}(R_b)F^{(3)}(R_b)R_b}{3F^{(2)}(R_b)^2} - \frac{F^{(1)}(R_b)}{3F^{(2)}(R_b)} + \frac{2F(R_b)F^{(3)}(R_b)}{3F^{(2)}(R_b)^2} - \frac{F^{(3)}(R_b)R_b}{3F^{(2)}(R_b)^2} < 0\,. \tag{519}$$

If the function $U(R_b)$ is positive, the time scale of the instability is given by the expression $1/\sqrt{U(R_b)}$. Then, if we consider the model $F(R) = R - \frac{\mu^4}{R}$ for example, the time scale becomes of the order 10^{-26}sec on the earth, and in effect the gravitational field would be very strong.

8. If we rewrite the $F(R)$ gravity in a scalar-tensor form, it can be seen that the corresponding scalar field σ has a direct coupling with matter. If the mass of the scalar field σ has the same order as the typical scale of the accelerating expansion of the Universe at present time, the propagation of the scalar field σ generates long range forces and hence it generates large corrections to the Newton law. In effect, if the Chameleon mechanism is used, the mass of the scalar field σ becomes large enough, and in effect, the force induced by the propagation of the scalar σ becomes short range. Hence, there would be no conflict with the present solar system experiments or observations. Therefore, the mass of the scalar field σ in Eq. (22), should become large enough owing to the Chameleon mechanism.

   It should be noted however, that a very large mass could be in conflict with the matter abundance, see for example Ref. [573]. If we quantize the scalar field σ, a scalar particle is produced, which may be created by the reheating process. If this scalar field is very heavy, the particle may decay to the standard model particles very excessively, which could be in conflict with observations.

If we require both the conditions of Eqs. (37) and (36), the extra, unstable de Sitter solution at $R = R_e$ $(0 < R_e < R_0)$ may occur. As it is known, the evolution of the Universe will stop at $R = R_0$, since the de Sitter solution $R = R_0$ should be stable, and therefore the curvature never becomes smaller than R_0. In effect, the extra de Sitter solution could not be realized in the evolution of the Universe.

In Ref. [333], an $F(R)$ gravity model was proposed, which satisfies all the above conditions, and it is the following,

$$\begin{aligned}\frac{F(R)}{R^2} =& \Big\{\left(X_m\left(R_I;R\right) - X_m\left(R_I;R_1\right)\right)\left(X_m\left(R_I;R\right) - X_m\left(R_I;R_0\right)\right)^{2n+2} \\ &+X_m\left(R_I;R_1\right)X_m\left(R_I;R_0\right)^{2n+2} + f_\infty^{2n+3}\Big\}^{\frac{1}{2n+3}} ,\end{aligned} \tag{520}$$

where,

$$X_m\left(R_I;R\right) \equiv \frac{(2m+1)\,R_I^{2m}}{\left(R-R_I\right)^{2m+1} + R_I^{2m+1}} . \tag{521}$$

We choose the parameters n and m to be integers greater or equal to unity, that is $n, m \geq 1$, and also the parameter R_1 is related to the curvature R_e as follows,

$$X\left(R_I;R_e\right) = \frac{(2n+2)\,X\left(R_I;R_1\right)X\left(R_I;R_1\right) + X\left(R_I;R_0\right)}{2n+3} . \tag{522}$$

In addition, it is compelling to assume that,

$$0 < R_1 < R_0 \ll R_I . \tag{523}$$

Let us see how the inflationary era can be realized by the model (520). The behavior of (38) in the limiting case when $R \sim R_I$, is given below,

$$\begin{aligned}f_{0I} =& \Big\{\left(X_m\left(R_I;R_I\right) - X_m\left(R_I;R_1\right)\right)\left(X_m\left(R_I;R_I\right) - X_m\left(R_I;R_0\right)\right)^{2n+2} \\ &+X_m\left(R_I;R_1\right)X_m\left(R_I;R_0\right)^{2n+2} + f_\infty^{2n+3}\Big\}^{\frac{1}{2n+3}} \\ f_{1I} =& \frac{2m+1}{R_I^{2m+2}}\Big\{\left(X_m\left(R_I;R_I\right) - X_m\left(R_I;R_1\right)\right)\left(X_m\left(R_I;R_I\right) - X_m\left(R_I;R_0\right)\right)^{2n+2} \\ &+X_m\left(R_I;R_1\right)X_m\left(R_I;R_0\right)^{2n+2} + f_\infty^{2n+3}\Big\}^{\frac{2(n+1)}{2n+3}}\Big\{\left(X_m\left(R_I;R_I\right) - X_m\left(R_I;R_0\right)\right)^{2n+2} \\ &+(2n+2)\left(X_m\left(R_I;R_I\right) - X_m\left(R_I;R_1\right)\right)\left(X_m\left(R_I;R_I\right) - X_m\left(R_I;R_0\right)\right)^{2n+1}\Big\} .\end{aligned} \tag{524}$$

Therefore, the condition (38) is satisfied. In effect, a de Sitter space-time solution exists, and this solution describes the inflationary era. Due to the fact that the condition (38) is satisfied, the de Sitter solution is quasi-stable, and specifically it is stable at large curvatures, but as the curvature decreases, it gradually becomes unstable. Due to the quasi-stability, the exit from inflation occurs in this theory, something guaranteed by the gradual decrease of the curvature during the inflationary era.

Now we demonstrate how the model at hand can realize the present-time acceleration of the Universe. In the limit $R \sim R_0$, we have,

$$\begin{aligned} f_{0L} =& \left\{ X_m\left(R_I; R_1\right) X_m\left(R_I; R_0\right)^{2n+2} + f_\infty^{2n+3} \right\}^{\frac{1}{2n+3}} , \\ f_{1L} =& \frac{1}{2m+3} \left\{ X_m\left(R_I; R_1\right) X_m\left(R_I; R_0\right)^{2n+2} + f_\infty^{2n+3} \right\}^{-\frac{2(n+1)}{2n+3}} \left(X_m\left(R_I; R_1\right) - X_m\left(R_I; R_0\right)\right) \\ & \times \frac{(2m+1)^{4(m+1)} \left\{R_I\left(R_0 - R_I\right)\right\}^{4m(m+1)}}{\left\{ \left(R_0 - R_I\right)^{2m+1} + R_I^{2m+1} \right\}^{4(m+1)}} . \end{aligned} \tag{525}$$

In effect, the condition of Eq. (37) is satisfied and therefore the stable de Sitter solution exists, which corresponds to the current accelerating expansion of the Universe.

We now discuss the conditions of Eqs. (30) or (36). The function $X\left(R_I; R\right)$ should be a monotonically decreasing function of the curvature R and we should also require that, in the limit $R \to 0$, the function $X\left(R_I; R\right)$ behaves as follows,

$$X\left(R_I; R\right) \to \frac{1}{R} . \tag{526}$$

Therefore, in the limit $R \to 0$, the $F(R)$ gravity in (520) actually reproduces the behavior of Eq. (36) and in addition it satisfies the condition (36).

An important issue must be discussed at this point, having to do with curvature singularities, generated when the matter density is large. In the large curvature limit, that is when $R \to \infty$, we find the following,

$$X\left(R_I; R\right) \to \frac{(2m+1)\, R_I^{2m}}{R^{2m+1}} \to 0 , \tag{527}$$

which tells that $F(R)$ behaves as (39) and therefore it is not possible for a curvature singularity to occur.

Also we can easily check the conditions for the absence of an antigravity era, namely (41) or (42), since the $F(R)$ function in (520) satisfies Eqs. (36) and (39) in the limiting cases $R \to 0$ or $R \to \infty$. In the interval $R_e < R < R_0$, Eq. (41) or (42) are trivially satisfied because the function $\frac{F(R)}{R^2}$ is a monotonically increasing function of the curvature R. Even in the interval $R_1 < R_0 \ll R \ll R_I$, Eq. (41) or (42) are again satisfied, since the function $\frac{F(R)}{R^2}$ behaves as (530). Therefore, an antigravity does not appear for all curvature values that are considered.

Now let us investigate whether the model is free from matter instabilities as in Ref. [301], an issue completely determined by the condition (519). Since $R_1 < R_0 \ll R \ll R_I$ on the earth or the sun, we find that the behavior of the $F(R)$ function is given as in Eq. (530) and hence we find the following expression for $U(R_b)$ in Eq. (519),

$$U(R_b) \sim -\frac{f_n}{6 f_\infty} < 0 , \tag{528}$$

which indicates that the matter instability does not occur for the $F(R)$ model at hand, due to the fact that the function $U(R_b)$ is negative.

At this point it would be interesting to check whether the Chameleon mechanism [298, 574] applies in the case at hand, or not. In order to see this, we should study the behavior of the mass of the scalar field σ in Eq. (22). Since we are interested for the mass of the scalar field σ in the present-time Universe, we may focus our investigation in the interval $R_1 < R_0 \ll R \ll R_I$, where we can approximate the function $X_m\left(R_I; R\right)$ as follows,

$$X_m\left(R_I; R\right) \sim \frac{1}{R} , \quad X_m\left(R_I; R_1\right) \sim \frac{1}{R_1} , \quad X_m\left(R_I; R_0\right) \sim \frac{1}{R_0} . \tag{529}$$

By using Eq. (529), we can approximate the functional form of the $F(R)$ gravity as follows,

$$\frac{F(R)}{R^2} \sim f_\infty + \frac{f_n}{R} , \quad f_n \equiv \frac{R_1 + (2n+2)\, R_0 f_\infty^{-2n-2}}{(2n+3)\, R_1 R_0^{2n+2}} \sim \frac{f_\infty^{-2n-2}}{R_0^{2n+2}} . \tag{530}$$

Note that we have assumed $R_1 \sim R_0$ in the last equation of (530). In the limit $f_\infty R \ll f_n$, that is,

$$\frac{1}{f_\infty^{2n+3}} \gg R R_0^{2n+2}\,, \tag{531}$$

by using Eq. (22), we obtain,

$$m_\sigma^2 \sim \frac{3}{4f_\infty}\,. \tag{532}$$

In the region inside the earth, since $1\,g \sim 6\times 10^{32}\,\mathrm{eV}$ and also, $1\,\mathrm{cm} \sim \left(2\times 10^{-5}\,\mathrm{eV}\right)^{-1}$, the density of the matter is approximately equal to $\rho_{\mathrm{matter}} \sim 1\mathrm{g/cm^3} \sim 5\times 10^{18}\,\mathrm{eV}^4$ and in effect, the curvature is approximated by $R \sim \kappa^2\rho_{\mathrm{matter}} \sim \left(10^{-19}\,\mathrm{eV}\right)^2$. On the other hand, in the atmosphere of the earth, we find, $\rho_{\mathrm{matter}} \sim 10^{-6}\mathrm{g/cm^3} \sim 10^{12}\,\mathrm{eV}^4$ and therefore $R \sim \kappa^2\rho_{\mathrm{matter}} \sim \left(10^{-25}\,\mathrm{eV}\right)^2$. In the solar system, where the interstellar gas exists, since one proton (or a hydrogen atom) is contained in a volume of the order $1\,\mathrm{cm}^3$ in the interstellar gas, we find $\rho_{\mathrm{matter}} \sim 10^{-5}\,\mathrm{eV}^4$, $R_0 \sim 10^{-61}\,\mathrm{eV}^2$. Then, we find that the condition of Eq. (531) is satisfied. As an example, we may choose $\frac{1}{f_\infty} \sim \mathrm{MeV}^2$, which yields a very small Compton length of the scalar field and in effect the resulting corrections to the Newton law are practically negligible.

Now we discuss in brief some general cosmological features of the model (520). Particularly, after the exit from inflation at $R = R_I$, radiation and matter could be produced, during the reheating process. It is possible that matter and radiation may dominate over the contribution from the $f(R)$ gravity (recall that the $f(R)$ gravity is the modified gravity part of $F(R) = R + f(R)$) and therefore the cosmological evolution thereafter is identical with that of the Einstein gravity. In effect, the radiation domination and the matter domination era of the Universe may be realized. Eventually, when the curvature of the Universe becomes of the order $R \sim R_0$, the late-time acceleration of the Universe is realized.

An interesting possibility of having a double-inflationary era could be realized, if the condition (40) holds true. Recall that the condition (40) expresses the fact that large curvatures cannot be generated, and hence an R^2 term may be generated in the very large curvature limit of the theory. It is well known that such R^2 terms are responsible for inflation. This new inflationary era might be added to the inflationary era which occurs when $R = R_I$, if the Universe started its evolution with a very high curvature, that is $R \gg R_I$.

In some models in the literature, the phenomenological implications are quite interesting, although not all the viability conditions are satisfied. For example, the following realistic model was proposed in Ref. [95]

$$f(R) = \frac{\alpha R^{2n} - \beta R^n}{1 + \gamma R^n}\,, \tag{533}$$

where, α, β, and γ are positive constants and the parameter n is a positive integer. The model of Eq. (533) has the de Sitter space-time solution, with its curvature being, [95]

$$R_0 = \left\{\left(\frac{1}{\gamma}\right)\left(1 + \sqrt{1 + \frac{\beta\gamma}{\alpha}}\right)\right\}^{1/n}\,, \tag{534}$$

and, therefore, the quantity $\tilde{R}_0$ in Eq. (29) takes the following form,

$$f(R_0) \sim -2\tilde{R}_0 = \frac{\alpha}{\gamma^2}\left(1 + \frac{\left(1 - \frac{\beta\gamma}{\alpha}\right)\sqrt{1 + \frac{\beta\gamma}{\alpha}}}{2 + \sqrt{1 + \frac{\beta\gamma}{\alpha}}}\right)\,. \tag{535}$$

Then we easily find that,

$$\alpha \sim 2\tilde{R}_0 R_0^{-2n}\,, \quad \beta \sim 4\tilde{R}_0^2 R_0^{-2n} R_I^{n-1}\,, \quad \gamma \sim 2\tilde{R}_0 R_0^{-2n} R_I^{n-1}\,. \tag{536}$$

As it was shown in [95], the correction to the Newton law is small for the model (533). This is due to the fact that the mass m_σ appearing in Eq. (18), is given by $m_\sigma^2 \sim 10^{-160+109n}\,\mathrm{eV}^2$ in the solar system and $m_\sigma^2 \sim 10^{-144+98n}\,\mathrm{eV}^2$ in the atmosphere of the earth, and therefore it is large enough [95].

Now we discuss another model which has some interesting cosmological features, which was proposed in Ref. [575] (see also [98] and [576]) which has the following form:

$$F(R) = R - 2\Lambda\left(1 - \mathrm{e}^{-\frac{R}{R_0}}\right) - \Lambda_i\left(1 - \mathrm{e}^{-\left(\frac{R}{R_i}\right)^n}\right) + \gamma R^\alpha\,. \tag{537}$$

For later convenience, we may define,

$$f_i = -\Lambda_i \left(1 - \mathrm{e}^{-\left(\frac{R}{R_i}\right)^n} \right) , \tag{538}$$

We choose the constants R_i and Λ_i to have the typical values of the scalar curvature and of the expected cosmological constant during the inflationary era, respectively, that is, $R_i,\, \Lambda_i \simeq 10^{20-38}\mathrm{eV}^2$. We also assume the parameter n to be $n > 1$. In the last term of (537), namely $\sim \gamma R^\alpha$, the parameter γ is assumed to be a positive constant and α is a real number. We need to note that the latter term is important in order to achieve the exit from inflation.

Let us first discuss how inflation and the exit from the inflation can be realized in the context of the model (537). Since the curvature R should be large during the inflationary era, we may assume that $R \gg R_i$. Then, f_i behaves as a cosmological constant $f_i \sim -\Lambda_i$, which effectively generates the early-time acceleration of the Universe. In order to satisfy the condition of Eq. (41), and to avoid antigravity, we must have,

$$R_i > \Lambda_i n \left(\frac{n-1}{n} \right)^{\frac{n-1}{n}} \mathrm{e}^{-\frac{n-1}{n}} . \tag{539}$$

Let us illustrate the previous arguments by using a simple example and by choosing $n = 4$, Eq. (539) is satisfied for $R_i > 1.522\,\Lambda_i$. Although f_i behaves as an effective cosmological constant, we shall investigate if the de Sitter space-time solution actually exists. We denote the de Sitter space-time curvature as $R = R_{\mathrm{dS}}$ and when the curvature R_{dS} is very large, that is, $R_{\mathrm{dS}} \gg R_i$, since $\Lambda_i \sim R_i$, the condition for the existence of the de Sitter solution, namely, $0 = 2F(R_{\mathrm{dS}}) - R_{\mathrm{dS}} F'(R_{\mathrm{dS}})$ has a non-trivial solution if $\Lambda_i \sim R_i$, and in turn this holds true when,

$$2 < \alpha < 3 . \tag{540}$$

By using Eq. (516), we find that the resulting solution describing the de Sitter space-time, is strongly unstable and due to this fact, the graceful exit from inflation comes as an outcome. Indeed, the instability of the de Sitter space-time is the following,

$$5/2 \leq \alpha < 3 , \tag{541}$$

and we can estimate the characteristic time of the instability as follows,

$$t_i \sim \frac{1}{\sqrt{R_{\mathrm{dS}}}} \sim 10^{-10} - 10^{-19}\,\mathrm{sec} . \tag{542}$$

Hence, the graceful exit from inflation surely occurs.

The inflationary era study of the model (537) has also been investigated in detail in Ref. [576] and the following estimates of the e-foldings number have been found,

$$N \simeq 107 \quad (\text{for} \quad \alpha = 5/2) , \quad N \simeq 64 \quad (\text{for} \quad \alpha = 8/3) , \quad N \simeq 51 \quad (\text{for} \quad \alpha = 11/4) . \tag{543}$$

More concretely, if we specify the parameters in Eq. (537) as follows,

$$n = 4 , \quad \alpha = \frac{5}{2} , \tag{544}$$

and also if we choose the value of the curvature R_i as,

$$R_i = 2\Lambda_i , \tag{545}$$

then we find that R_i satisfies the condition (539) and in effect, the antigravity era is avoided. The de Sitter space-time solution can be found by solving the equation, $0 = 2F(R_{\mathrm{dS}}) - R_{\mathrm{dS}} F'(R_{\mathrm{dS}})$ and this has as solution,

$$R_{\mathrm{dS}} = 4\Lambda_i . \tag{546}$$

Since we choose n to be large enough, the curvature R_{dS} becomes sufficiently large, when it is compared with R_i, and we find that $f_i(R_{\mathrm{dS}}) \simeq -\Lambda_i$. In addition, it can be seen that the only non-trivial solution of the equation $0 = 2F(R_{\mathrm{dS}}) - R_{\mathrm{dS}} F'(R_{\mathrm{dS}})$, is given in Eq. (546), and it can be seen that this solutions satisfies the de Sitter instability condition of Eq. (516). In turn, this indicates that the Universe ultimately exits from inflation, and tends towards a minimal attractor at $R = 0$. Alternatively, it is possible that the Universe might develop a singularity when $R \to \infty$. In order to avoid the curvature singularity, we need to require that the condition (38) for the quasi-stability is satisfied. In effect, the de Sitter solution is stable at large curvatures, but as the curvature gradually drops, the de Sitter solution becomes unstable.

Let us now discuss how the late-time acceleration era can be realized in the context of the $F(R)$ gravity model at hand. First let us point out that prior to the late-time acceleration era, the matter domination era occurs, and the parameter n in Eq. (538) was introduced in order to avoid the effects of inflation during the matter domination era. In fact, since $R \ll R_i$, if $n > 1$, we find

$$R \gg |f_i(R)| \simeq \frac{R^n}{R_i^{n-1}}\,, \tag{547}$$

and therefore f_i is suppressed during the matter domination era. The last term γR^α in Eq. (537) was introduced in order to generate the graceful exit from inflation, as we already discussed. If we choose $\gamma \sim 1/R_i^{\alpha-1}$ and $\alpha > 1$, the contribution from this term becomes practically negligible, when the small curvature regime is considered, and hence during the matter domination era, when $R \ll R_i$. In fact, we find that,

$$R \gg \frac{R^\alpha}{R_i^{\alpha-1}}\,. \tag{548}$$

It is easy to check that Eq. (30) or (36) are satisfied.

In order for the late-time acceleration to occur, it is compelling to have the stable asymptotic de Sitter solution in the theory. This is guaranteed if the condition in Eq. (518) is satisfied, and this happens when,

$$n > \alpha\,. \tag{549}$$

Also the de Sitter vacuum exists at late time, with curvature $R = R_{\mathrm{dS}}$ [576]. For the model (544), R_i satisfies the condition in Eq. (539), and hence the antigravity era is avoided.

A variant model of that developed in [575], was constructed in Ref. [98], in which case the $F(R)$ gravity contains only the first two terms of (537), namely, $R - 2\Lambda\left(1 - \mathrm{e}^{-\frac{R}{R_0}}\right)$. During the late-time era, the other terms in (537) can be neglected, and the model of Ref. [98] coincides with that in Eq. (537). In the limit $R \gg R_0$, we have $F(R) \simeq R - 2\Lambda$, and hence the model mimics the ΛCDM model. For simplicity, consider the model of Eq. (544). In the late-time era, we have,

$$F(R) = R - 2\Lambda(1 - \mathrm{e}^{-R/R_0})\,, \tag{550}$$

and therefore,

$$F'(R) = 1 - \frac{2\Lambda}{R_0}\mathrm{e}^{-R/R_0}\,, \tag{551}$$

$$F''(R) = \frac{2\Lambda}{R_0^2}\mathrm{e}^{-R/R_0}\,. \tag{552}$$

In the limit $R \gg R_0$, the second term in $F'(R)$ appearing in Eq. (551), can be safely neglected. In effect, we find $F'(R) \sim 1$, and therefore the antigravity is avoided in the present model at late times. Since $F(R) \simeq R$ in the limit $R \gg R_0$, we expect that the matter dominated era can be realized as in the Einstein-Hilbert action. At late times, a non-trivial solution of the equation $0 = 2F(R_{\mathrm{dS}}) - R_{\mathrm{dS}}F'(R_{\mathrm{dS}})$, is $R = 4\Lambda$, as in the case of (546). In contrast to the inflationary era de Sitter solution, the late-time de Sitter vacuum satisfies the condition (518) for small curvatures, and hence this solution is stable. Hence, this late-time de Sitter vacuum is the final and stable attractor of the model at hand.

By using the expression in Eq. (22), the mass of the scalar field during the late-time era is estimated to be equal to,

$$m_\sigma^2 \sim \frac{R_0^2 \mathrm{e}^{\frac{R}{R_0}}}{4\Lambda}\,, \tag{553}$$

which is positive, and hence the late-time asymptotic de Sitter space-time is stable. Since Λ in Eq. (537) plays the role of an effective cosmological constant in the late-time Universe, we have,

$$\Lambda \sim R_0 \sim \left(10^{-33}\,\mathrm{eV}\right)^2\,. \tag{554}$$

In the solar system, we find $R \sim 10^{-61}\,\mathrm{eV}^2$, and in effect we find that $m_\sigma^2 \sim 10^{1,000}\,\mathrm{eV}^2$, which is ultimately heavy. In effect, the corrections to the Newton law are negligible. Indeed, in the atmosphere of the earth, we have

$R \sim 10^{-50}\,\mathrm{eV}^2$, and even in the case we choose $1/b \sim R_0 \sim \left(10^{-33}\,\mathrm{eV}\right)^2$, we find that $m_\sigma^2 \sim 10^{10,000,000,000}\,\mathrm{eV}^2$. Therefore the corrections to the Newton law are negligible. There is an issue however having to do with the effects of having such a large mass, due to the excess of the particles produced by the scalar during the reheating, and this could be in conflict with current observations. In addition, if the Compton length of the scalar particle becomes smaller that the interatomic distance, the matter fluids cannot be approximated in the general relativistic way, that is as perfect fluids, and hence, the Chameleon mechanism cannot work in this case. This however could set an upper limit constraint on the mass of the scalar particle σ.

Let us work out some essential features of the dark energy era corresponding to the model of Eq. (537). The focus will be on the dark energy density evolution, which is equal to $\rho_{\mathrm{DE}} = \rho_{\mathrm{eff}} - \rho/F'(R)$. From the FRW equation we can identify the dark energy density as follows,

$$\frac{3}{\kappa^2}H^2 = \rho_{\mathrm{eff}} \equiv \frac{1}{F'(R)}\left\{\rho + \frac{1}{2\kappa^2}\left[\left(F'(R)R - F(R)\right) - 6H\dot{F}'(R)\right]\right\} . \tag{555}$$

We also use the new variable y_{H} defined by

$$y_{\mathrm{H}} \equiv \frac{\rho_{\mathrm{DE}}}{\rho_m^{(0)}} = \frac{H^2}{\tilde{m}^2} - a^{-3} - \chi a^{-4} , \tag{556}$$

where $\rho_m^{(0)}$ stands for the energy density of matter at present time, and $\tilde{m}^2$ is the mass scale

$$\tilde{m}^2 \equiv \frac{\kappa^2\rho_m^{(0)}}{3} \simeq 1.5 \times 10^{-67}\mathrm{eV}^2 . \tag{557}$$

Also we define χ as follows,

$$\chi \equiv \frac{\rho_r^{(0)}}{\rho_m^{(0)}} \simeq 3.1 \times 10^{-4} , \tag{558}$$

where $\rho_r^{(0)}$ is the energy density of radiation at present time.

By using the conservation law,

$$\frac{d\rho_{\mathrm{DE}}}{d\left(\ln a\right)} + 3\left(\rho_{\mathrm{DE}} + p_{\mathrm{DE}}\right) = 0 , \tag{559}$$

we obtain the following expression for the EoS dark energy EoS parameter w_{DE},

$$w_{\mathrm{DE}} = \frac{p_{\mathrm{DE}}}{\rho_{\mathrm{DE}}} = -1 - \frac{1}{3}\frac{1}{y_{\mathrm{H}}}\frac{dy_{\mathrm{H}}}{d\left(\ln a\right)} . \tag{560}$$

By combining Eq. (555) and the expression of the scalar curvature for the FRW Universe, which is, $R = 12H^2 + 6\dot{H}$, we find the following differential equation,

$$\frac{d^2 y_{\mathrm{H}}}{d(\ln a)^2} + J_1\frac{dy_{\mathrm{H}}}{d(\ln a)} + J_2 y_{\mathrm{H}} + J_3 = 0 , \tag{561}$$

which completely determines the evolution. The functions J_1, J_2, and J_3 are defined as follows,

$$\begin{aligned} J_1 &= 4 + \frac{1}{y_{\mathrm{H}} + a^{-3} + \chi a^{-4}}\frac{1 - F'(R)}{6\tilde{m}^2 F''(R)} , \quad J_2 = \frac{1}{y_{\mathrm{H}} + a^{-3} + \chi a^{-4}}\frac{2 - F'(R)}{3\tilde{m}^2 F''(R)} , \\ J_3 &= -3a^{-3} - \frac{(1 - F'(R))(a^{-3} + 2\chi a^{-4}) + (R - F(R))/(3\tilde{m}^2)}{y_{\mathrm{H}} + a^{-3} + \chi a^{-4}}\frac{1}{6\tilde{m}^2 F''(R)} . \end{aligned} \tag{562}$$

In Eq. (562), the scalar curvature R is expressed in terms of the quantity y_{H} as follows,

$$R = 3\tilde{m}^2\left(\frac{dy_{\mathrm{H}}}{d\ln a} + 4y_{\mathrm{H}} + a^{-3}\right) . \tag{563}$$

Also we choose the free parameters appearing in Eq. (537) as follows,

$$\Lambda = (7.93)\tilde{m}^2 , \quad \Lambda_i = 10^{100}\Lambda , \quad R_i = 2\Lambda_i , \qquad n = 4 ,$$

$$\alpha = \frac{5}{2}\,, \qquad \gamma = \frac{1}{(4\Lambda_i)^{\alpha-1}}\,, \quad R_0 = 0.6\Lambda\,, \quad 0.8\Lambda\,, \quad \Lambda\,. \tag{564}$$

In Ref. [575], Eq. (561) was solved numerically, in the range of $R_0 \ll R \ll R_i$, which includes the matter domination era and the current acceleration epoch. Then $y_{\rm H}$ can be expressed as a function of the redshift, which is equal to $z = \frac{1}{a} - 1$. For the numerical calculations, the initial conditions at $z = z_i$ were chosen in Ref. [575], as follows,

$$\left.\frac{dy_{\rm H}}{d(z)}\right|_{z_i} = 0\,, \quad y_{\rm H}|_{z_i} = \frac{\Lambda}{3\tilde{m}^2}\,, \tag{565}$$

which are practically the initial conditions of the ΛCDM model. By using these initial conditions, the cosmological evolution of the model at hand, is very close to that of the ΛCDM, when the large redshift regime is considered. We can choose the values of z_i, in such a way so that, $RF''(z = z_i) \sim 10^{-5}$, by simply assuming that $R = 3\tilde{m}^2(z+1)^3$. In effect, we have, $z_i = 1.5,\ 2.2,\ 2.5$ for $R_0 = 0.6\Lambda,\ 0.8\Lambda,\ \Lambda$, respectively. Note that the choice of the parameters were made in view of the latest WMAP results, in combination with Baryon Acoustic Oscillations and Supernovae surveys [577]. The results of the numerical analysis indicate that the dark energy EoS parameter is very close to the phantom divide line -1. At the present epoch, where $z = 0$, we find that $w_{\rm DE} = -0.994,\ -0.975,\ -0.950$ for $R_0 = 0.6\Lambda,\ 0.8\Lambda,\ \Lambda$ respectively. When R_0 decreases, the behavior of the model at hand (537), becomes indistinguishable from the ΛCDM model, in which case $w_{\rm DE} = -1$. Also the numerical analysis shows that the phantom divide crossing does not cause any serious phenomenological issues to the model at hand (537). Particularly, in the limit $z \to -1^+$, the scalar curvature R, becomes approximately equal to $12\tilde{m}^2 y_{\rm H}(z \to -1^+)$, and it acts like an effective cosmological constant. Practically, this effective cosmological constant is the stable late-time de Sitter attractor of the theory.

It is known that for some viable models, the dark energy density oscillates during the matter domination epoch, and the corresponding frequencies might diverge [576]. These dark energy oscillations were also pointed out in Ref. [578]. Due to the fact that the $F(R)$ gravity equations of motion include fourth order derivatives, the scalar modes oscillations occur, in addition to the graviton oscillations. In the scalar-tensor form of the $F(R)$ gravity, the corresponding oscillations are quantified in terms of the scalar field σ, and hence the frequency of the oscillations are affected by the mass of the scalar field σ. Practically, the scalar field σ is the scalar curvature R, and therefore any high-frequency oscillations occur, the background will also oscillate rapidly. This in effect could make the perturbative analysis inapplicable [576], due to the occurrence of non-linearities. In a later section we shall discuss the dark energy oscillations issue in a more concrete way.

D. Unification of Inflation with Dark Energy Era in Modified Gauss-Bonnet Gravity

We may use the formulation of Eq. (513) for the model that unifies the inflationary era with the accelerating expansion in the present Universe. Then, we may consider a model, for which the Hubble rate H becomes a large constant during early times, that is, for small $a = \mathrm{e}^N$, and at late times, the Hubble rate becomes a small constant. For example, consider the following model,

$$H(N) = \frac{H_e \mathrm{e}^{-N} + H_l \mathrm{e}^{N}}{\mathrm{e}^{-N} + \mathrm{e}^{N}}\,. \tag{566}$$

Then at early times, where $N \to -\infty$, we have $H \to H_e$. On the other hand, in at late times, where $N \to +\infty$, we have, $H \to H_l$. Then if we choose $H_e \gg H_l$, the inflationary era and the late-time acceleration can be realized. Then, by choosing $h(\phi)$ in Eq. (513) as follows,

$$h(\phi) = \frac{H_e \mathrm{e}^{-\phi} + H_l \mathrm{e}^{\phi}}{\mathrm{e}^{-\phi} + \mathrm{e}^{\phi}}\,, \tag{567}$$

the unification of early and late-time acceleration can be achieved easily for the Gauss-Bonnet gravity, by using the formalism we presented in sections II and III. We do not proceed in the details of this for brevity, since details of this can be found in the previous sections.

E. Phantom Dark Energy Era

An intriguing possibility for the late-time Universe is that the effective equation of state (EoS) of the Universe, namely $w_{\rm eff}$, ultimately crosses the phantom divide line which is $w_{\rm eff} = -1$. This possible evolution era of our Universe is known as phantom dark energy era, and in this section we shall present how this era can be realized in the context of $F(R)$ gravity. The motivation for studying a phantom dark energy era comes from the latest Planck data [579], but also from the earlier WMAP data [580] and from Type Ia supernovae gold dataset, where the possibility of crossing the phantom divide line is stressed. Particularly, with regard to the Planck data, the EoS parameter is constrained to be $w_{\text{eff}} = -1.54^{+0.62}_{-0.50}$, which is approximately 2σ in the phantom domain, so this clearly is a significant motivation to study a phantom dark energy era.

The possibility of having a phantom dark energy era in the context of modified gravity was stressed in Ref. [490], and in this section we shall present the most significant outcomes of this work. Apart from the modified gravity approach, there exist in the literature various different approaches with regard to the phantom dark energy era, for example in Ref. [249] a superacceleration era is realized, in Refs. [581, 582] a phantom dark energy era is realized in the context of scalar-tensor theories with non-minimal gravitational coupling between the scalar field and the scalar curvature, and in Refs. [354, 355] a non-minimal coupling between the scalar field and the Gauss-Bonnet term. Also in Ref. [583] the problem was addressed in the context of a scalar field with non-linear kinetic terms, and in Ref. [313] phantom dark energy was coupled with dark matter. Also in Ref. [250] a thermodynamical inhomogeneous dark energy model was introduced, and in Ref. [584] a multiple k-essence model was studied. Models with multiple scalar fields were studied in Refs. [585, 586], and in Refs. [252, 263, 587], quintom models were studied, which consisted of canonical and phantom scalar fields. Finally, string inspired models were studied in Refs. [588, 589].

Obviously the literature is rich on this issue, but in this section we shall briefly discuss how the phantom dark energy era can be realized by $F(R)$ gravity.

We start off our presentation by briefly mentioning the reconstruction method we shall use, which was developed in [293]. Consider an $F(R)$ gravity with action,

$$\mathcal{S} = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}F(R) + S_m(g_{\mu\nu}, \Psi_m)\,, \tag{568}$$

where S_m denotes the action of all the matter fluids present. For the FRW metric, the first FRW can be obtained by varying the action with respect to the metric, so it reads,

$$-18\left(4H(t)^2\dot{H}(t) + H(t)\ddot{H}(t)\right)F''(R) + 3\left(H^2(t) + \dot{H}(t)\right)F'(R) - \frac{F(R)}{2} + \kappa^2\rho = 0\,, \tag{569}$$

where $F'(R) = \frac{dF(R)}{dR}$. By using an auxiliary scalar field ϕ, the $F(R)$ gravity action (151) can be written as follows,

$$S = \int d^4x\sqrt{-g}\left(P(\phi)R + Q(\phi) + \mathcal{L}_{\text{matter}}\right)\,, \tag{570}$$

where the term $\mathcal{L}_m$ stands for the matter Lagrangian. The auxiliary functions $P(\phi)$ and $Q(\phi)$ will eventually enable us to determine the $F(R)$ gravity which realizes a specific cosmological evolution. By varying the action (570) with respect to the auxiliary scalar field ϕ, we obtain,

$$P'(\phi)R + Q'(\phi) = 0\,, \tag{571}$$

which will yield the function $\phi(R)$. Then, the resulting $F(R)$ gravity is,

$$F(\phi(R)) = P(\phi(R))R + Q(\phi(R))\,. \tag{572}$$

The function $P(\phi)$ satisfies the following differential equation,

$$\begin{aligned}
& -6H^2P(\phi(t)) - Q(\phi(t)) - 6H\frac{dP\left(\phi(t)\right)}{dt} + \rho_i = 0\,. \\
& \left(4\dot{H} + 6H^2\right)P(\phi(t)) + Q(\phi(t)) + 2\frac{d^2P(\phi(t))}{\mathrm{d}t^2} + \frac{dP(\phi(t))}{dt} + p_i = 0\,,
\end{aligned} \tag{573}$$

where $H(t)$ is the Hubble rate of a given evolution. By eliminating the function $Q(\phi(t))$, we obtain,

$$2\frac{d^2P(\phi(t))}{\mathrm{d}t^2} - 2H(t)P(\phi(t)) + 4\dot{H}\frac{dP(\phi(t))}{dt} + \rho_i + p_i = 0\,, \tag{574}$$

where ρ_i, p_i stand for the total energy density and pressure of the matter fluids which are present, respectively. As it is proved in the Appendix of Ref. [293], due to the mathematical equivalence of the actions (151) and (570), the auxiliary scalar field is identified with the cosmic time, that is $\phi = t$. Then, by assuming that the given cosmological evolution has the following scale factor,

$$a = a_0 \mathrm{e}^{g(t)}\,, \tag{575}$$

we can rewrite the differential equation (574) as follows,

$$2\frac{d^2 P(\phi(t))}{\mathrm{d}t^2} - 2g'(\phi)\frac{dP(\phi(t))}{dt} + 4g''(\phi)P(\phi(t)) + \sum_i (1+w_i)\rho_{i0} a_0^{-3(1+w_i)} \mathrm{e}^{-3(1+w_i)g(\phi)} = 0\,, \tag{576}$$

where w_i are the effective equation of state parameters for the perfect matter fluids that are present. Eventually, by finding the function $P(\phi)$, we may find easily $Q(\phi)$, by using Eq. (573), and its final form is,

$$Q(\phi) = -6g'(\phi)^2 P(\phi) - 6g'(\phi)\frac{dP(\phi)}{d\phi} + \sum_i (1+w_i)\rho_{i0} a_0^{-3(1+w_i)} \mathrm{e}^{-3(1+w_i)g(\phi)} \tag{577}$$

Let us apply this method in order to find the $F(R)$ gravity that realizes a cosmology which crosses the phantom divide line. Consider the vacuum $F(R)$ gravity case, in which case a solution to the differential equation (574) is,

$$P(\phi) = \mathrm{e}^{g(\phi)/2} p(\phi)\,, \quad g(\phi) = -10 \ln\left[\left(\frac{\phi}{t_0}\right)^{-\gamma} - C\left(\frac{\phi}{t_0}\right)^{\gamma+1}\right]\,,$$
$$p(\phi) = p_+ \phi^{\beta_+} + p_- \phi^{\beta_-}\,, \quad \beta_\pm = \frac{1 + \pm\sqrt{1 + 100\gamma(\gamma+1)}}{2}\,, \tag{578}$$

with γ and C being arbitrary positive constants, t_0 is the present time and $p_\pm$ being arbitrary constants. The function $g(\phi)$ diverges at the time instance t_s which is equal to,

$$t_s = t_0 C^{-1/(2\gamma+1)}\,, \tag{579}$$

so a Big Rip singularity occurs at t_s. Therefore we consider only the era $0 < t < t_s$, before the Big Rip singularity occurs. When $t \ll t_s$, the Hubble rate is approximately,

$$H(t) \sim \frac{10\gamma}{t}\,, \tag{580}$$

and by using the reconstruction method we just presented, the resulting $F(R)$ gravity for $t \ll t_s$ is [490],

$$F(R) \sim \left[\frac{(\frac{1}{t_0}\sqrt{60\gamma(20\gamma-1)R^{-1/2}})^\gamma}{1 - [\frac{1}{t_0}\sqrt{60\gamma(20\gamma-1)R^{-1/2}}]^{2\gamma+1}}\right]^5 R \sum_{j=\pm} \frac{5\gamma - 1 - \beta_j}{20\gamma - 1} p_j [60\gamma(20\gamma-1)]^{\beta_j/2} R^{-\beta_j/2}\,. \tag{581}$$

Accordingly, when $t \to t_s$, the Hubble rate reads,

$$H(t) \sim \frac{10}{t_s - t}\,, \tag{582}$$

and therefore the resulting $F(R)$ gravity is [490],

$$F(R) \sim \left(\frac{\frac{1}{t_0^\gamma}\left[t_s - 3\sqrt{140}R^{-1/2}\right]^\gamma}{1 - \left[1 - \frac{3\sqrt{140}R^{-1/2}}{t_s}\right]}\right)^5 R \sum_{j=\pm} p_j \left[t_s - 3\sqrt{140}R^{-1/2}\right]^{\beta_j}$$
$$\times \left[1 - \sqrt{\frac{20}{7}}\left[\frac{15}{84}t_s + (\beta_j - 15)R^{-1/2}\right]\frac{1}{t_s - 3\sqrt{140}R^{-1/2}}\right]\,. \tag{583}$$

It can be easily checked that the phantom divide is crossed for the above cosmology, since the EoS parameter $w_{\mathrm{eff}} = -1 - \frac{2\dot{H}}{H^2}$ for $t \to 0$ is equal to,

$$w_{\mathrm{eff}} = -1 + \frac{1}{15\gamma}\,, \tag{584}$$

which satisfies $w_{\text{eff}} > -1$. Accordingly, for $t > t_s$, the EoS is equal to $w_{\text{eff}} = -16/15$, which satisfies $w_{\text{eff}} < -1$, so the phantom divide line is crossed. It can be found that the phantom divide line is crossed at the time instance t_c, which is equal to,

$$t_c = t_s \left(-2\gamma + \sqrt{4\gamma^2 + \frac{\gamma}{\gamma+1}} \right)^{1/(2\gamma+1)} . \tag{585}$$

The phenomenological implications of a phantom scalar-tensor theory, when this is considered in the Jordan frame, are quite interesting and these were studied in [490], and as it was shown, a phantom scalar-tensor theory corresponds to a complex $F(R)$ gravity, when this theory is considered in the Jordan frame.

F. Dark Energy Oscillations in $F(R)$ Gravity Theories and Growth Index

In this section we shall address two important issues relevant to late-time dynamics, namely the dark energy oscillations issue [200, 576, 590–597] and the evolution of the growth index [576, 598], in the context of $F(R)$ gravity. Both issues are related to higher derivatives of the Hubble rate, so these are higher order effects in the late-time dynamics, and therefore these are non-trivial effects.

1. *Dark Energy Oscillations*

In order to reveal the dark energy oscillations issue, and to study it more formally, we rewrite the $F(R)$ gravity gravitational equations as follows,

$$\begin{aligned} 3F'H^2 &= k^2\rho_{\text{matter}} + \frac{1}{2}(F'R - F) - 3H\dot{F}' \\ -2F'\dot{H} &= k^2\left(\rho_{\text{matter}} + P_{\text{matter}}\right) + \ddot{F} - H\dot{F} , \end{aligned} \tag{586}$$

where ρ_{matter} is the total mass-energy density, which contains all the matter fluids present. In order to provide a generalized picture, we shall not restrict ourselves to perfect matter fluids, but we shall assume that the matter fluid consists of collisional matter [599, 600] and radiation, and therefore the total matter energy density is in the case at hand,

$$\rho_{\text{matter}} = \varepsilon_m + \rho_r^{(0)} a^{-4} , \tag{587}$$

where ε_m is,

$$\varepsilon_m = \rho_m \left[1 + \Pi_0 + w \ln\left(\frac{\rho_m}{\rho_m^{(0)}} \right) \right] , \tag{588}$$

and also ρ_m is,

$$\rho_m = \rho_{\text{matter}}^{(0)} \left(\frac{a_0}{a} \right)^3 . \tag{589}$$

It is conceivable that the non-collisional matter case is described by solely ρ_m. So by combining Eqs. (588) and (589), we eventually have,

$$\rho_{\text{matter}} = \rho_m^{(0)} a^{-3} \left[1 + \Pi_0 + 3w \ln(a) \right] + \rho_r^{(0)} a^{-4} . \tag{590}$$

Moreover, P_{matter} stands for the total pressure of the matter fluids present. We can rewrite the first equation in Eq. (586), as follows,

$$H^2 - (F' - 1)\left(H\frac{dH}{d\ln a} + H^2 \right) + \frac{1}{6}(F - R) + H^2 F'' \frac{dR}{d\ln a} = \frac{\rho_{\text{matter}}}{3} , \tag{591}$$

with R denoting the scalar curvature as usual, which can be rewrite as follows,

$$R = 12H^2 + 6H\frac{dH}{d\ln a} . \tag{592}$$

In order to provide a more generalized picture regarding the various types of collisional matter, we assume that ρ_m has the following form,

$$\rho_{\text{matter}} = \rho_m^{(0)} g(a) + \rho_r^{(0)} a^{-4} = \rho_m^{(0)} \left(g(a) + \chi a^{-4} \right) , \tag{593}$$

where $\rho_m^{(0)}$ and $\rho_r^{(0)} = \chi \rho_m^{(0)}$ denote the values of the mass energy density and of radiation, at present time. Also, $\chi \simeq 3.1 \times 10^{-4}$, and it is defined to be the ratio $\rho_r^{(0)}/\rho_m^{(0)}$. Effectively, the non-collisional nature of matter is quantified in terms of $g(a)$, and in the case of collisional non-relativistic matter, it is,

$$g(a) = a^{-3} \left[1 + \Pi_0 - 3w \ln(a) \right] . \tag{594}$$

The dark energy oscillations are higher order effects, and in order to perfectly quantify these effects, we need to introduce new variables in Eqs. (591), with the characteristic property that these new variables will vanish at high redshift regimes. During these regimes the $F(R)$ gravity modifications are expected to be negligible [596]. The new variables are,

$$y_H \equiv \frac{\rho_{\text{DE}}}{\rho_m^{(0)}} = \frac{H^2}{\tilde{m}^2} - g(a) - \chi a^{-4} , \tag{595a}$$

$$y_R \equiv \frac{R}{\tilde{m}^2} - \frac{dg(a)}{d \ln a} , \tag{595b}$$

where ρ_{DE} the total dark energy, energy density. Hence, the variable that quantifies the evolution is $y_H(z)$, which is the scaled dark energy density with the scaling factor being $\rho_m^{(0)}$. By using,

$$\frac{H^2}{\tilde{m}^2} - \frac{\rho_M}{2\tilde{m}^2} = \frac{H^2}{\tilde{m}^2} - g(a) - \chi a^{-4} = y_H , \tag{596}$$

from (595a), and by dividing Eqs. (591) by $\tilde{m}^2$, we get,

$$\frac{1}{\tilde{m}^2} \frac{dR}{d \ln a} = \left[-y_H + (F' - 1)\left(\frac{H}{\tilde{m}^2} \frac{dH}{d \ln a} + \frac{H^2}{\tilde{m}^2}\right) - \frac{1}{6\tilde{m}^2}(F - R) \right] \frac{1}{H^2 F''} . \tag{597}$$

Differentiating Eq. (595b), with respect to the variable $\ln a$ we obtain,

$$\frac{dy_R}{d \ln a} = \frac{1}{\tilde{m}^2} \frac{dR}{d \ln a} - \frac{d^2 g(a)}{d \ln a^2} , \tag{598}$$

and by using (597), we have,

$$\frac{dy_R}{d \ln a} = -\frac{d^2 g(a)}{d \ln a^2} + \left[-y_H + (F' - 1)\left(\frac{H}{\tilde{m}^2} \frac{dH}{d \ln a} + \frac{H^2}{\tilde{m}^2}\right) - \frac{1}{6\tilde{m}^2}(F - R) \right] \frac{1}{H^2 F''} . \tag{599}$$

Differentiating Eq. (595a) again with respect to the variable $\ln a$, we have,

$$\frac{H}{\tilde{m}^2} \frac{dH}{d \ln a} + \frac{H^2}{\tilde{m}^2} = \frac{1}{2} \frac{dy_H}{d \ln a} + \frac{1}{2} \frac{dg(a)}{d \ln a} + y_H + g(a) - \chi a^{-4} , \tag{600}$$

where we also used of Eq. (595a). By using Eqs. (600) and (599), we obtain,

$$\begin{aligned} \frac{dy_R}{d \ln a} = & -\frac{d^2 g(a)}{d \ln a^2} + \left[-y_H + (F' - 1) \left(\frac{1}{2} \frac{dy_H}{d \ln a} + \frac{1}{2} \frac{dg(a)}{d \ln a} + y_H + g(a) - \chi a^{-4} \right) \right. \\ & \left. - \frac{1}{6\tilde{m}^2}(F - R) \right] \frac{1}{\tilde{m}^2 F'' (y_H + g(a) + \chi a^{-4})} . \end{aligned} \tag{601}$$

So by differentiating Eq. (597) with respect to the variable $\ln a$, we obtain,

$$\frac{dy_H}{d \ln a} = \frac{2H}{\tilde{m}^2} \frac{dH}{d \ln a} - \frac{dg(a)}{d \ln a} + 4\chi a^{-4} . \tag{602}$$

Upon combining Eqs. (592) and (595a) we obtain,

$$\frac{2H}{\tilde{m}^2} \frac{dH}{d \ln a} = \frac{R}{3\tilde{m}^2} - \frac{4H^2}{\tilde{m}^2} = \frac{R}{3\tilde{m}^2} - 4y_H - 4g(a) - 4\chi a^{-4} , \tag{603}$$

and therefore Eq. (602) becomes,

$$\frac{dy_H}{d\ln a}v = \frac{R}{3\tilde{m}^2} - \frac{dg(a)}{d\ln a} - 4y_H - 4g(a)$$
$$= \frac{y_R}{3} - 4y_H - \frac{2dg(a)}{3d\ln a} - 4g(a)\,. \tag{604}$$

Moreover we can rewrite the scalar curvature as follows,

$$R = 3\tilde{m}^2\left(4y_H + 4g(a) + \frac{dy_H}{d\ln a} + \frac{dg(a)}{d\ln a}\right)\,. \tag{605}$$

By differentiating Eq. (604) with respect to the variable $\ln a$, we get,

$$\frac{d^2y_H}{d\ln a^2} = \frac{dy_R}{3d\ln a} - \frac{4dy_H}{d\ln a} - \frac{2}{3}\frac{d^2g(a)}{d\ln a^2} - 4\frac{dg(a)}{d\ln a}\,,$$

so by using Eq. (601), we get,

$$\frac{d^2y_H}{d\ln a^2} + \left(4 + \frac{1-F'}{6\tilde{m}^2F''(y_H + g(a) + \chi a^{-4})}\right)\frac{dy_H}{d\ln a} + \left(\frac{2-F'}{3\tilde{m}^2F''(y_H + g(a) + \chi a^{-4})}\right)y_H$$
$$+\left(\frac{d^2g(a)}{d\ln a^2} + 4\frac{dg(a)}{d\ln a} + \frac{(1-F')\left(3\frac{dg(a)}{d\ln a} + 6g(a) - 6\chi a^{-4}\right) + \frac{F-R}{\tilde{m}^2}}{18\tilde{m}^2F''(y_H + g(a) + \chi a^{-4})}\right) = 0\,. \tag{606}$$

By using the following relations,

$$\frac{d}{d\ln a} = -(z+1)\frac{d}{dz}\,, \tag{607a}$$
$$\frac{d^2}{d\ln a^2} = (z+1)\frac{d}{dz} + (z+1)^2\frac{d^2}{dz^2}\,. \tag{607b}$$

we can express all the previous results in terms of the redshift z, so by applying the above relations in Eq. (606), we obtain the following master differential equations, which dictates the way that the Universe evolves, which is,

$$\frac{d^2y_H}{dz^2} + \frac{1}{(z+1)}\left(-3 - \frac{F'(R)}{6\tilde{m}^2F''(R)(y_H + g(z) + \chi(z+1)^4)}\right)\frac{dy_H}{dz}$$
$$+\frac{1}{(z+1)^2}\frac{1-F'(R)}{3\tilde{m}^2F''(R)(y_H + g(Z) + \chi(z+1)^4)}y_H + \left(\frac{d^2g(z)}{dz^2} - \frac{3}{(z+1)}\frac{dg(z)}{dz}\right.$$
$$\left.-\frac{1}{(z+1)}\frac{F'(R)\left(-(z+1)\frac{dg(z)}{dz} + 2g(z) - 2\chi(z+1)^4\right) + \frac{F}{3\tilde{m}^2}}{6\tilde{m}^2F''(R)(y_H - g(z) + \chi(z+1)^4)}\right) = 0\,. \tag{608}$$

It is conceivable that the above differential equation is not easy to solve analytically, so we shall solve this numerically and we study the late-time evolution in terms of the resulting numerical solution. The matter profile shall be assumed to be that of Eq. (594), and we shall compare the cases that the matter is non-collisional to the collisional matter case. We assume that the $F(R)$ gravity is a modified exponential gravity of the form [576, 596],

$$F(R) = R - 2\Lambda\left(1 - \mathrm{e}^{\frac{R}{b\Lambda}}\right) - \tilde{\gamma}\Lambda\left(\frac{R}{3\tilde{m}^2}\right)^{1/3}\,, \tag{609}$$

where $\Lambda = 7.93\tilde{m}^2$ and $\tilde{\gamma} = 1/1000$ [576, 596, 597]. Also we assume that $\Omega_M = 0.279$, where Ω_M is the present value of the total energy density, and also that $w = 0.2$ and $w = 0.8$, so we study both these cases for collisional matter. Also the initial conditions are assumed to be [576, 596, 597],

$$y_H(z)\mid_{z=z_f} = \frac{\Lambda}{3\tilde{m}^2}\left(1 + \frac{z_f+1}{1000}\right)\,, \quad y'_{\mathrm{H}}(z)\mid_{z=z_f} = \frac{\Lambda}{3\tilde{m}^2}\frac{1}{1000}\,, \tag{610}$$

where $z_f = 10$ and also Λ is $\Lambda \simeq 11.89\mathrm{eV}^2$.

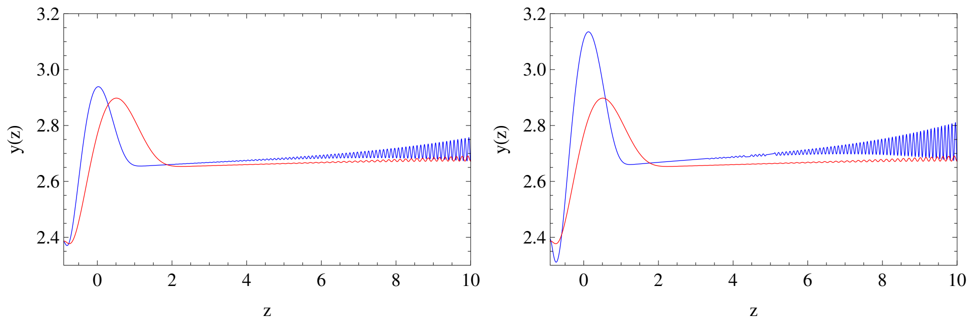

FIG. 1: Comparison of the scaled dark energy density $y_H(z) = \frac{\rho_{\rm DE}}{\rho_m^{(0)}}$ over z, for $w = 0.2$ (left) and $w = 0.8$ (right). The red line corresponds to non-collisional matter and the blue one to collisional matter.

We start our analysis with the numerical solution for $y_H(z)$, both for non-collisional and collisional matter, and we use the two values for the parameter w, namely $w = 0.2$ and $w = 0.8$. In Fig. 1 we compare the resulting $y_H(z)$ collisional and non-collisional matter cases. The blue curves correspond to collisional matter with $g(a)$ being given in Eq. (594), and the red curves correspond to non-collisional matter with $g(a) = a^{-3}$. The left plot corresponds to $w = 0.2$ and the right plot to $w = 0.8$. As it can be seen, the scaled dark energy $y_H(z) = \frac{\rho_{\rm DE}}{\rho_m^{(0)}}$ for the collisional matter, has a strong oscillatory behavior in comparison to the non-collisional matter. This behavior becomes more intense as the parameter w increases, and note that the two cases coincide when $w = 0$.

Let us now investigate the behavior of the dark energy oscillations, so we quantify these in terms of the dark energy equation of state parameter, which is defined as $\omega_{\rm DE} = P_{\rm DE}/\rho_{\rm DE}$. The dark energy density in terms of the redshift and also in terms of $y_H(z)$, is equal to,

$$\omega_{\rm DE}(z) = -1 + \frac{1}{3}(z+1)\frac{1}{y_H(z)}\frac{dy_H(z)}{dz}\,. \tag{611}$$

In Fig. 2, we have plotted the behavior of the dark energy equation of state parameter, for collisional matter (blue curves) and for non-collisional matter (red curves), for $w = 0.2$ (left plot) and $w = 0.8$ (right plot). In this case, the

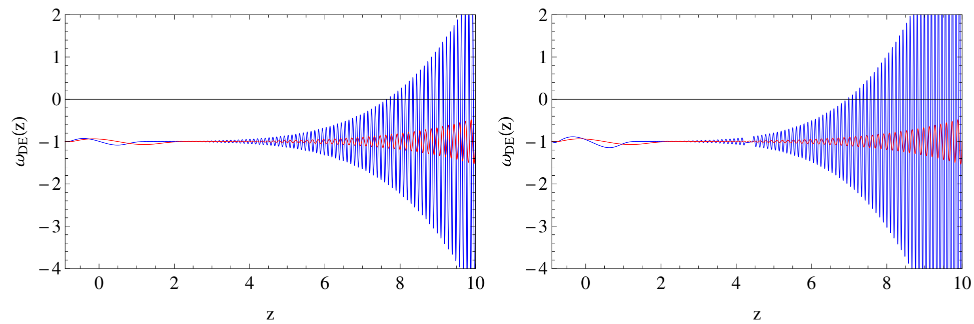

FIG. 2: Behavior of the dark energy equation of state parameter $\omega_{\rm DE}(z)$ as a function of the redshift z, for $w = 0.2$ (left plot) and $w = 0.8$ (right plot). The red curves correspond to non-collisional matter while the blue curves correspond to collisional matter.

same behavior as in the case of $y_H(z)$ occurs, so as the value of the parameter w increases, the oscillations become more pronounced.

Finally, let us see how the scalar curvature $R(z)$, the Hubble rate $H(z)$, and the total equation of state $\omega_{\rm eff}(z)$ behave as a function of z. Starting with the $H(z)$, this can be expressed in terms of $y_H(z)$ as follows,

$$H(z) = \sqrt{\tilde{m}^2 y_H(z) + g(a(z)) + \chi(z+1)^4}\,, \tag{612}$$

so by using the numerical solution we found for $y_H(z)$, in Fig. 3 (left plot), we plotted the evolution of the Hubble rate as a function of z, for $w = 0.8$, both for collisional (blue) and non-collisional matter (red). As it can be seen, the behavior of $H(z)$ in both the collisional and non-collisional matter, is the same. The same applies for the scalar

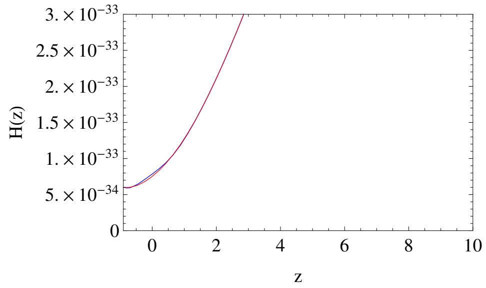

FIG. 3: Comparison of the Hubble parameter $H(z)$ over z for $w = 0.8$. The red line corresponds to non-collisional matter while the blue corresponds to collisional matter.

curvature, which can be written as,

$$R = 3\tilde{m}^2 \left(4y_H + 4g(z) - (z+1)\frac{dy_H}{dz} - (z+1)\frac{dg(z)}{dz} \right) . \tag{613}$$

Finally, let us discuss the behavior of the total equation of state parameter $\omega_{\mathrm{eff}}(z)$, and in Fig. (4), we compare the collisional (blue curves) and non-collisional (red curves) matter cases, for $w = 0.2$ (left plot) and $w = 0.8$ (right plots). Note that we used the formula,

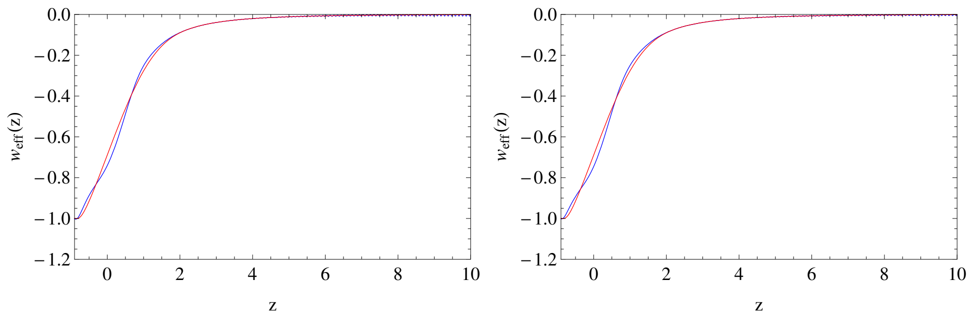

FIG. 4: Comparison of the effective equation of state parameter $\omega_{\mathrm{eff}}(z)$ over z, for $w = 0.2$ (left) and $w = 0.8$ (right). The red line corresponds to non-collisional matter while the blue corresponds to collisional matter.

$$\omega_{\mathrm{eff}}(z) = -1 + \frac{2(z+1)}{3H(z)}\frac{dH(z)}{dz} . \tag{614}$$

By looking at the behavior of the dark energy equation of state parameter and the total equation of state parameter, we can see that the total equation of state parameter never crosses the phantom divide line, however the dark energy equation of state parameter, has strong oscillations around the phantom divide line $w = -1$.

2. Growth Index Evolution

As a final issue, we shall study the evolution of the growth index in the context of $F(R)$ gravity, assuming a general profile for the matter fluids. It is known that the cosmological evolutions corresponding to various $F(R)$ gravities can be distinguished by using the cosmological perturbation theory. Particularly, it might happen that two $F(R)$ gravity models might generate similar cosmological evolutions in terms of the Hubble rate or in terms of the evolution of the scalar curvature in the two cases. In the previous section we came across with one situation like this, since the exponential $F(R)$ gravity model for collisional matter was indistinguishable from the non-collisional evolution, when the Hubble rate was considered. However, the cosmological perturbations may distinguish the various modified gravity models, since the cosmological perturbations differentiate the evolution caused by each model, from the background evolution [576, 601, 602]. In this section we shall focus on the matter density perturbations, in the subhorizon approximation, where the theory is consistent with the Newtonian limit [576, 601]. We shall address the matter perturbations issue in the context of $F(R)$ gravity with a general collisional mass profile. In the context of the subhorizon approximation, the wavelengths that are considered comoving with the spacelike hypersurface which describes the evolution, are much shorter in comparison to the Hubble radius $R_H = 1/(aH)$ of this hypersurface, that is, [576, 601],

$$\frac{k^2}{a^2} \gg H^2 \,, \tag{615}$$

with k and a being the wavenumber an the scale factor respectively. The subhorizon approximation breaks down before the matter domination era, so we focus on the matter domination era and onwards. In the case of a generalized collisional matter profile, the matter density perturbations are quantified in terms of $\delta = \frac{\delta\varepsilon_m}{\varepsilon_m}$, where ε_m is the collisional matter-energy density given in Eq. (588). The non-relativistic non-interacting matter case, can be easily obtained from the collisional case, if the self interactions are zero in (588), which means if Π_0 and w are zero in Eq. (588). The parameter δ satisfies the differential equation [576, 598, 603],

$$\ddot{\delta} + 2H\dot{\delta} - 4\pi G_{\text{eff}}(a,k)\varepsilon_m \delta = 0 \,, \tag{616}$$

where $G_{\text{eff}}(a,k)$ is the $F(R)$ gravity theory effective gravitational constant, which is,

$$G_{\text{eff}}(a,k) = \frac{G}{F'(R)}\left[1 + \frac{\frac{k^2}{a^2}\frac{F''(R)}{F'(R)}}{1 + 3\frac{k^2}{a^2}\frac{F''(R)}{F'(R)}}\right] \,, \tag{617}$$

where G is the Newtonian gravity gravitational constant. By using the quantity $f_g(z) = \frac{d\ln\delta}{d\ln a}$, and also the relations,

$$\begin{aligned} \dot{H} &= \frac{dH}{dz}(z+1)H(z)\,, \\ \dot{\delta} &= H\dot{f}_g\delta\,, \\ \ddot{\delta} &= \dot{H}f_g\delta + H\dot{f}_g\delta + Hf_g\dot{\delta}\,, \end{aligned} \tag{618}$$

we can express the differential equation (616) in terms of the redshift z,

$$\frac{df_g(z)}{dz} + \left(\frac{1+z}{H(z)}\frac{dH(z)}{dz} - 2 - f_g(z)\right)\frac{f_g(z)}{1+z} + \frac{4\pi}{G}\frac{G_{\text{eff}}(a(z),k)}{(z+1)H^2(z)}\varepsilon_m = 0 \,. \tag{619}$$

The term $G_{\text{eff}}(a(z),k)$ determines the effects of $F(R)$ gravity on the matter perturbations, and note that it depends explicitly on the wavenumber, something that does not happen in the general relativistic case. Now we numerically solve (619) and we use the same initial conditions we used in the previous section, with $z_{\text{fin}} = 10$. In order to respect the subhorizon approximation, we need to be cautious by choosing the values of the wavenumber, so by using the present time values of the Hubble rate and of the scale factor, we easily find that the wavenumber has to satisfy $k > 0.000156$, for non-collisional matter, while in the collisional case, we have $k > 0.0001174$. So by disregarding radiation, in which case,

$$g(a) = a^{-3}\left[1 + \Pi_0 + 3w\ln(a)\right] \,, \tag{620}$$

in Fig. 5 we plotted the growth factor $f_g(z)$ as a function of the redshift, for collisional (blue curve) and non-collisional matter (red curve), by using $k = 0.1\text{Mpc}^{-1}$ and $w = 0.8$ As we can see in Fig. (5), unlike in the Hubble rate comparison for collisional and non-collisional matter, where the two cases are indistinguishable, the matter perturbations differ significantly in the collisional and non-collisional cases. This justifies our statement that the matter perturbations are valuable since these can be used in order to distinguish the effects of different modified gravity theories.

Before closing we need to mention that in the case of mimetic $F(R)$ gravity, if the potential is appropriately chosen, the dark energy oscillations do not occur, see Refs. [454, 455] for more details on this issue.

Thus in this section we discussed the dark energy issue and how to describe inflation and dark energy in a unified way in the context of modified gravity.

We need to note that our presentation intended to describe the procedure of model building in the context of modified gravity, and how modified gravity models can harbor the two acceleration eras. So we did not attempt to solve any of the phenomenological models that the ΛCDM model suffers from, like the coincidence problem. This issue would require more advanced techniques, unknown so far, so this issue is out of the scope of this review. We believe

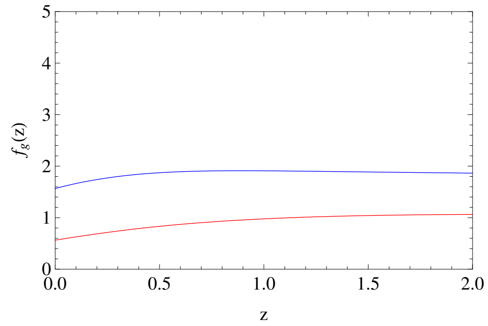

FIG. 5: Plots of the growth factor $f_g(z) = \frac{d\ln\delta}{d\ln a}$, as a function of the redshift z, for $k = 0.1\text{Mpc}^{-1}$ and $w = 0.8$. The blue curve corresponds to collisional matter and the red curve to non-collisional matter.

that a potential way to solve such intriguing problems such as the coincidence problem[4] is to use cosmographic principles[8, 19, 124, 397], or similar approaches, that may provide insightful hints. However, we mainly focused on describing certain aspects of late-time evolution and also to theorize how the unification of early and late-time acceleration can be achieved. It is conceivable that a the presentation of a complete cosmological model that describes all the evolution eras, is what most theoretical cosmologists seek for the moment, so it is not possible to present it in a review article, since this issue is the focus of the research of many cosmologists today (2017). Some insights may be provided by modified gravity, see for example Ref. [511], in which the unified description of all cosmological eras of our Universe was achieved in the context of $F(R)$ gravity, but in the model of Ref. [511], partial solutions were given due to the lack of analyticity. But the challenge in every cosmological theory is to find the ultimate theory which will combine all the features we just discussed. The path towards this theory is long and light will be shed by the observational data. As theoretical physics history dictates us, the cosmologists working today are particularly lucky since the existence of petabytes of cosmological data show the way on how to construct viable models, unlike what happens in string theories, due to the lack of experimental data.

V. ASTROPHYSICAL APPLICATIONS

A complete review on modified gravity could not be complete without even a small chapter devoted on astrophysical applications of modified gravity. Hence, although we deviate conceptually from the title of this review, in this chapter we shall briefly present the latest developments of modified gravity on astrophysical objects and solutions. We shall be interested in neutron and quark stars, the anti-evaporation phenomena in black holes and finally wormholes, in the context of $F(R)$ gravity.

A. Neutron and quark stars from $F(R)$ gravity

Modified gravity offers the possibility of having new types of neutron stars allowed, that cannot be described by the standard Einstein gravity. Recently, in Refs. [604–607], massive compact neutron stars with mass $M_{\text{NS}} \sim 2M_\odot$ have been found, where $M_\odot$ is the solar mass. In the contexts of the standard Einstein gravity and modern hadron physics, it is rather impossible to realize such super-massive neutron stars. The conceptual approaches towards understanding the neutron stars, are mainly realized by two streams of research, namely, the particle physics approach, and the gravitational theory approach. From the viewpoint of particle physics approach, the EoS of the matter which consists the neutron star, practically determines the physics of the neutron star. On the other hand, the gravitational approach may have new insights to offer since this is a much more fundamental approach, affecting the inner structure of the neutron star in a geometric way. Since the size of the compact stars is determined by the balance between two competing forces, the degeneracy force and the gravitation force, there are three approaches that may explain the mass of the neutron star, which we list here:

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[4] The coincidence problem refers to the question why dark energy and dark matter energy densities are similar in magnitude at present time.

(I) The repulsive force is considered to be stronger in comparison to the force corresponding to Quantum-Chromo-Dynamics originating equations of state. Such examples of EoS, can be due to new types of interactions, as in Refs. [608, 609].

(II) The attractive force is weaker in comparison to the one predicted from the standard Einstein gravity.

(III) Both cases (I) and (II) apply.

Here we shall consider the viewpoint of the approach (II), that is, we shall try to explain the massive neutron stars solutions, by using $F(R)$ gravity, as in Refs. [610–619] or by using the dRGT massive gravity coupled with matter, which is described by using standard equations of state [620].

We shall base the $F(R)$ gravity approach on neutron stars on Ref. [621], and for similar approaches on relativistic stars in the context of modified gravity, see Refs. [330, 622–631].

We consider the following form for the $F(R)$ gravity functions,

$$F(R) = R + \alpha h(R) . \tag{621}$$

At a later point we shall assume that α is small enough, but for the moment α is arbitrary. We shall consider a spherically symmetric and static metric of the form,

$$ds^2 = -\mathrm{e}^{2\phi}c^2dt^2 + \mathrm{e}^{2\lambda}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) . \tag{622}$$

Also the energy-momentum tensor corresponding to the perfect fluid which consists the neutron star, is $T_{\mu\nu} = \mathrm{diag}(\mathrm{e}^{2\phi}\rho c^2, \mathrm{e}^{2\lambda}p, r^2p, r^2\sin^2\theta p)$, were ρ and p are the energy density and the pressure of the perfect fluid respectively. In effect, we find the following components of the field equations,

$$\begin{aligned}
\frac{-8\pi G}{c^2}\rho = & - r^{-2} + \mathrm{e}^{-2\lambda}(1-2r\lambda')r^{-2} + \alpha h_R(-r^{-2} + \mathrm{e}^{-2\lambda}(1-2r\lambda')r^{-2}) \\
& - \frac{1}{2}\alpha(h - h_R R) + \mathrm{e}^{-2\lambda}\alpha\left[h_R' r^{-1}(2 - r\lambda') + h_R''\right] , \qquad (623) \\
\frac{8\pi G}{c^4}P = & - r^{-2} + \mathrm{e}^{-2\lambda}(1+2r\phi')r^{-2} + \alpha h_R(-r^{-2} + \mathrm{e}^{-2\lambda}(1+2r\phi')r^{-2}) \\
& - \frac{1}{2}\alpha(h - h_R R) + \mathrm{e}^{-2\lambda}\alpha h_R' r^{-1}(2 + r\phi') , \qquad (624)
\end{aligned}$$

where the prime denotes the derivative with respect to the radial coordinate r. We also introduce the function $M(r)$, which corresponds to the mass variable for the Schwarzschild space-time, which is equal to,

$$\mathrm{e}^{-2\lambda} = 1 - \frac{2GM(r)}{c^2 r} . \tag{625}$$

We shall identify the value of $M(r)$ on the surface of neutron star, to be the gravitational mass of the neutron star. By using the conservation of the energy momentum tensor $\nabla^\mu T_{\mu\nu} = 0$, we get in component form the continuity equation,

$$\frac{dp}{dr} = -(\rho + P/c^2)\frac{d\phi}{dr} . \tag{626}$$

In the same way as in the standard Einstein gravity, we can obtain the second Tolman-Oppenheimer-Volkov (TOV) equation by substituting $d\phi/dr$ from (626) in Eq. (624). In order to write down the TOV equation, for convenience reasons, we shall introduce the dimensionless variables in the following way,

$$M = mM_\odot , \quad r \to r_g r , \quad \rho \to \rho M_\odot / r_g^3 , \quad p \to pM_\odot c^2/r_g^3 , \quad R \to R/r_g^2 . \tag{627}$$

Recall that $M_\odot$ denotes the mass of the Sun and r_g is equal to $r_g = GM_\odot/c^2 = 1.47473$ km. Then, by using these new variables, Eqs. (623) and (624) can be rewritten as in the following way,

$$\left(1 + \alpha r_g^2 h_R + \frac{1}{2}\alpha r_g^2 h_R' r\right)\frac{dm}{dr} = 4\pi\rho r^2 - \frac{1}{4}\alpha r^2 r_g^2\left(h - h_R R - 2\left(1 - \frac{2m}{r}\right)\left(\frac{2h_R'}{r} + h_R''\right)\right) , \tag{628}$$

$$8\pi p = -2\left(1 + \alpha r_g^2 h_R\right)\frac{m}{r^3} - \left(1 - \frac{2m}{r}\right)\left(\frac{2}{r}(1 + \alpha r_g^2 h_R) + \alpha r_g^2 h_R'\right)(\rho + p)^{-1}\frac{dp}{dr}$$

$$-\frac{1}{2}\alpha r_g^2\left(h-h_R R-4\left(1-\frac{2m}{r}\right)\frac{h'_R}{r}\right). \tag{629}$$

For a non-vanishing α, it is compelling to include the equation corresponding to the trace of the field equations, which has the following form,

$$3\alpha r_g^2\left(\left(\frac{2}{r}-\frac{3m}{r^2}-\frac{dm}{rdr}-\left(1-\frac{2m}{r}\right)\frac{dp}{(\rho+p)dr}\right)\frac{d}{dr}+\left(1-\frac{2m}{r}\right)\frac{d^2}{dr^2}\right)h_R+\alpha r_g^2 h_R R-2\alpha r_g^2 h-R$$
$$=-8\pi(\rho-3p). \tag{630}$$

At this point it is necessary to specify the EoS inside the neutron star, in order to find a solution to the above equations. The simplest choice for an EoS is to use a polytropic EoS of the form $p\sim\rho^\gamma$, but we can use more realistic EoS for the matter of the neutron star.

We now perform a perturbation of the pressure, the mass density and the curvature in terms of the parameter α in (621) as follows,

$$p=p^{(0)}+\alpha p^{(1)}+\cdots,\quad \rho=\rho^{(0)}+\alpha\rho^{(1)}+\cdots,\quad m=m^{(0)}+\alpha m^{(1)}+\cdots,\quad R=R^{(0)}+\alpha R^{(1)}+\cdots. \tag{631}$$

Then by keeping terms of the order of $\mathcal{O}(\alpha)$, we obtain the following equation,

$$\begin{aligned}\frac{dm}{dr}=&4\pi\rho r^2-\alpha r^2\left(4\pi\rho^{(0)}h_R+\frac{1}{4}\left(h-h_R R\right)\right)\\&+\frac{1}{2}\alpha\left(\left(2r-3m^{(0)}-4\pi\rho^{(0)}r^3\right)\frac{d}{dr}+r(r-2m^{(0)})\frac{d^2}{dr^2}\right)h_R,\end{aligned} \tag{632}$$

$$\frac{r-2m}{\rho+p}\frac{dp}{dr}=4\pi r^2 p+\frac{m}{r}-\alpha r^2\left(4\pi p^{(0)}h_R+\frac{1}{4}\left(h-h_R R\right)\right)-\alpha\left(r-3m^{(0)}+2\pi p^{(0)}r^3\right)\frac{dh_R}{dr}. \tag{633}$$

Note that we disregard terms of the order $\mathcal{O}\left(\alpha^2\right)$ as $m=m^{(0)}+\alpha m^{(1)}$ and $p=p^{(0)}+\alpha p^{(1)}$. Since the scalar curvature appears only via the terms h_R and h, at order $\mathcal{O}(\alpha)$, we only need to keep terms of the order $\mathcal{O}(1)$, for R, in the following way,

$$R\approx R^{(0)}=8\pi(\rho^{(0)}-3p^{(0)}). \tag{634}$$

If the EoS for the nuclear matter for large densities is specified, the equations (632) and (633) can easily be solved. For example, the cases of SLy [632–634] and FPS [635] equations of state can be expressed in a unified manner,

$$\begin{aligned}\zeta=&\frac{a_1+a_2\xi+a_3\xi^3}{1+a_4\xi}f(a_5(\xi-a_6))+(a_7+a_8\xi)f(a_9(a_{10}-\xi))\\&+(a_{11}+a_{12}\xi)f(a_{13}(a_{14}-\xi))+(a_{15}+a_{16}\xi)f(a_{17}(a_{18}-\xi)),\end{aligned} \tag{635}$$

where ξ, ζ and $f(x)$ in the above equation are,

$$\zeta=\log(p/\mathrm{dyncm}^{-2}),\qquad \xi=\log(\rho/\mathrm{gcm}^{-3}),\qquad f(x)=\frac{1}{\exp(x)+1}. \tag{636}$$

The detailed form of the coefficients a_i for the SLy and FPS equations of state, can be found in Ref. [636]. We can also consider a neutron star model, with a quark core, in which case the EoS called Tis, is as follows,

$$p_Q=a(\rho-4B), \tag{637}$$

where a is a constant and the parameter B takes values from ~ 60 to 90 Mev/fm^3. The value of the parameter a in the case of quark matter, with a massless strange quark, is $a=1/3$, but here we shall assume that $a=0.28$, which corresponds to $m_s=250$ MeV. For the numerical calculations performed in Ref. [621], Eq. (637) was used under the assumption that $\rho\geq\rho_{\mathrm{tr}}$, with ρ_{tr} being the transition density for which the pressure of the quark matter is equal to the pressure of ordinary dense matter. For the case of an FPS EoS, the transition density is equal to $\rho_{\mathrm{tr}}=1.069\times10^{15}$ g/cm^3 ($B=80$ Mev/fm^3), while for a SLy EoS, it is $\rho_{\mathrm{tr}}=1.029\times10^{15}$ g/cm^3 ($B=60$ Mev/fm^3).

In Ref. [621], various $F(R)$ models for neutron stars have been studied, for example the exponential model in which case the $F(R)$ gravity is,

$$F(R)=R+\beta R(\exp(-R/R_0)-1). \tag{638}$$

In this case, for the neutron star models of (637), with a quark core assumed, there is no significant difference in comparison with the Einstein gravity. For the case of a simplified EoS as in Eq. (635), interesting results are found, although the model with a FPS EoS is ruled out by the latest observational data [607, 637]. In the case $\beta < 0$, and for large central densities, stable stellar configurations occur when $dM/d\rho_c > 0$. Studying the model with a SLy EoS, could potentially be more interesting, since the upper limit of neutron star mass is of the order $\sim 2M_{\odot}$. In addition, there exists a second branch of stable stellar configurations, in the large central density regime, which describes the observational data better, in comparison to the model with a SLy EoS, in the context of the Einstein gravity. Another interesting model studied in Ref. [621], has the following $F(R)$ gravity function,

$$F(R) = R + \alpha R^2 (1 + \gamma R) \,. \tag{639}$$

For this model, the neutron star configuration is stable, and neutron star solutions can be generated, which result to smaller radii in comparison to the Einstein gravity solutions. If a SLy EoS is assumed, the model at hand can describe neutron stars with mass $\sim 2M_{\odot}$.

Let us now discuss in brief the massive gravity description of neutron stars, in which case the corresponding Einstein equations are $G_{\mu\nu} + m_0^2 I_{\mu\nu} = \kappa^2 T_{\mu\nu}$. By using the Bianchi identity $\nabla^\mu G_{\mu\nu} = 0$ and in addition the conservation of the energy-momentum tensor $\nabla^\mu T_{\mu\nu}$, we obtain the constraint $\nabla^\mu I_{\mu\nu} = 0$. This constraint plays an important role in massive gravity and it drastically changes the qualitative behavior of static and spherically symmetric solutions.

We assume that the dynamical metric $g_{\mu\nu}$ and the reference metric $f_{\mu\nu}$ are both static and spherically symmetric, with the corresponding line elements having the following form,

$$\begin{aligned} g_{\mu\nu} dx^\mu dx^\nu &= -\,\mathrm{e}^{2\phi(\rho)} dt^2 + \mathrm{e}^{2\lambda(\rho)} d\rho^2 + D^2(\rho) \left(d\theta^2 + \sin^2\theta d\varphi^2 \right) \,, & (640) \\ f_{\mu\nu} dx^\mu dx^\nu &= -\, dt^2 + d\rho^2 + \rho^2 \left(d\theta^2 + \sin^2\theta d\varphi^2 \right) \,, & (641) \end{aligned}$$

where the radial coordinate is represented by using the variable ρ. By introducing a new variable r, defined in such a way so that $D(\rho) = r^2$, the line elements of Eqs. (640) and (641), can be rewritten as follows,

$$\begin{aligned} g_{\mu\nu} dx^\mu dx^\nu &= -\,\mathrm{e}^{2\phi} dt^2 + \mathrm{e}^{2\lambda} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2 \right) \,, & (642) \\ f_{\mu\nu} dx^\mu dx^\nu &= -\, dt^2 + \left(\chi'(r)\right)^2 dr^2 + \chi^2(r) \left(d\theta^2 + \sin^2\theta d\varphi^2 \right) \,. & (643) \end{aligned}$$

Note that the scalar function $\chi(r)$ quantifies the Stuckelberg field degree of freedom. In the case of minimal dRGT massive gravity, the parameters β_n are chosen in the following way,

$$\beta_0 = 3 \,, \quad \beta_1 = -1 \,, \quad \beta_2 = 0 \,, \quad \beta_3 = 0 \,. \tag{644}$$

Then the following equations are obtained,

$$m'(r) = 4\pi\tilde{\rho}(r) r^2 + \frac{1}{2}\alpha^2 \left(r_g M_{\odot}\right)^2 r^2 \left[3 - \frac{2\chi(r)}{r} - \chi'(r) \left(1 - \frac{2m(r)}{r} \right)^{1/2} \right] \,, \tag{645}$$

$$8\pi p(r) = -\frac{1}{r^2} + \frac{1}{r^2} \left[1 - 2(p+\tilde{\rho})^{-1} p' r \right] \left(1 - \frac{2m(r)}{r} \right) + \alpha^2 \left(r_g M_{\odot}\right)^2 \left(3 - \frac{2\chi(r)}{r} - \mathrm{e}^{\int (p+\tilde{\rho})^{-1} p' dr} \right) \,, \tag{646}$$

$$\begin{aligned} 8\pi p(r) = & \left[-\left((p+\tilde{\rho})^{-1} p' \right)' + \left((p+\tilde{\rho})^{-1} p' \right)^2 - \frac{1}{r} (p+\tilde{\rho})^{-1} p' \right] \left(1 - \frac{2m(r)}{r} \right) \\ & - \frac{1}{2} \left[(p+\tilde{\rho})^{-1} p' - \frac{1}{r} \right] \left(1 - \frac{2m(r)}{r} \right)' \\ & + \alpha^2 \left(r_g M_{\odot}\right)^2 \left(3 - \frac{\chi(r)}{r} - \chi'(r) \left(1 - \frac{2m(r)}{r} \right)^{1/2} - \mathrm{e}^{\int (p+\tilde{\rho})^{-1} p' dr} \right) \,. \end{aligned} \tag{647}$$

In addition, the constraint is given as follows,

$$0 = \left(-\left(p+\tilde{\rho}\right)^{-1} p' + \frac{2}{r} \right) \left(1 - \frac{2m(r)}{r} \right)^{1/2} - \frac{2}{r} \,. \tag{648}$$

A detailed numerical study can show that the mass-radius relation in the context of dRGT massive gravity, is more constrained in comparison to the standard Einstein gravity, and we showed that the maximal mass becomes smaller for quark and neutron stars. In effect, the compact massive neutron stars cannot be explained by using this specific version of dRGT massive gravity, a result which is in contrast with the $F(R)$ gravity case [613, 621]. We need to note that the results obtained in Ref. [620], do not always exclude the dRGT massive gravity description from being a viable one, due to the observational data. Indeed, for the calculations we took into account the standard EoS in order to find the maximal mass, and in addition the study was specialized by using a particular minimal model of massive gravity. It may be the case that a massive neutron star can be realized in the context of massive gravity, if alternative EoS are used, or more complicated versions of massive gravity are used, at the expense of having more free parameters in the theory.

B. Black holes in $F(R)$ Gravity and anti-evaporation

In this section we shall present some recent studies on black holes solutions, in the context of $F(R)$ gravity, based on Ref. [638]. For other astrophysical solutions and various metric solutions from $F(R)$ gravity and modified gravity in general, see for example Refs. [285, 330, 452, 617, 639–648].

In Ref. [638], it was demonstrated that anti-evaporation of the Nariai space-time may possibly occur in the context of $F(R)$ gravity. In the usual descriptions of black holes in the Einstein gravity, the horizon radius of the black hole, in vacuum, decreases in size, since the black hole looses energy via the Hawking radiation. However, as was demonstrated by Hawking and Bousso, if someone includes quantum corrections, then it is possible that the Nariai black hole [649, 650] may increase its size, with this phenomenon being known ever since as anti-evaporation of black holes [651]. The Nariai black hole is a limiting case of a Schwarzschild- de-Sitter space-time, and it can be obtained if the cosmological horizon of the Schwarzschild- de-Sitter black hole coincides with that of the black hole horizon. The metric of the Nariai space-time has the following line element,

$$ds^2 = \frac{1}{\Lambda^2 \cosh^2 x}\left(-dt^2 + dx^2\right) + \frac{1}{\Lambda^2} d\Omega^2 \,, \quad d\Omega^2 \equiv d\theta^2 + \sin^2\theta d\phi^2 \,. \tag{649}$$

The parameter Λ stands for a mass scale and also $d\Omega^2$ denotes the metric of the two dimensional unit sphere. Since the scalar curvature is equal to, $R = R_0 \equiv 4\Lambda^2$, by using Eq. (9), we obtain,

$$0 = F\left(4\Lambda^2\right) - 2\Lambda^2 F'\left(4\Lambda^2\right) \,. \tag{650}$$

We shall consider the perturbations of Eq. (650), so we assume,

$$ds^2 = \mathrm{e}^{2\rho(x,t)}\left(-dt^2 + dx^2\right) + \mathrm{e}^{-2\varphi(x,t)} d\Omega^2 \,. \tag{651}$$

By considering that the perturbations have the following form, $\rho = -\ln\left(\Lambda\cosh x\right) + \delta\rho$ and $\varphi = \ln\Lambda + \delta\varphi$, in effect we find the following perturbation equations,

$$\begin{aligned}
0 =& \frac{-F'\left(R_0\right) + 2\Lambda^2 F''\left(R_0\right)}{2\Lambda^2\cosh^2 x}\delta R - \frac{F\left(R_0\right)}{\Lambda^2\cosh^2 x}\delta\rho - F'\left(R_0\right)\left(-\delta\ddot{\rho} + 2\delta\ddot{\varphi} + \delta\rho'' + 2\tanh x\delta\varphi'\right) \\
&+ \tanh x F''\left(R_0\right)\delta R' + F''\left(R_0\right)\delta R'' \,, \\
0 =& -\frac{-F'\left(R_0\right) + 2\Lambda^2 F''\left(R_0\right)}{2\Lambda^2\cosh^2 x}\delta R + \frac{F\left(R_0\right)}{\Lambda^2\cosh^2 x}\delta\rho - F'\left(R_0\right)\left(\delta\ddot{\rho} + 2\delta\varphi'' - \delta\rho'' + 2\tanh x\delta\varphi'\right) \\
&+ F''\left(R_0\right)\delta\ddot{R} + \tanh x F''\left(R_0\right)\delta R' \,, \\
0 =& -2\left(\delta\dot{\varphi}' + \tanh x\delta\dot{\varphi}\right) + \frac{F''\left(R_0\right)}{F'\left(R_0\right)}\left(\delta\dot{R}' + \tanh x\delta\dot{R}\right) \,, \\
0 =& -\frac{-F'\left(R_0\right) + 2\Lambda^2 F''\left(R_0\right)}{2\Lambda^2}\delta R - \frac{F\left(R_0\right)}{\Lambda^2}\delta\varphi - \cosh^2 x F'\left(R_0\right)\left(-\delta\ddot{\varphi} + \delta\varphi''\right) \\
&- \cosh^2 x F''\left(R_0\right)\left(-\delta\ddot{R} + \delta R''\right) \,.
\end{aligned} \tag{652}$$

By setting $\delta\varphi = \varphi_0 \cosh\omega t \cosh^\beta x$ where φ_0, and ω are arbitrary constants, we find,

$$\omega = \pm\beta \,, \quad \beta = \frac{1}{2}\left(1 \pm \sqrt{\frac{19\alpha - 8}{3\alpha}}\right) \,, \quad \alpha \equiv \frac{2\Lambda^2 F''\left(R_0\right)}{F'\left(R_0\right)} = \frac{F\left(R_0\right)F''\left(R_0\right)}{F'\left(R_0\right)^2} \,. \tag{653}$$

The above relations indicate that if $\alpha < 0$ or $\alpha > \frac{8}{19}$, β and in effect ω are real. Since the real part of the parameter ω is always positive, the solution corresponding to the Nariai space-time is rendered always unstable.

The horizon is determined by the following equation,

$$g^{\mu\nu}\nabla_\mu\varphi\nabla_\nu\varphi = 0 \,. \tag{654}$$

Then we find

$$\delta\varphi = \delta\varphi_{\rm h} \equiv \varphi_0 \cosh^2 \beta t \,. \tag{655}$$

Since Eq. (651) indicates that $\mathrm{e}^{-\varphi}$ can be viewed as a radius coordinate, by using Eq. (649), the radius of the horizon may be defined by $r_{\rm h} = \mathrm{e}^{-\delta\varphi_{\rm h}}/\Lambda$ and then we obtain,

$$r_{\rm h} = \frac{\mathrm{e}^{-\varphi_0 \cosh^2 \beta t}}{\Lambda} \,. \tag{656}$$

In the case that $\varphi_0 < 0$, the radius $r_{\rm h}$ increases, and hence anti-evaporation occurs. If β and ω are complex, instead of $\delta\varphi = \varphi_0 \cosh \omega t \cosh^\beta x$, we obtain the following solution for φ

$$\delta\varphi = \Re\left\{ \left(C_+ \mathrm{e}^{\beta t} + C_+ \mathrm{e}^{-\beta t}\right) \mathrm{e}^{\beta x} \right\} \,, \tag{657}$$

where the parameters $C_\pm$ are complex numbers, and we expressed the real part of numbers as $\Re$. Since the real part of the parameter β is always positive, if $C_+ \neq 0$, when t increases, the parameter $\delta\varphi$ also increases and in effect the perturbations increase. As a result, the solution corresponding to the Nariai space-time is rendered unstable.

As a particular case, we may consider the following solution of Eq. (657), which is,

$$\delta\varphi = \delta\tilde\varphi_{\rm h} \equiv \delta\varphi_0 \left\{ \mathrm{e}^{\frac{t+x}{2}} \left(\cos\frac{\gamma\,(t+x)}{2} - \frac{1}{\gamma}\sin\frac{\gamma\,(t+x)}{2} \right) + \mathrm{e}^{\frac{-t+x}{2}} \left(\cos\frac{\gamma\,(t-x)}{2} + \frac{1}{\gamma}\sin\frac{\gamma\,(t-x)}{2} \right) \right\} \,. \tag{658}$$

It should be noted that the solution (658) is chosen to satisfy the initial condition $\delta\dot\varphi = 0$ at $t = 0$. Also we expressed the parameter β as,

$$\beta = \frac{1}{2}\left(1 \pm i\sqrt{\frac{8 - 19\alpha}{3\alpha}} \right) = \frac{1}{2}\left(1 + i\gamma\right) \,. \tag{659}$$

Then Eq. (654) takes the following form,

$$0 = \frac{\delta\varphi_0^2}{2}\gamma^2 A^2 \mathrm{e}^x \sin\frac{\gamma\,(t+x)}{2} \sin\frac{\gamma\,(t-x)}{2} \,, \tag{660}$$

and hence we find that an infinite numbers of horizons occur,

$$\text{(A) } x = -t + \frac{2n\pi}{\gamma} \text{ or (B) } x = t + \frac{2n\pi}{\gamma} \,. \tag{661}$$

On the horizon, we find that,

$$\text{(A) } \varphi = \delta\varphi_0(-1)^n \left\{ \mathrm{e}^{\frac{n\pi}{\gamma}} + \mathrm{e}^{-t+\frac{n\pi}{\gamma}} \left(\cos\left(\gamma t\right) + \frac{1}{\gamma}\sin\left(\gamma t\right) \right) \right\} \,, \tag{662}$$

$$\text{(B) } \varphi = \delta\varphi_0(-1)^n \left\{ \mathrm{e}^{\frac{n\pi}{\gamma}} + \mathrm{e}^{t+\frac{n\pi}{\gamma}} \left(\cos\left(\gamma t\right) - \frac{1}{\gamma}\sin\left(\gamma t\right) \right) \right\} \,. \tag{663}$$

Hence, the radius of horizon, which is defined by the condition $r_{\rm h} = \mathrm{e}^{-\delta\tilde\varphi_{\rm h}}/\Lambda$, oscillates, so we have a constant interplay of evaporation and anti-evaporation. Particularly, for the case (B) of Eq. (663), the amplitude of the oscillations gradually becomes larger.

C. Wormholes in $F(R)$ Gravity

As in most applications in modified gravity, the physical description of a phenomenon or quantity that modified gravity offers can be entirely different when compared to the Einstein-Hilbert gravitational description. In this section we shall consider an exotic and intriguing physical solution of the Einstein gravity, namely that of a wormhole. The wormhole solution [652–656] is a geometric space-time tunnel through which an observer can theoretically travel through it. The focus in this section will not be on the theoretical implications of a wormhole geometry per se, but we shall focus on the fact that modified gravity allows these solutions to exist without the need of having some extra exotic matter present. Indeed, in the ordinary Einstein-Hilbert description of wormholes, an exotic matter component is needed in order the solution is consistently defined, and this exotic matter is called exotic because it violates the null energy condition $T_{\mu\nu}k^\mu k^\nu \geq 0$, where k^μ is an arbitrary null vector. As it was shown in the literature [285], in the context of $F(R)$ gravity, wormhole geometries can consistently be realized. We shall base our analysis and notation on Ref. [285] where the $F(R)$ gravity realization of wormholes geometry was performed in detail. In the literature, several modified gravity and alternative approaches to the wormhole geometry existence can be found. For example in Refs. [265, 657–661] the possibility of having traversable wormholes was studied in various contexts, see also [662] for a quantum field theoretic approach on traversable wormholes. Also in Refs. [663–666] the stability of wormholes was studied in various theoretical contexts, and in Refs. [667, 668] the energy conditions of wormholes were examined. The possibility of having a wormhole supporting phantom energy was discussed in [669], or a phantom scalar [670], while in Ref. [671], various wormhole solutions were studied in the context of brane worlds. Finally in Refs. [672, 673] wormhole solutions were studied in the context of extra dimensions.

In the rest of this section we follow the presentation developed in Ref. [285]. Consider the spherically symmetric and static metric with the following line element,

$$ds^2 = \mathrm{e}^{-2\Phi(r)}dt^2 + \frac{1}{1-b(r)/r}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) , \tag{664}$$

with $\Phi(r)$ and $b(r)$ being arbitrary functions of the radius r, which are called the redshift and the shape function respectively. The radius value $r = r_0$ denotes the location of the throat, where $b(r_0) = r_0$. Two compelling conditions that need to be satisfied, in order for a wormhole solution to exist, are the following,

$$\frac{b(r) - b'(r)r}{b(r)^2} > 0 , \quad 1 - \frac{b(r)}{r} > 0 , \tag{665}$$

where the prime denotes differentiation with respect to r, and in addition, at the throat, the shape function must satisfy $b'(r_0) < 1$. It is exactly due to the imposition of the above conditions that in the ordinary Einstein-Hilbert description, an exotic form of matter, violating the null energy condition, is needed. As we shortly see, this violation of the null energy condition is not needed in the case of $F(R)$ gravity. For simplicity we set $\Phi(r) = \text{const}$, so that $\Phi'(r) = 0$. Consider an $F(R)$ gravity with matter fluids present, in which case the action is given as usual in (1), and the corresponding equations of motion can be found by varying the action with respect to the metric, and the result can be written as follows,

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = T^{\text{eff}}_{\mu\nu} , \tag{666}$$

with $T_{\mu\nu} = \frac{T_{\text{matter}\,\mu\nu}}{F'_R} + T^c_{\mu\nu}$, and $F_R = \frac{\partial F}{\partial R}$. The energy momentum tensor $T_{\text{matter}\,\mu\nu}$ corresponds to the ordinary matter fluids present, while $T^c_{\mu\nu}$ is equal to,

$$T^c_{\mu\nu} = \frac{1}{F_R}\left(\nabla_\mu\nabla_\nu F_R - \frac{1}{4}\left(RF'_R + \Box F_R + T\right)\right) , \tag{667}$$

where T is the trace of $T_{\text{matter}\,\mu\nu}$. In the case of modified gravity, the total energy momentum tensor of the matter and of the modified gravity that threads the wormhole is supposed to satisfy all the energy conditions, and this condition will eventually impose certain restrictions on the functional form of the $F(R)$ gravity. Particularly, the energy momentum tensor is assumed to have the following decomposition,

$$T_{\mu\nu} = (\rho + p_t)U_\mu U_\nu + p_t g_{\mu\nu} + (p_r - p_t)\chi_\mu\chi_\nu , \tag{668}$$

with U_μ being the comoving four-velocity, and χ_μ being the unit spacelike vector along the radial direction, that is, $\chi_\mu = \sqrt{1 - \frac{b(r)}{r}}\delta_\mu$. Also $\rho(r)$ is the energy density, $p_r(r)$ is the radial pressure along χ_μ and finally $p_t(r)$ is the transverse pressure along the orthogonal direction to χ_μ. In effect, the energy momentum is $T^\mu_\nu = \text{diag}[-\rho(r), p_r(r), p_t(r), p_t(r)]$, and the equations of motion become,

$$\begin{aligned}\frac{b'}{r^2} &= \frac{\rho + H}{F_R} ,\\ -\frac{b}{r^3} &= \frac{p_r}{F_R} + \frac{1}{F_R}\left(\left(1 - \frac{b}{r}\right)\left[F''_R - F'_R\frac{b'r - b}{2r^2\left(1 - \frac{b}{r}\right)}\right] - H\right) ,\end{aligned}$$

$$-\frac{b'r-b}{2r^3}=\frac{p_t}{F_R}+\frac{1}{F_R}\left[\left(1-\frac{b}{r}\frac{F_R'}{r}-H\right)\right], \tag{669}$$

where H is,

$$H(r)=\frac{1}{4}(F_R'R+\Box F_R'+T), \tag{670}$$

and the prime this time denotes differentiation with respect to the radius r. By using the line element (664) for $\Phi=\text{const}$, the scalar curvature reads $R=\frac{2b'}{r^2}$, and also the term $\Box F_R'$ reads,

$$\Box F_R=\left(1-\frac{b}{r}\right)\left[F_R''-F_R'\frac{b'r-b}{2r^2(1-\frac{b}{r})}+\frac{2F_R'}{r}\right]. \tag{671}$$

In the case at hand, by taking into account the gravitational equations, the null energy condition violation $T_{\mu\nu}k^\mu k^\nu<0$ takes the following form,

$$\frac{b'r-b}{r^3}<0, \tag{672}$$

which holds true due to relations (665). Also at the throat of the wormhole we have,

$$\left.\frac{\rho+p_r}{F_R}\right|_{r=r_0}+\left.\frac{1-b}{2r}\frac{F_R'}{F_R}\right|_{r=r_0}<0. \tag{673}$$

Therefore, in order for the conditions (665) (actually the second condition in Eq. (665)), to be satisfied, the functional form of the $F(R)$ gravity must be restricted at the location of the throat $r=r_0$ as follows,

$$F_R'|_{r=r_0}<-\left.\frac{2r(\rho+\rho_r)}{1-b'}\right|_{r=r_0}, \tag{674}$$

if $F_R'>0$, while in the case $F_R'<0$, the functional form of the $F(R)$ gravity is restricted as follows,

$$F_R'|_{r=r_0}>-\left.\frac{2r(\rho+\rho_r)}{1-b'}\right|_{r=r_0}. \tag{675}$$

By assuming that the matter fluids present, which are quantified by the energy momentum tensor $T_{\mu\nu}$, satisfy the null energy conditions, then, the functional form of the $F(R)$ gravity, must satisfy the following inequalities,

$$\frac{F_Rb'}{r^2}\geq 0,\quad \frac{(2F_R+rF_R')(b'r-b)}{2r^2}-F_R''\left(1-\frac{b}{r}\right)\geq 0. \tag{676}$$

Then if the shape function $b(r)$ is specified in a way that the above are respected, we may easily find the $F(R)$ gravity that satisfies the above conditions and also that it can generate the wormhole geometry (664). Indeed, by specifying the shape function $b(r)$, then the scalar curvature can be found as a function of r, that is $r(R)$, by using the relation $R=\frac{2b'}{r^2}$, and by also specifying the equation of state for ρ_r and p_r, we can obtain the functional form of $F_R(r)$ and by using the function $r(R)$ and integrating once with respect to the scalar curvature, we may obtain $F(R)$. In Ref. [285] several examples where worked out, and we shall briefly present one of them, for which the energy and pressure density functions satisfy $-\rho+p_r+2p_t=0$, so the resulting equations of motion become,

$$F_R''\left(1-\frac{b}{r}\right)-\frac{b'r+b-2r}{2r^2}F_R'-\frac{b'r-b}{2r^3}F_R=0. \tag{677}$$

Then, by assuming that the shape function is $b(r)=r_0^2/r$, the resulting $F_R(r)$ solution is,

$$F_R(r)=C_1\sinh\left[\sqrt{2}\arctan\left(\frac{r_0}{\sqrt{r^2-r_0^2}}\right)\right]+C_2\cosh\left[\sqrt{2}\arctan\left(\frac{r_0}{\sqrt{r^2-r_0^2}}\right)\right], \tag{678}$$

where C_i, $i = 1, 2$, are integration constants, and the energy and pressure density functions ρ, p_r and p_t, can be found by combing Eqs. (678) and (669). Then, by using $R = -2r_0^2/r^4$, we can find the resulting $F(R)$ gravity, which is [285],

$$F(R) = -C_1 R \sinh\left[\sqrt{2}\arctan\left(\frac{R_0}{\sqrt{R^2 - R_0^2}}\right)\right] + C_2 \cosh\left[\sqrt{2}\arctan\left(\frac{R_0}{\sqrt{R^2 - R_0^2}}\right)\right], \tag{679}$$

where $R_0 = -2/r_0^2$, so it is a negative value. In Ref. [285] many other examples were worked out, but the method is the same, so we refrain from going into further details. In the literature there exist various studies on wormholes in the context of modified gravity, and for important stream of papers we refer the reader to Refs. [674–684], where the $F(R)$ gravity case was studied, both in metric and Palatini formalisms, and in Refs. [373, 685–691] for other modified gravity and Brans-Dicke approaches.

VI. BOUNCING COSMOLOGIES FROM MODIFIED GRAVITY

A. General Features of Bouncing Cosmologies

The cosmological bounces [170–179, 182, 184, 216, 231, 233, 236, 692, 693], are quite appealing alternative scenarios to the standard inflationary paradigm, and the most appealing feature of these cosmologies is the absence of the initial singularity. In principle, a successful bouncing cosmology must solve all the problems that the inflationary scenario solved, and also it is required that a nearly scale invariant power spectrum is produced. There are many bouncing cosmologies in the literature, and in this chapter we shall discuss how these cosmologies can be realized by vacuum modified gravities. In particular we shall investigate how these cosmologies can be realized by $F(R)$, $f(\mathcal{G})$ and $F(T)$ gravities. For studies in the literature on this issue, see [549]. Before we proceed to this task, let us recall some essential features of bouncing cosmology.

One of the most important feature that a viable cosmology should have, is, the correct evolution of the Hubble horizon at early and late times. Particularly, at early times the Hubble horizon should contract and at intermediate and late times, the Hubble horizon should expand again. This is due to the fact that at early times, the Hubble horizon should contract in order for the primordial quantum fluctuations to exit the horizon. Then the Hubble horizon in the future should contract, so the initial modes will re-enter the horizon and will become physically relevant for present times observations.

The Universe's evolution in a general bouncing cosmology, consists of two eras, an era of contraction and an era of expansion. In between these two eras, the Universe reaches a bouncing point, at which the scale factor reaches a minimum, non-zero size, and eventually the Universe bounces off and starts to expand again. It is due to the fact the Universe reaches a non-zero minimum size, that the initial singularity problem is solved in the context of bouncing cosmologies. During the contracting period, the scale factor of the Universe decreases and it satisfies $\dot{a} < 0$. As the Universe reaches the bouncing point, the scale factor satisfies the condition $\dot{a} = 0$, and after that, and during the expanding period, the scale factor satisfies $\dot{a} > 0$. In terms of the Hubble rate $H(t)$, the conditions for a bounce to occur is that during the contraction, the Hubble rate satisfies $H(t) < 0$, at the bouncing point the Hubble rate is $H(t) = 0$, and as the Universe expands again, the Hubble rate is $H(t) > 0$. Suppose that the bouncing point occurs at $t = t_s$, then the conditions for a bounce to occur are summarized below,

$$\begin{aligned}
&\text{Before the bouncing point } t < t_s: \ \dot{a}(t) < 0, \quad H(t) < 0\,,\\
&\text{At the bouncing point } \ t = t_s: \ \dot{a}(t) = 0, \quad H(t) = 0\,,\\
&\text{After the bouncing point } \ t > t_s: \ \dot{a}(t) > 0, \quad H(t) > 0\,.
\end{aligned} \tag{680}$$

In the literature there exist various cosmological bounce scenarios, which we now briefly describe. First we consider the matter bounce scenario [171, 175, 185, 214, 226, 228, 235, 278, 279, 694], which occurs quite often in the context of LQC [695–699]. The matter bounce scale factor and the Hubble rate are the following,

$$a(t) = \left(\frac{3}{2}\rho_c t^2 + 1\right)^{\frac{1}{3}}, \quad H(t) = \frac{2t\rho_c}{2 + 3t^2\rho_c}, \tag{681}$$

where the parameter ρ_c is critical energy density related to the underlying LQC theory. The cosmic time dependence of the scale factor and of the Hubble rate can be found in Fig. 6. As it can be seen, the bouncing cosmology conditions are satisfied, since $H > 0$ for $t > 0$ and $H < 0$ for $t < 0$, and the bouncing point occurs at $t = 0$. In the matter bounce scenario, the primordial perturbations are generated early at the beginning of the contraction era. A complete study

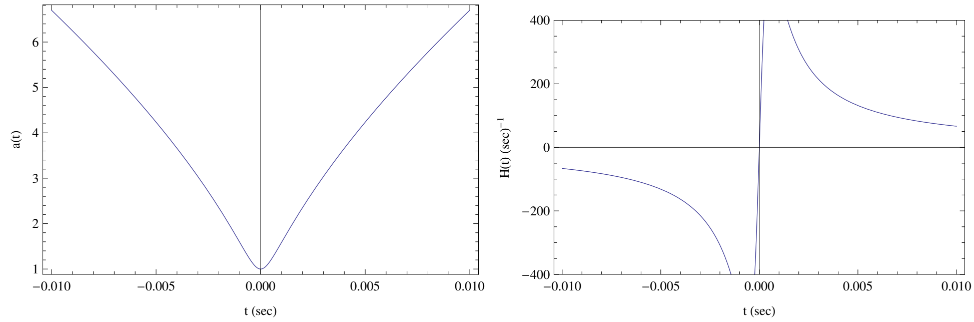

FIG. 6: The scale factor $a(t)$ (left plot) and the Hubble rate (right plot) as functions of the cosmic time t, for the matter bounce scenario $a(t) = \left(\frac{3}{2}\rho_c t^2 + 1\right)^{\frac{1}{3}}$.

of the Hubble horizon can describe this perfectly, so in Fig. 7 we can see the time dependence of the Hubble radius $R_H = 1/(aH)$, for $\rho_c = 2 \times 10^6\,\mathrm{sec}^{-2}$. As we can see, for $t < 0$ the Hubble radius decreases from an infinite size at $t \to -\infty$, until the bouncing point is reached. At the bouncing point the horizon reaches a minimal size and eventually it starts expanding again, for $t > 0$. Thus the primordial modes relevant for present day observations are generated at $t \to -\infty$, that is, at the beginning of the contracting phase. Another scenario with interesting phenomenology is the

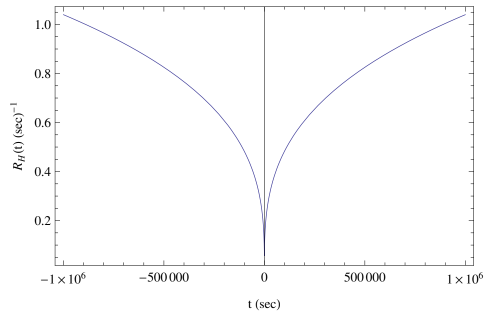

FIG. 7: The Hubble radius $R_H(t)$ as a function of the cosmological time t, for the matter bounce scenario $a(t) = \left(\frac{3}{2}\rho_c t^2 + 1\right)^{\frac{1}{3}}$.

superbounce scenario [193, 549, 700], which firstly appeared in the context of ekpyrotic cosmological scenarios [193]. For the superbounce scenario, the scale factor and the corresponding Hubble rate can be found below,

$$a(t) = (-t + t_s)^{\frac{2}{c^2}}\,, \quad H(t) = -\frac{2}{c^2(-t + t_s)}\,, \tag{682}$$

where the parameter c is an arbitrary parameter. Note that the bouncing point is at $t = t_s$. The cosmic time

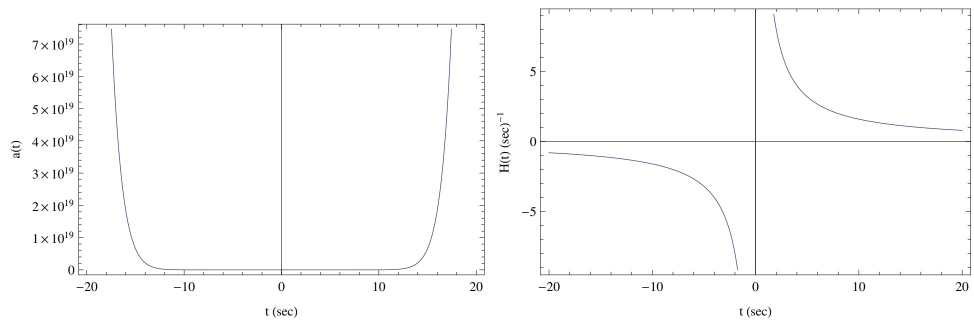

FIG. 8: The scale factor $a(t)$ (left plot) and the Hubble rate (right plot) as a function of the cosmological time t, for the superbounce scenario $a(t) = (-t + t_s)^{\frac{2}{c^2}}$.

dependence of the superbounce scenario can be found in Fig. 8, for $c = \sqrt{8}$ and $t_s = 10^{-36}$sec, and as it can be seen, the cosmological bounce conditions are satisfied in this case too, with the contraction occurring when $t < 0$ and the expansion occurring when $t > 0$. In addition, the Hubble horizon behaves in the same way as in the matter bounce case, since during the contraction phase it decreases, and during the expansion phase it increases.

Another interesting scenario which we now briefly mention is the singular bounce scenario, which was developed and thoroughly studied in [189, 191, 192, 562]. The new novel feature of this bouncing cosmology is that although the initial singularity is avoided, a Type IV singularity [192] occurs at the bouncing point. In the singular bounce scenario, the scale factor and the corresponding Hubble rate are,

$$a(t) = \mathrm{e}^{\frac{f_0}{\alpha+1}(t-t_s)^{\alpha+1}}\,, \quad H(t) = f_0\,(t-t_s)^{\alpha}\,, \tag{683}$$

where the parameter satisfies $f_0 > 0$. The condition in order to have a Type IV singularity at $t = t_s$, is that $\alpha > 1$, and also for physical consistency, α should be chosen as,

$$\alpha = \frac{2n+1}{2m+1}\,, \tag{684}$$

with the integers n and m being chosen in such a way, so that $\alpha > 1$. The singular bounce has a peculiar evolution of the Hubble radius, since for $t \to -\infty$, it has an infinite size and starts to decrease until near the bouncing point, where it blows up. After the bouncing point it starts to decrease again. As it was demonstrated in Refs. [189, 191, 192, 562], the power spectrum of the singular bounce for the modes near the bounce is not scale invariant, and therefore it has to be combined with alternative scenarios, like the ones studied in Refs. [223, 701] in order to be a viable cosmological bounce scenario.

In the following sections we shall present how the superbounce cosmology can be realized with $F(R)$, $f(\mathcal{G})$ and $F(T)$ gravity.

B. Bounce Cosmology from $F(R)$ Gravity

In order to realize the superbounce of Eq. (682), we shall use a reconstruction method introduced in [571] (for alternative methods see [702]), according to which the e-foldings number is used instead of the cosmic time, with the e-foldings number being related to the scale factor as follows,

$$\mathrm{e}^{-N} = \frac{a_0}{a}\,. \tag{685}$$

In terms of the variable N, we can re-express the $F(R)$ gravity equations of motion for the FRW Universe, as follows,

$$-18\left[4H^3(N)H'(N) + H^2(N)(H')^2 + H^3(N)H''(N)\right]F''(R) \tag{686}$$

$$+3\left[H^2(N) + H(N)H'(N)\right]F'(R) - \frac{F(R)}{2} + \kappa^2\rho = 0\,. \tag{687}$$

Introducing the function $G(N) = H^2(N)$, for the flat FRW Universe we eventually have that,

$$R = 3G'(N) + 12G(N)\,. \tag{688}$$

The above relation will enable us to determine the function $N(R)$. By writing the Hubble rate as a function of the scale factor we have,

$$H = \frac{2}{c^2}a^{-\frac{c^2}{2}}\,, \tag{689}$$

and therefore by expressing a in terms of N by making use of Eq. (685) we get,

$$G(N) = A\mathrm{e}^{-c^2 N}\,, \tag{690}$$

with the parameter A being $A = \frac{4}{c^4}a_0^{-c^2}$. Combining Eqs. (688) and (690), we can express N as a function of R, as follows,

$$N = -\frac{1}{c^2}\ln\left[\frac{R}{3A(4-c^2)}\right]\,, \tag{691}$$

and by substituting in (686), we can rewrite it as follows,

$$-9G(N(R))\left[4G'(N(R))+G''(N(R))\right]F''(R)+\left[3G(N)+\frac{3}{2}G'(N(R))\right]F'(R)-\frac{F(R)}{2}+\kappa^2\rho_{\mathrm{tot}}=0\,, \tag{692}$$

where $G'(N)=dG(N)/dN$ and $G''(N)=d^2G(N)/dN^2$. The total matter energy density ρ_{tot} should be expressed in terms of $N(R)$. By assuming that the perfect matter fluids have constant equation of state parameters, so these satisfy $\dot{\rho}_i+3H(1+w_i)\rho_i=0$, eventually we have,

$$\rho_{\mathrm{tot}}=\sum_i \rho_{i0}a_0^{-3(1+w_i)}\mathrm{e}^{-3N(R)(1+w_i)}\,. \tag{693}$$

By using the above relations, the FRW equation (692) becomes,

$$a_1R^2\frac{d^2F(R)}{dR^2}+a_2R\frac{dF(R)}{dR}-\frac{F(R)}{2}+\sum_i S_iR^{\frac{3(1+w_i)}{c^2}}=0\,, \tag{694}$$

where the parameters a_1 and a_2 are equal to,

$$a_1=\frac{c^2}{4-c^2}\,,\quad a_2=\frac{2-c^2}{2(4-c^2)}\,, \tag{695}$$

and also,

$$S_i=\frac{\kappa^2\rho_{i0}a_0^{-3(1+w_i)}}{\left[3A(4-c^2)\right]^{\frac{3(1+w_i)}{c^2}}}\,. \tag{696}$$

By solving the differential equation (694), we can obtain the exact $F(R)$ which can generate the superbounce cosmology (682).

We are interested in the vacuum $F(R)$ cosmology, so the differential equation (694) becomes a second order homogeneous Euler equation, which has the following solution,

$$F(R)=c_1R^{\rho_1}+c_2R^{\rho_2}\,, \tag{697}$$

with c_1, c_2 being arbitrary integration constants, and also ρ_1 and ρ_2 are,

$$\rho_1=\frac{-(a_2-a_1)+\sqrt{(a_2-a_1)^2+2a_1}}{2a_1}\,,\quad \rho_2=\frac{-(a_2-a_1)-\sqrt{(a_2-a_1)^2+2a_1}}{2a_1}\,. \tag{698}$$

In the case that $c\gg 1$, the $F(R)$ gravity becomes,

$$F(R)\simeq c_1R+c_2R^{-1/2}\,, \tag{699}$$

so the resulting $F(R)$ gravity is a modification of the Einstein gravity. Let us now consider the case that matter fluids are taken into account too. In this case, by solving (694), we obtain,

$$\begin{aligned}F(R)=&\left[\frac{c_2\rho_1}{\rho_2}-\frac{c_1\rho_1}{\rho_2(\rho_2-\rho_1+1)}\right]R^{\rho_2+1}+\sum_i\left[\frac{c_1S_i}{\rho_2(\delta_i+2+\rho_2-\rho_1)}\right]R^{\delta_i+2+\rho_2}\\&-\sum_i B_ic_2R^{\delta_i+\rho_2}+c_1R^{\rho_1}+c_2R^{\rho_2}\,,\end{aligned} \tag{700}$$

with c_1, c_2 being arbitrary integration constants, and also δ_i and B_i are defined as follows,

$$\delta_i=\frac{3(1+w_i)-2c^2}{c^2}-\rho_2+2\,,\quad B_i=\frac{S_i}{\rho_2\delta_i}\,. \tag{701}$$

In the limit $c\gg 1$, the $F(R)$ gravity above becomes,

$$F(R)\simeq R+\alpha R^2+c_1R^{-1/2}+\Lambda\,, \tag{702}$$

where we assumed that $c_2=1$, $\alpha=\frac{c_1}{3}-2+2c_1\sum_i\frac{S_i}{3}$ and $\Lambda=-\sum_i A_i$. Note that this resulting $F(R)$ gravity, resembles a higher order R^2 gravity. At early times it is approximated by the $R+\alpha R^2$ (Starobinsky inflation) gravity, while at late times it is $R-R^{-1/2}+\Lambda$, so it resembles the ΛCDM model.

1. Stability in the $F(R)$ reconstructions

Having at hand the $F(R)$ gravity solutions that realize the superbounce, now we shall examine the stability of the solutions. Particularly we are interested in the linear stability of the dynamical system that the FRW constitute. More details on this issue can be found in Ref. [549]. So we consider linear perturbations of the function $G(N)$, of the following form,

$$G(N) = g(N) + \delta g(N)\,, \tag{703}$$

and so by substituting this in the FRW equations (692) we obtain,

$$\begin{aligned} 0 =& g(N)\left.\frac{d^2F(R)}{dR^2}\right|_{R=R_1}\delta'' g(N) \\ &+\left\{3g(N)\left[4g'(N)+g''(N)\right]\left.\frac{d^3F(R)}{dR^3}\right|_{R=R_1}+\left[3g(N)-\frac{1}{2}g'(N)\right]\left.\frac{d^2F(R)}{dR^2}\right|_{R=R_1}\right\}\delta' g(N) \\ &+\left\{12g(N)\left[4g'(N)+g''(N)\right]\left.\frac{d^3F(R)}{dR^3}\right|_{R=R_1}\right. \\ &\left.+\left[-4g(N)+2g'(N)+g''(N)\right]\left.\frac{d^2F(R)}{dR^2}\right|_{R=R_1}+\frac{1}{3}\frac{dF(R)}{dR}\,|_{R=R_1}\right\}\delta g(N)\,, \end{aligned} \tag{704}$$

where $R_1 = 3g'(N) + 12g(N)$. The stability conditions for the perturbations of $G(N)$, are the following,

$$J_1 = \frac{6[4g'(N)+g''(N)]F'''(R)}{F''(R)} + 6 - \frac{g'(N)}{g(N)} > 0\,, \tag{705}$$

and also,

$$J_2 = \frac{36[4g'(N)+g''(N)]F'''(R)}{F''(R)} - 12 + \frac{6g'(N)}{g(N)} + \frac{3g''(N)}{g(N)} + \frac{F'(R)}{g(N)F''(R)} > 0\,. \tag{706}$$

For the superbounce realizing vacuum $F(R)$ gravity solution of Eq. (697), the stability conditions of Eq. (705) and (706) read,

$$\begin{aligned} J_1 =& 6 + c^2 - 2c^2\left[3^{\rho_1}c_1Q_1^{\rho_1}(\rho_1-2)(\rho_1-1)\rho_1 + 3^{\rho_2}c_2Q_1^{\rho_2}(\rho_2-2)(\rho_2-1)\rho_2\right] \\ &\times\left[3^{\rho_1}c_1Q_1^{\rho_1}(\rho_1-1)\rho_1 + 3^{\rho_2}c_2Q_1^{\rho_2}(\rho_2-1)\rho_2\right] > 0\,, \end{aligned} \tag{707}$$

with $Q_1 = -A(c^2-4)\mathrm{e}^{-c^2N}$, and

$$\begin{aligned} J_2 =& -12 - 6c^2 + 3c^4 + \mathrm{e}^{c^2N}\left(c_1Q_2^{\rho_1-1}\rho_1 + c_2Q_2^{\rho_2-1}\rho_2\right)A^{-1}Q_3^{-1} \\ &- 12c^2Q_2\left[c_1Q_2^{\rho_1-3}(\rho_1-2)(\rho_1-1)\rho_1 + c_2Q_2^{\rho_2-3}(\rho_2-2)(-1+\rho_2)\rho_2\right]Q_3^{-1} > 0\,, \end{aligned} \tag{708}$$

with $Q_2 = 3A\mathrm{e}^{-c^2N}(4-c^2)$ and $Q_3 = c_1Q_2^{\rho_1-2}(\rho_1-1)\rho_1 + c_2Q_2^{\rho_2-2}(\rho_2-1)\rho_2$. In the large-$c$ limit, the stability conditions read,

$$J_1 = 4c^2 + 6 > 0\,, \quad J_2 = 3c^4 + 18c^2 - 12 > 0\,, \tag{709}$$

and therefore the $F(R)$ gravity which realizes the superbounce is a stable solution.

Now we study the stability of the case that perfect matter fluids are present, so the $F(R)$ gravity has the form (700). In this case, the stability conditions of Eqs. (705) and (706), read,

$$\begin{aligned} J_1 =& 6 + c^2 - 6c^2Q_1\left[3^{\rho_1-3}c_1(\rho_1-2)(\rho_1-1)\rho_1Q_1^{\rho_1-3} + 3^{\rho_2-3}c_2(\rho_2-2)(\rho_2-1)\rho_2Q_1^{\rho_2-3}\right. \\ &+ 3^{\rho_2-2}q_1(\rho_2-1)\rho_2(\rho_2+1)Q_1^{\rho_2-2} - 3^{\delta_i+\rho_2-3}Q_1^{\delta_i+\rho_2-3}q_2(\delta_i+\rho_2-2)(\delta_i+\rho_2-1)(\delta_i+\rho_2) \\ &\left.+3^{\delta_i+\rho_2-1}Q_1^{\delta_i+\rho_2-1}q_2(\delta_i+\rho_2)(\delta_i+\rho_2+1)(\delta_i+\rho_2+2)\right] \end{aligned}$$

$$\times\Big[3^{\rho_1-2}c_1(\rho_1-1)\rho_1 Q_1^{\rho_1-2}+3^{\rho_2-2}c_2(\rho_2-1)\rho_2 Q_1^{\rho_2-2}+3^{\rho_2-1}q_1\rho_2(\rho_2+1)Q_1^{\rho_2-1}$$
$$-3^{\delta_i+\rho_2-2}q_2(\delta_i+\rho_2-1)(\delta_i+\rho_2)Q_1^{\delta_i+\rho_2-2}+3^{\delta_i+\rho_2}q_2(\delta_i+\rho_2+1)(\delta_i+\rho_2+2)Q_1^{\delta_i+\rho_2}\Big]^{-1}\,, \tag{710}$$

where $Q_1=-A(c^2-4)\mathrm{e}^{-c^2N}$, and also,

$$\begin{aligned}J_2=&-12-6c^2+3c^4+A\left(Q_3+Q_4\right)\Big\{\mathrm{e}^{c^2N}\Big[c_1\rho_1 Q_2^{\rho_1-1}+c_2\rho_2 Q_2^{\rho_2-1}\\&+Q_2^{\rho_2}q_1(1+\rho_2)-Q_2^{\delta_i+\rho_2-1}q_2(\delta_i+\rho_2)+Q_2^{\delta_i+\rho_2+1}q_2(2+\delta_i+\rho_2)\Big]\Big\}\\&+36Ac^2\mathrm{e}^{-c^2N}(c^2-4)\left(Q_3+Q_4\right)^{-1}\\&\times\Big[c_1Q_2^{\rho_1-3}(\rho_1-2)(\rho_1-1)\rho_1+c_2Q_2^{\rho_2-3}(\rho_2-2)(\rho_2-1)\rho_2\\&+Q_2^{\rho_2-2}q_1(\rho_2-1)\rho_2(\rho_2+1)-Q_2^{\delta_i+\rho_2-3}q_2(\delta_i+\rho_2-2)(\delta_i+\rho_2-1)(\delta_i+\rho_2)\\&+Q_2^{\delta_i+\rho_2-1}q_2(\delta_i+\rho_2)(\delta_i+\rho_2+1)(\delta_i+\rho_2+2)\Big]\,,\end{aligned} \tag{711}$$

with $Q_2=3A\mathrm{e}^{-c^2N}(4-c^2)$, $Q_3=c_1Q_2^{\rho_1-2}(\rho_1-1)\rho_1+c_2Q_2^{\rho_2-2}(\rho_2-1)\rho_2$ and $Q_4=Q_2^{\rho_2-1}q_1\rho_2(1+\rho_2)-Q_2^{\delta_i+\rho_2-2}q_2(\delta_i+\rho_2-1)(\delta_i+\rho_2)+Q_2^{\delta_i+\rho_2}q_2(1+\delta_i+\rho_2)(2+\delta_i+\rho_2)$. By taking the large-$c$ limit, we obtain,

$$J_1\approx 6c^2\,,\quad J_2\approx c^4\,, \tag{712}$$

so in this case too, the $F(R)$ gravity solution is stable.

C. Bounce Cosmology from $f(\mathcal{G})$ Gravity

In this section we shall investigate how the superbounce cosmology of Eq. (682) can be realized by using an $f(\mathcal{G})$ gravity. Using the reconstruction method we developed in the previous chapter, we introduce two functions $P(t)$ and $Q(t)$, which are chosen in such a way so these satisfy Eq. (376) and therefore the $f(\mathcal{G})$ the gravitational action (370) is given in Eq. (374). By varying with respect to t, we get the differential equation in Eq. (375), which when solved with respect to the function $t=t(\mathcal{G})$ it enables us to find the $f(\mathcal{G})$ function, by simply substituting $t=t(\mathcal{G})$ in Eq. (376). By combining Eqs. (376) and (373), we obtain (377) so by further combining Eqs. (376) and (377) we get (378) and hence we have the function $P(t)$. For the case at hand, the Hubble rate is, hence, the differential equation (378) reads,

$$(t-t_*)\frac{d^2P}{dt^2}-(2+a)\frac{dP}{dt}-\frac{c^4}{8}(t-t_*)=0\,, \tag{713}$$

which can be analytically solved, and the solution is,

$$P(t)=\frac{c^4\,t(2t_*-t)}{8(c^2+2)}\,. \tag{714}$$

By substituting the above in Eq. (377) we get the function $Q(t)$,

$$Q(t)=\frac{32(c^2-1)}{c^4(c^2+2)(t-t_*)^2}\,. \tag{715}$$

Combining Eqs. (375), (714), and (715) we get the following,

$$t_1(\mathcal{G})=t_*-\frac{\sqrt{2}\left[8(2-11c^2)G\right]^{1/4}}{c^2\sqrt{\mathcal{G}}}\,,\quad t_2(\mathcal{G})=t_*+\frac{\sqrt{2}\left[8(2-11c^2)\mathcal{G}\right]^{1/4}}{c^2\sqrt{\mathcal{G}}}\,. \tag{716}$$

Therefore, we substitute the functions t_1 and t_2 from Eq. (716) in Eq. (376), so finally we have the two $f(\mathcal{G})$ gravities that realize the superbounce cosmology.

$$F_1(\mathcal{G})=\frac{c^4\,t_*^2\,\mathcal{G}-8\sqrt{2(2-11c^2)\mathcal{G}}}{8(c^2+2)}\,,\quad F_2(\mathcal{G})=\frac{c^4\,t_*^2\,\mathcal{G}-4\sqrt{2(2-11c^2)\mathcal{G}}(1+\mathcal{G})}{8(2c^2+2)\mathcal{G}}\,. \tag{717}$$

1. Stability in the $f(\mathcal{G})$ reconstructions

Now we examine the linear stability of the gravitational equations for the solutions (717), so we adopt the method used in Ref. [549]. Consider the perturbation (703), so by inserting this in the gravitational equation (373), we obtain the following stability conditions,

$$\frac{J_2}{J_1} > 0\,, \quad \frac{J_3}{J_1} > 0\,, \tag{718}$$

with J_1 and J_2 being equal to,

$$J_1 = 288 g(N)^3 F''(\mathcal{G})\,, \tag{719}$$

$$\begin{aligned} J_2 =& 432 g(N)^2 \left\{(2g(N) + g'(N))F''(\mathcal{G}) \right.\\ &\left. + 8g(N)\left[g'(N)^2 + g(N)(4g'(N) + g''(N))\right] F''(\mathcal{G})\right\}\,, \end{aligned} \tag{720}$$

$$\begin{aligned} J_3 =& 6\left\{1 + 24g(N)\left\{-8g(N)^2 + 3g'(N)^2 + 6g(N)[3g'(N) + g''(N)]\right\} F''(\mathcal{G}) \right.\\ &\left. + 24g(N)[4g(N) + g'(N)]\left\{g'(N)^2 + g(N)[4g(N) + g''(N)]\right\} F''(\mathcal{G})\right\}\,. \end{aligned} \tag{721}$$

For the superbounce (682), we showed in the previous section that there are two $f(\mathcal{G})$ gravity solutions which realize this cosmology, namely $F_i(\mathcal{G})$, $i = 1, 2$, so for the function $F_1(\mathcal{G})$, we have,

$$\frac{J_2}{J_1} = \frac{3}{2}(c^2 - 2)\left(16A^2c^2\mathrm{e}^{-2c^2N} - 1\right) > 0\,, \tag{722}$$

and also,

$$\begin{aligned} \frac{J_3}{J_1} =& \frac{\left(-\frac{99}{2}c^6 + 108c^4 + 26c^2 - 8\right)}{2 - 11c^2} + 8A\mathrm{e}^{-c^2N} + \frac{Ac^2\mathrm{e}^{-c^2N}\left(11c^6 - 46c^4 + 30c^2 - 4\right)}{2 - 11c^2} \\ & - \sqrt{6}(4 - c^4)\sqrt{\frac{c^2 - 2}{11c^2 - 2}} > 0\,, \end{aligned} \tag{723}$$

where $A = \frac{4}{c^4}a_0^{-c^2}$. Hence, the solution $F_1(\mathcal{G})$ is conditionally stable. In the same way, for the function $F_2(\mathcal{G})$ we have,

$$\frac{J_2}{J_1} = \frac{3}{2}\left(c^2 - 2\right)\left(16A^2c^2\mathrm{e}^{-2c^2N} - 1\right) > 0\,, \tag{724}$$

and also,

$$\begin{aligned} \frac{J_3}{J_1} =& \frac{4A^2\left(16 - 60c^2 - 190c^4 + 207c^6\right)}{Q_a} \\ & + \frac{4A^3\mathrm{e}^{-c^2N}\left(-32 + 200c^2 - 152c^4 + 122c^6 - 68c^8 + 11c^{10}\right)}{Q_a} \\ & + \frac{A\mathrm{e}^{c^2N}\left(16 - 92c^2 + 30c^4 - 46c^6 + 11c^8\right)}{Q_a} + \frac{\mathrm{e}^{2c^2N}\left(-8 + 26c^2 + 117c^4 - 99c^6\right)}{Q_a} \\ & + \frac{16\sqrt{6}A^2(2 + c^2)(4 - 4c^2 + c^4)\sqrt{(2 - 11c^2)(c^2 - 2)}}{c^2Q_a} > 0\,, \end{aligned} \tag{725}$$

with $Q_a = (2 - 11c^2)\left[4A^2(c^2 - 2) + \mathrm{e}^{2c^2N}\right]$. Therefore, in this case too, conditional stability is ensured.

D. Bounce Cosmology from $F(T)$ Gravity

In this section we shall briefly investigate how the superbounce cosmology can be realized by an $F(T)$ gravity. We use the formalism we developed in chapter II, for $F(T)$, so for the flat FRW space-time, the field equations read,

$$\frac{TF_T}{3} - \frac{F}{6} + \frac{\kappa^2}{3}\rho_m = 0\,, \tag{726}$$

$$\dot{H}(F_T + 2TF_{TT}) = -\frac{\kappa^2}{2}(\rho_m + p_m)\,, \tag{727}$$

with ρ_m and p_m being the matter energy density and the pressure, and also for the flat FRW Universe we have,

$$T = -6H^2\,. \tag{728}$$

The Hubble rate as a function of the scale factor is $H = 2a^{-c^2/2}/2$. So we can express N as a function of T,

$$N = -\frac{1}{c^2}\ln\left(\frac{T}{\tilde{A}}\right)\,, \tag{729}$$

where,

$$\tilde{A} = -\frac{24}{c^4}a_0^{-c^2}\,. \tag{730}$$

In addition, by using the continuity equation for the matter fluids, we can write,

$$\rho_m = \sum_i \rho_{i0} a_0^{-3(1+w_i)} \mathrm{e}^{-3N(T)(1+w_i)}\,. \tag{731}$$

By substituting the above in Eq. (726) we have,

$$T\frac{dF(T)}{dT} - \frac{F(T)}{2} + \sum_i S_i T^{\frac{3(1+w_i)}{c^2}} = 0\,, \tag{732}$$

where S_i is defined by (696). Hence the solution of Eq. (732) is actually the exact $F(T)$ gravity that realizes the superbounce. Particularly, the vacuum $F(T)$ solution reads,

$$F(T) = c_1\,(-T)^{\frac{1}{2}}\,, \tag{733}$$

with c_1 being an arbitrary integration constant. In the case that the matter fluids are present, the solution reads,

$$F(T) = c_1\,(-T)^{\frac{1}{2}} - \sum_i Q_i\,(-T)^{\frac{3(1+w_i)}{c^2}}\,, \tag{734}$$

with

$$Q_i = \frac{2\kappa^2 c^2 \rho_{i0} a_0^{-3(1+w_i)}}{6(1+w_i) - c^2}\,. \tag{735}$$

As it can be seen, for $c = 3(1 + w_i)$ ($c = 3$ for dust matter), the superbounce cosmology can be realized by the teleparallel equivalent of General Relativity.

Hence in this chapter we demonstrated how bouncing cosmology scenarios can be realized in the context of $F(R)$, $f(\mathcal{G})$ and $F(T)$ gravity. We focused on the superbounce case and we also provided a stability analysis of the resulting modified gravities, with the stability referring to the dynamical system corresponding to the cosmological equations. For further details on these issues we refer the reader to Ref. [549].

Before closing this chapter we need to briefly discuss an important feature of the bouncing cosmology evolution which we did not address. Particularly, the reheating issue should be carefully addressed for bouncing cosmologies in the context of modified gravity. As we mentioned earlier the reheating era is extremely important for a cosmological evolution, since even in the case of bouncing cosmologies, the Universe should be reheated and in effect the particle content of a cosmological theory should be excited. However, in the literature this issue is not addressed appropriately in the context of the modified gravities we discussed in this review. To our knowledge, the only serious approach to the reheating issue in the context of bouncing cosmology was performed in Ref. [185] (see also section 6 of Ref. [175] for an interesting approach on the matter bounce scenario), where the authors used a two-field scalar model in order to realize the bounce. Particularly, the study was performed in the context of the ekpyrotic scenario, and some fields were assumed to have K-essence terms, see [185]. The reheating issue should be appropriately addressed in the context of the modified gravities we presented in this review, but this study is beyond the scopes of this work and should be studied in a focused research article.

VII. CONCLUSION

With this review we aimed to provide the latest developments in modified gravity, and also to provide a virtual toolbox for studying inflation, dark energy and bouncing cosmologies in the context of modified gravity. We tried to make the article self-contained, so we discussed most of the existing forms of modified gravity that appear in the literature and we presented in detail the theoretical framework of each modified gravity model. We also presented the essential information related to inflationary dynamics with various modified gravity theories, and also we demonstrated how various bouncing cosmologies can be realized by using various popular modified gravity theories. Now the question is, what can be the future in modified gravity theories, did we solved every existing problem, so that modified gravity can be considered a theory very well studied? The answer is categorically no, since even after many years of extensive research, we believe that we are still in the start of a long journey, and the theoretical challenges that need firstly to be understood, and then correctly modeled, are quite many. Practically, where the ΛCDM model fails to provide a consistent description of the observable Universe, modified gravity could fill in the gap. The ΛCDM model has many successes in being compatible with the observational data, at least in the pre-2000 observational data, since firstly, it predicted successfully the location and the existence of the baryon acoustic oscillations. Secondly, the ΛCDM model fits very well the statistics of weak gravitational lensing. Thirdly, the polarization of the Cosmic Microwave Background predicted by the ΛCDM model, fits very well the 2015 Planck data [48]. Particularly, the temperature power spectrum (TT) peak, the temperature polarization cross spectrum peaks (TE), and the polarization spectrum (EE) peaks, are very conveniently accommodated in the ΛCDM model predictions.

However, there are many challenging open problems and observational data, that the ΛCDM model does not suffice to offer a successful description for these [703, 704]. Mainly, the dark matter and dark energy problems haunt the ΛCDM model, since no dark matter particle has yet been observed in a particle accelerators or indirectly from direct dark matter searches. Also, the dark energy seems to be undetectable by particle accelerators or from any other laboratory apparatus, at least for the moment. Moreover, the ΛCDM model predictions on large scales are very successful, however, on sub-galactic scales, many issues occur with regards to the predictions of many dwarf galaxies and an excess of dark matter present in the innermost regions of the galaxies. However, it is still debatable if these issues are actually problems of the model itself or a problem of statistical analysis of the current research approach on these issues. In addition, there are several issues that are still in question with regard to the ΛCDM model, since these are measured by taking into account several imposed conditions. It is therefore possible that a modified gravity extension of the ΛCDM model might offer consistent answers to these problems. For example the redshift magnitude of type Ia supernovae, depends strongly on the properties of stars that exploded during the early-time era of the Universe, at least the first stages of the evolution. Also the gravitational dynamics of galaxies, gas and stars, and especially their relative motion, strongly depends on the assumption that the light coming from stars is the source of mass in the Universe. The same assumption applies when one considers gravitational lensing. Hence in some sense, the current data that fit the ΛCDM model offer an accurate cosmological description, but not a precision cosmology. It is thus possible that future data might not fit so well the ΛCDM model. Also a future challenging task for modified gravity is to explain why the cosmological constant is very small, which conveniently allows for galaxies to form and not to tear apart due to the gravitational acceleration.

There are two highly non-trivial problems that eventually modified gravity might provide consistent answers. The first has to do with the physics before the inflationary era, and in some studies, this issue has already been addressed, since it is speculated that a bounce preexisted the inflationary era. This issue is however highly non-trivial and it requires much effort in order to find a consistent answer on this. The second issue is related to the question if the cosmos comes to a crushing end eventually. In modified gravity it is possible that the Universe might abruptly end its evolution on a crushing type finite-time singularity, like a Big Rip or a Big Crush. The effects of these singularities on galactic scales might be evident even some million years before the occurrence of the singularity [247], so the question is if such a scenario is possible.

In addition, modified gravity provides an alternative view of classical particle physics problems, like the baryogenesis issue. Particularly, it is possible to generate non-zero baryon to entropy ratio in the Universe by using the gravitational baryogenesis mechanism [705]. Then, in the context of modified gravity it is possible to generalize the gravitational baryogenesis mechanism, and various proposals towards this issue have appeared in the literature [562, 706–717]. It is conceivable that the phenomenology provided by modified gravity is quite rich and research on this topic is still ongoing.

Moreover, modified gravity is one of the few theoretical frameworks for which inflation and dark energy can be described in an unified way, and we believe that the most successful model will be the one which is most compatible with observations and theoretical predictions.

Also a mysterious era to current cosmological research is the pre-inflationary era. In the existing literature there exist various proposal describing this pre-inflationary era, for example there are considerations for a superinflationary era [243] which occurs before the slow-roll inflationary era. Also a bouncing phase before the inflationary era is also an appealing alternative to the superinflationary scenario [243]. In both these scenarios, the initial singularity is absent, so these are quite appealing alternative scenarios to the standard Big Bang cosmology. The motivation to look for a pre-inflationary era is supported by certain features of the cosmic microwave background, and particularly from the large scale power deficit of the TT-mode of the cosmic microwave background. This issue is verified by the latest Planck data [48], as was pointed out in Ref. [243]. Actually the large scale anomalies can occur due to an existing contracting or expanding phase before the slow-roll era, or some sort of an superinflationary phase. The corresponding power spectrum of primordial curvature perturbations admit a large scale cutoff, which can potentially explain the power deficit of the TT-mode. The bounces preceding the inflationary phase, were proposed in Refs. [243–245, 718–720], and these are known as the "bounce inflation scenario". An interesting proposal that appeared in the literature was the singular bounce [192] in which case a soft Type IV singularity occurs at the bounce point. The power spectrum in the context of $F(R)$ gravity is not scale invariant, so this scenario could be combined with a slow-roll inflationary scenario which occurs after the bounce. In these cases modified gravity offers a conceptually simple description, so it is interesting to look for combined scenarios in the context of modified gravity. It is interesting to note that in the context of $f(\mathcal{G})$ gravity, the singular bounce yields a nearly scale invariant power spectrum [191], so this is intriguing, since a different modified gravity description of the same cosmological evolution yields entirely different results.

Moreover, with regard to the mysterious dark matter component of our Universe, it needs to be verified whether it is a particle, in which case the lightest supersymmetric particle is a viable candidate [721] or it has a modified gravity geometric origin [722–724]. Among many possible experimental ways that may determine the nature of dark matter, one promising to our opinion is the direct dark matter determination [725] and especially the experiments that use collisions of dark matter onto nuclei [726–728]. The latter method has the appealing feature of being costless in comparison to other dark matter experiments. In some cases, it is possible that direct dark matter searches may constraint the Big Bounce cosmology [728, 729], therefore the scientific future of dark matter is full of surprises for cosmologists.

Another interesting perspective comes from the future observational data on gravitational waves, which might bring new physics along, related to modified gravity to some extend. As it was noted in Refs. [730, 731] (see also [732–738]), any newly discovered degrees of freedom of gravitational waves, may in some sense reveal the modified gravity theory which may govern such new degrees of freedom. It is certainly an arena for future gravity tests, and for alternative gravity tests at a cosmological scale, see [739] and also Ref. [733] for a study on massive relics of $F(R)$ gravity gravitational waves. Along with primordial gravity waves and the $F(R)$ gravity degrees of freedom, the issue of primordial magnetic fields can be viewed as a consequence of $F(R)$ gravity, since magnetic fields are known to become amplified in $F(R)$ gravity [740]. It is important to note that a cosmological bounce and also the Big-Bang inflationary scenario are just two out of many proposed cosmological scenarios, like for example the emergent Universe scenario [741, 742]. The future cosmological observations will indicate in a consistent way which cosmological scenario fits the observational data in a successful way and also will determine how inflation and dark energy are constrained [587, 743].

We need to note that one of the ultimate aims and challenges for theoretical physicists at least the last 100 years is to find the theory that unifies gravity with all other interactions. This could be equivalent to finding a consistent quantum gravity theory, which for some physicists is a rather intangible dream or a wishful thought. To our opinion, nature represents its simplicity in various physical systems, so it could be that microphysics systems, like condensed matter physics, might eventually show the way on how gravity manifests its quantum self. It can be that the micro-geometry of space-time mimics to some extend microsystems and modified gravity might play an important role in this interplay of disciplines, see for example Refs. [744, 745] for an insightful study on this. In principle we could go deeper from the level of curvature fluctuations, and we could speak for and understand principles like the metric fluctuations. There is a long way for completely understanding these principles, so the future might bring surprises. Also, the existence of the mysterious cold spots [746–748] in the cosmic microwave background spectrum is an intriguing feature of the late-time observations of our Universe, and the possible explanations [746] offer a fruitful background for theoretical cosmology research. Eventually, modified gravity might play an important role towards the understanding of the underlying physics.

We believe that modified gravity offers aspects of possible realities for our Universe, but in general it is an established research principle in cosmology that the Universe will eventually surprise us. Hence what we can do is to humbly try to find pieces of the reality of our Universe. The time is right for new studies in the cosmology of modified gravity, since the theoretical cosmologists confront the same situation as the theoretical physicists in the beginning of the twentieth century did, that is, too many data need to be correctly interpreted. Also at the time being, modified gravity is mainly a phenomenological theory aiming for describing the evolution of the Universe. However, as we saw in chapter II (super)string theory quite frequently induces modifications to General Relativity. Furthermore, after the discovery of gravitational waves, we also tend to believe in a quantum gravity theory yet to be found. However, any quantum gravity differs from General Relativity and in effect, it may be considered as a sort of modified gravity. Also, by reconsidering standard approaches in cosmology, like for example the slow-roll era, and by replacing these with other conceptual approaches, like for example an era of constant-roll during inflation, may bring new insights in modified gravity cosmology [749]. In effect, modified gravity cosmology is a timely theoretical arena for the young scientists minds.

Finally, in this work we reviewed some aspects of modified gravity which are mainly related with our research interests. Clearly, quite a number of topics and models were not discussed or addressed properly. This is not strange, as it is impossible to cover all the aspects of a vast and still quickly growing subject such as modified gravity, in a single review. We aimed to address some important features of inflation, dark energy and bouncing cosmology, and we hope that we contributed positively in the field.

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[1] A. G. Riess *et al.* [Supernova Search Team], Astron. J. **116** (1998) 1009 [astro-ph/9805201].
[2] S. M. Carroll, V. Duvvuri, M. Trodden and M. S. Turner, Phys. Rev. D **70** (2004) 043528 [astro-ph/0306438].
[3] P. J. E. Peebles and B. Ratra, Rev. Mod. Phys. **75** (2003) 559 [astro-ph/0207347].
[4] R. Durrer and R. Maartens, Dark Energy: Observational & Theoretical Approaches, ed. P Ruiz-Lapuente (Cambridge UP, 2010), pp48 - 91 [arXiv:0811.4132 [astro-ph]].
[5] R. R. Caldwell and M. Kamionkowski, Ann. Rev. Nucl. Part. Sci. **59** (2009) 397 [arXiv:0903.0866 [astro-ph.CO]].
[6] R. Caldwell and M. Kamionkowski, Nature **458** (2009) 587.
[7] M. Li, X. D. Li, S. Wang and Y. Wang, Commun. Theor. Phys. **56** (2011) 525 [arXiv:1103.5870 [astro-ph.CO]].
[8] K. Bamba, S. Capozziello, S. Nojiri and S. D. Odintsov, Astrophys. Space Sci. **342** (2012) 155 [arXiv:1205.3421 [gr-qc]].
[9] F. S. N. Lobo, Dark Energy-Current Advances and Ideas [arXiv:0807.1640 [gr-qc]].
[10] G. Gubitosi, F. Piazza and F. Vernizzi, JCAP **1302** (2013) 032 [JCAP **1302** (2013) 032] [arXiv:1210.0201 [hep-th]].
[11] J. K. Bloomfield, E. E. Flanagan, M. Park and S. Watson, JCAP **1308** (2013) 010 [arXiv:1211.7054 [astro-ph.CO]].
[12] J. Gleyzes, D. Langlois, F. Piazza and F. Vernizzi, JCAP **1308** (2013) 025 [arXiv:1304.4840 [hep-th]].
[13] M. Li, X. D. Li, S. Wang and Y. Wang, Front. Phys. (Beijing) **8** (2013) 828 [arXiv:1209.0922 [astro-ph.CO]].
[14] M. Sami, Curr. Sci. **97** (2009) 887 [arXiv:0904.3445 [hep-th]].
[15] A. B. Balakin, Symmetry **8** (2016) no.7, 56 [arXiv:1606.06331 [gr-qc]].
[16] N. N. Weinberg and M. Kamionkowski, Mon. Not. Roy. Astron. Soc. **341** (2003) 251 [astro-ph/0210134].
[17] L. Perivolaropoulos, AIP Conf. Proc. **848** (2006) 698 [astro-ph/0601014].
[18] A. Dobado and A. L. Maroto, Astrophys. Space Sci. **320** (2009) 167 [arXiv:0802.1873 [astro-ph]].
[19] P. K. S. Dunsby and O. Luongo, Int. J. Geom. Meth. Mod. Phys. **13** (2016) no.03, 1630002 [arXiv:1511.06532 [gr-qc]].
[20] I. Zlatev, L. M. Wang and P. J. Steinhardt, Phys. Rev. Lett. **82** (1999) 896 [astro-ph/9807002].
[21] S. M. Carroll, Phys. Rev. Lett. **81** (1998) 3067 [astro-ph/9806099].
[22] L. M. Wang and P. J. Steinhardt, Astrophys. J. **508** (1998) 483 [astro-ph/9804015].
[23] T. Chiba, Phys. Rev. D **60** (1999) 083508 [gr-qc/9903094].
[24] T. Barreiro, E. J. Copeland and N. J. Nunes, Phys. Rev. D **61** (2000) 127301 [astro-ph/9910214].
[25] T. Chiba, T. Okabe and M. Yamaguchi, Phys. Rev. D **62** (2000) 023511 [astro-ph/9912463].
[26] Z. Haiman, J. J. Mohr and G. P. Holder, Astrophys. J. **553** (2000) 545 [astro-ph/0002336].
[27] S. Capozziello, Int. J. Mod. Phys. D **11** (2002) 483 [gr-qc/0201033].
[28] V. Sahni, Class. Quant. Grav. **19** (2002) 3435 [astro-ph/0202076].
[29] S. Capozziello, V. F. Cardone, S. Carloni and A. Troisi, Int. J. Mod. Phys. D **12** (2003) 1969 [astro-ph/0307018].
[30] J. G. Hao and X. Z. Li, Phys. Lett. B **606** (2005) 7 [astro-ph/0404154].
[31] B. A. Bassett, P. S. Corasaniti and M. Kunz, Astrophys. J. **617** (2004) L1 [astro-ph/0407364].
[32] L. Perivolaropoulos, Phys. Rev. D **71** (2005) 063503 [astro-ph/0412308].
[33] X. Zhang, Mod. Phys. Lett. A **20** (2005) 2575 [astro-ph/0503072].
[34] G. Olivares, F. Atrio-Barandela and D. Pavon, Phys. Rev. D **71** (2005) 063523 [astro-ph/0503242].
[35] Z. K. Guo, N. Ohta and Y. Z. Zhang, Phys. Rev. D **72** (2005) 023504 [astro-ph/0505253].
[36] R. R. Caldwell and E. V. Linder, Phys. Rev. Lett. **95** (2005) 141301 [astro-ph/0505494].
[37] R. J. Scherrer and A. A. Sen, Phys. Rev. D **77** (2008) 083515 [arXiv:0712.3450 [astro-ph]].
[38] P. Creminelli, G. D'Amico, J. Norena and F. Vernizzi, JCAP **0902** (2009) 018 [arXiv:0811.0827 [astro-ph]].
[39] S. Capozziello, S. Carloni and A. Troisi, Recent Res. Dev. Astron. Astrophys. **1** (2003) 625 [astro-ph/0303041].
[40] A. Y. Kamenshchik, U. Moschella and V. Pasquier, Phys. Lett. B **511** (2001) 265 [gr-qc/0103004].
[41] N. Banerjee and D. Pavon, Phys. Rev. D **63** (2001) 043504 [gr-qc/0012048].

[42] V. Sahni and L. M. Wang, Phys. Rev. D **62** (2000) 103517 [astro-ph/9910097].
[43] F. Perrotta, C. Baccigalupi and S. Matarrese, Phys. Rev. D **61** (1999) 023507 [astro-ph/9906066].
[44] V. Faraoni, Phys. Rev. D **62** (2000) 023504 [gr-qc/0002091].
[45] D. F. Torres, Phys. Rev. D **66** (2002) 043522 [astro-ph/0204504].
[46] V. Pettorino and C. Baccigalupi, Phys. Rev. D **77** (2008) 103003 [arXiv:0802.1086 [astro-ph]].
[47] O. Bertolami and P. J. Martins, Phys. Rev. D **61** (2000) 064007 [gr-qc/9910056].
[48] P. A. R. Ade *et al.* [Planck Collaboration], Astron. Astrophys. **594** (2016) A20 [arXiv:1502.02114 [astro-ph.CO]].
[49] P. A. R. Ade *et al.* [BICEP2 and Keck Array Collaborations], Phys. Rev. Lett. **116** (2016) 031302 [arXiv:1510.09217 [astro-ph.CO]].
[50] A. D. Linde, Lect. Notes Phys. **738** (2008) 1 [arXiv:0705.0164 [hep-th]].
[51] D. S. Gorbunov and V. A. Rubakov, "Introduction to the theory of the early universe: Cosmological perturbations and inflationary theory," Hackensack, USA: World Scientific (2011) 489 p
[52] D. H. Lyth and A. Riotto, Phys. Rept. **314** (1999) 1 [hep-ph/9807278].
[53] A. D. Linde, Phys. Lett. **129B** (1983) 177.
[54] A. D. Linde, Phys. Lett. **162B** (1985) 281.
[55] A. Albrecht and P. J. Steinhardt, Phys. Rev. Lett. **48** (1982) 1220.
[56] A. D. Linde, Phys. Rev. D **49** (1994) 748 [astro-ph/9307002].
[57] M. Sasaki and E. D. Stewart, Prog. Theor. Phys. **95** (1996) 71 [astro-ph/9507001].
[58] N. Turok, Class. Quant. Grav. **19** (2002) 3449.
[59] A. D. Linde, Prog. Theor. Phys. Suppl. **163** (2006) 295 [hep-th/0503195].
[60] S. Kachru, R. Kallosh, A. D. Linde, J. M. Maldacena, L. P. McAllister and S. P. Trivedi, JCAP **0310** (2003) 013 [hep-th/0308055].
[61] R. Brandenberger, arXiv:1601.01918 [hep-th].
[62] K. Bamba and S. D. Odintsov, Symmetry **7** (2015) no.1, 220 [arXiv:1503.00442 [hep-th]].
[63] J. Martin, C. Ringeval and V. Vennin, Phys. Dark Univ. **5-6** (2014) 75 [arXiv:1303.3787 [astro-ph.CO]].
[64] J. Martin, C. Ringeval, R. Trotta and V. Vennin, JCAP **1403** (2014) 039 [arXiv:1312.3529 [astro-ph.CO]].
[65] D. Baumann and L. McAllister, arXiv:1404.2601 [hep-th].
[66] D. Baumann, arXiv:0907.5424 [hep-th].
[67] A. Linde, arXiv:1402.0526 [hep-th].
[68] E. Pajer and M. Peloso, Class. Quant. Grav. **30** (2013) 214002 [arXiv:1305.3557 [hep-th]].
[69] M. Yamaguchi, Class. Quant. Grav. **28** (2011) 103001 [arXiv:1101.2488 [astro-ph.CO]].
[70] C. T. Byrnes and K. Y. Choi, Adv. Astron. **2010** (2010) 724525 [arXiv:1002.3110 [astro-ph.CO]].
[71] A. H. Guth, Phys. Rev. D **23** (1981) 347.
[72] S. Nojiri and S. D. Odintsov, Int. J. Geom. Meth. Mod. Phys. **11** (2014) 1460006 [arXiv:1306.4426 [gr-qc]].
[73] S. Nojiri and S. D. Odintsov, eConf C **0602061** (2006) 06 [Int. J. Geom. Meth. Mod. Phys. **4** (2007) 115] [hep-th/0601213].
[74] S. Capozziello and M. De Laurentis, Phys. Rept. **509** (2011) 167 [arXiv:1108.6266 [gr-qc]].
[75] V. Faraoni and S. Capozziello, Fundam. Theor. Phys. **170** (2010).
[76] S. Capozziello, M. De Laurentis and V. Faraoni, Open Astron. J. **3** (2010) 49 [arXiv:0909.4672 [gr-qc]].
[77] S. Nojiri and S. D. Odintsov, Phys. Rept. **505** (2011) 59 [arXiv:1011.0544 [gr-qc]].
[78] T. Clifton, P. G. Ferreira, A. Padilla and C. Skordis, Phys. Rept. **513** (2012) 1 [arXiv:1106.2476 [astro-ph.CO]].
[79] M. Ferraris, M. Francaviglia and I. Volovich, Class. Quant. Grav. **11** (1994) 1505 [gr-qc/9303007].
[80] X. H. Meng and P. Wang, Phys. Lett. B **584** (2004) 1 [hep-th/0309062].
[81] M. Amarzguioui, O. Elgaroy, D. F. Mota and T. Multamaki, Astron. Astrophys. **454** (2006) 707 [astro-ph/0510519].
[82] E. E. Flanagan, Phys. Rev. Lett. **92** (2004) 071101 [astro-ph/0308111].
[83] T. Koivisto and H. Kurki-Suonio, Class. Quant. Grav. **23** (2006) 2355 [astro-ph/0509422].
[84] S. Capozziello, T. Harko, T. S. Koivisto, F. S. N. Lobo and G. J. Olmo, JCAP **1304** (2013) 011 [arXiv:1209.2895 [gr-qc]].
[85] S. Capozziello, T. Harko, F. S. N. Lobo and G. J. Olmo, Int. J. Mod. Phys. D **22** (2013) 1342006 [arXiv:1305.3756 [gr-qc]].
[86] G. J. Olmo, Int. J. Mod. Phys. D **20** (2011) 413 [arXiv:1101.3864 [gr-qc]].
[87] G. J. Olmo, H. Sanchis-Alepuz and S. Tripathi, Phys. Rev. D **80** (2009) 024013 [arXiv:0907.2787 [gr-qc]].
[88] T. Harko, T. S. Koivisto, F. S. N. Lobo and G. J. Olmo, Phys. Rev. D **85** (2012) 084016 [arXiv:1110.1049 [gr-qc]].
[89] A. N. Makarenko, S. Odintsov and G. J. Olmo, Phys. Rev. D **90** (2014) 024066 [arXiv:1403.7409 [hep-th]].
[90] S. Capozziello, S. Nojiri and S. D. Odintsov, Phys. Lett. B **632** (2006) 597 [hep-th/0507182].
[91] S. Nojiri and S. D. Odintsov, Gen. Rel. Grav. **38** (2006) 1285 [hep-th/0506212].
[92] B. M. N. Carter and I. P. Neupane, JCAP **0606** (2006) 004 [hep-th/0512262].
[93] A. R. Liddle and L. A. Urena-Lopez, Phys. Rev. Lett. **97** (2006) 161301 [astro-ph/0605205].
[94] B. Chen, M. Li and Y. Wang, Nucl. Phys. B **774** (2007) 256 [astro-ph/0611623].
[95] S. Nojiri and S. D. Odintsov, Phys. Lett. B **657** (2007) 238 [arXiv:0707.1941 [hep-th]].
[96] S. A. Appleby and R. A. Battye, Phys. Lett. B **654** (2007) 7 [arXiv:0705.3199 [astro-ph]].
[97] S. Nojiri and S. D. Odintsov, Phys. Rev. D **77** (2008) 026007 [arXiv:0710.1738 [hep-th]].
[98] G. Cognola, E. Elizalde, S. Nojiri, S. D. Odintsov, L. Sebastiani and S. Zerbini, Phys. Rev. D **77** (2008) 046009 [arXiv:0712.4017 [hep-th]].
[99] G. Cognola, E. Elizalde, S. D. Odintsov, P. Tretyakov and S. Zerbini, Phys. Rev. D **79** (2009) 044001 [arXiv:0810.4989 [gr-qc]].
[100] T. S. Koivisto and N. J. Nunes, Phys. Rev. D **80** (2009) 103509 [arXiv:0908.0920 [astro-ph.CO]].

[101] T. Koivisto and D. F. Mota, JCAP **0808** (2008) 021 [arXiv:0805.4229 [astro-ph]].
[102] J. Q. Xia and X. Zhang, Phys. Lett. B **660** (2008) 287 [arXiv:0712.2570 [astro-ph]].
[103] A. R. Liddle, C. Pahud and L. A. Urena-Lopez, Phys. Rev. D **77** (2008) 121301 [arXiv:0804.0869 [astro-ph]].
[104] E. Elizalde and D. Saez-Gomez, Phys. Rev. D **80** (2009) 044030 [arXiv:0903.2732 [hep-th]].
[105] E. Elizalde, S. Nojiri, S. D. Odintsov and D. Saez-Gomez, Eur. Phys. J. C **70** (2010) 351 [arXiv:1006.3387 [hep-th]].
[106] A. N. Makarenko, S. D. Odintsov and G. J. Olmo, Phys. Lett. B **734** (2014) 36 [arXiv:1404.2850 [gr-qc]].
[107] J. de Haro and E. Elizalde, Gen. Rel. Grav. **48** (2016) no.6, 77 [arXiv:1602.03433 [gr-qc]].
[108] S. Bahamonde, C. G. Bohmer, F. S. N. Lobo and D. Saez-Gomez, Universe **1** (2015) no.2, 186 [arXiv:1506.07728 [gr-qc]].
[109] E. Elizalde, J. Quiroga Hurtado and H. I. Arcos, Int. J. Mod. Phys. D **17** (2008) 2159 [arXiv:0708.0591 [gr-qc]].
[110] S. Nojiri, S. D. Odintsov and P. V. Tretyakov, Phys. Lett. B **651** (2007) 224 [arXiv:0704.2520 [hep-th]].
[111] D. Huterer and E. V. Linder, Phys. Rev. D **75** (2007) 023519 [astro-ph/0608681].
[112] S. Nojiri, S. D. Odintsov and H. Stefancic, Phys. Rev. D **74** (2006) 086009 [hep-th/0608168].
[113] S. Capozziello, V. F. Cardone and A. Troisi, Phys. Rev. D **71** (2005) 043503 [astro-ph/0501426].
[114] A. Borowiec, W. Godlowski and M. Szydlowski, eConf C **0602061** (2006) 09 [Int. J. Geom. Meth. Mod. Phys. **4** (2007) 183] [astro-ph/0607639].
[115] S. Nojiri, S. D. Odintsov and O. G. Gorbunova, J. Phys. A **39** (2006) 6627 [hep-th/0510183].
[116] M. Szydlowski, A. Kurek and A. Krawiec, Phys. Lett. B **642** (2006) 171 [astro-ph/0604327].
[117] A. Borowiec, W. Godlowski and M. Szydlowski, Phys. Rev. D **74** (2006) 043502 [astro-ph/0602526].
[118] G. Allemandi, A. Borowiec and M. Francaviglia, Phys. Rev. D **70** (2004) 103503 [hep-th/0407090].
[119] D. A. Easson, Int. J. Mod. Phys. A **19** (2004) 5343 [astro-ph/0411209].
[120] S. Carloni, P. K. S. Dunsby, S. Capozziello and A. Troisi, Class. Quant. Grav. **22** (2005) 4839 [gr-qc/0410046].
[121] T. Clifton and J. D. Barrow, Phys. Rev. D **72** (2005) no.10, 103005 Erratum: [Phys. Rev. D **90** (2014) no.2, 029902] [gr-qc/0509059].
[122] A. K. Sanyal, Phys. Lett. B **645** (2007) 1 [astro-ph/0608104].
[123] S. A. Appleby and R. A. Battye, JCAP **0805** (2008) 019 [arXiv:0803.1081 [astro-ph]].
[124] S. Capozziello, V. F. Cardone and V. Salzano, Phys. Rev. D **78** (2008) 063504 [arXiv:0802.1583 [astro-ph]].
[125] J. D. Evans, L. M. H. Hall and P. Caillol, Phys. Rev. D **77** (2008) 083514 [arXiv:0711.3695 [astro-ph]].
[126] S. Capozziello and M. Francaviglia, Gen. Rel. Grav. **40** (2008) 357 [arXiv:0706.1146 [astro-ph]].
[127] B. Li, J. D. Barrow and D. F. Mota, Phys. Rev. D **76** (2007) 044027 [arXiv:0705.3795 [gr-qc]].
[128] O. Bertolami, C. G. Boehmer, T. Harko and F. S. N. Lobo, Phys. Rev. D **75** (2007) 104016 [arXiv:0704.1733 [gr-qc]].
[129] B. Li and J. D. Barrow, Phys. Rev. D **75** (2007) 084010 [gr-qc/0701111].
[130] Y. S. Song, W. Hu and I. Sawicki, Phys. Rev. D **75** (2007) 044004 [astro-ph/0610532].
[131] E. V. Arbuzova, A. D. Dolgov and L. Reverberi, JCAP **1202** (2012) 049 [arXiv:1112.4995 [gr-qc]].
[132] M. Artymowski and Z. Lalak, JCAP **1409** (2014) 036 [arXiv:1405.7818 [hep-th]].
[133] S. Fay, S. Nesseris and L. Perivolaropoulos, Phys. Rev. D **76** (2007) 063504 [gr-qc/0703006 [GR-QC]].
[134] A. Yurov, Eur. Phys. J. Plus **126** (2011) 132 [astro-ph/0305019].
[135] Y. S. Piao and Y. Z. Zhang, Phys. Rev. D **70** (2004) 063513 [astro-ph/0401231].
[136] P. F. Gonzalez-Diaz and J. A. Jimenez-Madrid, Phys. Lett. B **596** (2004) 16 [hep-th/0406261].
[137] K. Izumi and S. Mukohyama, JCAP **1006** (2010) 016 [arXiv:1004.1776 [hep-th]].
[138] C. J. Feng, X. Z. Li and E. N. Saridakis, Phys. Rev. D **82** (2010) 023526 [arXiv:1004.1874 [astro-ph.CO]].
[139] Z. G. Liu and Y. S. Piao, Phys. Lett. B **713** (2012) 53 [arXiv:1203.4901 [gr-qc]].
[140] S. Nojiri and S. D. Odintsov, Phys. Rev. D **68** (2003) 123512 [hep-th/0307288].
[141] M. Rinaldi, Class. Quant. Grav. **32** (2015) 045002 [arXiv:1404.0532 [astro-ph.CO]].
[142] S. Nojiri, S. D. Odintsov and V. K. Oikonomou, Phys. Rev. D **94** (2016) no.10, 104050 [arXiv:1608.07806 [gr-qc]].
[143] A. Borowiec, [arXiv:0812.4383 [gr-qc]].
[144] E. Elizalde and S. I. Vacaru, Gen. Rel. Grav. **47** (2015) 64 [arXiv:1310.6868 [math-ph]].
[145] Y. F. Cai and J. Wang, Class. Quant. Grav. **25** (2008) 165014 [arXiv:0806.3890 [hep-th]].
[146] C. Q. Geng, C. C. Lee, E. N. Saridakis and Y. P. Wu, Phys. Lett. B **704** (2011) 384 [arXiv:1109.1092 [hep-th]].
[147] S. I. Vacaru, Eur. Phys. J. C **75** (2015) no.4, 176 [arXiv:1504.04346 [gr-qc]].
[148] G. J. Olmo, Phys. Rev. Lett. **98** (2007) 061101 [gr-qc/0612002].
[149] J. Santos, J. S. Alcaniz, M. J. Reboucas and F. C. Carvalho, Phys. Rev. D **76** (2007) 083513 [arXiv:0708.0411 [astro-ph]].
[150] K. Bamba, Int. J. Geom. Meth. Mod. Phys. **13** (2016) no.06, 1630007 [arXiv:1604.02632 [gr-qc]].
[151] M. Zubair, F. Kousar and S. Bahamonde, Phys. Dark Univ. **14** (2016) 116 [arXiv:1604.07213 [gr-qc]].
[152] G. J. Olmo, Phys. Rev. Lett. **95** (2005) 261102 [gr-qc/0505101].
[153] V. Faraoni, Phys. Rev. D **74** (2006) 023529 [gr-qc/0607016].
[154] P. J. Zhang, Phys. Rev. D **76** (2007) 024007 [astro-ph/0701662].
[155] G. J. Olmo, Phys. Rev. D **75** (2007) 023511 [gr-qc/0612047].
[156] G. Allemandi and M. L. Ruggiero, Gen. Rel. Grav. **39** (2007) 1381 [astro-ph/0610661].
[157] A. L. Erickcek, T. L. Smith and M. Kamionkowski, Phys. Rev. D **74** (2006) 121501 [astro-ph/0610483].
[158] W. T. Lin, J. A. Gu and P. Chen, Int. J. Mod. Phys. D **20** (2011) 1357 [arXiv:1009.3488 [astro-ph.CO]].
[159] L. Iorio, JCAP **1105** (2011) 019 [arXiv:1012.0226 [gr-qc]].
[160] G. J. Olmo, Phys. Rev. D **72** (2005) 083505 [gr-qc/0505135].
[161] S. Capozziello, A. Stabile and A. Troisi, Phys. Rev. D **76** (2007) 104019 [arXiv:0708.0723 [gr-qc]].
[162] V. Faraoni, Phys. Rev. D **72** (2005) 124005 [gr-qc/0511094].

[163] V. Faraoni, Phys. Rev. D **74** (2006) 104017 [astro-ph/0610734].
[164] G. Cognola, M. Gastaldi and S. Zerbini, Int. J. Theor. Phys. **47** (2008) 898 [gr-qc/0701138].
[165] I. Sawicki and W. Hu, Phys. Rev. D **75** (2007) 127502 [astro-ph/0702278].
[166] R. C. Nunes, S. Pan, E. N. Saridakis and E. M. C. Abreu, arXiv:1610.07518 [astro-ph.CO].
[167] V. Faraoni, Annals Phys. **317** (2005) 366 [gr-qc/0502015].
[168] S. Carloni and J. P. Mimoso, arXiv:1701.00231 [gr-qc].
[169] R. C. Tolman, Phys. Rev. **37** (1931) 1639. doi:10.1103/PhysRev.37.1639
[170] R. H. Brandenberger, arXiv:1206.4196 [astro-ph.CO].
[171] R. Brandenberger and P. Peter, arXiv:1603.05834 [hep-th].
[172] D. Battefeld and P. Peter, Phys. Rept. **571** (2015) 1 [arXiv:1406.2790 [astro-ph.CO]].
[173] M. Novello and S. E. P. Bergliaffa, Phys. Rept. **463** (2008) 127 [arXiv:0802.1634 [astro-ph]].
[174] Y. F. Cai, Sci. China Phys. Mech. Astron. **57** (2014) 1414 doi:10.1007/s11433-014-5512-3 [arXiv:1405.1369 [hep-th]].
[175] J. de Haro and Y. F. Cai, Gen. Rel. Grav. **47** (2015) no.8, 95 [arXiv:1502.03230 [gr-qc]].
[176] J. L. Lehners, Class. Quant. Grav. **28** (2011) 204004 [arXiv:1106.0172 [hep-th]].
[177] J. L. Lehners, Phys. Rept. **465** (2008) 223 [arXiv:0806.1245 [astro-ph]].
[178] Y. K. E. Cheung, C. Li and J. D. Vergados, arXiv:1611.04027 [astro-ph.CO].
[179] Y. F. Cai, A. Marciano, D. G. Wang and E. Wilson-Ewing, Universe **3** (2016) no.1, 1 doi:10.3390/universe3010001 [arXiv:1610.00938 [astro-ph.CO]].
[180] C. Cattoen and M. Visser, Class. Quant. Grav. **22** (2005) 4913 [gr-qc/0508045].
[181] V. F. Mukhanov and R. H. Brandenberger, Phys. Rev. Lett. **68** (1992) 1969. doi:10.1103/PhysRevLett.68.1969
[182] C. Li, R. H. Brandenberger and Y. K. E. Cheung, Phys. Rev. D **90** (2014) no.12, 123535 [arXiv:1403.5625 [gr-qc]].
[183] D. Brizuela, G. A. D. Mena Marugan and T. Pawlowski, Class. Quant. Grav. **27** (2010) 052001 [arXiv:0902.0697 [gr-qc]].
[184] Y. F. Cai, E. McDonough, F. Duplessis and R. H. Brandenberger, JCAP **1310** (2013) 024 [arXiv:1305.5259 [hep-th]].
[185] J. Quintin, Y. F. Cai and R. H. Brandenberger, Phys. Rev. D **90** (2014) no.6, 063507 [arXiv:1406.6049 [gr-qc]].
[186] Y. F. Cai, R. Brandenberger and P. Peter, Class. Quant. Grav. **30** (2013) 075019 [arXiv:1301.4703 [gr-qc]].
[187] N. J. Poplawski, Phys. Rev. D **85** (2012) 107502 [arXiv:1111.4595 [gr-qc]].
[188] M. Koehn, J. L. Lehners and B. Ovrut, Phys. Rev. D **93** (2016) no.10, 103501 [arXiv:1512.03807 [hep-th]].
[189] S. D. Odintsov and V. K. Oikonomou, Phys. Rev. D **92** (2015) no.2, 024016 [arXiv:1504.06866 [gr-qc]].
[190] S. Nojiri, S. D. Odintsov and V. K. Oikonomou, Phys. Rev. D **93** (2016) no.8, 084050 [arXiv:1601.04112 [gr-qc]].
[191] V. K. Oikonomou, Phys. Rev. D **92** (2015) no.12, 124027 [arXiv:1509.05827 [gr-qc]].
[192] S. D. Odintsov and V. K. Oikonomou, arXiv:1512.04787 [gr-qc].
[193] M. Koehn, J. L. Lehners and B. A. Ovrut, Phys. Rev. D **90** (2014) no.2, 025005 [arXiv:1310.7577 [hep-th]].
[194] L. Battarra and J. L. Lehners, JCAP **1412** (2014) no.12, 023 [arXiv:1407.4814 [hep-th]].
[195] J. Martin, P. Peter, N. Pinto Neto and D. J. Schwarz, Phys. Rev. D **65** (2002) 123513 [hep-th/0112128].
[196] J. Khoury, B. A. Ovrut, P. J. Steinhardt and N. Turok, Phys. Rev. D **64** (2001) 123522 [hep-th/0103239].
[197] E. I. Buchbinder, J. Khoury and B. A. Ovrut, Phys. Rev. D **76** (2007) 123503 [hep-th/0702154].
[198] M. G. Brown, K. Freese and W. H. Kinney, JCAP **0803** (2008) 002 [astro-ph/0405353].
[199] J. C. Hackworth and E. J. Weinberg, Phys. Rev. D **71** (2005) 044014 [hep-th/0410142].
[200] S. Nojiri and S. D. Odintsov, Phys. Lett. B **637** (2006) 139 [hep-th/0603062].
[201] M. C. Johnson and J. L. Lehners, Phys. Rev. D **85** (2012) 103509 [arXiv:1112.3360 [hep-th]].
[202] P. Peter and N. Pinto-Neto, Phys. Rev. D **66** (2002) 063509 [hep-th/0203013].
[203] M. Gasperini, M. Giovannini and G. Veneziano, Phys. Lett. B **569** (2003) 113 [hep-th/0306113].
[204] P. Creminelli, A. Nicolis and M. Zaldarriaga, Phys. Rev. D **71** (2005) 063505 [hep-th/0411270].
[205] J. L. Lehners and E. Wilson-Ewing, JCAP **1510** (2015) no.10, 038 [arXiv:1507.08112 [astro-ph.CO]].
[206] J. Mielczarek, M. Kamionka, A. Kurek and M. Szydlowski, JCAP **1007** (2010) 004 [arXiv:1005.0814 [gr-qc]].
[207] J. L. Lehners and P. J. Steinhardt, Phys. Rev. D **87** (2013) no.12, 123533 [arXiv:1304.3122 [astro-ph.CO]].
[208] Y. F. Cai, J. Quintin, E. N. Saridakis and E. Wilson-Ewing, JCAP **1407** (2014) 033 [arXiv:1404.4364 [astro-ph.CO]].
[209] P. Laguna, Phys. Rev. D **75** (2007) 024033 [gr-qc/0608117].
[210] A. Corichi and P. Singh, Phys. Rev. Lett. **100** (2008) 161302 [arXiv:0710.4543 [gr-qc]].
[211] M. Bojowald, Gen. Rel. Grav. **40** (2008) 2659 [arXiv:0801.4001 [gr-qc]].
[212] P. Singh, K. Vandersloot and G. V. Vereshchagin, Phys. Rev. D **74** (2006) 043510 [gr-qc/0606032].
[213] G. Date and G. M. Hossain, Phys. Rev. Lett. **94** (2005) 011302 [gr-qc/0407074].
[214] J. de Haro, JCAP **1211** (2012) 037 [arXiv:1207.3621 [gr-qc]].
[215] F. Cianfrani and G. Montani, Phys. Rev. D **82** (2010) 021501 [arXiv:1006.1814 [gr-qc]].
[216] Y. F. Cai and E. Wilson-Ewing, JCAP **1403** (2014) 026 [arXiv:1402.3009 [gr-qc]].
[217] J. Mielczarek and M. Szydlowski, Phys. Rev. D **77** (2008) 124008 [arXiv:0801.1073 [gr-qc]].
[218] J. Mielczarek, T. Stachowiak and M. Szydlowski, Phys. Rev. D **77** (2008) 123506 [arXiv:0801.0502 [gr-qc]].
[219] P. Diener, B. Gupt and P. Singh, Class. Quant. Grav. **31** (2014) 105015 [arXiv:1402.6613 [gr-qc]].
[220] J. Haro, A. N. Makarenko, A. N. Myagky, S. D. Odintsov and V. K. Oikonomou, Phys. Rev. D **92** (2015) no.12, 124026 [arXiv:1506.08273 [gr-qc]].
[221] X. Zhang and Y. Ma, Phys. Rev. D **84** (2011) 064040 [arXiv:1107.4921 [gr-qc]].
[222] X. Zhang and Y. Ma, Phys. Rev. Lett. **106** (2011) 171301 [arXiv:1101.1752 [gr-qc]].
[223] Y. F. Cai and E. Wilson-Ewing, JCAP **1503** (2015) no.03, 006 [arXiv:1412.2914 [gr-qc]].
[224] E. Wilson-Ewing, JCAP **1303** (2013) 026 [arXiv:1211.6269 [gr-qc]].

[225] F. Finelli and R. Brandenberger, Phys. Rev. D **65** (2002) 103522 [hep-th/0112249].
[226] Y. F. Cai, R. Brandenberger and X. Zhang, Phys. Lett. B **703** (2011) 25 [arXiv:1105.4286 [hep-th]].
[227] J. Haro and J. Amoros, PoS FFP **14** (2016) 163 [arXiv:1501.06270 [gr-qc]].
[228] Y. F. Cai, R. Brandenberger and X. Zhang, JCAP **1103** (2011) 003 [arXiv:1101.0822 [hep-th]].
[229] J. Haro and J. Amoros, JCAP **1412** (2014) no.12, 031 [arXiv:1406.0369 [gr-qc]].
[230] R. Brandenberger, Phys. Rev. D **80** (2009) 043516 [arXiv:0904.2835 [hep-th]].
[231] J. de Haro and J. Amoros, JCAP **1408** (2014) 025 [arXiv:1403.6396 [gr-qc]].
[232] S. D. Odintsov and V. K. Oikonomou, Phys. Rev. D **90** (2014) no.12, 124083 [arXiv:1410.8183 [gr-qc]].
[233] T. Qiu and K. C. Yang, JCAP **1011** (2010) 012 [arXiv:1007.2571 [astro-ph.CO]].
[234] V. K. Oikonomou, Gen. Rel. Grav. **47** (2015) no.10, 126 [arXiv:1412.8195 [gr-qc]].
[235] K. Bamba, J. de Haro and S. D. Odintsov, JCAP **1302** (2013) 008 [arXiv:1211.2968 [gr-qc]].
[236] Y. F. Cai, T. Qiu, Y. S. Piao, M. Li and X. Zhang, JHEP **0710** (2007) 071 [arXiv:0704.1090 [gr-qc]].
[237] Y. F. Cai and E. N. Saridakis, Class. Quant. Grav. **28** (2011) 035010 [arXiv:1007.3204 [astro-ph.CO]].
[238] P. P. Avelino and R. Z. Ferreira, Phys. Rev. D **86** (2012) 041501 [arXiv:1205.6676 [astro-ph.CO]].
[239] J. D. Barrow, D. Kimberly and J. Magueijo, Class. Quant. Grav. **21** (2004) 4289 [astro-ph/0406369].
[240] J. Haro and E. Elizalde, JCAP **1510** (2015) no.10, 028 [arXiv:1505.07948 [gr-qc]].
[241] E. Elizalde, J. Haro and S. D. Odintsov, Phys. Rev. D **91** (2015) no.6, 063522 [arXiv:1411.3475 [gr-qc]].
[242] Y. F. Cai, E. N. Saridakis, M. R. Setare and J. Q. Xia, Phys. Rept. **493** (2010) 1 doi:10.1016/j.physrep.2010.04.001 [arXiv:0909.2776 [hep-th]].
[243] Y. Cai, Y. T. Wang and Y. S. Piao, Phys. Rev. D **92** (2015) no.2, 023518 [arXiv:1501.01730 [astro-ph.CO]].
[244] Y. S. Piao, B. Feng and X. m. Zhang, Phys. Rev. D **69** (2004) 103520 [hep-th/0310206].
[245] Y. S. Piao, Phys. Rev. D **71** (2005) 087301 [astro-ph/0502343].
[246] S. D. Odintsov and V. K. Oikonomou, Phys. Rev. D **94** (2016) no.6, 064022 [arXiv:1606.03689 [gr-qc]].
[247] R. R. Caldwell, M. Kamionkowski and N. N. Weinberg, Phys. Rev. Lett. **91** (2003) 071301 [astro-ph/0302506].
[248] B. McInnes, JHEP **0208** (2002) 029 [hep-th/0112066].
[249] S. Nojiri and S. D. Odintsov, Phys. Lett. B **562** (2003) 147 [hep-th/0303117].
[250] S. Nojiri and S. D. Odintsov, Phys. Rev. D **72** (2005) 023003 [hep-th/0505215].
[251] V. Gorini, A. Kamenshchik and U. Moschella, Phys. Rev. D **67** (2003) 063509 [astro-ph/0209395].
[252] E. Elizalde, S. Nojiri and S. D. Odintsov, Phys. Rev. D **70** (2004) 043539 [hep-th/0405034].
[253] V. Faraoni, Int. J. Mod. Phys. D **11** (2002) 471 [astro-ph/0110067].
[254] P. Singh, M. Sami and N. Dadhich, Phys. Rev. D **68** (2003) 023522 [hep-th/0305110].
[255] C. Csaki, N. Kaloper and J. Terning, Annals Phys. **317** (2005) 410 [astro-ph/0409596].
[256] P. X. Wu and H. W. Yu, Nucl. Phys. B **727** (2005) 355 [astro-ph/0407424].
[257] S. Nesseris and L. Perivolaropoulos, Phys. Rev. D **70** (2004) 123529 [astro-ph/0410309].
[258] M. Sami and A. Toporensky, Mod. Phys. Lett. A **19** (2004) 1509 [gr-qc/0312009].
[259] H. Stefancic, Phys. Lett. B **586** (2004) 5 [astro-ph/0310904].
[260] L. P. Chimento and R. Lazkoz, Phys. Rev. Lett. **91** (2003) 211301 [gr-qc/0307111].
[261] L. P. Chimento and R. Lazkoz, Mod. Phys. Lett. A **19** (2004) 2479 [gr-qc/0405020].
[262] E. Babichev, V. Dokuchaev and Y. Eroshenko, Class. Quant. Grav. **22** (2005) 143 [astro-ph/0407190].
[263] X. F. Zhang, H. Li, Y. S. Piao and X. M. Zhang, Mod. Phys. Lett. A **21** (2006) 231 [astro-ph/0501652].
[264] M. P. Dabrowski and T. Stachowiak, Annals Phys. **321** (2006) 771 [hep-th/0411199].
[265] F. S. N. Lobo, Phys. Rev. D **71** (2005) 084011 [gr-qc/0502099].
[266] R. G. Cai, H. S. Zhang and A. Wang, Commun. Theor. Phys. **44** (2005) 948 [hep-th/0505186].
[267] I. Y. Aref'eva, A. S. Koshelev and S. Y. Vernov, Phys. Rev. D **72** (2005) 064017 [astro-ph/0507067].
[268] H. Q. Lu, Z. G. Huang and W. Fang, hep-th/0504038.
[269] W. Godlowski and M. Szydlowski, Phys. Lett. B **623** (2005) 10 [astro-ph/0507322].
[270] B. Guberina, R. Horvat and H. Nikolic, Phys. Rev. D **72** (2005) 125011 [astro-ph/0507666].
[271] M. P. Dabrowski, C. Kiefer and B. Sandhofer, Phys. Rev. D **74** (2006) 044022 [hep-th/0605229].
[272] L. P. Chimento and M. G. Richarte, Phys. Rev. D **92** (2015) no.4, 043511 [arXiv:1506.05823 [gr-qc]].
[273] J. D. Barrow and S. Z. W. Lip, Phys. Rev. D **80** (2009) 043518 [arXiv:0901.1626 [gr-qc]].
[274] A. V. Yurov, A. V. Astashenok and P. F. Gonzalez-Diaz, Grav. Cosmol. **14** (2008) 205 [arXiv:0705.4108 [astro-ph]].
[275] M. Bouhmadi-Lopez, P. F. Gonzalez-Diaz and P. Martin-Moruno, Int. J. Mod. Phys. D **17** (2008) 2269 [arXiv:0707.2390 [gr-qc]].
[276] M. Bouhmadi-Lopez, P. F. Gonzalez-Diaz and P. Martin-Moruno, Phys. Lett. B **659** (2008) 1 [gr-qc/0612135].
[277] F. Briscese, E. Elizalde, S. Nojiri and S. D. Odintsov, Phys. Lett. B **646** (2007) 105 [hep-th/0612220].
[278] K. Bamba, A. N. Makarenko, A. N. Myagky, S. Nojiri and S. D. Odintsov, JCAP **1401** (2014) 008 [arXiv:1309.3748 [hep-th]].
[279] C. Barragan, G. J. Olmo and H. Sanchis-Alepuz, Phys. Rev. D **80** (2009) 024016 [arXiv:0907.0318 [gr-qc]].
[280] H. Farajollahi and F. Milani, Mod. Phys. Lett. A **25** (2010) 2349 [arXiv:1004.3512 [gr-qc]].
[281] A. Escofet and E. Elizalde, Mod. Phys. Lett. A **31** (2016) no.17, 1650108 [arXiv:1510.05848 [gr-qc]].
[282] C. Barragan and G. J. Olmo, Phys. Rev. D **82** (2010) 084015 [arXiv:1005.4136 [gr-qc]].
[283] T. S. Koivisto, Phys. Rev. D **82** (2010) 044022 [arXiv:1004.4298 [gr-qc]].
[284] M. Komada, S. Nojiri and T. Katsuragawa, Phys. Lett. B **755** (2016) 31 [arXiv:1409.1663 [hep-th]].
[285] F. S. N. Lobo and M. A. Oliveira, Phys. Rev. D **80** (2009) 104012 [arXiv:0909.5539 [gr-qc]].

[286] G. Cognola, E. Elizalde, S. Nojiri, S. D. Odintsov and S. Zerbini, JCAP **0502** (2005) 010 [hep-th/0501096].
[287] S. Nojiri and S. D. Odintsov, AIP Conf. Proc. **1241** (2010) 1094 [arXiv:0910.1464 [hep-th]].
[288] S. Nojiri and S. D. Odintsov, Gen. Rel. Grav. **36** (2004) 1765 [hep-th/0308176].
[289] S. Nojiri and S. D. Odintsov, J. Phys. Conf. Ser. **66** (2007) 012005 [hep-th/0611071].
[290] A. A. Starobinsky, JETP Lett. **86** (2007) 157 [arXiv:0706.2041 [astro-ph]].
[291] S. Nojiri and S. D. Odintsov, Phys. Lett. B **652** (2007) 343 [arXiv:0706.1378 [hep-th]].
[292] S. Nojiri and S. D. Odintsov, Mod. Phys. Lett. A **19** (2004) 627 [hep-th/0310045].
[293] S. Nojiri and S. D. Odintsov, Phys. Rev. D **74** (2006) 086005 [hep-th/0608008].
[294] S. Capozziello, S. Nojiri, S. D. Odintsov and A. Troisi, Phys. Lett. B **639** (2006) 135 [astro-ph/0604431].
[295] S. Bahamonde, S. D. Odintsov, V. K. Oikonomou and P. V. Tretyakov, arXiv:1701.02381 [gr-qc].
[296] S. Bahamonde, S. D. Odintsov, V. K. Oikonomou and M. Wright, Annals Phys. **373** (2016) 96 [arXiv:1603.05113 [gr-qc]].
[297] W. Hu and I. Sawicki, Phys. Rev. D **76** (2007) 064004 [arXiv:0705.1158 [astro-ph]].
[298] J. Khoury and A. Weltman, Phys. Rev. D **69** (2004) 044026 [astro-ph/0309411].
[299] E. V. Linder, Phys. Rev. D **80** (2009) 123528 [arXiv:0905.2962 [astro-ph.CO]].
[300] K. Bamba, C. Q. Geng and C. C. Lee, JCAP **1008** (2010) 021 [arXiv:1005.4574 [astro-ph.CO]].
[301] A. D. Dolgov and M. Kawasaki, Phys. Lett. B **573** (2003) 1 [astro-ph/0307285].
[302] L. M. Sokolowski, Class. Quant. Grav. **24** (2007) 3391 [gr-qc/0702097 [GR-QC]].
[303] A. W. Brookfield, C. van de Bruck and L. M. H. Hall, Phys. Rev. D **74** (2006) 064028 [hep-th/0608015].
[304] M. Abdelwahab, S. Carloni and P. K. S. Dunsby, Class. Quant. Grav. **25** (2008) 135002 [arXiv:0706.1375 [gr-qc]].
[305] V. K. Oikonomou, Gen. Rel. Grav. **45** (2013) 2467 [arXiv:1304.4089 [gr-qc]].
[306] R. R. Caldwell, Phys. Lett. B **545** (2002) 23 [astro-ph/9908168].
[307] S. Nojiri and S. D. Odintsov, Phys. Lett. B **571** (2003) 1 [hep-th/0306212].
[308] P. F. Gonzalez-Diaz, Phys. Lett. B **586** (2004) 1 [astro-ph/0312579].
[309] P. F. Gonzalez-Diaz, TSPU Bulletin **44N7** (2004) 36 [hep-th/0408225].
[310] E. Elizalde, S. Nojiri, S. D. Odintsov and P. Wang, Phys. Rev. D **71** (2005) 103504 [hep-th/0502082].
[311] I. Y. Aref'eva, A. S. Koshelev and S. Y. Vernov, Theor. Math. Phys. **148** (2006) 895 [Teor. Mat. Fiz. **148** (2006) 23] [astro-ph/0412619].
[312] E. M. Barbaoza, Jr. and N. A. Lemos, Gen. Rel. Grav. **38** (2006) 1609 [gr-qc/0606084].
[313] S. Nojiri, S. D. Odintsov and S. Tsujikawa, Phys. Rev. D **71** (2005) 063004 [hep-th/0501025].
[314] J. D. Barrow and C. G. Tsagas, Class. Quant. Grav. **22** (2005) 1563 [gr-qc/0411045].
[315] J. D. Barrow, Class. Quant. Grav. **21** (2004) 5619 [gr-qc/0409062].
[316] J. D. Barrow, Class. Quant. Grav. **21** (2004) L79 [gr-qc/0403084].
[317] L. Fernandez-Jambrina and R. Lazkoz, Phys. Rev. D **70** (2004) 121503 [gr-qc/0410124].
[318] M. P. Dabrowski, Phys. Rev. D **71** (2005) 103505 [gr-qc/0410033].
[319] K. Lake, Class. Quant. Grav. **21** (2004) L129 [gr-qc/0407107].
[320] S. Nojiri and S. D. Odintsov, Phys. Lett. B **595** (2004) 1 [hep-th/0405078].
[321] J. de Haro, J. Amoros and E. Elizalde, Phys. Rev. D **85** (2012) 123527 [arXiv:1204.4039 [gr-qc]].
[322] J. Beltrán Jiménez, D. Rubiera-Garcia, D. Sáez-Gómez and V. Salzano, Phys. Rev. D **94** (2016) no.12, 123520 [arXiv:1607.06389 [gr-qc]].
[323] S. Nojiri and S. D. Odintsov, Phys. Rev. D **78** (2008) 046006 [arXiv:0804.3519 [hep-th]].
[324] M. C. B. Abdalla, S. Nojiri and S. D. Odintsov, Class. Quant. Grav. **22** (2005) L35 [hep-th/0409177].
[325] K. Bamba, S. Nojiri and S. D. Odintsov, JCAP **0810** (2008) 045 [arXiv:0807.2575 [hep-th]].
[326] S. Capozziello, M. De Laurentis, S. Nojiri and S. D. Odintsov, Phys. Rev. D **79** (2009) 124007 [arXiv:0903.2753 [hep-th]].
[327] K. Bamba, S. D. Odintsov, L. Sebastiani and S. Zerbini, Eur. Phys. J. C **67** (2010) 295 [arXiv:0911.4390 [hep-th]].
[328] T. Kobayashi and K. i. Maeda, Phys. Rev. D **79** (2009) 024009 [arXiv:0810.5664 [astro-ph]].
[329] I. Thongkool, M. Sami, R. Gannouji and S. Jhingan, Phys. Rev. D **80** (2009) 043523 [arXiv:0906.2460 [hep-th]].
[330] E. Babichev and D. Langlois, Phys. Rev. D **81** (2010) 124051 [arXiv:0911.1297 [gr-qc]].
[331] S. A. Appleby and J. Weller, JCAP **1012** (2010) 006 [arXiv:1008.2693 [astro-ph.CO]].
[332] S. Nojiri and S. D. Odintsov, Phys. Lett. B **686** (2010) 44 [arXiv:0911.2781 [hep-th]].
[333] S. Nojiri and S. D. Odintsov, Prog. Theor. Phys. Suppl. **190** (2011) 155 [arXiv:1008.4275 [hep-th]].
[334] L. Lombriser, A. Slosar, U. Seljak and W. Hu, Phys. Rev. D **85** (2012) 124038 [arXiv:1003.3009 [astro-ph.CO]].
[335] F. Schmidt, A. Vikhlinin and W. Hu, Phys. Rev. D **80** (2009) 083505 [arXiv:0908.2457 [astro-ph.CO]].
[336] S. Nojiri and S. D. Odintsov, Phys. Lett. B **599** (2004) 137 [astro-ph/0403622].
[337] G. Allemandi, A. Borowiec, M. Francaviglia and S. D. Odintsov, Phys. Rev. D **72** (2005) 063505 [gr-qc/0504057].
[338] O. Bertolami, J. Paramos, T. Harko and F. S. N. Lobo, arXiv:0811.2876 [gr-qc].
[339] S. Capozziello, G. Lambiase and H. J. Schmidt, Annalen Phys. **9** (2000) 39 [gr-qc/9906051].
[340] O. Bertolami, F. S. N. Lobo and J. Paramos, Phys. Rev. D **78** (2008) 064036 [arXiv:0806.4434 [gr-qc]].
[341] T. Harko, F. S. N. Lobo, S. Nojiri and S. D. Odintsov, Phys. Rev. D **84** (2011) 024020 [arXiv:1104.2669 [gr-qc]].
[342] M. Jamil, D. Momeni, M. Raza and R. Myrzakulov, Eur. Phys. J. C **72** (2012) 1999 [arXiv:1107.5807 [physics.gen-ph]].
[343] M. Sharif and M. Zubair, JCAP **1203** (2012) 028 Erratum: [JCAP **1205** (2013) E01] [arXiv:1204.0848 [gr-qc]].
[344] M. J. S. Houndjo, Int. J. Mod. Phys. D **21** (2012) 1250003 [arXiv:1107.3887 [astro-ph.CO]].
[345] F. G. Alvarenga, A. de la Cruz-Dombriz, M. J. S. Houndjo, M. E. Rodrigues and D. Saez-Gomez, Phys. Rev. D **87** (2013) no.10, 103526 Erratum: [Phys. Rev. D **87** (2013) no.12, 129905] [arXiv:1302.1866 [gr-qc]].
[346] Z. Haghani, T. Harko, F. S. N. Lobo, H. R. Sepangi and S. Shahidi, Phys. Rev. D **88** (2013) no.4, 044023 [arXiv:1304.5957

[gr-qc]].
[347] S. D. Odintsov and D. Saez-Gomez, Phys. Lett. B **725** (2013) 437 [arXiv:1304.5411 [gr-qc]].
[348] M. J. S. Houndjo and O. F. Piattella, Int. J. Mod. Phys. D **21** (2012) 1250024 [arXiv:1111.4275 [gr-qc]].
[349] S. Nojiri and S. D. Odintsov, Phys. Lett. B **631** (2005) 1 [hep-th/0508049].
[350] G. Cognola, E. Elizalde, S. Nojiri, S. D. Odintsov and S. Zerbini, Phys. Rev. D **73** (2006) 084007 [hep-th/0601008].
[351] E. Elizalde, R. Myrzakulov, V. V. Obukhov and D. Saez-Gomez, Class. Quant. Grav. **27** (2010) 095007 [arXiv:1001.3636 [gr-qc]].
[352] K. Izumi, Phys. Rev. D **90** (2014) no.4, 044037 [arXiv:1406.0677 [gr-qc]].
[353] S. Capozziello, [gr-qc/0412088].
[354] S. Nojiri, S. D. Odintsov and M. Sasaki, Phys. Rev. D **71** (2005) 123509 [hep-th/0504052].
[355] M. Sami, A. Toporensky, P. V. Tretjakov and S. Tsujikawa, Phys. Lett. B **619** (2005) 193 [hep-th/0504154].
[356] G. Calcagni, S. Tsujikawa and M. Sami, Class. Quant. Grav. **22** (2005) 3977 [hep-th/0505193].
[357] D. G. Boulware and S. Deser, Phys. Lett. B **175** (1986) 409.
[358] S. Nojiri, S. D. Odintsov and M. Sami, Phys. Rev. D **74** (2006) 046004 [hep-th/0605039].
[359] F. W. Hehl, P. Von Der Heyde, G. D. Kerlick and J. M. Nester, Rev. Mod. Phys. **48** (1976) 393.
[360] K. Hayashi and T. Shirafuji, Phys. Rev. D **19** (1979) 3524 Addendum: [Phys. Rev. D **24** (1982) 3312].
[361] Y. F. Cai, S. Capozziello, M. De Laurentis and E. N. Saridakis, Rept. Prog. Phys. **79** (2016) no.10, 106901 [arXiv:1511.07586 [gr-qc]].
[362] E. E. Flanagan and E. Rosenthal, Phys. Rev. D **75** (2007) 124016 [arXiv:0704.1447 [gr-qc]].
[363] J. W. Maluf, Annalen Phys. **525** (2013) 339 [arXiv:1303.3897 [gr-qc]].
[364] J. Garecki, arXiv:1010.2654 [gr-qc].
[365] R. Ferraro and F. Fiorini, Phys. Rev. D **78** (2008) 124019 [arXiv:0812.1981 [gr-qc]].
[366] G. R. Bengochea and R. Ferraro, Phys. Rev. D **79** (2009) 124019 [arXiv:0812.1205 [astro-ph]].
[367] E. V. Linder, Phys. Rev. D **81** (2010) 127301 Erratum: [Phys. Rev. D **82** (2010) 109902] [arXiv:1005.3039 [astro-ph.CO]].
[368] Y. F. Cai, S. H. Chen, J. B. Dent, S. Dutta and E. N. Saridakis, Class. Quant. Grav. **28** (2011) 215011 [arXiv:1104.4349 [astro-ph.CO]].
[369] S. H. Chen, J. B. Dent, S. Dutta and E. N. Saridakis, Phys. Rev. D **83** (2011) 023508 [arXiv:1008.1250 [astro-ph.CO]].
[370] K. Bamba, C. Q. Geng, C. C. Lee and L. W. Luo, JCAP **1101** (2011) 021 [arXiv:1011.0508 [astro-ph.CO]].
[371] Y. Zhang, H. Li, Y. Gong and Z. H. Zhu, JCAP **1107** (2011) 015 [arXiv:1103.0719 [astro-ph.CO]].
[372] G. G. L. Nashed, Adv. High Energy Phys. **2015** (2015) 680457 [arXiv:1403.6937 [gr-qc]].
[373] C. G. Boehmer, T. Harko and F. S. N. Lobo, Phys. Rev. D **85** (2012) 044033 [arXiv:1110.5756 [gr-qc]].
[374] G. G. L. Nashed and W. El Hanafy, Eur. Phys. J. C **74** (2014) 3099 [arXiv:1403.0913 [gr-qc]].
[375] P. A. Gonzalez, E. N. Saridakis and Y. Vasquez, JHEP **1207** (2012) 053 [arXiv:1110.4024 [gr-qc]].
[376] K. Karami and A. Abdolmaleki, JCAP **1204** (2012) 007 [arXiv:1201.2511 [gr-qc]].
[377] K. Bamba, R. Myrzakulov, S. Nojiri and S. D. Odintsov, Phys. Rev. D **85** (2012) 104036 [arXiv:1202.4057 [gr-qc]].
[378] M. E. Rodrigues, M. J. S. Houndjo, D. Saez-Gomez and F. Rahaman, Phys. Rev. D **86** (2012) 104059 [arXiv:1209.4859 [gr-qc]].
[379] S. Capozziello, P. A. Gonzalez, E. N. Saridakis and Y. Vasquez, JHEP **1302** (2013) 039 [arXiv:1210.1098 [hep-th]].
[380] S. Chattopadhyay and A. Pasqua, Astrophys. Space Sci. **344** (2013) 269 [arXiv:1211.2707 [physics.gen-ph]].
[381] K. Izumi and Y. C. Ong, JCAP **1306** (2013) 029 [arXiv:1212.5774 [gr-qc]].
[382] J. T. Li, C. C. Lee and C. Q. Geng, Eur. Phys. J. C **73** (2013) no.2, 2315 [arXiv:1302.2688 [gr-qc]].
[383] Y. C. Ong, K. Izumi, J. M. Nester and P. Chen, Phys. Rev. D **88** (2013) 024019 [arXiv:1303.0993 [gr-qc]].
[384] G. Otalora, JCAP **1307** (2013) 044 [arXiv:1305.0474 [gr-qc]].
[385] G. G. L. Nashed, Phys. Rev. D **88** (2013) 104034 [arXiv:1311.3131 [gr-qc]].
[386] G. Kofinas and E. N. Saridakis, Phys. Rev. D **90** (2014) 084044 [arXiv:1404.2249 [gr-qc]].
[387] T. Harko, F. S. N. Lobo, G. Otalora and E. N. Saridakis, Phys. Rev. D **89** (2014) 124036 [arXiv:1404.6212 [gr-qc]].
[388] W. El Hanafy and G. L. Nashed, Astrophys. Space Sci. **361** (2016) no.6, 197 [arXiv:1410.2467 [hep-th]].
[389] E. L. B. Junior, M. E. Rodrigues, I. G. Salako and M. J. S. Houndjo, Class. Quant. Grav. **33** (2016) no.12, 125006 [arXiv:1501.00621 [gr-qc]].
[390] M. L. Ruggiero and N. Radicella, Phys. Rev. D **91** (2015) 104014 [arXiv:1501.02198 [gr-qc]].
[391] W. El Hanafy and G. G. L. Nashed, Eur. Phys. J. C **75** (2015) 279 [arXiv:1409.7199 [hep-th]].
[392] R. C. Nunes, A. Bonilla, S. Pan and E. N. Saridakis, arXiv:1608.01960 [gr-qc].
[393] M. Krssak and E. N. Saridakis, Class. Quant. Grav. **33** (2016) no.11, 115009 doi:10.1088/0264-9381/33/11/115009 [arXiv:1510.08432 [gr-qc]].
[394] K. Bamba, S. D. Odintsov and D. Saez-Gomez, Phys. Rev. D **88** (2013) 084042 [arXiv:1308.5789 [gr-qc]].
[395] J. B. Dent, S. Dutta and E. N. Saridakis, JCAP **1101** (2011) 009 [arXiv:1010.2215 [astro-ph.CO]].
[396] R. J. Yang, Eur. Phys. J. C **71** (2011) 1797 [arXiv:1007.3571 [gr-qc]].
[397] S. Capozziello, V. F. Cardone, H. Farajollahi and A. Ravanpak, Phys. Rev. D **84** (2011) 043527 [arXiv:1108.2789 [astro-ph.CO]].
[398] C. Q. Geng, C. C. Lee and E. N. Saridakis, JCAP **1201** (2012) 002 [arXiv:1110.0913 [astro-ph.CO]].
[399] H. Farajollahi, A. Ravanpak and P. Wu, Astrophys. Space Sci. **338** (2012) 23 [arXiv:1112.4700 [physics.gen-ph]].
[400] V. F. Cardone, N. Radicella and S. Camera, Phys. Rev. D **85** (2012) 124007 [arXiv:1204.5294 [astro-ph.CO]].
[401] S. Bahamonde, C. G. Bohmer and M. Wright, Phys. Rev. D **92** (2015) no.10, 104042 [arXiv:1508.05120 [gr-qc]].
[402] A. Paliathanasis, S. Basilakos, E. N. Saridakis, S. Capozziello, K. Atazadeh, F. Darabi and M. Tsamparlis, Phys. Rev. D

89 (2014) 104042 [arXiv:1402.5935 [gr-qc]].
[403] R. Ferraro and F. Fiorini, Phys. Rev. D **75** (2007) 084031 [gr-qc/0610067].
[404] V. K. Oikonomou, Phys. Rev. D **95** (2017) no.8, 084023 [arXiv:1703.10515 [gr-qc]].
[405] V. K. Oikonomou, Mod. Phys. Lett. A **32** (2017) 1750067 [arXiv:1703.06713 [gr-qc]].
[406] L. Iorio and E. N. Saridakis, Mon. Not. Roy. Astron. Soc. **427** (2012) 1555 [arXiv:1203.5781 [gr-qc]].
[407] J. Amoros, J. de Haro and S. D. Odintsov, Phys. Rev. D **87** (2013) 104037 [arXiv:1305.2344 [gr-qc]].
[408] M. Jamil, D. Momeni and R. Myrzakulov, Eur. Phys. J. C **72** (2012) 2122 [arXiv:1209.1298 [gr-qc]].
[409] S. Basilakos, S. Capozziello, M. De Laurentis, A. Paliathanasis and M. Tsamparlis, Phys. Rev. D **88** (2013) 103526 [arXiv:1311.2173 [gr-qc]].
[410] K. Bamba, S. Nojiri and S. D. Odintsov, Phys. Lett. B **725** (2013) 368 [arXiv:1304.6191 [gr-qc]].
[411] K. Bamba and S. D. Odintsov, PoS KMI **2013** (2014) 023 [arXiv:1402.7114 [hep-th]].
[412] M. Fierz and W. Pauli, Proc. Roy. Soc. Lond. A **173** (1939) 211. doi:10.1098/rspa.1939.0140
[413] H. van Dam and M. J. G. Veltman, Nucl. Phys. B **22** (1970) 397. doi:10.1016/0550-3213(70)90416-5
[414] V. I. Zakharov, JETP Lett. **12** (1970) 312 [Pisma Zh. Eksp. Teor. Fiz. **12** (1970) 447].
[415] A. I. Vainshtein, Phys. Lett. **39B** (1972) 393. doi:10.1016/0370-2693(72)90147-5
[416] D. G. Boulware and S. Deser, Annals Phys. **89** (1975) 193. doi:10.1016/0003-4916(75)90302-4
[417] C. de Rham and G. Gabadadze, Phys. Rev. D **82** (2010) 044020 doi:10.1103/PhysRevD.82.044020 [arXiv:1007.0443 [hep-th]].
[418] C. de Rham, G. Gabadadze and A. J. Tolley, Phys. Rev. Lett. **106** (2011) 231101 doi:10.1103/PhysRevLett.106.231101 [arXiv:1011.1232 [hep-th]].
[419] S. F. Hassan and R. A. Rosen, JHEP **1202** (2012) 126 doi:10.1007/JHEP02(2012)126 [arXiv:1109.3515 [hep-th]].
[420] K. Hinterbichler, Rev. Mod. Phys. **84** (2012) 671 doi:10.1103/RevModPhys.84.671 [arXiv:1105.3735 [hep-th]].
[421] C. de Rham, Living Rev. Rel. **17** (2014) 7 doi:10.12942/lrr-2014-7 [arXiv:1401.4173 [hep-th]].
[422] Y. F. Cai, F. Duplessis and E. N. Saridakis, Phys. Rev. D **90** (2014) no.6, 064051 [arXiv:1307.7150 [hep-th]].
[423] S. Nojiri and S. D. Odintsov, Phys. Lett. B **716** (2012) 377 [arXiv:1207.5106 [hep-th]].
[424] S. Nojiri, S. D. Odintsov and N. Shirai, JCAP **1305** (2013) 020 [arXiv:1212.2079 [hep-th]].
[425] J. Kluson, S. Nojiri and S. D. Odintsov, Phys. Lett. B **726** (2013) 918 [arXiv:1309.2185 [hep-th]].
[426] A. H. Chamseddine and V. Mukhanov, JHEP **1311** (2013) 135 [arXiv:1308.5410 [astro-ph.CO]].
[427] A. H. Chamseddine, V. Mukhanov and A. Vikman, JCAP **1406** (2014) 017 [arXiv:1403.3961 [astro-ph.CO]].
[428] A. Golovnev, Phys. Lett. B **728** (2014) 39 [arXiv:1310.2790 [gr-qc]].
[429] D. Momeni, A. Altaibayeva and R. Myrzakulov, Int. J. Geom. Methods Mod. Phys. 11, 1450091 (2014) [arXiv:1407.5662 [gr-qc]].
[430] N. Deruelle and J. Rua, JCAP **1409** (2014) 002 [arXiv:1407.0825 [gr-qc]].
[431] M. Chaichian, J. Kluson, M. Oksanen and A. Tureanu, JHEP **1412** (2014) 102 [arXiv:1404.4008 [hep-th]].
[432] L. Sebastiani, S. Vagnozzi and R. Myrzakulov, arXiv:1612.08661 [gr-qc].
[433] S. Nojiri and S. D. Odintsov, Mod. Phys. Lett. A **29** (2014) no.40, 1450211 [arXiv:1408.3561 [hep-th]].
[434] S. D. Odintsov and V. K. Oikonomou, Phys. Rev. D **93** (2016) no.2, 023517 [arXiv:1511.04559 [gr-qc]].
[435] S. D. Odintsov and V. K. Oikonomou, Astrophys. Space Sci. **361** (2016) no.5, 174 [arXiv:1512.09275 [gr-qc]].
[436] S. D. Odintsov and V. K. Oikonomou, Annals Phys. **363** (2015) 503 [arXiv:1508.07488 [gr-qc]].
[437] G. Leon and E. N. Saridakis, JCAP **1504** (2015) no.04, 031 [arXiv:1501.00488 [gr-qc]].
[438] R. Myrzakulov and L. Sebastiani, Astrophys. Space Sci. **361** (2016) no.6, 188 [arXiv:1601.04994 [gr-qc]].
[439] D. Momeni, R. Myrzakulov and E. Gudekli, Int. J. Geom. Meth. Mod. Phys. **12** (2015) no.10, 1550101 [arXiv:1502.00977 [gr-qc]].
[440] A. V. Astashenok, S. D. Odintsov and V. K. Oikonomou, Class. Quant. Grav. **32** (2015) no.18, 185007 [arXiv:1504.04861 [gr-qc]].
[441] R. Myrzakulov, L. Sebastiani and S. Vagnozzi, Eur. Phys. J. C **75** (2015) 444 [arXiv:1504.07984 [gr-qc]].
[442] G. Cognola, R. Myrzakulov, L. Sebastiani, S. Vagnozzi and S. Zerbini, Class. Quant. Grav. **33** (2016) no.22, 225014 [arXiv:1601.00102 [gr-qc]].
[443] F. Arroja, N. Bartolo, P. Karmakar and S. Matarrese, JCAP **1509** (2015) 051 [arXiv:1506.08575 [gr-qc]].
[444] A. Ijjas, J. Ripley and P. J. Steinhardt, Phys. Lett. B **760** (2016) 132 [arXiv:1604.08586 [gr-qc]].
[445] H. Saadi, Eur. Phys. J. C **76** (2016) no.1, 14 [arXiv:1411.4531 [gr-qc]].
[446] J. Matsumoto, S. D. Odintsov and S. V. Sushkov, Phys. Rev. D **91** (2015) no.6, 064062 [arXiv:1501.02149 [gr-qc]].
[447] R. Myrzakulov, L. Sebastiani, S. Vagnozzi and S. Zerbini, Fund. J. Mod. Phys. **8** (2015) 119 [arXiv:1505.03115 [gr-qc]].
[448] Y. Rabochaya and S. Zerbini, Eur. Phys. J. C **76** (2016) no.2, 85 [arXiv:1509.03720 [gr-qc]].
[449] R. Myrzakulov, L. Sebastiani, S. Vagnozzi and S. Zerbini, Class. Quant. Grav. **33** (2016) no.12, 125005 [arXiv:1510.02284 [gr-qc]].
[450] V. K. Oikonomou, Universe **2** (2016) no.2, 10 [arXiv:1511.09117 [gr-qc]].
[451] R. Myrzakulov and L. Sebastiani, Gen. Rel. Grav. **47** (2015) no.8, 89 [arXiv:1503.04293 [gr-qc]].
[452] V. K. Oikonomou, Int. J. Mod. Phys. D **25** (2016) no.07, 1650078 [arXiv:1605.00583 [gr-qc]].
[453] A. V. Astashenok and S. D. Odintsov, Phys. Rev. D **94** (2016) no.6, 063008 [arXiv:1512.07279 [gr-qc]].
[454] S. D. Odintsov and V. K. Oikonomou, Phys. Rev. D **94** (2016) no.4, 044012 [arXiv:1608.00165 [gr-qc]].
[455] V. K. Oikonomou, Mod. Phys. Lett. A **31** (2016) no.33, 1650191 [arXiv:1609.03156 [gr-qc]].
[456] S. Nojiri, S. D. Odintsov and V. K. Oikonomou, Class. Quant. Grav. **33** (2016) no.12, 125017 [arXiv:1601.07057 [gr-qc]].
[457] S. D. Odintsov and V. K. Oikonomou, Astrophys. Space Sci. **361** (2016) no.7, 236 [arXiv:1602.05645 [gr-qc]].

[458] E. A. Lim, I. Sawicki and A. Vikman, JCAP **1005** (2010) 012 [arXiv:1003.5751 [astro-ph.CO]].
[459] S. Capozziello, A. N. Makarenko and S. D. Odintsov, Phys. Rev. D **87** (2013) no.8, 084037 [arXiv:1302.0093 [gr-qc]].
[460] S. Capozziello, J. Matsumoto, S. Nojiri and S. D. Odintsov, Phys. Lett. B **693** (2010) 198 [arXiv:1004.3691 [hep-th]].
[461] C. Gao, Y. Gong, X. Wang and X. Chen, Phys. Lett. B **702** (2011) 107 [arXiv:1003.6056 [astro-ph.CO]].
[462] A. N. Makarenko, Int. J. Geom. Meth. Mod. Phys. **13** (2016) no.05, 1630006.
[463] J. L. Anderson and D. Finkelstein, Am. J. Phys. **39** (1971) 901.
[464] W. Buchmuller and N. Dragon, Phys. Lett. B **207** (1988) 292.
[465] M. Henneaux and C. Teitelboim, Phys. Lett. B **222** (1989) 195.
[466] W. G. Unruh, Phys. Rev. D **40** (1989) 1048.
[467] Y. J. Ng and H. van Dam, J. Math. Phys. **32** (1991) 1337.
[468] D. R. Finkelstein, A. A. Galiautdinov and J. E. Baugh, J. Math. Phys. **42** (2001) 340 [gr-qc/0009099].
[469] E. Alvarez, JHEP **0503** (2005) 002 [hep-th/0501146].
[470] E. Alvarez, D. Blas, J. Garriga and E. Verdaguer, Nucl. Phys. B **756** (2006) 148 [hep-th/0606019].
[471] A. H. Abbassi and A. M. Abbassi, Class. Quant. Grav. **25** (2008) 175018 [arXiv:0706.0451 [gr-qc]].
[472] G. F. R. Ellis, H. van Elst, J. Murugan and J. P. Uzan, Class. Quant. Grav. **28** (2011) 225007 [arXiv:1008.1196 [gr-qc]].
[473] P. Jain, Mod. Phys. Lett. A **27** (2012) 1250201 [arXiv:1209.2314 [astro-ph.CO]].
[474] N. K. Singh, Mod. Phys. Lett. A **28** (2013) 1350130 [arXiv:1205.5151 [astro-ph.CO]].
[475] J. Kluson, Phys. Rev. D **91** (2015) no.6, 064058 [arXiv:1409.8014 [hep-th]].
[476] A. Padilla and I. D. Saltas, Eur. Phys. J. C **75** (2015) no.11, 561 [arXiv:1409.3573 [gr-qc]].
[477] C. Barcelo, R. Carballo-Rubio and L. J. Garay, Phys. Rev. D **89** (2014) no.12, 124019 [arXiv:1401.2941 [gr-qc]].
[478] C. Barcelo, R. Carballo-Rubio and L. J. Garay, arXiv:1406.7713 [gr-qc].
[479] D. J. Burger, G. F. R. Ellis, J. Murugan and A. Weltman, arXiv:1511.08517 [hep-th].
[480] E. Alvarez, S. Gonzalez-Martin, M. Herrero-Valea and C. P. Martin, JHEP **1508** (2015) 078 [arXiv:1505.01995 [hep-th]].
[481] P. Jain, A. Jaiswal, P. Karmakar, G. Kashyap and N. K. Singh, JCAP **1211** (2012) 003 [arXiv:1109.0169 [astro-ph.CO]].
[482] P. Jain, P. Karmakar, S. Mitra, S. Panda and N. K. Singh, JCAP **1205** (2012) 020 [arXiv:1108.1856 [gr-qc]].
[483] I. Cho and N. K. Singh, Class. Quant. Grav. **32** (2015) no.13, 135020 [arXiv:1412.6205 [gr-qc]].
[484] A. Basak, O. Fabre and S. Shankaranarayanan, Gen. Rel. Grav. **48** (2016) no.10, 123 [arXiv:1511.01805 [gr-qc]].
[485] C. Gao, R. H. Brandenberger, Y. Cai and P. Chen, JCAP **1409** (2014) 021 [arXiv:1405.1644 [gr-qc]].
[486] A. Eichhorn, JHEP **1504** (2015) 096 [arXiv:1501.05848 [gr-qc]].
[487] I. D. Saltas, Phys. Rev. D **90** (2014) no.12, 124052 [arXiv:1410.6163 [hep-th]].
[488] P. Chaturvedi, N. K. Singh and D. V. Singh, arXiv:1610.07661 [gr-qc].
[489] S. Nojiri, S. D. Odintsov and V. K. Oikonomou, JCAP **1605** (2016) no.05, 046 [arXiv:1512.07223 [gr-qc]].
[490] K. Bamba, C. Q. Geng, S. Nojiri and S. D. Odintsov, Phys. Rev. D **79** (2009) 083014 [arXiv:0810.4296 [hep-th]].
[491] J. Haro, J. Amoros and E. Elizalde, Phys. Rev. D **83** (2011) 123528.
[492] S. Nesseris and L. Perivolaropoulos, JCAP **0701** (2007) 018 [astro-ph/0610092].
[493] X. m. Chen, Y. g. Gong and E. N. Saridakis, JCAP **0904** (2009) 001 [arXiv:0812.1117 [gr-qc]].
[494] B. M. Leith and I. P. Neupane, JCAP **0705** (2007) 019 [hep-th/0702002].
[495] V. Faraoni, Phys. Rev. D **68** (2003) 063508 [gr-qc/0307086].
[496] E. N. Saridakis and S. V. Sushkov, Phys. Rev. D **81** (2010) 083510 [arXiv:1002.3478 [gr-qc]].
[497] S. V. Sushkov and S. W. Kim, Gen. Rel. Grav. **36** (2004) 1671 [gr-qc/0404037].
[498] E. O. Kahya, V. K. Onemli and R. P. Woodard, Phys. Rev. D **81** (2010) 023508 [arXiv:0904.4811 [gr-qc]].
[499] V. K. Onemli and R. P. Woodard, Phys. Rev. D **70** (2004) 107301 [gr-qc/0406098].
[500] V. Faraoni, Class. Quant. Grav. **22** (2005) 3235 [gr-qc/0506095].
[501] M. P. Dabrowski, T. Stachowiak and M. Szydlowski, Phys. Rev. D **68** (2003) 103519 [hep-th/0307128].
[502] J. de Haro, JCAP **1207** (2012) 007 [arXiv:1204.5604 [gr-qc]].
[503] I. H. Brevik, Int. J. Mod. Phys. D **15** (2006) 767 [gr-qc/0601100].
[504] V. F. Mukhanov, H. A. Feldman and R. H. Brandenberger, Phys. Rept. **215** (1992) 203.
[505] A. A. Starobinsky, Phys. Lett. **117B** (1982) 175.
[506] F. L. Bezrukov and M. Shaposhnikov, Phys. Lett. B **659** (2008) 703 [arXiv:0710.3755 [hep-th]].
[507] A. R. Liddle, P. Parsons and J. D. Barrow, Phys. Rev. D **50** (1994) 7222 [astro-ph/9408015].
[508] E. J. Copeland, E. W. Kolb, A. R. Liddle and J. E. Lidsey, Phys. Rev. D **48** (1993) 2529 [hep-ph/9303288].
[509] A. R. Liddle and D. H. Lyth, Phys. Lett. B **291** (1992) 391 [astro-ph/9208007].
[510] S. D. Odintsov and V. K. Oikonomou, Phys. Rev. D **92** (2015) no.12, 124024 [arXiv:1510.04333 [gr-qc]].
[511] S. D. Odintsov and V. K. Oikonomou, Class. Quant. Grav. **33** (2016) no.12, 125029 [arXiv:1602.03309 [gr-qc]].
[512] S. D. Odintsov and V. K. Oikonomou, Phys. Rev. D **92** (2015) no.2, 024058 [arXiv:1507.05273 [gr-qc]].
[513] S. D. Odintsov and V. K. Oikonomou, Class. Quant. Grav. **32** (2015) no.23, 235011 [arXiv:1504.01772 [gr-qc]].
[514] K. Bamba, S. Nojiri and S. D. Odintsov, Phys. Lett. B **737** (2014) 374 [arXiv:1406.2417 [hep-th]].
[515] M. Sasaki and T. Tanaka, Prog. Theor. Phys. **99** (1998) 763 [gr-qc/9801017].
[516] D. I. Kaiser, E. A. Mazenc and E. I. Sfakianakis, Phys. Rev. D **87** (2013) 064004 [arXiv:1210.7487 [astro-ph.CO]].
[517] S. Renaux-Petel and K. Turzynski, JCAP **1506** (2015) no.06, 010 [arXiv:1405.6195 [astro-ph.CO]].
[518] D. I. Kaiser and E. I. Sfakianakis, Phys. Rev. Lett. **112** (2014) no.1, 011302 [arXiv:1304.0363 [astro-ph.CO]].
[519] H. Noh and J. c. Hwang, Phys. Rev. D **69** (2004) 104011.
[520] J. c. Hwang and H. Noh, Phys. Rev. D **66** (2002) 084009 [hep-th/0206100].
[521] J. c. Hwang and H. Noh, Class. Quant. Grav. **19** (2002) 527 [astro-ph/0103244].

[522] J. c. Hwang and H. Noh, Phys. Lett. B **495** (2000) 277 [astro-ph/0009268].
[523] J. c. Hwang and H. Noh, Phys. Rev. D **54** (1996) 1460.
[524] L. Sebastiani and R. Myrzakulov, Int. J. Geom. Meth. Mod. Phys. **12** (2015) no.9, 1530003 [arXiv:1506.05330 [gr-qc]].
[525] H. Farajollahi, M. Setare, F. Milani and F. Tayebi, Gen. Rel. Grav. **43** (2011) 1657 [arXiv:1005.2026 [physics.gen-ph]].
[526] S. D. Odintsov and V. K. Oikonomou, arXiv:1611.00738 [gr-qc].
[527] R. Kallosh and A. Linde, JCAP **1307** (2013) 002 [arXiv:1306.5220 [hep-th]].
[528] S. Ferrara, R. Kallosh, A. Linde and M. Porrati, Phys. Rev. D **88** (2013) no.8, 085038 [arXiv:1307.7696 [hep-th]].
[529] R. Kallosh, A. Linde and D. Roest, JHEP **1311** (2013) 198 [arXiv:1311.0472 [hep-th]].
[530] M. Galante, R. Kallosh, A. Linde and D. Roest, Phys. Rev. Lett. **114** (2015) no.14, 141302 [arXiv:1412.3797 [hep-th]].
[531] J. J. M. Carrasco, R. Kallosh and A. Linde, JHEP **1510** (2015) 147 [arXiv:1506.01708 [hep-th]].
[532] A. Linde, JCAP **1505** (2015) 003 [arXiv:1504.00663 [hep-th]].
[533] D. Roest and M. Scalisi, Phys. Rev. D **92** (2015) 043525 [arXiv:1503.07909 [hep-th]].
[534] R. Kallosh, A. Linde and D. Roest, JHEP **1408** (2014) 052 [arXiv:1405.3646 [hep-th]].
[535] L. Sebastiani, G. Cognola, R. Myrzakulov, S. D. Odintsov and S. Zerbini, Phys. Rev. D **89** (2014) no.2, 023518 [arXiv:1311.0744 [gr-qc]].
[536] Y. F. Cai, J. O. Gong and S. Pi, Phys. Lett. B **738** (2014) 20 [arXiv:1404.2560 [hep-th]].
[537] Z. Yi and Y. Gong, Phys. Rev. D **94** (2016) no.10, 103527 [arXiv:1608.05922 [gr-qc]].
[538] S. Carloni, P. K. S. Dunsby and A. Troisi, Phys. Rev. D **77** (2008) 024024 [arXiv:0707.0106 [gr-qc]].
[539] D. I. Kaiser, [astro-ph/9507048].
[540] V. Faraoni, Phys. Rev. D **75** (2007) 067302 [gr-qc/0703044 [GR-QC]].
[541] G. Domenech and M. Sasaki, Int. J. Mod. Phys. D **25** (2016) no.13, 1645006 [arXiv:1602.06332 [gr-qc]].
[542] D. J. Brooker, S. D. Odintsov and R. P. Woodard, Nucl. Phys. B **911** (2016) 318 [arXiv:1606.05879 [gr-qc]].
[543] D. I. Kaiser, Phys. Rev. D **52** (1995) 4295 [astro-ph/9408044].
[544] K. Bamba, R. Myrzakulov, S. D. Odintsov and L. Sebastiani, Phys. Rev. D **90** (2014) no.4, 043505
[545] H. Noh and J. c. Hwang, Phys. Lett. B **515** (2001) 231 [astro-ph/0107069].
[546] J. c. Hwang and H. r. Noh, Phys. Rev. D **65** (2002) 023512 [astro-ph/0102005].
[547] J. c. Hwang and H. Noh, Phys. Lett. B **506** (2001) 13 [astro-ph/0102423].
[548] K. Bamba, Z. K. Guo and N. Ohta, Prog. Theor. Phys. **118** (2007) 879 [arXiv:0707.4334 [hep-th]].
[549] S. D. Odintsov, V. K. Oikonomou and E. N. Saridakis, Annals Phys. **363** (2015) 141 [arXiv:1501.06591 [gr-qc]].
[550] J. D. Barrow and P. Saich, Phys. Lett. B **249** (1990) 406.
[551] J. D. Barrow and A. R. Liddle, Phys. Rev. D **47** (1993) no.12, R5219 [astro-ph/9303011].
[552] K. Rezazadeh, K. Karami and P. Karimi, JCAP **1509** (2015) no.09, 053 [arXiv:1411.7302 [gr-qc]].
[553] J. D. Barrow, M. Lagos and J. Magueijo, Phys. Rev. D **89** (2014) no.8, 083525 [arXiv:1401.7491 [astro-ph.CO]].
[554] R. Herrera, M. Olivares and N. Videla, Int. J. Mod. Phys. D **23** (2014) no.10, 1450080 [arXiv:1404.2803 [gr-qc]].
[555] M. Jamil, D. Momeni and R. Myrzakulov, Int. J. Theor. Phys. **54** (2015) no.4, 1098 [arXiv:1309.3269 [gr-qc]].
[556] R. Herrera and N. Videla, Eur. Phys. J. C **67** (2010) 499 [arXiv:1003.5645 [astro-ph.CO]].
[557] A. D. Rendall, Class. Quant. Grav. **22** (2005) 1655 [gr-qc/0501072].
[558] A. De Felice and T. Suyama, JCAP **0906** (2009) 034 [arXiv:0904.2092 [astro-ph.CO]].
[559] J. D. Barrow and A. A. H. Graham, Phys. Rev. D **91** (2015) no.8, 083513 [arXiv:1501.04090 [gr-qc]].
[560] S. Nojiri, S. D. Odintsov and V. K. Oikonomou, Phys. Rev. D **91** (2015) no.8, 084059 [arXiv:1502.07005 [gr-qc]].
[561] S. Nojiri, S. D. Odintsov, V. K. Oikonomou and E. N. Saridakis, JCAP **1509** (2015) 044 [arXiv:1503.08443 [gr-qc]].
[562] V. K. Oikonomou, Int. J. Geom. Meth. Mod. Phys. **13** (2016) no.03, 1650033 [arXiv:1512.04095 [gr-qc]].
[563] S. Nojiri, S. D. Odintsov and V. K. Oikonomou, Phys. Rev. D **92** (2015) no.12, 124059 [arXiv:1511.06776 [gr-qc]].
[564] K. Kleidis and V. K. Oikonomou, Astrophys. Space Sci. **361** (2016) no.10, 326 [arXiv:1609.00848 [gr-qc]].
[565] S. Nojiri, S. D. Odintsov and V. K. Oikonomou, Phys. Lett. B **747** (2015) 310 [arXiv:1506.03307 [gr-qc]].
[566] M. A. Amin, M. P. Hertzberg, D. I. Kaiser and J. Karouby, Int. J. Mod. Phys. D **24** (2014) 1530003 [arXiv:1410.3808 [hep-ph]].
[567] L. Kofman, A. D. Linde and A. A. Starobinsky, Phys. Rev. D **56** (1997) 3258 [hep-ph/9704452].
[568] P. B. Greene, L. Kofman, A. D. Linde and A. A. Starobinsky, Phys. Rev. D **56** (1997) 6175 [hep-ph/9705347].
[569] L. Kofman, A. D. Linde and A. A. Starobinsky, Phys. Rev. Lett. **73** (1994) 3195 [hep-th/9405187].
[570] M. B. Mijic, M. S. Morris and W. M. Suen, Phys. Rev. D **34** (1986) 2934.
[571] S. Nojiri, S. D. Odintsov and D. Saez-Gomez, Phys. Lett. B **681** (2009) 74 [arXiv:0908.1269 [hep-th]].
[572] P. K. S. Dunsby, E. Elizalde, R. Goswami, S. Odintsov and D. S. Gomez, Phys. Rev. D **82** (2010) 023519 [arXiv:1005.2205 [gr-qc]].
[573] T. Katsuragawa and S. Matsuzaki, arXiv:1610.01016 [gr-qc].
[574] P. Brax, C. van de Bruck, A. C. Davis, J. Khoury and A. Weltman, Phys. Rev. D **70** (2004) 123518 [astro-ph/0408415].
[575] E. Elizalde, S. Nojiri, S. D. Odintsov, L. Sebastiani and S. Zerbini, Phys. Rev. D **83** (2011) 086006 [arXiv:1012.2280 [hep-th]].
[576] K. Bamba, A. Lopez-Revelles, R. Myrzakulov, S. D. Odintsov and L. Sebastiani, Class. Quant. Grav. **30** (2013) 015008 [arXiv:1207.1009 [gr-qc]].
[577] E. Komatsu *et al.* [WMAP Collaboration], Astrophys. J. Suppl. **180** (2009) 330 [arXiv:0803.0547 [astro-ph]].
[578] J. Matsumoto, Phys. Rev. D **87** (2013) no.10, 104002
[579] P. A. R. Ade *et al.* [Planck Collaboration], Astron. Astrophys. **594** (2016) A13 [arXiv:1502.01589 [astro-ph.CO]].
[580] D. N. Spergel *et al.* [WMAP Collaboration], Astrophys. J. Suppl. **170** (2007) 377 [astro-ph/0603449].

[581] L. Perivolaropoulos, JCAP **0510** (2005) 001 [astro-ph/0504582].
[582] M. z. Li, B. Feng and X. m. Zhang, JCAP **0512** (2005) 002 [hep-ph/0503268].
[583] A. Vikman, Phys. Rev. D **71** (2005) 023515 [astro-ph/0407107].
[584] L. P. Chimento and R. Lazkoz, Phys. Lett. B **639** (2006) 591 [astro-ph/0604090].
[585] E. Elizalde, S. Nojiri, S. D. Odintsov, D. Saez-Gomez and V. Faraoni, Phys. Rev. D **77** (2008) 106005 [arXiv:0803.1311 [hep-th]].
[586] R. R. Caldwell and M. Doran, Phys. Rev. D **72** (2005) 043527 [astro-ph/0501104].
[587] B. Feng, X. L. Wang and X. M. Zhang, Phys. Lett. B **607** (2005) 35 [astro-ph/0404224].
[588] Y. f. Cai, M. z. Li, J. X. Lu, Y. S. Piao, T. t. Qiu and X. m. Zhang, Phys. Lett. B **651** (2007) 1 [hep-th/0701016].
[589] A. de la Cruz-Dombriz and A. Dobado, Phys. Rev. D **74** (2006) 087501 [gr-qc/0607118].
[590] E. V. Linder, Astropart. Phys. **25** (2006) 167 [astro-ph/0511415].
[591] A. Kurek, O. Hrycyna and M. Szydlowski, Phys. Lett. B **659** (2008) 14 [arXiv:0707.0292 [astro-ph]].
[592] J. Liu, H. Li, J. Xia and X. Zhang, JCAP **0907** (2009) 017 [arXiv:0901.2033 [astro-ph.CO]].
[593] D. Jain, A. Dev and J. S. Alcaniz, Phys. Lett. B **656** (2007) 15 [arXiv:0709.4234 [astro-ph]].
[594] O. Gorbunova and D. Saez-Gomez, Open Astron. J. **3** (2010) 73 [arXiv:0909.5113 [gr-qc]].
[595] R. Lazkoz, V. Salzano and I. Sendra, Phys. Lett. B **694** (2011) 198 [arXiv:1003.6084 [astro-ph.CO]].
[596] E. Elizalde, S. D. Odintsov, L. Sebastiani and S. Zerbini, Eur. Phys. J. C **72** (2012) 1843 [arXiv:1108.6184 [gr-qc]].
[597] V. K. Oikonomou, N. Karagiannakis and M. Park, Phys. Rev. D **91** (2015) no.6, 064029 [arXiv:1411.3199 [gr-qc]].
[598] H. Motohashi, A. A. Starobinsky and J. Yokoyama, JCAP **1106** (2011) 006 [arXiv:1101.0744 [astro-ph.CO]].
[599] V. K. Oikonomou and N. Karagiannakis, Class. Quant. Grav. **32** (2015) no.8, 085001 [arXiv:1408.5353 [gr-qc]].
[600] K. Kleidis and N. K. Spyrou, Astron. Astrophys. **529** (2011) A26 [arXiv:1104.0442 [gr-qc]].
[601] J. Matsumoto, Phys. Rev. D **83** (2011) 124040 [arXiv:1105.1419 [astro-ph.CO]].
[602] X. Fu, P. Wu and H. W. Yu, Eur. Phys. J. C **68** (2010) 271 [arXiv:1012.2249 [gr-qc]].
[603] A. de la Cruz-Dombriz, A. Dobado and A. L. Maroto, Phys. Rev. D **77** (2008) 123515 [arXiv:0802.2999 [astro-ph]].
[604] O. Barziv, L. Kaper, M. H. van Kerkwijk, J. H. Telting and J. van Paradijs, Astron. Astrophys. **377** (2001) 925
[605] M. L. Rawls, J. A. Orosz, J. E. McClintock, M. A. P. Torres, C. D. Bailyn and M. M. Buxton, Astrophys. J. **730** (2011) 25 [arXiv:1101.2465 [astro-ph.SR]].
[606] D. J. Nice, E. M. Splaver, I. H. Stairs, O. Loehmer, A. Jessner, M. Kramer, 2 and J. M. Cordes, Astrophys. J. **634** (2005) 1242 [astro-ph/0508050].
[607] P. Demorest, T. Pennucci, S. Ransom, M. Roberts and J. Hessels, Nature **467** (2010) 1081 [arXiv:1010.5788 [astro-ph.HE]].
[608] T. Miyatsu, M. K. Cheoun and K. Saito, Phys. Rev. C **88** (2013) no.1, 015802
[609] K. Tsubakihara and A. Ohnishi, Nucl. Phys. A **914** (2013) 438
[610] S. Capozziello, M. De Laurentis, S. D. Odintsov and A. Stabile, Phys. Rev. D **83** (2011) 064004 [arXiv:1101.0219 [gr-qc]].
[611] A. V. Astashenok, S. Capozziello and S. D. Odintsov, Phys. Rev. D **89** (2014) no.10, 103509
[612] A. V. Astashenok, S. Capozziello and S. D. Odintsov, JCAP **1501** (2015) no.01, 001 [arXiv:1408.3856 [gr-qc]].
[613] S. Capozziello, M. De Laurentis, R. Farinelli and S. D. Odintsov, Phys. Rev. D **93** (2016) no.2, 023501
[614] S. S. Yazadjiev, D. D. Doneva and K. D. Kokkotas, Phys. Rev. D **91** (2015) no.8, 084018
[615] K. V. Staykov, D. D. Doneva, S. S. Yazadjiev and K. D. Kokkotas, JCAP **1410** (2014) no.10, 006 [arXiv:1407.2180 [gr-qc]].
[616] P. Fiziev and K. Marinov, Bulg. Astron. J. **23** (2015) 3 [arXiv:1412.3015 [gr-qc]].
[617] K. Henttunen, T. Multamaki and I. Vilja, Phys. Rev. D **77** (2008) 024040 [arXiv:0705.2683 [astro-ph]].
[618] M. Aparicio Resco, A. de la Cruz-Dombriz, F. J. Llanes Estrada and V. Zapatero Castrillo, Phys. Dark Univ. **13** (2016) 147 [arXiv:1602.03880 [gr-qc]].
[619] T. Multamaki and I. Vilja, Phys. Lett. B **659** (2008) 843 [arXiv:0709.3422 [astro-ph]].
[620] T. Katsuragawa, S. Nojiri, S. D. Odintsov and M. Yamazaki, Phys. Rev. D **93** (2016) 124013 [arXiv:1512.00660 [gr-qc]].
[621] A. V. Astashenok, S. Capozziello and S. D. Odintsov, JCAP **1312** (2013) 040 [arXiv:1309.1978 [gr-qc]].
[622] H. Alavirad and J. M. Weller, Phys. Rev. D **88** (2013) no.12, 124034
[623] M. De Laurentis and I. De Martino, Mon. Not. Roy. Astron. Soc. **431** (2014) 741
[624] A. de la Cruz-Dombriz and D. Saez-Gomez, Entropy **14** (2012) 1717 [gr-qc]].
[625] S. Capozziello, M. De Laurentis, I. De Martino, M. Formisano and S. D. Odintsov, Phys. Rev. D **85** (2012) 044022 [arXiv:1112.0761 [gr-qc]].
[626] C. Gungor and K. Y. Eksi, arXiv:1108.2166 [astro-ph.SR].
[627] K. Bamba, S. Nojiri and S. D. Odintsov, Phys. Lett. B **698** (2011) 451
[628] A. S. Arapoglu, C. Deliduman and K. Y. Eksi, JCAP **1107** (2011) 020 [arXiv:1003.3179 [gr-qc]].
[629] A. Cooney, S. DeDeo and D. Psaltis, Phys. Rev. D **82** (2010) 064033 [arXiv:0910.5480 [astro-ph.HE]].
[630] E. Babichev and D. Langlois, Phys. Rev. D **80** (2009) 121501 Erratum: [Phys. Rev. D **81** (2010) 069901] 10.1103/PhysRevD.80.121501 [arXiv:0904.1382 [gr-qc]].
[631] T. Kobayashi and K. i. Maeda, Phys. Rev. D **78** (2008) 064019 [arXiv:0807.2503 [astro-ph]].
[632] E. Chabanat, P. Bonche, P. Haensel, J. Meyer and R. Schaeffer, Nucl. Phys. A **635** (1998) 231 Erratum: [Nucl. Phys. A **643** (1998) 441]. 10.1016/S0375-9474(98)00180-8
[633] F. Douchin and P. Haensel, Phys. Lett. B **485** (2000) 107
[634] F. Douchin and P. Haensel, Astron. Astrophys. **380** (2001) 151
[635] V.R. Pandharipande, D.G. Ravenhall D.G., *Hot nuclear matter* in Nuclear Matter and Heavy Ion Collisions, Eds. M. Soyeur, H. Flocard, B.Tamain, and M. Porneuf (Dordrecht: Reidel), 103?132 (1989).

[636] M. Camenzind, *Compact Objects in Astrophysics*, Springer (2007).
[637] J. Antoniadis *et al.*, Science **340** (2013) 6131 [arXiv:1304.6875 [astro-ph.HE]].
[638] S. Nojiri and S. D. Odintsov, Class. Quant. Grav. **30** (2013) 125003 [arXiv:1301.2775 [hep-th]].
[639] S. Capozziello, M. De Laurentis and A. Stabile, Class. Quant. Grav. **27** (2010) 165008 [arXiv:0912.5286 [gr-qc]].
[640] V. Reijonen, arXiv:0912.0825 [gr-qc].
[641] M. Sharif and M. F. Shamir, Class. Quant. Grav. **26** (2009) 235020 [arXiv:0910.5787 [gr-qc]].
[642] V. Faraoni, Class. Quant. Grav. **26** (2009) 195013 [arXiv:0909.0514 [gr-qc]].
[643] A. M. Nzioki, S. Carloni, R. Goswami and P. K. S. Dunsby, Phys. Rev. D **81** (2010) 084028 [arXiv:0908.3333 [gr-qc]].
[644] A. de la Cruz-Dombriz, A. Dobado and A. L. Maroto, Phys. Rev. D **80** (2009) 124011 Erratum: [Phys. Rev. D **83** (2011) 029903] [arXiv:0907.3872 [gr-qc]].
[645] E. V. Arbuzova and A. D. Dolgov, Phys. Lett. B **700** (2011) 289 [arXiv:1012.1963 [astro-ph.CO]].
[646] A. Addazi and S. Capozziello, Mod. Phys. Lett. A **31** (2016) no.09, 1650054 [arXiv:1602.00485 [gr-qc]].
[647] K. Kainulainen, J. Piilonen, V. Reijonen and D. Sunhede, Phys. Rev. D **76** (2007) 024020 [arXiv:0704.2729 [gr-qc]].
[648] A. Addazi, arXiv:1610.04094 [gr-qc].
[649] H. Nariai, Sci. Rep. Tohoku Univ. **34** (1950) 160.
[650] H. Nariai Sci. Rep. Tohoku Univ. **35** (1951) 62.
[651] R. Bousso and S. W. Hawking, Phys. Rev. D **57** (1998) 2436 [hep-th/9709224].
[652] C. W. Misner, Phys. Rev. **118** (1960) 1110.
[653] M. S. Morris and K. S. Thorne, Am. J. Phys. **56** (1988) 395.
[654] M. S. Morris, K. S. Thorne and U. Yurtsever, Phys. Rev. Lett. **61** (1988) 1446.
[655] I. R. Klebanov, L. Susskind and T. Banks, Nucl. Phys. B **317** (1989) 665.
[656] S. W. Hawking, Phys. Rev. D **37** (1988) 904.
[657] M. Visser, Phys. Rev. D **39** (1989) 3182 [arXiv:0809.0907 [gr-qc]].
[658] C. Barcelo and M. Visser, Class. Quant. Grav. **17** (2000) 3843 [gr-qc/0003025].
[659] M. Visser, S. Kar and N. Dadhich, Phys. Rev. Lett. **90** (2003) 201102 [gr-qc/0301003].
[660] D. Hochberg and M. Visser, Phys. Rev. D **56** (1997) 4745 [gr-qc/9704082].
[661] S. Nojiri, O. Obregon, S. D. Odintsov and K. E. Osetrin, Phys. Lett. B **458** (1999) 19 [gr-qc/9904035].
[662] L. H. Ford and T. A. Roman, Phys. Rev. D **53** (1996) 5496 [gr-qc/9510071].
[663] C. Armendariz-Picon, Phys. Rev. D **65** (2002) 104010 [gr-qc/0201027].
[664] F. S. N. Lobo, Phys. Rev. D **71** (2005) 124022 [gr-qc/0506001].
[665] E. Poisson and M. Visser, Phys. Rev. D **52** (1995) 7318 [gr-qc/9506083].
[666] S. V. Sushkov and Y. Z. Zhang, Phys. Rev. D **77** (2008) 024042 [arXiv:0712.1727 [gr-qc]].
[667] D. Hochberg and M. Visser, Phys. Rev. Lett. **81** (1998) 746 [gr-qc/9802048].
[668] D. Hochberg and M. Visser, Phys. Rev. D **58** (1998) 044021 [gr-qc/9802046].
[669] S. V. Sushkov, Phys. Rev. D **71** (2005) 043520 [gr-qc/0502084].
[670] S. V. Bolokhov, K. A. Bronnikov and M. V. Skvortsova, Class. Quant. Grav. **29** (2012) 245006 [arXiv:1208.4619 [gr-qc]].
[671] K. A. Bronnikov and S. W. Kim, Phys. Rev. D **67** (2003) 064027 [gr-qc/0212112].
[672] K. A. Bronnikov, P. A. Korolyov, A. Makhmudov and M. V. Skvortsova, arXiv:1612.04205 [gr-qc].
[673] K. A. Bronnikov and M. V. Skvortsova, Grav. Cosmol. **22** (2016) no.4, 316 [arXiv:1608.04974 [gr-qc]].
[674] T. Harko, F. S. N. Lobo, M. K. Mak and S. V. Sushkov, Phys. Rev. D **87** (2013) no.6, 067504 [arXiv:1301.6878 [gr-qc]].
[675] C. Bambi, A. Cardenas-Avendano, G. J. Olmo and D. Rubiera-Garcia, Phys. Rev. D **93** (2016) no.6, 064016 [arXiv:1511.03755 [gr-qc]].
[676] F. S. N. Lobo, J. Martinez-Asencio, G. J. Olmo and D. Rubiera-Garcia, Phys. Rev. D **90** (2014) no.2, 024033 [arXiv:1403.0105 [hep-th]].
[677] S. Capozziello, T. Harko, T. S. Koivisto, F. S. N. Lobo and G. J. Olmo, Phys. Rev. D **86** (2012) 127504 [arXiv:1209.5862 [gr-qc]].
[678] K. A. Bronnikov, M. V. Skvortsova and A. A. Starobinsky, Grav. Cosmol. **16** (2010) 216 [arXiv:1005.3262 [gr-qc]].
[679] H. Saiedi, Mod. Phys. Lett. A **27** (2012) 1250220 [arXiv:1409.2179 [gr-qc]].
[680] S. Habib Mazharimousavi and M. Halilsoy, Mod. Phys. Lett. A **31** (2016) no.37, 1650203 [arXiv:1209.2015 [gr-qc]].
[681] A. DeBenedictis and D. Horvat, Gen. Rel. Grav. **44** (2012) 2711 [arXiv:1111.3704 [gr-qc]].
[682] F. Darabi, Theor. Math. Phys. **173** (2012) 1734 [arXiv:1110.5485 [gr-qc]].
[683] H. Saeidi and B. N. Esfahani, Mod. Phys. Lett. A **26** (2011) 1211 [arXiv:1409.2176 [physics.gen-ph]].
[684] C. Bejarano, F. S. N. Lobo, G. J. Olmo and D. Rubiera-Garcia, arXiv:1607.01259 [gr-qc].
[685] T. Azizi, Int. J. Theor. Phys. **52** (2013) 3486 [arXiv:1205.6957 [gr-qc]].
[686] N. M. Garcia and F. S. N. Lobo, Phys. Rev. D **82** (2010) 104018 [arXiv:1007.3040 [gr-qc]].
[687] S. V. Sushkov and R. Korolev, Class. Quant. Grav. **29** (2012) 085008 [arXiv:1111.3415 [gr-qc]].
[688] L. Battarra, G. Lavrelashvili and J. L. Lehners, Phys. Rev. D **90**, no. 12, 124015 (2014) [arXiv:1407.6026 [hep-th]].
[689] S. I. Vacaru, Eur. Phys. J. C **74** (2014) 2781 [arXiv:1403.1815 [gr-qc]].
[690] F. S. N. Lobo and M. A. Oliveira, Phys. Rev. D **81** (2010) 067501 [arXiv:1001.0995 [gr-qc]].
[691] S. V. Sushkov and S. M. Kozyrev, Phys. Rev. D **84** (2011) 124026 [arXiv:1109.2273 [gr-qc]].
[692] T. Qiu, Class. Quant. Grav. **27** (2010) 215013 [arXiv:1007.2929 [hep-ph]].
[693] Y. F. Cai, D. A. Easson and R. Brandenberger, JCAP **1208** (2012) 020 [arXiv:1206.2382 [hep-th]].
[694] J. de Haro, Europhys. Lett. **107** (2014) 29001 [arXiv:1403.4529 [gr-qc]].
[695] A. Ashtekar and P. Singh, Class. Quant. Grav. **28** (2011) 213001 [arXiv:1108.0893 [gr-qc]].

[696] A. Ashtekar, Nuovo Cim. B **122** (2007) 135 [gr-qc/0702030].
[697] A. Corichi and P. Singh, Phys. Rev. D **80** (2009) 044024 [arXiv:0905.4949 [gr-qc]].
[698] P. Singh, Class. Quant. Grav. **26** (2009) 125005 [arXiv:0901.2750 [gr-qc]].
[699] M. Bojowald, Class. Quant. Grav. **26** (2009) 075020 [arXiv:0811.4129 [gr-qc]].
[700] V. K. Oikonomou, Astrophys. Space Sci. **359** (2015) no.1, 30 [arXiv:1412.4343 [gr-qc]].
[701] S. D. Odintsov and V. K. Oikonomou, Phys. Rev. D **91** (2015) no.6, 064036 [arXiv:1502.06125 [gr-qc]].
[702] S. Carloni, R. Goswami and P. K. S. Dunsby, Class. Quant. Grav. **29** (2012) 135012 [arXiv:1005.1840 [gr-qc]].
[703] L. Perivolaropoulos, arXiv:0811.4684 [astro-ph].
[704] S. Mendoza and G. J. Olmo, Astrophys. Space Sci. **357** (2015) no.2, 133 [arXiv:1401.5104 [gr-qc]].
[705] H. Davoudiasl, R. Kitano, G. D. Kribs, H. Murayama and P. J. Steinhardt, Phys. Rev. Lett. **93** (2004) 201301 [hep-ph/0403019].
[706] G. Lambiase and G. Scarpetta, Phys. Rev. D **74** (2006) 087504 [astro-ph/0610367].
[707] S. D. Odintsov and V. K. Oikonomou, Phys. Lett. B **760** (2016) 259 [arXiv:1607.00545 [gr-qc]].
[708] K. Saaidi and H. Hossienkhani, Astrophys. Space Sci. **333** (2011) 305 [arXiv:1010.4966 [gr-qc]].
[709] H. M. Sadjadi, Phys. Rev. D **76** (2007) 123507 [arXiv:0709.0697 [gr-qc]].
[710] G. Lambiase, Phys. Lett. B **642** (2006) 9 [hep-ph/0612212].
[711] M. C. Bento, R. Gonzalez Felipe and N. M. C. Santos, Phys. Rev. D **71** (2005) 123517 [hep-ph/0504113].
[712] B. Feng, H. Li, M. z. Li and X. m. Zhang, Phys. Lett. B **620** (2005) 27 [hep-ph/0406269].
[713] V. K. Oikonomou and E. N. Saridakis, Phys. Rev. D **94** (2016) no.12, 124005 [arXiv:1607.08561 [gr-qc]].
[714] V. K. Oikonomou, S. Pan and R. C. Nunes, arXiv:1610.01453 [gr-qc].
[715] S. D. Odintsov and V. K. Oikonomou, Europhys. Lett. **116** (2016) no.4, 49001 [arXiv:1610.02533 [gr-qc]].
[716] E. V. Arbuzova and A. D. Dolgov, arXiv:1612.06206 [gr-qc].
[717] J. A. S. Lima and D. Singleton, Phys. Lett. B **762** (2016) 506 [arXiv:1610.01591 [gr-qc]].
[718] T. Saidov and A. Zhuk, Phys. Rev. D **81** (2010) 124002 [arXiv:1002.4138 [hep-th]].
[719] Y. F. Cai, T. t. Qiu, J. Q. Xia and X. Zhang, Phys. Rev. D **79** (2009) 021303 [arXiv:0808.0819 [astro-ph]].
[720] K. Bamba, G. G. L. Nashed, W. El Hanafy and S. K. Ibraheem, Phys. Rev. D **94** (2016) no.8, 083513 [arXiv:1604.07604 [gr-qc]].
[721] G. Jungman, M. Kamionkowski and K. Griest, Phys. Rept. **267** (1996) 195 [hep-ph/9506380].
[722] S. Capozziello, V. F. Cardone and A. Troisi, JCAP **0608** (2006) 001 [astro-ph/0602349].
[723] S. Nojiri and S. D. Odintsov, TSPU Bulletin N **8(110)** (2011) 7 [arXiv:0807.0685 [hep-th]].
[724] C. G. Boehmer, T. Harko and F. S. N. Lobo, Astropart. Phys. **29** (2008) 386 [arXiv:0709.0046 [gr-qc]].
[725] Y. Bai, P. J. Fox and R. Harnik, JHEP **1012** (2010) 048 [arXiv:1005.3797 [hep-ph]].
[726] G. Prezeau, A. Kurylov, M. Kamionkowski and P. Vogel, Phys. Rev. Lett. **91** (2003) 231301 [astro-ph/0309115].
[727] V. K. Oikonomou, J. D. Vergados and C. C. Moustakidis, Nucl. Phys. B **773**, 19 (2007) [hep-ph/0612293].
[728] Y. K. E. Cheung and J. D. Vergados, JCAP **1502** (2015) no.02, 014 [arXiv:1410.5710 [hep-ph]].
[729] C. Li, JCAP **1609** (2016) no.09, 038 [arXiv:1512.06794 [astro-ph.CO]].
[730] S. Capozziello, R. Cianci, M. De Laurentis and S. Vignolo, Eur. Phys. J. C **70** (2010) 341 [arXiv:1007.3670 [gr-qc]].
[731] G. Basini, S. Capozziello and M. De Laurentis, Int. J. Geom. Meth. Mod. Phys. **13** (2016) no.03, 1650034.
[732] T. L. Smith, M. Kamionkowski and A. Cooray, Phys. Rev. D **73** (2006) 023504 [astro-ph/0506422].
[733] C. Corda, Eur. Phys. J. C **65** (2010) 257 [arXiv:1007.4077 [gr-qc]].
[734] M. Kamionkowski and A. Kosowsky, Phys. Rev. D **57** (1998) 685 [astro-ph/9705219].
[735] S. Capozziello, M. De Laurentis, S. Nojiri and S. D. Odintsov, Gen. Rel. Grav. **41** (2009) 2313 [arXiv:0808.1335 [hep-th]].
[736] S. Bellucci, S. Capozziello, M. De Laurentis and V. Faraoni, Phys. Rev. D **79** (2009) 104004 [arXiv:0812.1348 [gr-qc]].
[737] K. N. Ananda, S. Carloni and P. K. S. Dunsby, Phys. Rev. D **77** (2008) 024033 [arXiv:0708.2258 [gr-qc]].
[738] M. Kamionkowski, A. Kosowsky and A. Stebbins, Phys. Rev. Lett. **78** (1997) 2058 [astro-ph/9609132].
[739] B. Jain and J. Khoury, Annals Phys. **325** (2010) 1479 [arXiv:1004.3294 [astro-ph.CO]].
[740] G. Lambiase, S. Mohanty and G. Scarpetta, JCAP **0807** (2008) 019.
[741] G. F. R. Ellis, J. Murugan and C. G. Tsagas, Class. Quant. Grav. **21** (2004) no.1, 233 [gr-qc/0307112].
[742] Y. F. Cai, M. Li and X. Zhang, Phys. Lett. B **718** (2012) 248 [arXiv:1209.3437 [hep-th]].
[743] J. Q. Xia, H. Li and X. Zhang, Phys. Rev. D **88** (2013) 063501 [arXiv:1308.0188 [astro-ph.CO]].
[744] F. S. N. Lobo, G. J. Olmo and D. Rubiera-Garcia, Phys. Rev. D **91** (2015) no.12, 124001 [arXiv:1412.4499 [hep-th]].
[745] F. R. Klinkhamer, Phys. Rev. D **81** (2010) 043006 [arXiv:0904.3276 [gr-qc]].
[746] M. Cruz, E. Martinez-Gonzalez, P. Vielva, J. M. Diego, M. Hobson and N. Turok, Mon. Not. Roy. Astron. Soc. **390** (2008) 913 [arXiv:0804.2904 [astro-ph]].
[747] L. Rudnick, S. Brown and L. R. Williams, Astrophys. J. **671** (2007) 40 [arXiv:0704.0908 [astro-ph]].
[748] D. L. Larson and B. D. Wandelt, Astrophys. J. **613** (2004) L85 [astro-ph/0404037].
[749] S. Nojiri, S. D. Odintsov and V. K. Oikonomou, arXiv:1704.05945 [gr-qc].