RESEARCH REPORTS

ON THE OSCILLATIONS OF "SUMMED AND DIFFERENTIAL TYPES" UNDER THE PARAMETRIC EXCITATION

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In a vibratory system having multiple degree-of-freedom and under parametric excitation of frequency ω, oscillations of "summed and differential types" with frequencies $\omega_i(\doteqdot p_i)$, $\omega_j(\doteqdot p_j)$ take place, when ω becomes nearly equal to sum of and difference in two natural frequencies p_i, p_j, *i.e.*, $\omega=p_i \pm p_j=p_{ij}$, in which p_{ij} is the resonant frequency.

In the present paper, the characteristics of these oscillations are studied theoretically and experimentally, and it is seen that solutions of first approximation obtained through rather simple analysis grasp sufficiently these oscillatory phenomena. Futhermore, it is cleared up that unstable oscillations can occur only in summed type and not in differential type.

1. Introduction

In a vibratory system having multiple degree-of-freedom and under parametric excitation, oscillations of the so-called "Summed and Differential Types" occur with ordinary unstable oscillations appearing at $\omega \doteqdot 2p_r$. These oscillations consist of two oscillations having frequencies ω_i and ω_j which satisfy the following relations:

$$\omega_i \pm \omega_j=\omega, \qquad \omega_i \doteqdot p_i, \qquad \omega_j \doteqdot p_j, \tag{1}$$

where ω is the frequency of parametric excitation and p_i, p_j ($i \neq j$; $p_i > p_j$; $i, j= 1, 2, \ldots, k$; k=number of degrees of freedom) are two natural frequencies of the system. Since Eq. (1) results in a relation $\omega \doteqdot p_i \pm p_j$, it is seen that the oscillations of summed and differential types take place when a frequency ω of parametric excitation becomes nearly equal to the resonant frequency

$$p_{ij}=p_i \pm p_j. \tag{2}$$

Some studies only on the possibility of occurrence of these kinds of unstable vibrations have been made[1], and it seems that both detailed proposition of solutions and discussions of characteristics of these kinds of oscillations have not been carried out. In the present paper, the properties of oscillations, *i.e.*, the frequencies, the phase angles, the amplitude ratios between two oscillations, the negative damping coefficients, the location and width of unstable region in which two unstable oscillations of frequencies ω_i and ω_j build up, and the relation between initial conditions and solutions of the oscillations are studied in detail. Results obtained by a first approximate analysis are compared with experimental results of double pendulums and of analog computer.

2. Equation of motion and preliminary analysis

In the present paper, the vibratory system without damping is treated, and the theoretical analysis of the first approximation in which higher powers of small quantities are neglected, is carried out.

The vibratory system of k degree-of-freedom without damping and under parametric excitation of frequency ω is governed by the following equation of motions:

$$\sum_{s=1}^{k}\{a_{rs}\ddot{x}_s + \alpha_{rs}x_s\} = \varepsilon_r' x_r \cos \omega t. \qquad (r=1, 2, \ldots, k) \tag{3}$$

In a dynamic system, the left side of Eq. (3) are terms of inertia and spring force, respectively, and the right side represents parametric excitation. In Eq. (3) magnitude of parametric excitation ε_r' is assumed as small quantity. The frequency equation of the system is

$$A = |(\alpha_{rs} - a_{rs}p^2)| = 0, \tag{4}$$

where p is the natural frequency. Let cofactor of A when $p=p_m$ be $A_{m,rs}$, and putting

$$d_{rs} = \frac{A_{s,rs}}{\sqrt{\sum_{l,m}^{k} a_{lm} A_{s,ls} A_{s,ms}}}, \tag{5}$$

transformation from generalized coordinate x_s into normal coordinate X_s is performed by

$$x_r = \sum_{s=1}^{k} d_{rs} X_s, \qquad (r=1, 2, \ldots, k). \tag{6}$$

Substitution of Eq. (6) into Eq. (3) attains the following equation of motion expressed by normal coordinate:

$$\ddot{X}_r + p_r^2 X_r = \sum_{s=1}^{k} \varepsilon_{rs} X_s \cos \omega t, \tag{7}$$

where

$$\varepsilon_{rs} = \sum_{l=1}^{k} d_{lr} d_{ls} \varepsilon_l' = \varepsilon_{sr}. \tag{8}$$

The i th and j th equations of Eq. (7) can be rewritten as

$$\left.\begin{aligned}\ddot{X}_i + \omega_i^2 X_i &= (\omega_i^2 - p_i^2) X_i + \sum_{s=1}^{k} \varepsilon_{is} X_s \cos \omega t = f_i,\\ \ddot{X}_j + \omega_j^2 X_j &= (\omega_j^2 - p_j^2) X_j + \sum_{s=1}^{k} \varepsilon_{js} X_s \cos \omega t = f_j.\end{aligned}\right\} \tag{9}$$

When the frequencies ω_i, ω_j satisfy the relations of Eq. (1), all terms in the right side of Eq. (9), *i.e.*, f_i and f_j become small quantities, and hence approximate analysis can be carried out. In this paper, a first approximate analysis through the similar procedure to Kryloff and Bogoliuboff's method[2] is employed. Accordingly the first step consists in taking for $X_{i,j}$ and $\dot{X}_{i,j}$ in the following form respectively:

$$X_{i,j} = a_{i,j} \sin(\omega_{i,j} t + \varphi_{i,j}), \quad \dot{X}_{i,j} = a_{i,j}\omega_{i,j} \cos(\omega_{i,j} t + \varphi_{i,j}), \tag{10}$$

and it is assumed to consider amplitudes $a_{i,j}$ and phase angles $\varphi_{i,j}$ in the above equations as functions of time t. Differentiation of the first equation of Eq. (10) results in

$$\dot{X}_{i,j} = \dot{a}_{i,j} \sin(\omega_{i,j} t + \varphi_{i,j}) + a_{i,j}(\omega_{i,j} + \dot{\varphi}_{i,j}) \cos(\omega_{i,j} t + \varphi_{i,j}),$$

and substitution of $\dot{X}_{i,j}$ in Eq. (10) into the above equation leads to

$$\dot{a}_{i,j} \sin(\omega_{i,j} t + \varphi_{i,j}) + a_{i,j}\dot{\varphi}_{i,j} \cos(\omega_{i,j} t + \varphi_{i,j}) = 0. \tag{11}$$

Differentiating the second equation of Eq. (10) and inserting it into Eq. (9), we get

$$\dot{a}_{i,j}\omega_{i,j} \cos(\omega_{i,j} t + \varphi_{i,j}) - a_{i,j}\omega_{i,j}\dot{\varphi}_{i,j} \sin(\omega_{i,j} t + \varphi_{i,j}) = f_{i,j}. \tag{12}$$

Through Eqs. (11), (12), the following differential equations are derived

$$\dot{a}_{i,j} = \frac{f_{i,j}}{\omega_{i,j}} \cos(\omega_{i,j} t + \varphi_{i,j}), \quad \dot{\varphi}_{i,j} = -\frac{f_{i,j}}{a_{i,j}\omega_{i,j}} \sin(\omega_{i,j} t + \varphi_{i,j}). \tag{13}$$

Applying an approximate procedure of "average method" to Eq. (13), that is, substituting Eqs. (9), (10) into $f_{i,j}$ of the right side of Eq. (13) and eliminating all terms except for constant terms, Eq. (13) is rewritten as follows:

$$\left.\begin{aligned}\dot{a}_{i,j} &= \frac{\varepsilon_{ij} a_{j,i}}{4\,\omega_{i,j}} \sin(\varphi_{j,i} \pm \varphi_{i,j}),\\ \dot{\varphi}_{i,j} &= -\frac{(\omega_{i,j}^2 - p_{i,j}^2)}{2\,\omega_{i,j}} \pm \frac{\varepsilon_{ij} a_{j,i}}{4\,\omega_{i,j} a_{i,j}} \cos(\varphi_{j,i} \pm \varphi_{i,j}).\end{aligned}\right\} \tag{14}$$

Since a reciplocal relation $\varepsilon_{ij} = \varepsilon_{ji}$ holds as shown in Eq. (8), a common coefficient ε_{ij} is used for both equations with suffixes i and j in Eq. (14). Assuming that solutions of oscillations are normal solutions, we should put

$$\dot{\varphi}_i = \dot{\varphi}_j = 0. \tag{15}$$

Consequently Eq. (14) is reduced to

$$\dot{a}_i = \pm \frac{\varepsilon_{ij} a_j}{4\,\omega_i} \sin \varphi_{ij}, \qquad \dot{a}_j = \frac{\varepsilon_{ij} a_i}{4\,\omega_j} \sin \varphi_{ij}, \tag{16}$$

$$2\,\Delta_i = \pm\,\varepsilon_{ij}\frac{a_j}{a_i}\cos\varphi_{ij}, \quad 2\,\Delta_j = \pm\,\varepsilon_{ij}\frac{a_i}{a_j}\cos\varphi_{ij}, \tag{17}$$

in which

$$\varphi_{ij} = \varphi_i \pm \varphi_j, \qquad \Delta_{i,j} = \omega_{i,j}^2 - p_{i,j}^2. \tag{18}$$

In Eqs. (1), (2), (14), (16), (17) and (18), the upper and lower signs of $\pm$ are adopted for oscillations in summed and differential types severally. It is seemed that only a sum of or difference in two phase angles φ_i, φ_j, *i.e.*, only φ_{ij} may have physical meaning. In Eq. (17), the detuning $\Delta_{i,j}$ are small quantities because of Eq. (1).

3. Oscillations of summed type

Referring the following relations given from Eq. (17):

$$\cos^2\varphi_{ij} = \frac{4\,\Delta_i\Delta_j}{\varepsilon_{ij}^2}, \tag{19}$$

$$\frac{\varepsilon_{ij}^2}{16\,\omega_i\omega_j}\sin^2\varphi_{ij} = \frac{\varepsilon_{ij}^2}{16\,\omega_i\omega_j}(1-\cos^2\varphi_{ij}) = \frac{\varepsilon_{ij}^2 - 4\,\Delta_i\Delta_j}{16\,\omega_i\omega_j} = \mu^2, \tag{20}$$

and adopting the upper sign of $\pm$ in Eq. (16), we have

$$\ddot{a}_i = \mu^2 a_i, \qquad \ddot{a}_j = \mu^2 a_j, \tag{21}$$

which leads to

$$a_{i,j} = A_{i,j}e^{\pm\mu t}. \tag{22}$$

Inserting Eq. (22) into Eq. (16), amplitude ratio for two oscillations of frequencies ω_i and ω_j is given as follows:

$$a_i/a_j = A_i/A_j = \sqrt{\omega_j/\omega_i}\,. \tag{23}$$

Although we have $a_i/a_j = A_i/A_j = \pm\sqrt{\omega_j/\omega_i}$, the upper sign $+$ is employed here as shown in the above equation, because difference between both signs can be canceled by difference of $\pm\pi$ in φ_{ij}. Substituting Eq. (23) into Eq. (17) in which the upper sign of $\pm$ is adopted, we have

$$\frac{\Delta_i}{\Delta_j} = \frac{\omega_i^2 - p_i^2}{\omega_j^2 - p_j^2} = \frac{\omega_i}{\omega_j}, \tag{24}$$

and from Eq. (24) and the relation of $\omega = \omega_i + \omega_j$, a cubic equation determining frequencies ω_i and ω_j is given as follows:

$$2\,\omega_{i,j}^3 - 3\,\omega\omega_{i,j}^2 + (\omega^2 - p_i^2 - p_j^2)\,\omega_{i,j} + p_{i,j}^2\,\omega = 0. \tag{25}$$

So far as a first approximation, through Eqs. (1), (18), (24) we obtain the following approximate equations by which ω_i and ω_j can be determined more easily than by Eq. (25)

$$\omega_{i,j} = p_{i,j} + \frac{\nabla_{i,j}}{2}, \tag{26}$$

in which

$$\nabla_{i,j} = \omega - (p_i + p_j) = \omega - p_{ij}. \tag{27}$$

Similar process with the above yields approximate values of the detunings $\Delta_{i,j}$, the phase angle φ_{ij} and the negative damping coefficient μ, as follows:

$$\Delta_{i,j} = p_{i,j}\nabla_{ij}, \tag{28}$$

$$\cos\varphi_{ij} = \frac{\nabla_{ij}}{E_{ij}}, \tag{29}$$

$$\mu = \frac{1}{2}\sqrt{E_{ij}^2 - \nabla_{ij}^2}, \tag{30}$$

where

$$E_{ij} = \frac{\varepsilon_{ij}}{2\sqrt{p_i p_j}}. \tag{31}$$

3. 1 Oscillations of summed type within unstable region

When the frequency ω of parametric excitation comes near to the resonant frequency p_{ij} where $\omega_i = p_i$, $\omega_j = p_j$, and detunings $\Delta_{i,j}$ and ∇_{ij} become so small that the relation

$$\varepsilon_{ij}^2 \geqq 4\,\Delta_i\,\Delta_j \geqq 0, \qquad i.e.,\ E_{ij}^2 \geqq \nabla_{ij}^2 \tag{32}$$

holds, conditions of $\cos^2\varphi_{ij} \leqq 1$, $1 \geqq \sin^2\varphi_{ij} \geqq 0$ and $\mu^2 \geqq 0$ are satisfied as seen from Eqs. (19), (20) or Eqs. (29), (30) and both φ_{ij} and μ become real numbers. Denoting φ_{ij} with a value between 0 and π as ψ_{ij}, we have from Eq. (29)

$$\cos\psi_{ij} = \frac{\nabla_{ij}}{|E_{ij}|}. \tag{33}$$

and it follows that when $E_{ij}>0$, $+\psi_{ij}$ and $-\psi_{ij}$ are used for $+\mu t$ and $-\mu t$ in Eq. (22) separately and vice versa when $E_{ij}<0$. Through Eqs. (10), (18), (22), (23) and (33), solutions for X_i, X_j are given as follows:

$$\left.\begin{aligned} X_i &= Ae^{\mu t}\sin(\omega_i t + \varphi_i) + Be^{-\mu t}\sin(\omega_i t + \varphi_i'), \\ X_j &= \sqrt{\frac{\omega_i}{\omega_j}}\{Ae^{\mu t}\sin(\omega_j t \pm \psi_{ij} - \varphi_i) + Be^{-\mu t}\sin(\omega_j t \mp \psi_{ij} - \varphi_i')\}, \end{aligned}\right\} \tag{34}$$

in which the upper and lower signs correspond to conditions $E_{ij}>0$ and $E_{ij}<0$, respectively, and A, B, φ_i, φ_i' are all arbitrary constants determined by initial conditions. Once ω or ∇_{ij} is given, frequencies ω_i and ω_j, negative damping coefficient μ and phase angle ψ_{ij} are determined by Eq. (26), Eq. (30) and Eq. (33) severally, thus the solutions are settled. It should be noticed that both unstable oscillations of frequencies ω_i and ω_j have a common negative damping coefficient μ.

By adoption of equal sign in Eq. (32) or putting $\mu=0$ in Eq. (30), critical frequencies ω_{c1}, ω_{c2} or critical detunings ∇_{c1}, ∇_{c2} which decide boundaries between stable and unstable regions are derived as follows:

$$\omega_{c1} = p_{ij} + |E_{ij}|,\ \omega_{c2} = p_{ij} - |E_{ij}|\ :\ \nabla_{c1} = |E_{ij}|,\ \nabla_{c2} = -|E_{ij}|, \tag{35}$$

in which ω_{c1} and ∇_{c1} give an upper boundary and ω_{c2} and ∇_{c2} a lower boundary. As seen in Eq. (33), when $E_{ij}>0$ phase angle ψ_{ij} takes values π, $\pi/2$ and 0 at $\omega=\omega_{c2}$, p_{ij}, ω_{c1} respectively, and when $E_{ij}<0$, 0, $\pi/2$, π, it follows that by means of the Routh's method, the stability of the oscillations on these boundaries can be ensured.

It is shown from the above discussion that when the frequency ω of parametric excitation comes near the resonant frequency $p_{ij}\,(=p_i+p_j)$ and takes any value between critical frequencies ω_{c1} and ω_{c2}, two oscillations of frequencies $\omega_i\,(\fallingdotseq p_i)$ and $\omega_j\,(\fallingdotseq p_j)$ build up simultaneously and thus the unstable oscillations of summed type take place. In the system with k degree-of-freedom, there are $k(k-1)/2$ unstable regions of unstable oscillations of summed type.

Incidentally, analysis for ordinary unstable oscillations appearing in the neighborhood of $\omega=2p_r$ $(r=1, 2, \ldots, k)$ is discussed here. Through a similar procedure with above mentioned, we obtain the following equations of motion in place of Eqs. (16), (17):

$$\dot{a}_r = \frac{\varepsilon_{rr} a_r}{2\omega}\sin 2\varphi_r, \quad \omega^2 - 4p_r^2 = 2\varepsilon_{rr}\cos 2\varphi_r. \tag{36}$$

From the above equations, the solution X_r, negative damping coefficient μ_r, phase angle ψ_r and critical frequencies ω_{c1}, ω_{c2} of these kinds of unstable oscillations are given respectively as follows:

$$\left.\begin{aligned}
&X_r = Ae^{\mu_r t}\sin\left(\frac{1}{2}\omega t \pm \psi_r\right) + Be^{-\mu_r t}\sin\left(\frac{1}{2}\omega t \mp \psi_r\right),\\
&\quad \text{(the upper and lower signs of } \pm \text{ correspond to}\\
&\quad \varepsilon_{rr}>0 \text{ and } \varepsilon_{rr}<0 \text{ separately.)}\\
&\mu_r = \frac{1}{2}\sqrt{\frac{\varepsilon_{rr}^2}{4p_r^2} - \nabla_r^2}\cdot \quad (\nabla_r = \omega - 2p_r),\\
&\cos 2\psi_r = \frac{2p_r\nabla_r}{\varepsilon_{rr}}, \quad \left(\frac{\pi}{2} \geqq \psi_r \geqq 0\right).\\
&\omega_{c1} = 2p_r + \frac{|\varepsilon_{rr}|}{2p_r}, \quad \omega_{c2} = 2p_r - \frac{|\varepsilon_{rr}|}{2p_r}.
\end{aligned}\right\} \tag{37}$$

Results of Eq. (37) coincide with those of one degree-of-freedom system.

Furthermore, it can be concluded that unstable oscillations of summed type of higher order in the neighborhood of $\omega=(p_i+p_j)/s$ (s: positive integer) as well as ordinary unstable oscillations of higher order do not appear, in so far as a first approximation.

3. 2. Oscillations in stable region

When ω goes far off from p_{ij} and passes ω_{c1}, ω_{c2}, *i.e.*, the relations

$$\varepsilon_{ij}^2 < 4\,\Delta_i\,\Delta_j\ :\ E_{ij}^2 < \nabla_{ij}^2 \tag{32 a}$$

are to be held, φ_{ij} and μ can now be not real because of $\cos^2\varphi_{ij}>1$, $\sin^2\varphi_{ij}<0$

and $\mu^2<0$, as is seen through Eqs. (19), (20). By putting

$$\nu^2 = -\mu^2 = \frac{4\,\Delta_i\,\Delta_j - \varepsilon_{ij}^2}{16\,\omega_i\,\omega_j} \tag{20 a}$$

and expressing approximately ν as

$$\nu = \frac{1}{2}\sqrt{\nabla_{ij}^2 - E_{ij}^2}\,, \tag{30 a}$$

Eq. (22) is reduced to

$$a_{i,j} = A_{i,j}e^{\pm i\nu t}, \qquad (i=\sqrt{-1}) \tag{22 a}$$

and from Eqs. (16), (22 a) the phase angle φ_{ij} is shown in the form

$$\sin\varphi_{ij} = \pm\frac{4\,i\nu}{\varepsilon_{ij}}\sqrt{\omega_i\,\omega_j} \fallingdotseq \pm\frac{2\,i\nu}{E_{ij}} \tag{38}$$

where the upper and lower signs correspond to those in Eq. (22 a). Referring Eqs. (34), (22 a) and (38), and rewriting to equations expressed by real numbers, we attain the following equations for solutions X_i, X_j in stable region:

$$\left.\begin{aligned}
X_i &= A\sin(\omega_i+\nu)t + B\cos(\omega_i+\nu)t + C\sin(\omega_i-\nu)t + D\cos(\omega_i-\nu)t,\\
X_j &= A'\sin(\omega_j-\nu)t + B'\cos(\omega_j-\nu)t + C'\sin(\omega_j+\nu)t + D'\cos(\omega_j+\nu)t\\
&= \frac{\sqrt{\omega_i/\omega_j}}{E_{ij}}[(\nabla_{ij}+2\,\nu)\{A\sin(\omega_j-\nu)t - B\cos(\omega_j-\nu)t\}\\
&\qquad + (\nabla_{ij}-2\,\nu)\{C\sin(\omega_j+\nu)t - D\cos(\omega_j+\nu)t\}],
\end{aligned}\right\} \tag{34 a}$$

where A, B, C and D are all arbitrary constants decided by initial conditions. As represented in Eq. (34 a), in the stable region ($\omega>\omega_{c1}$, $\omega<\omega_{c2}$) free oscillations with frequencies $\omega_{i,j}\pm\nu$ take place and there is no unstable oscillation. Observing Eq. (34 a), it is seen that two oscillations of frequencies $\omega_i+\nu$ and $\omega_j-\nu$ as well as $\omega_i-\nu$ and $\omega_j+\nu$ make a pair, and regardless of ν a sum of frequencies in pair is still equal to $\omega=\omega_i+\omega_j$. Further it is noticeable that amplitude ratios between two oscillations making a pair are fixed independently of arbitrary constants, *i.e.*, initial conditions, as shown in Eq. (34 a) or the following equations:

$$\left.\begin{aligned}
\frac{\sqrt{A^2+B^2}}{\sqrt{A'^2+B'^2}} &= \frac{|A|}{|A'|} = \frac{|B|}{|B'|} = \frac{|E_{ij}|}{\sqrt{\omega_i/\omega_j}\,|\nabla_{ij}+2\,\nu|},\\
\frac{\sqrt{C^2+D^2}}{\sqrt{C'^2+D'^2}} &= \frac{|C|}{|C'|} = \frac{|D|}{|D'|} = \frac{|E_{ij}|}{\sqrt{\omega_i/\omega_j}\,|\nabla_{ij}-2\nu|}.
\end{aligned}\right\} \tag{39}$$

Through the similar analysis with the above, we find the solutions for the ordinary stable oscillations in the neighborhood of $\omega=2\,p_r$ as follows:

$$\left.\begin{aligned}
X_r &= A\sin\left(\frac{1}{2}\omega+\nu_r\right)t + B\cos\left(\frac{1}{2}\omega+\nu_r\right)t\\
&\qquad + A\sin\left(\frac{1}{2}\omega-\nu_r\right)t - B\cos\left(\frac{1}{2}\omega-\nu_r\right)t,\\
\nu_r &= \frac{1}{2}\sqrt{\nabla_r^2 - \frac{\varepsilon_{rr}^2}{4\,p_r^2}}.
\end{aligned}\right\} \tag{37 a}$$

4. Oscillations of differential type

Adopting the lower sign in Eqs. (16), (17), we have

$$\ddot{a}_{i,j} = -\mu^2 a_{i,j}, \tag{21 b}$$

$$a_{i,j} = A_{i,j} e^{\pm i\mu t}. \tag{22 b}$$

By a similar procedure to summed type, we get the following relations:

$$a_i/a_j = A_i/A_j = i\sqrt{\omega_j/\omega_i}\,, \tag{23 b}$$

$$\Delta_i/\Delta_j = (\omega_i^2 - p_i^2)/(\omega_j^2 - p_j^2) = -\omega_i/\omega_j, \tag{24 b}$$

$$\left.\begin{aligned} &2\,\omega_i^3 - 3\,\omega\omega_i^2 + (\omega^2 - p_i^2 - p_j^2)\,\omega_i + p_i^2\omega = 0, \\ &2\,\omega_j^3 + 3\,\omega\omega_j^2 + (\omega^2 - p_i^2 - p_j^2)\,\omega_j - p_j^2\omega = 0, \end{aligned}\right\} \tag{25 b}$$

$$\omega_i = p_i + \frac{\nabla_{ij}}{2}, \quad \omega_j = p_j - \frac{\nabla_{ij}}{2}, \tag{26 b}$$

$$\nabla_{ij} = \omega - (p_i - p_j) = \omega - p_{ij}, \tag{27 b}$$

$$\Delta_i = p_i \nabla_{ij}, \quad \Delta_j = -p_j \nabla_{ij}, \tag{28 b}$$

$$\mu = \frac{1}{2}\sqrt{E_{ij}^2 + \nabla_{ij}^2}, \tag{30 b}$$

which correspond to Eqs. (23), (24), (25), (26), (27), (28), (30) of summed type, respectively. Reffering that μ is always a real number as shown in Eq. (30 b) and that φ_{ij} is not real as being of the form

$$\cos\varphi_{ij} = i\nabla_{ij}/E_{ij}, \tag{29 b}$$

the solutions of differential type are written as follows:

$$\left.\begin{aligned} X_i &= A\sin(\omega_i+\mu)t + B\cos(\omega_i+\mu)t + C\sin(\omega_i-\mu)t + D\cos(\omega_i-\mu)t, \\ X_j &= A'\sin(\omega_j+\mu)t + B'\cos(\omega_j+\mu)t + C'\sin(\omega_j-\mu)t + D'\cos(\omega_j-\mu)t \\ &= -\frac{\sqrt{\omega_i/\omega_j}}{E_{ij}}[(\nabla_{ij}+2\mu)\{A\sin(\omega_j+\mu)t + B\cos(\omega_j+\mu)t\} \\ &\qquad + (\nabla_{ij}-2\mu)\{C\sin(\omega_j-\mu)t + D\cos(\omega_j-\mu)t\}]. \end{aligned}\right\} \tag{34 b}$$

Eq. (34 b) being similar to Eq. (34 a) of the stable oscillations of summed type represents free oscillations with frequencies $\omega_{i,j} \pm \mu$, so that there is no unstable oscillation of differential type. In the oscillations of differential type, two oscillations of frequencies $\omega_i+\mu$ and $\omega_j+\mu$ as well as $\omega_i-\mu$ and $\omega_j-\mu$ make a pair, and a difference of frequencies in pair is still equal to $\omega = \omega_i - \omega_j$, and the amplitude ratios are found as follows:

$$\left.\begin{aligned} \frac{\sqrt{A^2+B^2}}{\sqrt{A'^2+B'^2}} &= \frac{|A|}{|A'|} = \frac{|B|}{|B'|} = \frac{|E_{ij}|}{\sqrt{\omega_i/\omega_j}\,|\nabla_{ij} - 2\mu|}, \\ \frac{\sqrt{C^2+D^2}}{\sqrt{C'^2+D'^2}} &= \frac{|C|}{|C'|} = \frac{|D|}{|D'|} = \frac{|E_{ij}|}{\sqrt{\omega_i/\omega_j}\,|\nabla_{ij} + 2\mu|}. \end{aligned}\right\} \tag{39 b}$$

It is concluded that, in so far as a first approximation, no unstable oscillation of differential type of higher order in the neighborhood of $\omega=(p_i-p_j)/s$ (s : positive integer) can appear.

5. Verification of analytical results through experiments and analog computer

In this section, experiments and calculations by analog computer are performed for the oscillations of summed and differential type which take place in an oscillatory system of double pendulums with two degree-of-freedom as shown Fig. 1, where the first and second pendulums of, lengthes l_1, l_2 and mass m_1, m_2, are supported at the points A_1, A_2; I_1, I_2 are moments of inertia about the supporting points A_1, A_2 and b_1, b_2 are distances between A_1 and gravitational center G_1 and between A_2 and G_2, respectively; there are springs having spring constants k_1 and k_2 at the ends of both pendulums. When supporting point A_1 oscillates vertically with amplitude e and frequency ω, the parametric excitation of frequency ω is induced and the system shown in Fig. 1 is governed by Eq. (3). If only number of degrees of freedom k, the suffixes i and j are replaced by 2, 1 and 2 separately, all results obtained up to now can be applied for this system. Various coefficients in Eq. (3) for this system are given as follows:

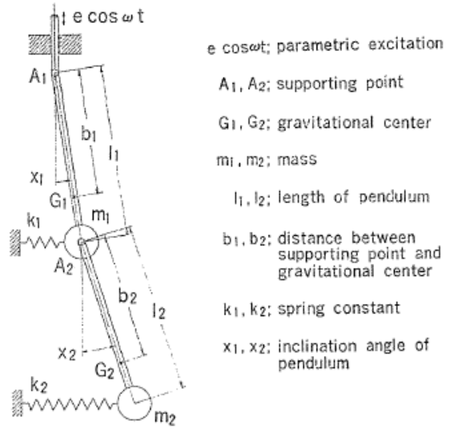

FIG. 1. Vibratory system of double pendulums.

$$\left.\begin{aligned} &a_{11}=I_1+m_2l_1^2,\ \ a_{12}=a_{21}=m_2l_1b_2,\ \ a_{22}=I_2,\\ &\alpha_{11}=(k_1+k_2)l_1^2+(m_1b_1+m_2l_1)g,\ \ \alpha_{12}=\alpha_{21}=k_2l_1l_2,\\ &\alpha_{22}=k_2l_2^2+m_2b_2g,\ \ \varepsilon_1'=e\omega^2(m_1b_1+m_2l_1),\ \ \varepsilon_2'=e\omega^2m_2b_2,\end{aligned}\right\}\tag{40}$$

where g is gravitational acceleration.

5. 1. Experimental apparatus and block diagram of analog computer

Experiments are carried out by the experimental apparatus shown in Fig. 2. Through two stepless transmissions ⑱, ⑯, rubbercoupling ⑰, pulleies ⑮, ⑬ and shaft ⑩, rotation of motor ⑲ is transmitted to rotor ⑧ which consists of eccentric shaft ⑥. Eccentricity e of the shaft ⑥ can be changed by screw ⑦. Rotation of eccentric shaft ⑥ is transformed to vertical rectilinear oscillation $e\cos\omega t$ of the supporting point A_1 of the first pendulum ① through bearing ⑤, joint ④ and guide ③. Further, on the both pendulum ends with mass ㉔, ㉕, coil springs ⑳, ㉑ are attached, in order to adjust magnitude of the natural frequencies of the system, and horizontal motions of steel edges ㉒, ㉓ at the ends of the first and second pendulums are recorded optically on oscillograph paper, thus oscillations referring to generalized coordinates x_1, x_2 are obtained experimentally. In this

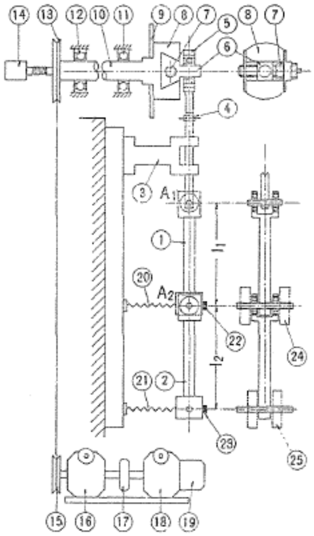

FIG. 2. Experimental apparatus.

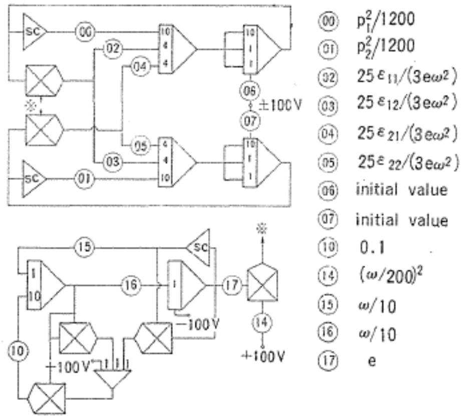

FIG. 3. Block diagram of the analog computer. (SC=sign changer)

experimental apparatus, various dimensions are as follows:

$$m_1g=1.910\ \text{kg},\ m_2g=1.730\ \text{kg},\ I_1=0.963\ \text{kg cm s}^2,\ I_2=0.786\ \text{kg cm s}^2,$$
$$l_1=l_2=30.0\ \text{cm},\ b_1=18.85\ \text{cm},\ b_2=17.35\ \text{cm},\ k_1=k_2=6.4\ \text{kg/cm},$$

and hence the natural frequencies of the apparatus are

$$p_1=13.86\ \text{c/s},\qquad p_2=9.85\ \text{c/s}.$$

Block diagram of analog computer used to obtain calculations for oscillations referring to normal coordinates X_1 and X_2 is shown in Fig. 3, where sinusoidal function generator to yield excitation $e\cos\omega t$ is shown in the lower figure and it is inserted into the place shown the mark ※ of the upper figure.

5. 2. Comparisons of analytical conclusions with results of experiments and analog computer

The $\mu-\omega$ diagram for $e=0.152$ cm ($E_{12}=0.955$ rad/s) is shown in Fig. 4, where not only magnitude of negative damping coefficient μ but width of unstable region of the unstable oscillations of summed type are furnished and μ of the ordinary unstable oscillations appearing in the neighborhood of $\omega=2p_{1,2}$ are additionally illustrated for comparison. In Fig. 4, the resonant frequencies $\omega=p_1+p_2=p_{12}$ and $\omega=2p_{1,2}$ are indicated by vertical chain lines. Curves of chain line in Fig. 4 are negative damping coefficients μ obtained from Eqs. (30), (37), and symboles ▲, ● show results of experiments and analog computer separately, which are given by the following equation

$$\mu=\frac{\ln(a_t/a)}{t}\ \text{rad/s}, \tag{41}$$

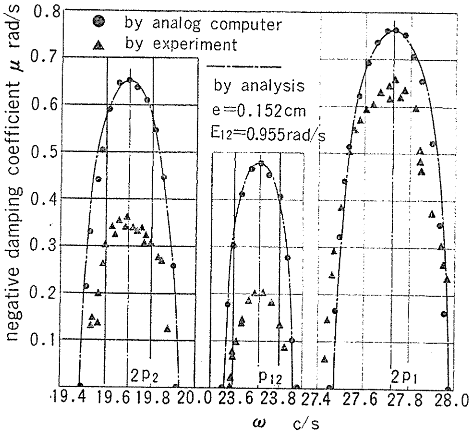

FIG. 4. $\mu-\omega$ diagram for unstable oscillations of summed and ordinary types. (μ=negative damping coefficient; ω=frequency of parametric excitation; e, E_{12}= magnitude of parametric excitation; p_1=13.86 c/s, p_2= 9.85 c/s, p_1+p_2=23.71 c/s)

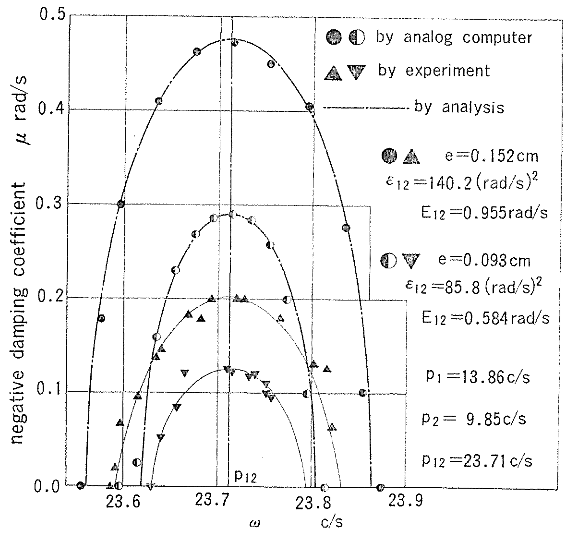

FIG. 5. Negative damping coefficient for unstable oscillations of summed type. (μ= negative damping coefficient; ω=frequency of parametric excitation; e, ε_{12}, E_{12}=magnitude of parametric excitation; $p_{12}=p_1+p_2$= 23.71 c/s)

where t is time, and a, a_t are amplitudes of the oscillatory waves obtained by experiments and analog computer when $t=0$ and $t=t$ severally. In Fig. 4, results of analysis entirely agree with results of analog computer, while experimental results give rather smaller values of μ because of inevitable damping in the apparatus.

Negative damping coefficients μ of unstable oscillations of summed type for $e=0.152$ cm ($E_{12}=0.955$ rad/s) and $e=0.093$ cm ($E_{12}=0.584$ rad/s) given by Eq. (30) are shown in Fig. 5, where the larger e, *i.e.*, E_{12} results in the larger μ and the wider unstable region.

Boundaries ω_{c1} and ω_{c2} of unstable region given by Eq. (35) (broken line), experiments (symbols $\triangle$, ∇; ∇: lower limit of unstable region) and analog computer (symbol ○) are indicated in Fig. 6, where difference in experimental results from those of analysis and analog computer are also caused by existence of inevitable damping in experimental apparatus.

Frequencies $\omega_{1,2}\pm\nu$ and $\omega_{1,2}$ of stable and unstable oscillations obtained by Eqs. (26), (30 a) (curves of broken and chain line), experiments (symbols $\triangle$, ▲) and analog computer (symbols ○, ●) are shown in Fig. 7, where the curves I indicate frequencies $\omega_{1,2}\pm\nu$ of stable oscillations and its upper and lower branches correspond to $\omega_{1,2}+\nu$ and $\omega_{1,2}-\nu$ respectively, and curves II represent frequencies ω_1 and ω_2 of

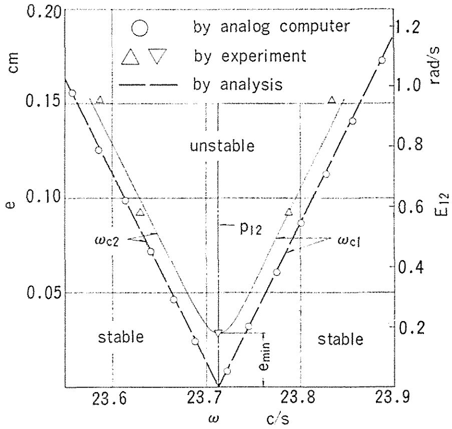

FIG. 6. Boundaries of unstable region for oscillations of summed type. (ω = frequency of parametric excitation; $\omega_{c_1}, \omega_{c_2}$ = critical frequencies giving the boundaries; e, E_{12} = magnitude of parametric excitation; $p_{12}=p_1+p_2=23.71$ c/s)

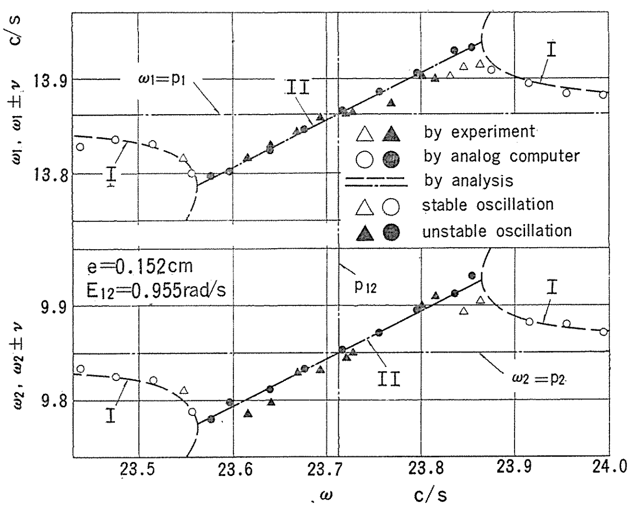

FIG. 7. Frequencies of stable and unstable oscillations of summed type. (ω = frequency of parametric excitation; e, E_{12} = magnitude of parametric excitation; $\omega_{1,2}$ = frequencies of unstable oscillations; $\omega_{1,2}\pm\nu$ = frequencies of stable oscillations; $p_1=13.86$ c/s, $p_2=9.85$ c/s, $p_{12}=p_1+p_2=23.71$ c/s)

unstable oscillations. In the stable region, only the frequencies $\omega_{1,2}-\nu$ on the higher frequency side (the left side) and the frequencies $\omega_{1,2}+\nu$ on the lower side (the right side) which are nearer to the natural frequencies $p_{1,2}$ between two frequencies making a pair are obtained by experiments and analog computer, because the amplitudes with these frequencies are larger than another, as shown in Fig. 8 and Eq. (39). In Fig. 7, the results of analog computer also agree with curves of broken and chain lines given by analysis, and the experimental results differ from them slightly.

Amplitude ratios of stable and unstable oscillations of summed type are given in Fig. 8, where curve I induced by Eq. (23) and symbols ▲, ● give amplitude ratios of unstable oscillations and curves II through Eq. (39) and symbol ○ are amplitude ratios between two stable oscillations having frequencies $\omega_1+\nu$ and $\omega_2-\nu$, and curves II′ from Eq. (39) and symbol ⊖ express ratios of $\omega_1-\nu$ to $\omega_2+\nu$. Inevitable damping in experimental apparatus results in some differences between symbols ▲ and curve I.

An example of no occurrence of unstable oscillations of differential type is illustrated experimentally in Fig. 9. Although there is a resonant point $p_{12}=$

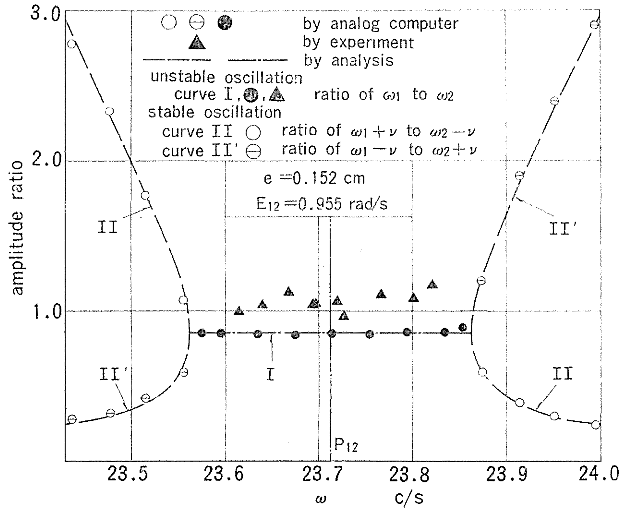

FIG. 8. Amplitude ratio of stable and unstable oscillations of summed type. (ω=frequency of parametric excitation; e, E_{12}=magnitude of parametric excitation; $p_{12}=p_1+p_2=23.71$ c/s)

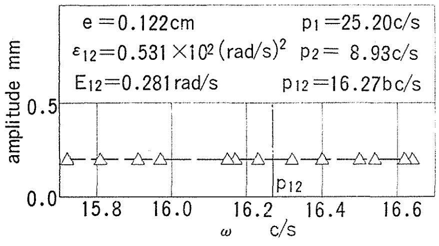

FIG. 9. Experimental results of no occurrence of unstable oscillation of differential type. (ω=frequency of parametric excitation; e, ε_{12}, E_{12}= magnitude of parametric excitation; $p_{12}=p_1-p_2=16.27$ c/s)

$p_1-p_2=16.27$ c/s in the experimental apparatus of Fig. 9, only random free oscillations having small amplitudes due to disturbance take place and no unstable oscillation appears in the neighborhood of the resonant point p_{12} as shown in Fig. 9.

The some results are also obtained by analog computer, and there is nothing but free oscillations furnished by initial conditions. Frequencies and amplitude ratios of these stable free oscillations of differential type are shown in Fig. 10, 11, severally, where symbols ○ indicate results of analog computer and broken line curves represent analytical results through Eqs. (26 b), (30 b((in Fig. 10) and Eq. (39 b) (in Fig. 11). It is seen in Fig. 10 that the relation $\omega=\omega_1-\omega_2$ are always satisfied between two frequencies $\omega_{1,2}\pm\nu$ making a pair.

Oscillatory waves of unstable oscillations of summed type appeared on the experimental apparatus of double pendulums are illustrated in Fig. 12 where the upper and lower photographs give oscillatory waves of the first and second pendulums respectively and frequency ω of parametric excitation is known by vertical black lines. Observing oscillatory waves, it is found that the relation $\omega:\omega_1:\omega_2=41:24:17$, and hence $\omega=\omega_1+\omega_2$ holds.

Oscillatory waves of analog computer are shown in Fig. 13, in which the oscillations of frequencies ω_1 and ω_2 appear separately because of normal coordinate.

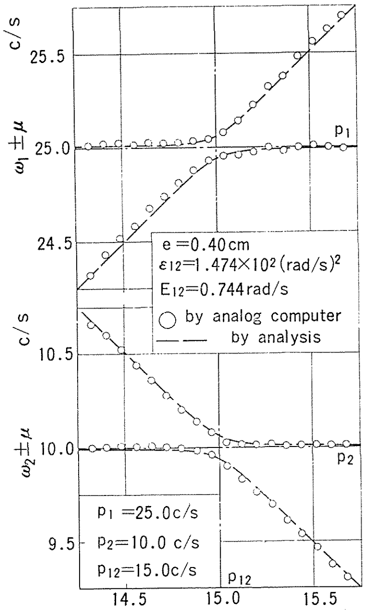

FIG. 10. Frequencies of stable oscillations of differential type. (ω = frequency of parametric excitation; e, ε_{12}, E_{12} = magnitude of parametric excitation; $\omega_{1,2}\pm\mu$ = frequencies of stable oscillations; p_1 = 25.0 c/s, p_2 = 10.0 c/s, $p_{12}=p_1-p_2=15.0$ c/s)

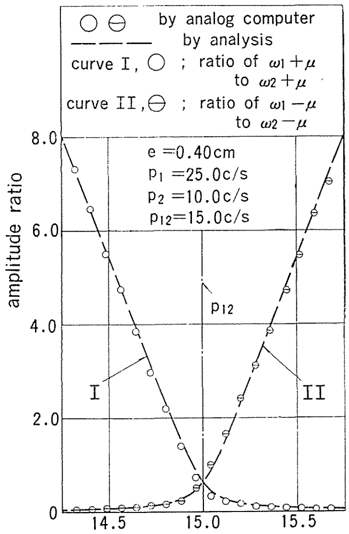

FIG. 11. Amplitude ratio of stable oscillations of differential type. (ω = frequency of parametric excitation; e = magnitude of parametric excitation; $p_{12}=p_1-p_2=15.0$ c/s)

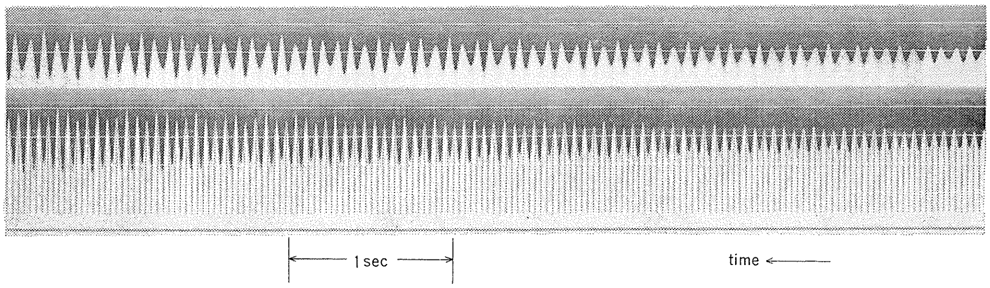

FIG. 12. Oscillatory waves of unstable oscillations of summed type by experiment of double pendulums. ($e=0.152$ cm, $\omega=23.67$ c/s, $\mu=0.19$ rad/s, $\omega:\omega_1:\omega_2=41:24:17$, $p_1=13.86$ c/s, $p_2=9.85$ c/s, $p_{12}=p_1+p_2=23.71$ c/s)

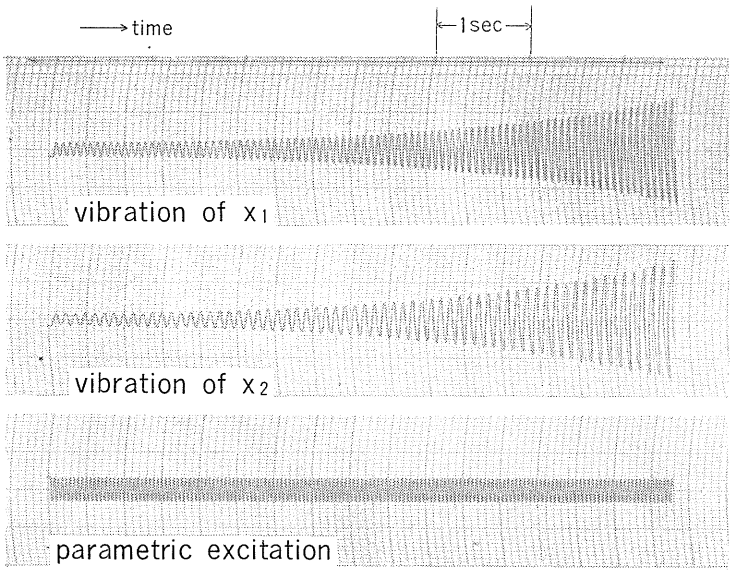

FIG. 13. Oscillatory waves of unstable oscillations of summed type by analog computer. ($e=0.152$ cm, $\omega=23.64$ c/s, $\mu=0.41$ rad/s, $\omega:\omega_1:\omega_2=41:24:17$, $p_1=13.86$ c/s, $p_2=9.85$ c/s, $p_{12}=p_1+p_2=23.71$ c/s)

8. Conclusions

Obtained results may be summarized as follows:

1) In oscillatory system with multiple degree-of-freedom and under parametric excitation, two unstable oscillations with frequencies $\omega_i\,(\fallingdotseq p_i)$ and $\omega_j\,(\fallingdotseq p_j)$ can simultaneously take place in the neighborhood of the resonant point $p_{ij}=p_i+p_j$, that is, unstable oscillations of summed type can occur.

2) Sum of frequencies ω_i and ω_j of unstable oscillations is equal to frequency ω of parametric excitation, *i.e.*, $\omega_i+\omega_j=\omega$.

3) Solutions of this kind of unstable oscillations are obtained through Eq. (34) and amplitude ratio, frequencies, negative damping coefficient, phase angle and

unstable region are given by Eqs. (23), (26), (30), (33) and (35), separately.

4) In stable region of summed type, two free oscillations with frequencies $\omega_{i,j} \pm \nu$ appear, and those solutions are found by Eq. (34 a) and frequencies by Eqs. (26), (30 a) and amplitude ratio by Eq. (39).

5) Oscillations on the boundaries between stable and unstable regions are always stable.

6) Oscillations of differential type consist of two free oscillations with frequencies $\omega_{i,j} \pm \mu$ as shown by Eq. (34 b), and frequencies are obtained by Eqs. (26 b), (30 b) and amplitude ratio by Eq. (39 b).

7) So far as magnitude of parametric excitation is somewhat small as in this paper, solutions of the first approximation are enough to grasp exactly the oscillatory phenomena due to parametric excitation and they show good agreement with the results of experiment and analog computer.

8) Analytic caluculation is rather simple by virtue of the approximation.

9) There is no unstable oscillation of summed and differential type of higher order in so far as the first approximation, and it can not appear experimentally

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