ON A GENERALIZED SAMPLING THEOREM

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In the theory of information, or in the statistical theory of communication, Shannon's sampling theorem plays an important rôle in case of treating the continuous channel of information systems. In the present paper the authors give a generalized sampling theorem, which includes Shannon's theorem as a special case. Being based on the generalized theorem presented here, five new formulae are given as special examples. The theorem is also conveniently applied to the physical analysis.

Let $f(t)$ belong to L_p, and let the following reciprocity relations hold:

$$F(s) = \mathscr{L}_s \cdot f \equiv \int K(s,t) f(t)\,dt, \tag{1}$$

and

$$f(t) = \mathscr{L}_t^{-1} \cdot F \equiv \int \tilde{K}(t,s) F(s)\,ds. \tag{2}$$

Further we assume that

$$F(s) = 0, \qquad \text{for } s \leq \alpha \text{ and } \beta \leq s \tag{3}$$

and that $F(s)$ can be expanded in a complete orthogonal system of functions:

$$\{\phi_n(s)\,;\ \int \phi_m(s)\phi_n(s)\,ds = \gamma_m \cdot \delta_{m,n} \quad (m, n = \text{integers})\}, \tag{4}$$

in the interval $\alpha \leq s \leq \beta$, *i.e.*

$$F(s) = \sum_n a_n \phi_n(s). \quad \text{for } \alpha \leq s \leq \beta \tag{5}$$

The expression (5), being multiplied by $\phi_m(s)$ and integrated over s, gives:

$$\int_\alpha^\beta F(s)\phi_m(s)\,ds = \sum_n a_n \cdot \int_\alpha^\beta \phi_m(s)\phi_n(s)\,ds = \sum_n a_n \cdot \gamma_m \delta_{m,n} = \gamma_m a_m. \tag{6}$$

From (2), (3), (4), (5) and (6), we obtain

$$f(t) = \mathscr{L}_t^{-1} \cdot F = \sum_n a_n \cdot \mathscr{L}_t^{-1} \cdot \phi_n =$$

$$= \sum_n \left(\frac{1}{\gamma_n} \cdot \int_\alpha^\beta F(s)\phi_n(s)\,ds \right) \cdot \int \tilde{K}(t,s)\phi_n(s)\,ds. \tag{7}$$

If the functions $f(s)$, $\{\phi_n(s)\}$, and the integral kernels $K(s,t)$ and $\tilde{K}(t,s)$, are given, then we can construct an expansion in series of $f(t)$ by means of (7).

If we can take

$$\phi_n(s) = \tilde{K}(\lambda_n, s), \qquad \text{for } \alpha \leq s \leq \beta \tag{8}$$

with constants λ_n, *i.e.* if the kernel $\tilde{K}(\lambda_n, s)$ can be put equal to $\phi_n(s)$, then the expression (7) is simplified into:

$$\begin{aligned} f(t) &= \sum_n \left(\frac{1}{\gamma_n} \cdot \int_\alpha^\beta F(s)\tilde{K}(\lambda_n, s)ds\right) \cdot \int_\alpha^\beta \tilde{K}(t,s)\tilde{K}(\lambda_n, s)ds \\ &= \sum_n \frac{1}{\gamma_n} f(\lambda_n) \cdot \int_\alpha^\beta \tilde{K}(t,s)\tilde{K}(\lambda_n, s)ds, \end{aligned} \tag{9}$$

by means of (2). The expression (9) gives a generalized sampling theorem. The points at the variable t:

$$t = \lambda_n, \qquad (n = \text{integers}) \tag{10}$$

are called *sampling points*, and the function of t:

$$\mathscr{L}_t^{-1} \cdot \phi_n = \int_\alpha^\beta \tilde{K}(t,s)\phi_n(s)ds = \int_\alpha^\beta \tilde{K}(t,s)\tilde{K}(\lambda_n, s)ds, \qquad (n = \text{integers}) \tag{11}$$

is the *sampling function.*

The idea of the generalized sampling theorem was suggested by one of the present authors, Takizawa[4]. The importance of the expression (7) and (9) in the physical analysis is made clear in the following examples.

Example 1

We shall take the Fourier transform for (1) and (2). Let $f(t) \in L_2$, and let

$$F(s) = \mathscr{L}_s \cdot f = \int_{-\infty}^{+\infty} K(s,t)f(t)dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \exp[ist] \cdot f(t)dt, \tag{12}$$

$$f(t) = \mathscr{L}_t^{-1} \cdot F = \int_{-\infty}^{+\infty} \tilde{K}(t,s)\, F(s)ds = \int_{-\infty}^{+\infty} \exp[-its] \cdot F(s)ds, \tag{13}$$

$$\{\phi_n(s)\} = \{\exp[-i\lambda_n s];\ \lambda_n = n\pi/\beta,\ n = \text{integers}\}, \tag{14}$$

and

$$\alpha = -\beta, \tag{15}$$

then we obtain, from (13),

$$f(t) = \int_{-\infty}^{+\infty} \exp[-its] \cdot F(s)ds = \int_{-\beta}^{\beta} \exp[-its] \cdot F(s)ds. \tag{16}$$

We expand $F(s)$ in $\{\phi_n(s)\}$ in the interval $[-\beta, \beta]$, *i.e.*

$$F(s) = \sum_{n=-\infty}^{+\infty} a_n\phi_n(s) = \sum_{n=-\infty}^{+\infty} a_n \cdot \exp\left[-i\frac{n\pi}{\beta}s\right], \tag{17}$$

with

$$\gamma_n = 2\beta, \qquad (n = \text{integers}) \tag{18}$$

and

$$a_n = \frac{1}{2\beta}\int_{-\beta}^{\beta} F(s)\cdot\exp\left[+i\frac{n\pi}{\beta}s\right]ds. \tag{19}$$

In this case, the expression (7) becomes to:

$$\begin{aligned} f(t) = \mathscr{L}_t^{-1}\cdot F &= \frac{1}{2\beta}\sum_n\left(\int_{-\beta}^{\beta}F(s)\cdot\exp\left[+i\frac{n\pi}{\beta}s\right]ds\right)\cdot\int_{-\beta}^{\beta}\exp[-its]\cdot\exp\left[-i\frac{n\pi}{\beta}s\right]ds \\ &= \sum_{n=-\infty}^{+\infty} f\left(-\frac{n\pi}{\beta}\right)\cdot\frac{\sin(\beta t+n\pi)}{\beta t+n\pi} \\ &= \sum_{n=-\infty}^{+\infty} f\left(\frac{n\pi}{\beta}\right)\cdot\frac{\sin(\beta t-n\pi)}{\beta t-n\pi}, \end{aligned} \tag{20}$$

which is nothing but Shannon's sampling theorem[1)2)3)]. The expression (20) gives $f(t)$ in terms of the sampling function $\sin(\beta t-n\pi)/(\beta t-n\pi)$, with values of $f(n\pi/\beta)$ at sampling points $t=n\pi/\beta$ (n=integers).

Example 2

We shall take the Fourier cosine transforms for (1) and (2), *i.e.*

$$F(s) = \mathscr{L}_s\cdot f = \int K(s,t)f(t)dt = \frac{2}{\pi}\int_0^{+\infty}\cos(st)f(t)dt, \tag{21}$$

and

$$f(t) = \mathscr{L}_t^{-1}\cdot F = \int \tilde{K}(t,s)F(s)ds = \int_0^{+\infty}\cos(ts)F(s)ds. \tag{22}$$

The function $F(s)$ is assumed to be expanded in an orthogonal cosine series in the interval $0\leq s\leq\beta$, *i.e.*

$$F(s) = \sum_n a_n\phi_n(s) = \sum_n a_n\cos(\lambda_n s), \qquad \text{for } 0\leq s\leq\beta \tag{23}$$

with

$$\{\phi_n(s)\} = \{\cos(\lambda_n s)\ ;\ \int_0^{\beta}\cos(\lambda_m s)\cos(\lambda_n s)\,ds = \gamma_m\cdot\delta_{m,n} \quad (n = \text{integers})\}, \tag{24}$$

and

$$\gamma_n = \frac{\beta}{2}\left\{1+\frac{\sin(2\lambda_n\beta)}{2\lambda_n\beta}\right\}, \tag{25}$$

where λ_n are the roots (arranged in ascending order of magnitude) of the equation:

$$\lambda_n\tan(\lambda_n\beta) = A, \tag{26}$$

with a constant A

Then the expression (9) takes form:

$$f(t) = \frac{2}{\beta}\sum_n \frac{1}{1+\dfrac{\sin(2\lambda_n\beta)}{2\lambda_n\beta}}\left(\int_0^\beta F(s)\cos(\lambda_n s)ds\right)\cdot\int_0^\beta \cos(ts)\cos(\lambda_n s)ds$$

$$= \frac{2}{\beta}\sum_n f(\lambda_n)\frac{\cos(\lambda_n\beta)}{1+\dfrac{\sin(2\lambda_n\beta)}{2\lambda_n\beta}}\cdot\frac{t\tan(\beta t)-A}{t^2-\lambda_n^2}\cos(\beta t), \tag{27}$$

with sampling points $t=\lambda_n$ (n=integers), and the sampling function:

$$\frac{t\tan(\beta t)-A}{t^2-\lambda_n^2}\cos(\beta t). \tag{28}$$

Example 3

We shall take the Fourier sine transforms for (1) and (2), *i.e.*

$$F(s) = \mathscr{L}_s\cdot f = \int K(s,t)f(t)dt = \frac{2}{\pi}\int_0^{+\infty}\sin(st)f(t)dt, \tag{29}$$

and

$$f(t) = \mathscr{L}_t^{-1}\cdot F = \int \tilde{K}(t,s)F(s)ds = \int_0^{+\infty}\sin(ts)F(s)ds. \tag{30}$$

The function $F(s)$ is assumed to be expanded in an orthogonal sine series in the interval $0\leq s\leq\beta$, *i.e.*

$$F(s) = \sum_n a_n\phi_n(s) = \sum_n a_n\sin(\lambda_n s), \qquad \text{for } 0\leq s\leq\beta \tag{31}$$

with

$$\{\phi_n(s)\} = \left\{\sin(\lambda_n s)\,;\int_0^\beta \cos(\lambda_m s)\cos(\lambda_n s)ds = \gamma_m\cdot\delta_{m,n}\quad (n=\text{integers})\right\}, \tag{32}$$

and

$$\gamma_n = \frac{\beta}{2}\left\{1-\frac{\sin(2\lambda_n\beta)}{2\lambda_n\beta}\right\}, \tag{33}$$

where λ_n are the roots (arranged in ascending order of magnitude) of the equation:

$$\lambda_n\cot(\lambda_n\beta) = B, \tag{34}$$

with a constant B.

Then the expression (9) becomes to:

$$f(t) = \frac{2}{\beta}\sum_n \frac{1}{1-\dfrac{\sin(2\lambda_n\beta)}{2\lambda_n\beta}}\left(\int_0^\beta F(s)\sin(\lambda_n s)ds\right)\cdot\int_0^\beta \sin(ts)\sin(\lambda_n s)ds$$

$$= -\frac{2}{\beta}\sum_n f(\lambda_n)\frac{\sin(\lambda_n\beta)}{1-\dfrac{\sin(2\lambda_n\beta)}{2\lambda_n\beta}}\cdot\frac{t\cot(\beta t)-B}{t^2-\lambda_n^2}\sin(\beta t), \tag{35}$$

with sampling points $t=\lambda_n$ (n=integers), and the sampling function:

$$\frac{t\cot(\beta t)-B}{t^2-\lambda_n^2}\sin(\beta t). \tag{36}$$

The formulae (27) and (35) were obtained by Kroll[9] in connection with an integral equation.

Example 4

Let us take the Hankel transforms of order ν ($\nu\geq-1/2$) for (1) and (2), with

$$\int_0^{+\infty}\sqrt{t}\,f(t)dt<+\infty,$$

absolutely convergent, *i.e.*

$$F(s)=\int_0^{+\infty}K(s,t)f(t)dt=\int_0^{+\infty}tJ_\nu(st)f(t)dt, \tag{37}$$

and

$$f(t)=\int_0^{+\infty}\widetilde{K}(t,s)\,F(s)ds=\int_0^{+\infty}sJ_\nu(ts)F(s)ds. \tag{38}$$

The function $F(s)$ is expanded in the Fourier-Bessel series[5] in the interval $0\leq s\leq\beta$:

$$F(s)=\sum_n a_n\phi_n(s)=\sum_{n=1}^{+\infty}a_nJ_\nu(j_ns),\qquad \text{for } 0\leq s\leq\beta \tag{39}$$

with

$$\{\phi_n(s)\}=\left\{J_\nu(j_ns)\ ;\ \int_0^\beta sJ_\nu(j_ms)J_\nu(j_ns)ds=\gamma_m\cdot\delta_{m,n}\right\}, \tag{40}$$

and

$$\gamma_n=\frac{\beta^2}{2}J_{\nu+1}^2(j_n\beta), \tag{41}$$

where $j_n\beta$ ($n=1,2,3,\ldots$) are the positive zeros of $J_\nu(z)$, being arranged in ascending order of magnitude, *i.e.*

$$J_\nu(j_n\beta)=0. \tag{42}$$

The expression (9) takes form:

$$\begin{aligned} f(t)&=\frac{2}{\beta^2}\sum_{n=1}^{+\infty}\frac{1}{J_{\nu+1}^2(j_n\beta)}\left(\int_0^\beta sF(s)J_\nu(j_ns)ds\right)\cdot\int_0^\beta sJ_\nu(ts)J_\nu(j_ns)ds\\ &=-\frac{2}{\beta}\sum_{n=1}^{+\infty}f(j_n)\frac{j_n}{J_{\nu+1}(j_n\beta)}\cdot\frac{J_\nu(\beta t)}{t^2-j_n^2}, \end{aligned} \tag{43}$$

with sampling points $t=j_n$ (n=integers), and the sampling function:

$$\frac{J_\nu(\beta t)}{t^2 - j_n^2}. \tag{44}$$

The formula (43) was previously given by the present authors[8].

Example 5

We shall take the Hankel transforms (37) and (38). The function $F(s)$ is assumed to be expanded in the Deni expansion[6] in the interval $0 \leq s \leq \beta$, *i.e.*

$$F(s) = \sum_n a_n \phi_n(s) = \sum_{n=1}^{+\infty} a_n J_\nu(\lambda_n s), \qquad \text{for } 0 \leq s \leq \beta \tag{45}$$

with

$$\{\phi_n(s)\} = \left\{ J_\nu(\lambda_n s)\,;\, \int_0^\beta s J_\nu(\lambda_m s) J_\nu(\lambda_n s)\, ds = \gamma_m \cdot \delta_{m,n} \right\}, \tag{46}$$

and

$$\gamma_n = \frac{1}{2\lambda_n^2}\{(\lambda_n^2 + h^2)\beta^2 - \nu^2\} J_\nu(\lambda_n \beta), \tag{47}$$

where λ_n ($n=1, 2, 3, \ldots$) are the positive roots (arranged in ascending order of magnitude) of the following equation:

$$\lambda_n J'_\nu(\lambda_n \beta) + h J_\nu(\lambda_n \beta) = 0, \tag{48}$$

with a constant h.

Then the expression (9) becomes to:

$$\begin{aligned} f(t) &= 2\sum_{n=1}^{+\infty} \frac{\lambda_n^2}{\{(\lambda_n^2 + h^2)\beta^2 - \nu^2\} J_n^2(\lambda_n \beta)} \left(\int_0^\beta s F(s) J_\nu(\lambda_n s)\, ds \right) \cdot \int_0^\beta s J_\nu(ts) J_\nu(\lambda_n s)\, ds \\ &= -2\beta \sum_{n=1}^{+\infty} f(\lambda_n) \frac{\lambda_n^2}{\{(\lambda_n^2 + h^2)\beta^2 - \nu^2\} J_\nu(\lambda_n \beta)} \cdot \frac{t J'_\nu(\beta t) + h J_\nu(\beta t)}{t^2 - \lambda_n^2}, \end{aligned} \tag{49}$$

with sampling points $t = \lambda_n$ (n=integers), and the sampling function:

$$\frac{t J'_\nu(\beta t) + h J_\nu(\beta t)}{t^2 - \lambda_n^2}. \tag{50}$$

The formula (49) was also obtained by Kroll[9].

Letting h tend to infinity in the expressions (48) and (49), we see that they reduce to the following expressions, respectively:

$$J_\nu(\lambda_n \beta) = 0, \tag{51}$$

and

$$f(t) = -\frac{2}{\beta} \sum_{n=1}^{+\infty} f(\lambda_n) \frac{\lambda_n}{J_{\nu+1}(\lambda_n \beta)} \cdot \frac{J_\nu(\beta t)}{t^2 - \lambda_n^2}. \tag{52}$$

The expressions (51) and (52) are nothing but the expressions (42) and (43), respectively. Accordingly, the expression (43) is a limiting case of (49).

Example 6

We shall take the other Hankel transforms[7] for (1) and (2), namely

$$F(s)=\int K(s,t)f(t)dt=\int_{\alpha}^{\beta} tT_{\nu}(t\alpha, st)f(t)dt, \tag{53}$$

and

$$B_{\nu}(t\alpha)\cdot f(t)=\int \widetilde{K}(t,s)F(s)ds=\int_{\alpha}^{+\infty} sT_{\nu}(t\alpha, ts)F(s)ds, \qquad \text{for } \nu\geq -1/2 \tag{54}$$

with

$$T_{\mu}(x,z)=Y_{\nu}(x)J_{\mu}(z)-J_{\nu}(x)Y_{\mu}(z), \tag{55}$$

and

$$B_{\nu}(z)=J_{\nu}^{2}(z)+Y_{\nu}^{2}(z), \tag{56}$$

under the condition that the integral

$$\int_{\alpha}^{+\infty} tf(t)dt<+\infty,$$

is absolutely convergent.

We expand the function $F(s)$ in the orthogonal series $\{T_{\nu}(\lambda_n\alpha, \lambda_n s)\}$ in the interval $0<\alpha\leq s\leq\beta$, *i.e.*

$$F(s)=\sum_{n=1}^{+\infty} a_n\phi_n(s)=\sum_{n=1}^{+\infty} a_n T_{\nu}(\lambda_n\alpha, \lambda_n s), \qquad \text{for } 0<\alpha\leq s\leq\beta \tag{57}$$

and

$$\{\phi_n(s)\}=\left\{T_{\nu}(\lambda_n\alpha, \lambda_n s)\ ;\ \int_{\alpha}^{\beta} sT_{\nu}(\lambda_m\alpha, \lambda_m s)T_{\nu}(\lambda_n\alpha, \lambda_n s)ds=\gamma_m\cdot\delta_{m,n}\right\}, \tag{58}$$

with

$$\begin{aligned}\gamma_n&=\frac{\beta^2}{2}T_{\nu+1}^{2}(\lambda_n\alpha, \lambda_n\beta)-\frac{\alpha^2}{2}T_{\nu+1}^{2}(\lambda_n\alpha, \lambda_n\alpha)\\&=\frac{2}{\pi^2\lambda_n^2}\left[\frac{J_{\nu}^{2}(\lambda_n\alpha)}{J_{\nu}^{2}(\lambda_n\beta)}-1\right],\end{aligned} \tag{59}$$

where λ_n are the positive roots of the equation:

$$T_{\nu}(\lambda_n\alpha, \lambda_n\beta)=0, \tag{60}$$

i.e.

$$Y_{\nu}(\lambda_n\alpha)J_{\nu}(\lambda_n\beta)=J_{\nu}(\lambda_n\alpha)Y_{\nu}(\lambda_n\beta). \tag{61}$$

Accordingly, we obtain, from (9), the following expression:

$$
\begin{aligned}
B_\nu(t\alpha)\cdot f(t) &= \sum_{n=1}^{+\infty}\left(\frac{1}{\gamma_n}\int_\alpha^\beta sT_\nu(\lambda_n\alpha,\ \lambda_n s)F(s)ds\right)\cdot\int_\alpha^\beta sT_\nu(t\alpha,\ ts)T_\nu(\lambda_n\alpha,\ \lambda_n s)ds \\
&= \sum_{n=1}^{+\infty}\frac{\pi^2\lambda_n^2}{2}\cdot\frac{J_\nu^2(\lambda_n\beta)}{J_\nu^2(\lambda_n\alpha)-J_\nu^2(\lambda_n\beta)}\cdot B_\nu(\lambda_n\alpha)f(\lambda_n)\cdot\beta\frac{d}{d\beta}T_\nu(\lambda_n\alpha,\ \lambda_n\beta)\cdot\frac{T_\nu(\alpha t,\ \beta t)}{t^2-\lambda_n^2} \\
&= -\pi\sum_{n=1}^{+\infty}f(\lambda_n)\frac{\lambda_n^2B_\nu(\lambda_n\alpha)J_\nu(\lambda_n\alpha)J_\nu(\lambda_n\beta)}{J_\nu^2(\lambda_n\alpha)-J_\nu^2(\lambda_n\beta)}\cdot\frac{T_\nu(\alpha t,\ \beta t)}{t^2-\lambda_n^2},
\end{aligned}
\tag{62}
$$

with sampling points $t=\lambda_n$ $(n=1, 2, 3, \ldots)$, and the sampling function:

$$
\frac{T_\nu(\alpha t,\ \beta t)}{t^2-\lambda_n^2}. \tag{63}
$$

The expressions (27), (35), (43), (49), and (62), are examples of the new sampling formulae derived from our generalized sampling theorem (9).

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