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# CONSTITUTIVE EQUATIONS OF CREEP AND CREEP DAMAGE IN POLYCRYSTALLINE METALS*

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## Abstract

Constitutive and evolution equations of creep and creep damage in polycrystalline metals are formulated by taking account of their microstructural mechanisms. The representation of the microstructural changes induced by creep damage in terms of a tensorial internal state variable and the oriented nature of the creep damage are first discussed. Then, by assuming that the effect of material damage in creep consists in the magnification of the stress effect due to internal fissures and voids, the constitutive and evolution equations for creep and creep damage are expressed as a set of tensorial functions of a net stress tensor and a damage effect tensor of rank two. These equations are specified in conformity with the experimental results reported so far. The relation between the present theory and the previous ones is also discussed. Finally, the present theory is applied to the creep damage analysis of a thin-walled tube subjected to nonsteady combined tension and torsion.

## 1. Introduction

Modern technology often requires inelastic analyses of structural components subject to complicated history of loading in order to assure their reliable and safe operation. One of the major difficulties of these analyses consists in finding accurate constitutive relations which describe the material response under various

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service histories in non-conventional environments. Since inelastic deformation is usually accompanied with microstructural changes of materials, the development of such constitutive equations should be performed by taking account of the principal mechanisms of the microstructural change due to the inelastic deformation, and should be done on the basis of mathematically consistent methods.

Because the material deterioration in the process of creep is one of the most important phenomena in the design of high-temperature equipments of longer service life, the problems of creep damage theory have been an objective of many papers[1~4]. Kachanov[5] first proposed a phenomenological theory of creep damage in uniaxial state of stress, and his theory has been extended to multiaxial state of stress by several researchers. However, the majority of them have been formulated on the assumption of the isotropy of the damage, or formulated by taking account of the damage anisotropy but only with respect to principal stress coordinates fixed in the material[5~8]. Though Kachanov[2] and Hayhurst and Storåkers[9] proposed creep damage theories by representing the damage state on a plane in a material by a vector in the corresponding direction, these theories have not been formulated in a form of a tensorial relation and hence are not applicable to non-steady multiaxial state of stress.

According to the experimental investigations reported so far, it has been observed that the creep damage in usual polycrystalline metals advances mainly due to the nucleation and the growth of voids and fissures at grain boundaries perpendicular to the direction of maximum principal stress at each instant[10~13]. Therefore, in order to formulate an accurate theory of creep damage, it is necessary to incorporate these structural changes of oriented nature in the condition of multiaxial and non-steady state of stress.

In the present paper, a continuum theory of creep and creep damage in polycrystalline metals is formulated by representing the microstructural change due to creep damage in terms of a symmetric tensor of rank two as an internal state variable. After elucidating the relation to the previous theories, the present constitutive equation is applied to the creep damage analysis of a thin-walled tube under non-steady combined state of stress.

## 2. Constitutive and Evolution Equations of Creep and Creep Damage*

### *2. 1. Damage Tensor and Constitutive Equation*

Materials subjected to creep at elevated temperature for long time is usually accompanied with time-dependent internal deterioration called creep damage. The creep damage in polycrystalline metals and alloys is induced by the nucleation and the growth of microscopic round voids (r-type voids) or wedge-shaped cracks (w-type cracks) on grain boundaries, mostly perpendicular to the direction of the maximum principal stress[10~13]. The coalescence of these cavities into macroscopic cracks leads to the final fracture of the materials.

In order to develop an elaborate theory of creep and creep damage in metals, it is necessary first to express the effect of these cavities in terms of appropriate macroscopic variables, and then to formulate the relations which govern the evolu-

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* See the footnote of p. 188.

tion of these variables.

When Vakulenko and Kachanov developed a continuum theory of cracked elastic media, they represented an assembly of these cracks by a *crack-density tensor* of rank two as follows[14]:

$$\boldsymbol{\Omega}=\frac{1}{V}\sum_{k=1}^{N}\int_{S(k)}\boldsymbol{b}^{(k)}\otimes\boldsymbol{n}^{(k)}dS_{(k)} \tag{2.1}$$

where $V$ is the volume occupied by a material element in the initial state, $S_{(k)}$ denotes the initial surface of $k$-th crack in $V$, $\boldsymbol{n}^{(k)}$ and $\boldsymbol{b}^{(k)}$ are a unit normal vector and a relative discontinuous displacement at a point on $S_{(k)}$ due to crack opening, and the symbol $\otimes$ stands for the tensor product.

Since equation (2.1) holds for an assembly of cracks, it is applicable also to creep damage induced by w-type cracks, because the surface element of grain boundary before crack initiation and the relative discontinuous displacement to crack opening correspond to $S_{(k)}$ and $\boldsymbol{b}^{(k)}$ in equation (2.1), respectively. For r-type cavities, on the other hand, $S_{(k)}$ and $\boldsymbol{b}^{(k)}$ might not be defined readily, because the mechanism of cavity growth which occurs due to the condensation of vacancies at the void surface[10~12] is different from that of crack initiation. However, since r-type cavities grow also on boundaries perpendicular to the applied stress, the creep damage caused by the r-type cavities will have the similar oriented nature to that of w-type cracks. Thus, let us define $S_{(k)}$ and $\boldsymbol{b}^{(k)}$ as the initial element of grain boundary occupied by $k$-th cavity and the normal vector to $S_{(k)}$ with the magnitude equal to the thickness of the cavity measured along the normal to $S_{(k)}$, respectively. Then, equation (2.1) applies also to r-type cavities.

Since the component of $\boldsymbol{b}^{(k)}$ parallel to $\boldsymbol{n}^{(k)}$ corresponds to the crack opening displacement and will be more effective in creep damage than that perpendicular to $\boldsymbol{n}^{(k)}$, we allow for only the former component. Then, $\boldsymbol{\Omega}$ in equation (2.1) reduces to a symmetric tensor.

Since creep rate and rate of damage growth may depend on history of inelastic strain in the material matrix as well as the current state of stress and material deterioration, they may be specified as

$$\boldsymbol{D}=\boldsymbol{G}(\boldsymbol{\sigma},\ \boldsymbol{\Omega},\ \kappa) \tag{2.2}$$

$$\overset{\circ}{\boldsymbol{\Omega}}=\boldsymbol{H}(\boldsymbol{\sigma},\ \boldsymbol{\Omega},\ \kappa) \tag{2.3}$$

where $\boldsymbol{D}$, $\boldsymbol{\sigma}$ and $\kappa$ are rate of deformation tensor, Cauchy stress tensor and a certain parameter representing the microstructural changes of the matrix, respectively. The rate of damage tensor $\overset{\circ}{\boldsymbol{\Omega}}=\dot{\boldsymbol{\Omega}}+\boldsymbol{\Omega}\boldsymbol{W}-\boldsymbol{W}\boldsymbol{\Omega}$ is the Jaumann derivative of $\boldsymbol{\Omega}$, where a superposed dot and $\boldsymbol{W}$ denote material time derivative and a material spin tensor.

### *2. 2. Specification of Constitutive and Evolution Equations*

The effect of material damage in creep can be interpreted as the magnification of the stress effect owing to the existence of internal voids and fissures. If the increased effect of stress due to the damage can be represented by a *net stress tensor* $\boldsymbol{S}$, it can be expressed by a tensor function of $\boldsymbol{\sigma}$ and $\boldsymbol{\Omega}$:

$$\boldsymbol{S}=\boldsymbol{K}(\boldsymbol{\sigma},\ \boldsymbol{\Omega}) \tag{2.4}$$

It is supposed that the mechanism of stress increase due to the damage mainly consists of the reduction of the effective area in the material, and of the stress concentration at cavities. If we take the former one as the principal mechanism of stress increase, the tensor function $\boldsymbol{K}$ may be a linear transformation from $\boldsymbol{\sigma}$ to $\boldsymbol{S}$, and $\boldsymbol{S}$ should coincide with $\boldsymbol{\sigma}$ for vanishing $\boldsymbol{\Omega}$.

Then, if we expand the tensor function $\boldsymbol{K}$ by applying the representation theorem for an isotropic tensor function of two symmetric tensors[15], and retain the linear terms with respect to $\boldsymbol{\sigma}$, we have the expression

$$\boldsymbol{S}=(1/2)(\boldsymbol{\Phi}\boldsymbol{\sigma}+\boldsymbol{\sigma}\boldsymbol{\Phi}) \tag{2.5}$$

$$\boldsymbol{\Phi}=\alpha_0\boldsymbol{I}+\alpha_1\boldsymbol{\Omega}+\alpha_2\boldsymbol{\Omega}^2 \tag{2.6}$$

In the above equation, coefficients $\alpha_i$ $(i=0, 1, 2)$ are scalar functions of $tr\boldsymbol{\Omega}$, $tr\boldsymbol{\Omega}^2$, $tr\boldsymbol{\Omega}^3$, and $\boldsymbol{I}$ denotes the identity tensor. In order that $\boldsymbol{S}$ may coincide with $\boldsymbol{\sigma}$ for vanishing damage, $\alpha_0=1$ should hold for $\boldsymbol{\Omega}=0$.

Since the tensor $\boldsymbol{\Omega}$ was assumed to be symmetric, the tensor $\boldsymbol{\Phi}$ is also symmetric and hence has three real principal values $\Phi^{(i)}$ $(i=1, 2, 3)$. If we take a rectangular coordinate system $x_i$ $(i=1, 2, 3)$ whose coordinate axes coincide with the principal axes of $\boldsymbol{\Phi}$, the components of $\boldsymbol{S}$ can be expressed as

$$S_{ij}=(1/2)(\Phi^{(i)}+\Phi^{(j)})\sigma_{ij} \quad (i, j\text{: no sum}) \tag{2.7}$$

It will be observed that the normal component of the Cauchy stress tensor $\sigma_{ii}$ ($i$: no sum) and its shear component $\sigma_{ij}$ $(i\neq j)$ are magnified by the factors $\Phi^{(i)}$ and $(\Phi^{(i)}+\Phi^{(j)})/2$, respectively. In other words, the stress components acting on the $i$-th principal plane are magnified by the principal value $\Phi^{(i)}$. Thus, $1/\Phi^{(i)}$ can be interpreted as the ratio of the reduction of the effective area in the $i$-th principal plane of $\boldsymbol{\Phi}$.

On the other hand, if we take another system of rectangular coordinates $y_\alpha$ $(\alpha=1, 2, 3)$ which coincide with the principal axes of $\boldsymbol{\sigma}$, equation (2.5) has the component form

$$S_{\alpha\beta}=(1/2)\Phi_{\alpha\beta}(\sigma^{(\alpha)}+\sigma^{(\beta)}) \quad (\alpha, \beta\text{: no sum}) \tag{2.8}$$

In this case, not only the principal stress $\sigma^{(\alpha)}$ is magnified by $\Phi_{\alpha\alpha}$, shear component $(1/2)\Phi_{\alpha\beta}\sigma^{(\alpha)}$ occurs on the principal plane of $\boldsymbol{\sigma}$. Namely, the principal directions of $\boldsymbol{S}$ generally deviate from those of $\boldsymbol{\sigma}$ due to the effect of creep damage. Thus, $\boldsymbol{\Phi}$ in equation (2.5) stands for the effect of material damage on $\boldsymbol{\sigma}$, and hence we will call it a *damage effect tensor.*

Since the inverse of the damage effect tensor $\boldsymbol{\Psi}=\boldsymbol{\Phi}^{-1}$ has been interpreted as the fraction of the effective area diminished by the material damage, it would be more appropriate to consider $\mathring{\boldsymbol{\Psi}}$ as the rate of damage rather than $\mathring{\boldsymbol{\Phi}}$ and $\mathring{\boldsymbol{\Omega}}$. If the damage effect would consist only of the stress increase due to the reduction of the effective area, $\boldsymbol{D}$ and $\mathring{\boldsymbol{\Psi}}$ could be prescribed in terms of $\boldsymbol{S}$ and $\kappa$. However, as already mentioned, there will be also the effect of stress concentration at the fissures or voids. Though this non-linear effect of $\boldsymbol{\sigma}$ might be represented by taking additional terms in the expression (2.5), the simple interpretation of $\boldsymbol{S}$ would become difficult in this case. Therefore, in order to incorporate the effects of stress concentration, we will represent $\boldsymbol{D}$ and $\mathring{\boldsymbol{\Psi}}$ as function of $\boldsymbol{\Phi}$ as well as $\boldsymbol{S}$

and $\kappa$:

$$D=\overline{G}(S,\ \Phi,\ \kappa) \tag{2.9}$$

$$\overset{\circ}{\Psi}=\overline{H}(S,\ \Phi,\ \kappa),\quad \Psi=\Phi^{-1} \tag{2.10}$$

As observed from equation (2.5), the initial value of $\Psi$ is the identity tensor $I$.

If we apply the representation theorem for the isotropic tensor function of two symmetric tensors[15], the function $\overline{G}$ of equation (2.9) is expressed as a tensor polynomial with respect to $S$ and $\Phi$:

$$D=\beta_0 I+\beta_1\Phi+\beta_2\Phi^2+\beta_3 S+\beta_4(\Phi S+S\Phi)+\beta_5(\Phi^2 S+S\Phi^2)$$
$$+\beta_6 S^2+\beta_7(\Phi S^2+S^2\Phi) \tag{2.11}$$

where $\beta_i$ ($i=0, 1, \ldots, 7$) are functions of $\kappa$ and the following ten basic invariants of $\Phi$ and $S$:

$$\begin{gathered} tr\Phi,\quad tr\Phi^2,\quad tr\Phi^3,\quad trS,\quad trS^2,\quad trS^3, \\ tr(\Phi S),\quad tr(\Phi^2 S),\quad tr(\Phi S^2),\quad tr(\Phi^2 S^2) \end{gathered} \tag{2.12}$$

By substituting equations (2.5) and (2.6) into equation (2.11) and comparing the resulting equation with the expanded form of equation (2.2), it is shown that all terms of equation (2.11) are not tensorially independent with respect to $\sigma$ and $\Omega$. If we retain only the tensorially independent terms, equation (2.11) reduces to a simpler expression

$$D=\beta_0 I+\beta_1\Phi+\beta_3 S+B_6 S^2+\beta_7(\Phi S^2+S^2\Phi) \tag{2.13}$$

If the dependence of $D$ on $\Phi$ is disregarded furthermore, the above equation is simplified as

$$D=\beta_0 I+\beta_3 S+\beta_6 S^2 \tag{2.14}$$

From the implication of equation (2.5), the inverse of the damage effect tensor $\Psi$ should be prescribed so that it may have principal values less than unity. Moreover, in view of the mechanism of the material damage, the rate of the tensor $\Psi$ may not be a continuous function of stress. Therefore it would be impossbile to discuss the specific form of function $\overline{H}$ in equation (2.10) by expanding it into a tensor polynomial as was done in equation (2.9).

According to the experimental observations on the polycrystalline metals under multiaxial states of stress, there are two classes of rupture behaviour; one is governed by the maximum principal tensile stress, while the other by the octahedral shear stress[16]. The copper, for example, follows the former stress criterion, and the fissures develop mainly on the plane perpendicular to the direction of maximum principal tensile stress[10~12, 16]. On the other hand, though the aluminium conforms to the octahedral shear stress criterion, it was found that cracks and voids in the material develop in the same manner as in copper[9,16]. Thus, the effect of these cavities can be described by a principal component of $\Psi$ in the tensile direction. This implies that the tensor $\Psi$ develops in the direction $\nu^{(i)}$ and $\nu_D^{(i)}$ of positive principal value of $S$ and its deviator $S_D$, respectively. Since these cavities are not completely flat, they have also isotropic effect more or less. Thus, in view of that $\Psi$ diminishes according to the damage growth, equation (2. 10) may have the following form:

$$\overset{\circ}{\Psi}=-\gamma I-\sum_{i=1}^{3}M^{(i)}[\nu^{(i)}\otimes\nu^{(i)}]-\sum_{j=1}^{3}N^{(j)}[\nu_D{}^{(j)}\otimes\nu_D{}^{(j)}] \tag{2. 15}$$

where $\gamma$, $M^{(i)}$ and $N^{(j)}$ are a scalar valued function and tensor valued functions of rank four of $S$, $\Phi$ and $\kappa$, respectively. Furthermore, $M^{(i)}[\nu^{(i)}\otimes\nu^{(i)}]$ of equation (2. 15) stands for $M^{(i)}_{pqrs}\,\nu_r^{(i)}\,\nu_s^{(i)}$, and the indices $i$ and $j$ are relevant only when principal values $S^{(i)}$ and $S_D{}^{(j)}$ of $S$ and $S_D$ are positive.

As a consequence, the tensile creep damage may be described by the first and the second term of equation (2. 15), if $\gamma$ and $M^{(i)}$ are specified properly as functions of maximum principal tensile stress or the octahedral shear stress of $S$. In the case of shear stress, on the other hand, Hayhurst and Storåkers[9] observed in their experiments on copper and aluminium disks under torsion extensive formation of grain boundary voids on plane perpendicular to the direction of principal stress. Therefore, equation (2. 15) may apply also to the creep damage induced by shearing stress. Creep damage in the condition of compressive stress can be expressed in the third term of the right hand side of equation (2. 15). However, this term is supposed to be small, because significant microstructural change has not been observed in this condition[11~16].

### *2. 3. Example of Constitutive and Evolution Equations*

For the practical application, the equations (2. 11) and (2. 15) must be specified further. The process of creep damage is not incompressible by nature. However, the volumetric change will be insignificant until immediately before the rupture, and hence we will assume the isochoric deformation. If the creep theory of von Mises type and the strain hardening hypothesis hold in the case of vanishing damage, equation (2. 14) can be expressed as

$$\begin{aligned}&D=A\kappa^m(3\bar{J}_2)^{(n-1)/2}S_D,\\&\bar{J}_2=(1/2)tr(S_D)^2,\qquad \kappa=\int(trD^2)^{1/2}dt\end{aligned} \tag{2. 16}$$

where $\bar{J}_2$ denotes the second invariant of tensor $S_D$, and $A$ and $n$ are material constants.

By taking the term $M^{(1)}[\nu^{(1)}\otimes\nu^{(1)}]$ from equation (2. 15), we have a simple anisotropic (non-homogeneous) creep damage law in a multiaxial stress state as follows:

$$\begin{aligned}&\overset{\circ}{\Psi}=-B[tr(\Psi\nu^{(1)}\otimes\nu^{(1)})]^k[\alpha S^{(1)}+(1-\alpha)\sqrt{3\bar{J}_2}]^l\nu^{(1)}\otimes\nu^{(1)}\\&S^{(1)}=<\max[S^{(i)}]>\quad(i=1,\ 2,\ 3)\end{aligned} \tag{2. 17}$$

where creep damage is assumed to develop only on the plane perpendicular to the direction of $S^{(1)}$, and the symbol $<\ >$ stands for the Macauley bracket which indicates the operation $<A>=A$ if $A>0$ and $<A>=0$ if $A\leq 0$. By taking the first term $\gamma I$ of equation (2. 15), on the other hand, we obtain an isotropic (homogeneous) creep damage law:

$$\mathring{\boldsymbol{\Psi}}=\dot{\Psi}\boldsymbol{I}, \quad \dot{\Psi}=-B\Psi^{k}[\alpha S^{(1)}+(1-\alpha)\sqrt{3\bar{J}_2}]^{l} \tag{2.18}$$

In equations (2.17) and (2.18), $B$, $k$, $l$ and $\alpha$ are material constants. A special form of equation (2.17) and (2.18) together with equation (2.16) was already employed by the present authors to analyse the creep damage process of thin-walled tubes[17].

The creep rupture may be governed by the current states of stress $\boldsymbol{\sigma}$, damage $\boldsymbol{\Phi}$ or $\boldsymbol{\Psi}$ and the parameter $\kappa$ as well as by the instantaneous rupture strength of the material. However, the above equations of creep damage are usually highly nonlinear with respect to $\boldsymbol{S}$, and hence $\boldsymbol{S}$ increases rapidly immediately before the rupture. Consequently, the creep rupture time may not be affected largely, even if we do not take account of the instantaneous rupture strength. Thus, we assume that the rupture occurs on a principal plane when one of the principal values of $\boldsymbol{\Phi}$ or $\boldsymbol{\Psi}$ attains to infinite or to zero, respectively.

## 3. Comparison with Previous Theories

In order to show the validity and the applicability of the present theory, we will now explicate its relation with the previous theories and apply it to the creep damage analysis of a thin-walled tube under non-steady and combined state of stress.

### 3. 1. *Creep in Uniaxial State of Stress*

Let us consider the simplest case of uniaxial tension. In view of equation (2.5), equations (2.16) through (2.18) reduce to

$$D=A(\sigma/\Psi)^{n} \tag{3.1}$$

$$\dot{\Psi}=-B\Psi^{k}(\sigma/\Psi)^{l} \tag{3.2}$$

where $D$, $\Psi$ and $\sigma$ denote the components of $\boldsymbol{D}$, $\boldsymbol{\Psi}$ and $\boldsymbol{\sigma}$ in the tensile direction, respectively. From equations (3.1) and (3.2), it is readily seen that in this case our theory reduces to that of Rabotnov[1]. The special case of $k=0$ in equation (3.2), in particular, corresponds to Kachanov's theory[5].

### 3. 2. *Creep in Multiaxial State of Stress*

In order to show the relation between the present theory and those proposed in the previous papers, let us now consider multiaxial creep where principal axes of stress are fixed in the material.

If we write the anisotropic creep damage law (2.17) in regard to the principal axes of $\boldsymbol{\sigma}$, the rate of creep damage can be written as follows:

$$\begin{aligned}\dot{\Psi}^{(1)}&=-B\Psi^{(1)k}[\alpha\sigma^{(1)}/\Psi^{(1)}+(1-\alpha)\sqrt{3\bar{J}_2}]^{l}\\ \dot{\Psi}^{(2)}&=\dot{\Psi}^{(3)}=0\end{aligned} \tag{3.3}$$

$$\bar{J}_2=(1/6)[(\sigma^{(1)}/\Psi^{(1)}-\sigma^{(2)})^2+(\sigma^{(2)}-\sigma^{(3)})^2+(\sigma^{(3)}-\sigma^{(1)}/\Psi^{(1)})^2] \tag{3.4}$$

where $\sigma^{(1)}>\sigma^{(2)}>\sigma^{(3)}$ and $\sigma^{(1)}>0$. If we assume the steady-state creep (*i. e.* $m=0$), equation (2. 16) leads to

$$\begin{aligned}
D^{(1)} &= A(3\bar{J}_2)^{(n-1)/2}[\sigma^{(1)}/\Psi^{(1)} - (1/2)(\sigma^{(2)} + \sigma^{(3)})] \\
D^{(2)} &= A(3\bar{J}_2)^{(n-1)/2}[\sigma^{(2)} - (1/2)(\sigma^{(3)} + \sigma^{(1)}/\Psi^{(1)})] \\
D^{(3)} &= A(3\bar{J}_2)^{(n-1)/2}[\sigma^{(3)} - (1/2)(\sigma^{(1)}/\Psi^{(1)} + \sigma^{(2)})]
\end{aligned} \tag{3. 5}$$

In the case of isotropic creep damage law (2. 18), on the other hand, we have the corresponding relations

$$\dot{\Psi} = -B[\alpha\sigma^{(1)} + (1-\alpha)\sqrt{3J_2}\,]^l/\Psi^{-k+l} \tag{3. 6}$$

$$\begin{aligned}
D^{(1)} &= A(3J_2)^{(n-1)/2}[\sigma^{(1)} - (1/2)(\sigma^{(2)} + \sigma^{(3)})]/\Psi^n \\
D^{(2)} &= A(3J_2)^{(n-1)/2}[\sigma^{(2)} - (1/2)(\sigma^{(3)} + \sigma^{(1)})]/\Psi^n \\
D^{(3)} &= A(3J_2)^{(n-1)/2}[\sigma^{(3)} - (1/2)(\sigma^{(1)} + \sigma^{(2)})]/\Psi^n
\end{aligned} \tag{3. 7}$$

$$J_2 = (1/6)[(\sigma^{(1)} - \sigma^{(2)})^2 + (\sigma^{(2)} - \sigma^{(3)})^2 + (\sigma^{(3)} - \sigma^{(1)})^2] \tag{3. 8}$$

It will be observed that equations (3. 3) through (3. 5) and (3. 6) through (3. 8) correspond to the non-homogeneous and the homogeneous case of the creep damage theories of Hayhurst and Leckie[6] and Goel[8]. It should be noticed that, though these theories are valid only for fixed principal stress directions, our original theory (2. 16) through (2. 18) are applicable also to the case of rotating principal stress axes.

### *3. 3. Thin-Walled Tubes Subject to Tension Followed by Torison*

So far we have shown that the theory developed in this paper comprises important creep damage theories in the literatures as its special cases. However, since the theories of Kachanov[2] and Hayhurst and Storåkers[9] based on a vector damage variable are not tensorial equations, they cannot be obtained from the framework of our theory. Then, we will analyse a simple example of combined loading solved by Kachanov[2], and discuss the difference between these theories. Hereafter the assumption of small deformation and the creep damage law (2. 17) with $k=0$ will be employed.

Let us consider a thin-walled tube which is subjected to axial tension $\sigma$ for the time interval $0 \leqq t \leqq t^*$, and then to torsion $\tau$ for $t \geqq t^*$. The circumferential and the axial coordinates will be denoted by $x$ and $y$, respectively.

Equation (3. 2) gives the value of $\Psi_{yy}$ at $t=t^*$:

$$\Psi^*_{yy} = [1 - B(l+1)\sigma^l t^*]^{1/(l+1)} \tag{3. 9}$$

By use of equations (2. 5) and (2. 17), non-vanishing components of $S$ in the process of torsion are written as

$$\begin{aligned}
S_{xx} &= S_{yy} = -\tau\Psi_{xy}/(\Psi_{xx}\Psi_{yy} - \Psi^2_{xy}), \\
S_{xy} &= (\tau/2)(\Psi_{xx} + \Psi_{yy})/(\Psi_{xx}\Psi_{yy} - \Psi^2_{xy})
\end{aligned} \tag{3. 10}$$

As observed from these relations, the principal direction of $S^{(1)}$ inclines always from the $x$-axis by 45° counter-clockwisely. Expressing the equation (2.17) with respect to the rectangular coordinate system $0$-$\xi\eta$ rotated from $0$-$xy$ system by 45° counter-clockwisely, we obtain

$$\begin{aligned} \dot{\Psi}_{\xi\xi} &= -B[\alpha S^{(1)} + (1-\alpha)\sqrt{3\bar{J}_2}]^l \\ \dot{\Psi}_{\xi\eta} &= \dot{\Psi}_{\eta\eta} = 0 \end{aligned} \tag{3.11}$$

The initial conditions of these equations at $t=t^*$ can be written as

$$\begin{aligned} \Psi^*_{\xi\xi} &= \Psi^*_{\eta\eta} = (1/2)(1+\Psi^*_{yy}) \\ \Psi^*_{\xi\eta} &= -(1/2)(1-\Psi^*_{yy}) \end{aligned} \tag{3.12}$$

Since creep rupture occurs when the minimum principal value of $\Psi$ attains to zero, the condition $\det\Psi=0$ together with (3.11) and (3.12) determines the following value of $\Psi_{\xi\xi}$ at the creep rupture time $t_R$:

$$(\Psi_{\xi\xi})_R = (1-\Psi^*_{yy})^2/2(1+\Psi^*_{yy}) \tag{3.13}$$

Then, the angle $\phi$ between the $x$-axis and the normal to the rupture plane are

$$\begin{aligned} \phi &= \pi/4 + (1/2)\ \arctan\ [2\Psi^*_{\xi\eta}/\{(\Psi_{\xi\xi})_R - \Psi^*_{\eta\eta}\}] \\ &= \pi/4 + (1/2)\ \arctan\ [(1-\Psi^{*2}_{yy})/2\Psi^*_{yy}] \end{aligned} \tag{3.14}$$

From equation (3.11), furthermore, we obtain the creep rupture time $t_R$ as follows:

$$\int_{(\Psi_{\xi\xi})_R}^{(\Psi^*_{\xi\xi})} [\alpha S^{(1)} + (1-\alpha)\sqrt{3\bar{J}_2}]^{-l} d\Psi_{\xi\xi} = B(t_R - t^*) \tag{3.15}$$

$$\begin{aligned} S^{(1)} &= S_{\xi\xi} = \tau\Psi^*_{\eta\eta}/[\Psi_{\xi\xi}\Psi^*_{\eta\eta} - (\Psi^*_{\xi\eta})^2] \\ \bar{J}_2 &= (\tau^2/3)[(\Psi_{\xi\xi})^2 + (\Psi^*_{\eta\eta})^2 - \Psi_{\xi\xi}\Psi^*_{\eta\eta}]/[\Psi_{\xi\xi}\Psi^*_{\eta\eta} - (\Psi^*_{\xi\eta})^2]^2 \end{aligned} \tag{3.16}$$

In the case of $\alpha=1$, equation (3.15) can be integrated analytically as follows:

$$t_R = t^* + 1/[B(l+1)\tau^l][2\Psi^*_{yy}/(1+\Psi^*_{yy})]^{l+1} \tag{3.17}$$

Fig. 1 shows the relation between the creep rupture time $t_R$ and the time of stress change $t^*$ from tension to torsion in the case of $\alpha=1$ and $\sigma=\tau$, where $t_\sigma$ denotes the rupture time for uniaxial tension:

$$t_\sigma = [B(l+1)\sigma^l]^{-1} \tag{3.18}$$

The solid and the dashed lines represent the result of the present analysis and that of Kachanov calculated from the following equation[2]:

$$\dot{\Psi}_\nu = -B(\sigma_\nu/\Psi_\nu)^l \tag{3.19}$$

where the subscript $\nu$ denotes the direction of the relevant plane, and $\Psi_\nu$ and $\sigma_\nu$ are the damage variable and the normal stress with respect to the plane $\nu$. Equation (3.19) is exactly the same as that of Hayhurst and Storåkers[9].

As will be seen from the figure, Kachanov's result which can be approximated by a relation $t_R/t_\sigma = t^*/t_\sigma + 1$ for larger values of $l$ gives longer rupture times than those of the present theory. This fact implies that the damage process of torsion due to Kachanov's theory proceeds almost independently of the damage accumulated in the preceeding process of tension. This is obviously attributable to the simplified assumption in equation (3.19) that the rate of damage on the plane $\nu$ does not depend on the damage state of other planes. In the present theory, on the other hand, since creep damage has been expressed in terms of tensorial quantity damage developed in the process of tension has significant effect on the subsequent process of torsion.

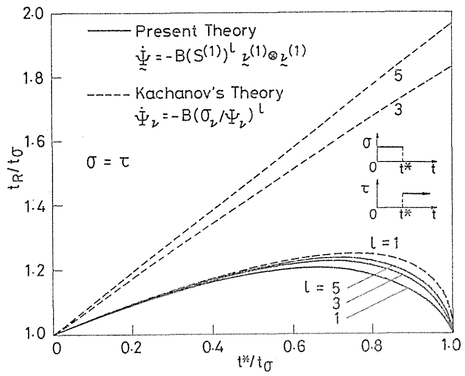


Fig. 1. Relation between the rupture time and the time of stress change.

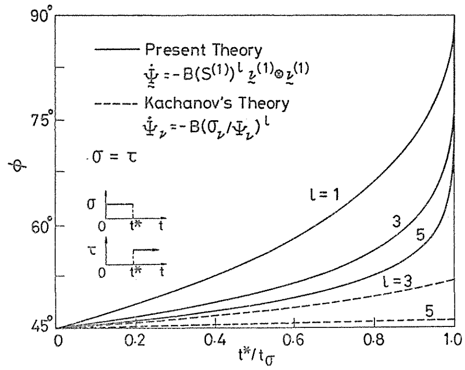


Fig. 2. Relation between the direction of fracture surface and the time of stress change.

The variation of the rupture plane in relation to the time of stress change is shown in Fig. 2. It will be noticed that the rupture plane predicted by Kachanov's theory is closer to that of torsion ($\phi = 45°$) than that of the present theory. This is again attributable to the above mentioned feature of equation (3.19).

## 4. Conclutions*

By representing the microstructural change of material in the process of creep damage by a symmetric tensor of rank two, constitutive and evolution equations of creep and creep damage were developed from the continuum mechanics point of

* Recently, the present authors developed a more elaborate and consistent theory of anisotropic creep damage in polycrystalline metals and alloys[18]. By postulating that the principal mechanical effects of creep damage consist in the net area reduction caused by cavity formation in the materials, they described the damage states by means of another second rank symmetric damage tensor specified by the three-dimensional cavity-area density. Further development of this theory will be found in the papers[19~21].

view. Specific forms of these equations were derived by taking account of the results of experimental observations reported so far. It was shown that most creep damage laws in the literature can be obtained from the present theory as its special cases.

Since the present theory was developed by incorporating the oriented nature of creep damage and formulated in the framework of the non-linear tensor theory, it has much more generality in comparison with the past empirical theories. Besides the validity of the proposed theory, we should emphasize its feasibility in creep damage analyses of structural components under general state of stress. Another important feature of the present theory will be its versatility to develop more elaborate constitutive equations.

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