1. Introduction

In dynamically typed languages like Scheme, types are not asserted at compile time. In general, dynamically typed languages are more expressive than statically typed languages; programmers can use functions or variables to range over any values of any types. For example, the following Scheme program defines the function SAT that computes satisfiability of curried boolean functions of any arity:

\[
\text{(define (SAT x) (if (boolean? x) x (or (SAT (x #t)) (SAT (x #f)))))}
\]

A typical input of SAT is

\[
t_1 := (\text{lambda} (y) (\text{lambda} (z) (or (not y) z)))
\]

SAT works in a straightforward way; if the argument x is a boolean value, then it is satisfiable if \(x = \#t\) so x is returned. Otherwise SAT considers x to be a boolean function, and returns true if either \((x \#t)\) or \((x \#f)\) is satisfiable. The expression \((\text{SAT } t_1)\) evaluates to \#t because \((t_1 \#f) \#t\) evaluates to \#t. Such programming is not allowed in usual statically typed languages, because the variable x does not have a fixed type.

The price for the expressiveness is the difficulty in debugging; erroneous expressions are detected only when their evaluation fails with runtime errors. In the example above, for uncurried version of \(t_1\):

\[
t_2 := (\text{lambda} (y z) (or (not y) z))
\]

the expression \((\text{SAT } t_2)\) always raises a runtime error, but it will not be rejected by the compiler. Thus, to produce bug-free programs, developers need a careful test scenario to detect all possible runtime errors.

Our work is motivated to help programmers of a dynamically typed language to make bug-free programs by (1) accepting expressions whose evaluation never causes runtime errors, (2) rejecting expressions whose evaluation always causes runtime errors, and (3) by still retaining the expressiveness of dynamically typed languages. In this paper, we will present a sound type system that provides enough expressive power for this triple goal. We concentrate on the theory, and a type inference algorithm will not be presented here.

The key idea of our work is to allow types to express the situation where runtime errors occur. For this purpose we introduce a special error type, denoted by \(E\), representing runtime errors. Using \(E\) in type expressions, erroneous situations can be expressed; e.g., \(\tau \rightarrow E\) is the type for functions that cause runtime errors for inputs of type \(\tau\). The type system assures that (1) evaluation never causes errors for expressions typed with any type that does not contain \(E\) (defined later precisely). (2) evaluation always causes runtime errors (or it diverges) for expressions typed with \(E\). To retain the expressiveness of dynamically typed languages, (3) untyped expressions should be evaluated with runtime checks as usual.

To bring out the full benefit of \(E\), our system has a certain expressiveness including intersection type \([7]\), \([20]\) \(\sigma \cap \tau\), set-theoretic union type \([3]\), \([17]\) \(\sigma \cup \tau\), complement type \([11]\), \([24]\) \(\tau^C\), universal quantification \([13]\), \([14]\), \([18]\) \(\forall \sigma. \tau\), existential quantification \([1\)]{0.1111111111111111] \(\exists \sigma. \tau\), and recursive type \([17]\) \(\mu \sigma. \tau\), together with sub-

\footnote{We follow MacQueen et al. \([17]\) and quantifications do not have a construct such as pack or open.}
typing $\sigma \subseteq \tau$. Using these type constructions, SAT is typed as follows:

$$\text{SAT} : (\tau_1 \to \text{Bool}) \cap (\tau_1^2 \to \text{E})$$

where $\tau_1 = \mu\alpha.\text{Bool} \cup (\text{Bool} \to \alpha)$. This typing means that SAT returns a boolean value for any curried boolean functions and boolean inputs, and always causes a runtime error for other inputs.

In the existing formalization employing semantics for types and/or expressions, it is essential for soundness that erroneous expressions should not have a type. Since we give the new type $\text{E}$ for erroneous expressions, a different approach is required to prove the soundness of our system.

The syntactic approach inspired by Wright et al. [26] does not depend on semantics; however, their approach cannot be easily extended for subtyping. Suppose we are trying to prove some predicate $P$ on functional types by induction on the height of the subtyping proof. Transitivity of subtyping allows a deduction such as:

$$\frac{\Gamma \vdash \sigma \subseteq \rho \quad \Gamma \vdash \rho \subseteq \tau'}{\Gamma \vdash \sigma \subseteq \tau'} \quad (\subseteq-\text{TRANS})$$

The induction hypothesis applies for $P(\sigma \to \tau)$, but not for $P(\rho)$ because $\rho$ is not always a functional type. Even worse, $\rho$ may be more complex than $\sigma' \to \tau'$ e.g., $(\sigma \to \tau)\rho, (\sigma \to \tau)\rho(\sigma \to \tau), \mu\alpha.(\sigma \to \tau) \cap \rho \cup (\sigma \to \tau)$, and so on.

To overcome this problem, we introduce a non-transitive relation $\subseteq$ called intransitive subtyping, and define subtyping $\subseteq$ as the transitive closure of $\subseteq$. Under this definition, we show that syntactically complex subtypes can be ignored for typing values, which is the key to our soundness proof.

The rest of this paper is organized as follows: Section 2 presents our target functional language and Section 3 introduces the type system. Section 4 proves the subject reduction property of the system and Section 5 presents soundness theorems. Section 6 demonstrates how the system works using the SAT example. Section 7 describes related works, and Section 8 gives concluding remarks.

2. Expressions

In this section, we define the functional language we consider.

Definition 1 Let $X$ be a set of variables, $C$ be a set of constants, $F$ be a set of built-in functions, and $\emptyset$ be a special symbol representing a runtime error. The set $\Lambda$ of expressions and the set $\text{Val}$ of values are defined by the following grammar:

$$t ::= x \mid c \mid f \mid (t,t) \mid t \mid Lx.t \mid \emptyset \quad (\Lambda)$$

$$v ::= c \mid f \mid (v,v) \mid Lx.t \mid \emptyset \quad (\text{Val})$$

We use meta-variables $x,y,z$ for variables, $e$ for a constant, $f$ for a function, $r,s,t$ for expressions, and $u,v$ for values.

The set $FV(t)$ of free variables of $t$ is defined as usual. We assume bound variables are renamed to avoid capture. The substitution of $s$ by $s$ in $t$ is defined as usual and written $t[x \mapsto s]$.

Definition 2 The one-step reduction relation $\rightarrow$ is defined in Fig. 1. The reduction relation $\rightarrow^*$ is the reflexive transitive closure of $\rightarrow$.

The rule $(\beta)$ represents the call-by-value $\beta$-reduction. The rule $(\delta)$ reduces applications of built-in functions whose interpretations are given by a total function $\delta : F \times \text{Val} \to \text{Val}$. Note that $\delta$ is total; if $f$ should be interpreted as a partial function which is not defined for input $v$, then $\delta(f,v)$ should be defined to be the error symbol $\emptyset$. An application of non-function is also reduced to $\emptyset$ by $(\epsilon)$.

Note that our definition allows values to contain errors. In order to exclude such situations, we introduce the notion of safe values:

Definition 3 (Safe Values) The set $\text{Val}_{\text{safe}}$ of safe values is the least set satisfying:

- $c, f, Lx.t \in \text{Val}_{\text{safe}}$
- $(u,v) \in \text{Val}_{\text{safe}} \Rightarrow (u,v) \in \text{Val}_{\text{safe}}$

3. The Type System

Definition 4 Let $B$ be a set of base types and $X_F$ be a set of type variables. The set $\mathcal{T}$ of types is defined by the following grammar:

$$\tau ::= a \mid \tau \mid \tau \times \tau \mid \tau \cup \tau \mid \tau^C \mid \forall a. \tau \mid 3a. \tau \mid \mu a. \tau \mid \text{E}$$

We use meta-variables $\alpha, \beta$ for type variables, $\iota$ for a base type and $\rho, \sigma, \tau$ for types.

Parentheses are added to avoid ambiguity with precedence in order $\sigma^C, x, \&, \cup, \to$ and binding operators. Notions of free type variables, renaming of bound type variables and type substitution are defined analogously to expressions.

Definition 5 For every base type $\iota$, we assume the set $\text{C}_\iota \subseteq \text{C}$ is given. A subtyping environment $\Gamma$ is a set of subtyping assumptions written $a_1 \preceq a_1' \cdots \preceq a_n \preceq a_n'$ for distinct $a_i$ and $a_i'$. A formula in form $\Gamma \vdash \tau \preceq \tau'$ is called intransitive subtyping, whose validity is defined by the rules in Fig. 2. Subtyping $\Gamma \vdash \tau \subseteq \tau'$ is defined as the transitive closure of the intransitive subtyping.

We write the type $3\alpha.\iota$ by $\iota$, and $\forall a.\alpha \cup \iota$ by $\iota$. $\iota$ is the maximum and $\perp$ is the minimum type w.r.t. subtyping.

Definition 6 For each $f \in F$, we assume a set $\text{Ty}(f)$ of types in form $\sigma \to \tau$ is given. A type environment $\Delta$ is a set of assumptions in form $x_1 : \tau_1, \cdots, x_n : \tau_n$ for distinct $x_i$. A formula in form $\Delta \vdash t : \tau$ is called a type judgment, whose validity is defined by the rules in Fig. 3.

The function $\text{Ty}$ gives the basis for typing built-in functions. To ensure type preservation under $\delta$-reduction, we assume the following:

Assumption 7 $(\delta$-typability) For all $f \in F$ and $\sigma \rightarrow \tau \in \text{Ty}(f)$, we assume
Remark 9 One may wonder why we distinguish ≤ and ⊆ despite the admissibility of (⊆). The question is: Is \( t \vdash \tau \), derived in our system if and only if it is derived in the system where ≤ and ⊆ are identified? This is not trivial because of the rule (≤-\( C \)C) where a condition of free type variables must be satisfied in one-step of ≤. Though we expect a positive result, the problem is left open at the time of writing.

Remark 10 The following rule (∪E) is proposed by MacQueen et al. [17]:

The system adopts (∪L), which Barbanera [3] showed to be strictly weaker than (∪E). Nonetheless, a restricted form of (∪E), where \( s \) is assumed to be a value, can be derived with help of Lemma 13 latter stated. This result is enough to prove type preservation under call-by-value \( \beta \) -reduction. The restriction seems to be reasonable for further extension, e.g., (∪E) is shown to break subject reduction property in conjunctive-dissjunctive \( \lambda \)-calculus [9].

### 4. Subject Reduction Property

In this section we prove the main property to the soundness of our system, i.e., the subject reduction: If \( t \vdash \tau \) and \( t \longrightarrow t' \), then \( t' \vdash \tau \).

As mentioned earlier, subtyping becomes a problem in a straightforward approach. The problem is (⊆), where the premise may be more complex in syntax than the conclusion, because of the rules (⊆-\( C \)C), (⊆-∪E), (⊆-∪), and (⊆-VE). By following the lemmas we show that such inversion can be converted under the
If $t = r'$ or $t = r''$ with $r \rightarrow r'$ or $s \rightarrow s'$, it is obvious from the I.H.
Otherwise $r$ must be a value. By the form and Lemma 12, we have to consider only $(\varepsilon \rightarrow \varepsilon^C), i.e., \varepsilon$.
Then the proof proceeds to a case analysis of $t \rightarrow t'$.

**Case $\beta_1$**. $r = \lambda x.t'$ and $t' = t''[x \mapsto s]$. By Lemma 14, we have $\Delta, x : \sigma \rightarrow t'' : \tau^C$. Using $\langle z \rangle$ with $\sigma \leq \tau$, we get $\Delta, x : \sigma \rightarrow t' : \tau^C$, and using $\langle z \rangle$ with $\tau^C \subseteq \Sigma$, we get $\Delta, x : \sigma \rightarrow t' : \Sigma$. By Lemma 13, $\Delta \vdash t'.

**Case $\delta$**. This case is ensured by $\delta$-typability.

**Case $\epsilon$**. We have $t' = \varepsilon$ and $\Delta \vdash t' : E$ by (E).

### 5. Types of Expressions That Always/Never Cause Runtime Errors

As a corollary of the subject reduction theorem, the strong soundness is immediately obtained:

**Theorem 17 (Strong Soundness)** If $\vdash t : \tau$ and $t \rightarrow v$, then $v : \tau$.

The following error soundness ensures that expressions typed with $E$ always cause runtime errors whenever they are evaluated.

**Theorem 18 (Error Soundness)** If $\vdash t : E$ and $t \rightarrow v$, then $v = \varepsilon$.

**Proof** By the strong soundness theorem, we have $\Delta \vdash v : E$. Lemma 15–4 denies every possible form of $v$ but $\varepsilon$.

Since our system types erroneous expressions, the usual reasoning of "typed expressions are safe" is of course unsound. To ensure the safety of typed expressions, we must restrict their types to safe types, which are not inhabited by $\varepsilon$ and pairs containing $\varepsilon$.

**Definition 19 (Safe Types)** The set $T_{safe}$ of safe types is the least set satisfying:

- $\tau, \sigma \rightarrow \tau \in T_{safe}$ for any type $\sigma$ and $\tau$.
- $\tau_1, \tau_2 \in T_{safe} \Rightarrow \tau_1 \land \tau_2, \tau_1 \lor \tau_2 \in T_{safe}$.
- $\tau \in T_{safe} \Rightarrow \mu.\tau, \forall.\tau \in T_{safe}$.

Note that we consider $\sigma \rightarrow \tau$ to be always safe, because any function does not raise a runtime error until it is applied. On the other hand, since a pair containing an error should be treated as an error, $\tau_1 \land \tau_2$ is safe only if both $\tau_1$ and $\tau_2$ are safe.

**Lemma 20** If $\tau$ is safe, then $\tau[\forall \rightarrow \sigma]$ is also safe.

**Proof** Obvious by structural induction on $\tau$.

**Lemma 21** If $\vdash v : \tau$ for a safe type $\tau$, then $v$ is a safe value.

**Proof** By induction on the height of the proof of $\vdash v : \tau$. By the definition we have following cases of the form of $\tau$:

- $t \lor u \rightarrow \sigma$. By Lemma 15, $v$ must be in form $c \lor u$, thus $v \in T_{safe}$.
- $\sigma \land \tau_2$ with $\sigma_1$ and $\tau_2$. In this case $v$ must be in form $(u_1, u_2)$ with $\tau_1 : \sigma_1$ for both $i \in \{1, 2\}$. By the I.H. $u_i$ is a safe value, thus so is $(u_1, u_2)$.
- $\sigma_1 \lor \tau_2$ with $\sigma_1$ and $\tau_2$. By Lemma 12 we get a shorter proof of $\vdash v : \tau'$ for either or both of $\tau' = \sigma_1$ and $\tau_2$. Thus by the I.H., $v$ is safe.
- $\mu.\sigma$ or $\forall.\sigma$ with $\sigma$ safe. By Lemma 12 we have a shorter proof of $\vdash v : \tau'$ for $\tau' = \sigma[\forall \rightarrow \mu.\sigma]$ or $\sigma[\forall \rightarrow \rho]$. In either case, $\tau'$ is also safe by Lemma 20, and $v$ is safe by the I.H.

**Theorem 22 (Weak Soundness)** If $\vdash t : \tau$ and $t \rightarrow v$ for a safe type $\tau$, then $v$ is a safe value.
Proof] By the strong soundness, we have \( \vdash v : \tau \). Since \( \tau \) is assumed safe, Lemma 21 ensures that \( v \) is a safe value.

6. Examples

In this section we demonstrate how our type system detects runtime errors by taking the Scheme function SAT in the introduction, for example. To express SAT in our language, we need a fixpoint operator \( Y := \lambda x. (\lambda x. f(x))x \).

**Proposition 23** The following axiom is admissible:

\[
\Delta \vdash Y : (\tau \rightarrow \tau) \rightarrow \tau
\]

Proof] This is originally shown by MacQueen et al.\[17\] The proof tree for our system is shown in Fig. 4.

Using \( Y \), SAT is represented in \( \Lambda \) by \( \lambda x. \text{af}(x) \).

\[
\text{ty}(\text{af}) = \{ \text{true} \rightarrow (\forall \alpha. \alpha \rightarrow \alpha), \text{false} \rightarrow (\forall \alpha. \alpha \rightarrow \alpha) \}
\]

\[
\text{ty}(\text{af})_2 = \{ \text{true} \rightarrow (\forall \alpha. \alpha \rightarrow \alpha), \text{false} \rightarrow (\forall \alpha. \alpha \rightarrow \alpha) \}
\]

The subproof for \( \vdash \text{SAT} : (\tau_1 \rightarrow \text{Bool}) \rightarrow (\tau_1^C \rightarrow \text{E}) \) is presented in Fig. 5. As described in the introduction, this typing means that SAT returns a boolean value for any curried boolean function and boolean input, and always causes a runtime error for other inputs.

An uncurried binary boolean function \( \tau_2 \) should have following type \( \tau_2 \):

\[
\tau_2 = (\text{Bool} \times \text{Bool} \rightarrow \text{Bool}) \cap ((\text{Bool} \times \text{Bool})^C \rightarrow \text{E})
\]

We have \( \vdash \tau_2 \subseteq \tau_2^C \) as shown in Fig. 6, thus we can deduce \( \vdash \text{SAT} : (\tau_1 \rightarrow \text{Bool}) \cap (\tau_1^C \rightarrow \text{E}) \), which means that the evaluation of SAT \( \tau_2 \) causes a runtime error.

7. Related Works

Intersection types were introduced by Coppo et al.\[7\] and Pottinger\[20\]. They showed that reduction preserves typing (often called Subject Reduction after Curry), and that an expression is
typable if and only if it is strongly normalizing. This means that a complete type inference algorithm does not exist for intersection types.

Milner [18] presented a soundness theorem for Hindley/Milner style polymorphism by means of denotational semantics. Types are interpreted as downward-closed and directed-complete sets (ideals) in the semantic domain of expressions. He also presented the famous type inference algorithm $W$ which is complete w.r.t. first rank polymorphism.

MacQueen et al. [17] extended the ideal model for recursive types, and additionally introduced unquoted existential quantification and set-theoretic union types. They formalized the semantics of a recursive type as the unique fixpoint of corresponding contractive mapping on ideals. They restricted the types to be formally contractive in order to ensure the fixpoint is unique. (This harmless restriction was eliminated anyway in our system.)

Subtyping on intersection types was presented by Barendregt et al. [4], and later extended for intersection and union types by Barbanera et al. [3].

Amadio et al. [2] formalized subtyping on recursive types with the help of regular tree expressions, and the system was extended for union, intersection and recursive types by Damm [8]. Subtyping on union and recursive types is also presented by Cartwright et al. [5].

Frisch et al. [11], [12] introduced semantic subtyping. They presented a set-theoretic model of types independent from the semantics of expressions, and defined the subtyping relation by the set inclusion relation on this model. The semantic subtyping method allowed set-theoretic interpretation of complement type $\tau^c$, which was not possible in the ideal model [6]. Hosoya et al. [15] extended the semantic subtyping for parametric polymorphism.

Our system covers all the notions described above, except that we have not yet proposed a type inference algorithm. These results are summarized in Table 1.

Several studies have been motivated to cover the disadvantages of dynamically typed languages by means of static typing.

**Soft typing** [1], [5], [25] Soft typing introduces a static type check for dynamically typed languages. If the static type check succeeds, the program is assured to be type-safe. In order not to restrict the expressiveness of dynamically typed languages, programs will not be rejected even though the static check fails; instead, runtime checks are inserted which are unknown to fail or not on execution. That is, soft typing does not detect runtime errors; programmers must manually check whether a ‘softly’ rejected program contains a bug, or not.

**Complete typing** [23] Complete typing rejects provably erroneous programs at compile time, and is expected to help detect real bugs from softly rejected programs. Complete typing requires an inference system that is different from the usual sound system, and it requires an algorithm testing disjointness of types (i.e. emptiness of intersection types), which is known to be undecidable [21].

**Hybrid typing** [10], [16] Hybrid typing is an extension of soft typing with refinement types, and it rejects some programs at compile time if the type check is proved to fail. However, hybrid type checking is not suitable for our purpose because it was originally designed for a statically typed language, and type check failure does not immediately imply a runtime error.

### 8. Conclusions and Future Work

We have presented a type system that accepts expressions whose evaluation never causes runtime errors, and rejects expressions whose evaluation always causes runtime errors. The system is proved to be sound by several soundness theorems:

- **Subject Reduction** If an expression has a type, then its reduction also has the same type. (Theorem 16)
- **Strong Soundness** If an expression has a type and its evaluation terminates, then the value has the same type. (Theorem 17)
- **Weak Soundness** If an expression has a safe type, then its evaluation never causes a runtime error. That is, the expression should be accepted by the type checker. (Theorem 22)
- **Error Soundness** If an expression has the type $E$, then its evaluation must cause a runtime error. That is, the expression should be rejected by the type checker. (Theorem 18)

Though we have presented the type system, we have not yet presented a type inference algorithm. Since our system has in-
intersection types and higher rank polymorphism where a complete type inference is impossible for each system [19], [22], a complete type inference is apparently impossible. Thus, we need to discover a fragment of our system where type inference becomes possible in practical time.

Reference


Appendix

A.1 Detailed Proofs

Lemma 11 Let α, α’ ∈ FV(Γ), α ∈ FV(τ) and α’ ∈ FV(τ). If Γ ⊢ σ ≤ σ’ and Γ, α ⊢ α’ ∨ τ ≤ τ’ then Γ ⊢ τ[α[τ[α’[σ[σ’]].

Proof By induction on the height of the proof of Γ, α ⊢ α’ ∨ τ ≤ τ’.

Case (±-assoc). The following two subcases are possible:

Case τ = α and τ’ = α’. Then the assumption Γ ⊢ σ ≤ σ’ is equivalent to Γ ⊢ τ[α[τ[α[σ[σ’]].

Case (β ≤ β’) ∈ Γ with τ = β and τ’ = β’. Since α, α’ ∈ FV(Γ), by (±-assoc) we get Γ ⊢ β[β[α[τ[τ’]] ≤ β[β’[α[τ’]]].

Case (μβ) ∈ Γ, α ≤ α’ ≤ β ≤ β’ and τ’ = μβ.ρ. Then the assumption Γ, β ⊢ β’ ≤ β[ρ[ρ’]. From ρ ∈ FV(ρ’), by (I.H.) we have Γ, β ⊢ β[β’ ≤ β[ρ[ρ’] ≤ β’[ρ[ρ’]]. Applying (±μC), we get Γ ⊢ β’[ρ[ρ’] ≤ μβ[ρ[ρ’]] ≤ ρ[ρ’][α[τ’]].

Others. Obvious because they are independent from the subtyping environment and closed under type substitution.

Lemma 12 If Δ ⊢ v : τ has a proof with height h, then there exists a proof shorter than h for:

(1) both Δ ⊢ v : τ₁ and Δ ⊢ v : τ₂ if τ = τ₁ ∪ τ₂,

(2) either Δ ⊢ v : τ₁ or Δ ⊢ v : τ₂ if τ = τ₁ ∪ τ₂,

(3) Δ ⊢ v : σ[α[μ.σ[σ’.σ]], if τ = μ.σ,

(4) Δ ⊢ v : σ[α[ρ[ρ’.σ]], if τ = ν[σ[ρ[ρ’.σ]].

Proof By mutual induction on the proof of Δ ⊢ v : τ.

Case (∃(Π)) Δ ⊢ v : τ₁ ∪ τ₂. This case applies for statement 1 and it is trivial.

Case (∃(∪)) Δ ⊢ v : τ. All the statements are immediate from the I.H.

Case (∀(V)) Δ ⊢ v : σ[α[μ.σ[σ’.σ]], with α ∈ FV(Δ). This case applies for statement 4. By replacing α to ρ in the proof of Δ ⊢ v : σ[α[μ.σ[σ’.σ]], we can make a proof of height h – 1 for Δ ⊢ v : σ[α[ρ[ρ’.σ]].

Case (≤(Π)) Δ ⊢ v : τ₁ ≤ τ. Proof proceeds to case analysis of τ₁ ≤ τ.

• The following subcases apply for all statements:

Case (≤(Π)) Δ ⊢ v : τ₁ ≤ τ. By I.H. 2, Δ ⊢ v : τ has a proof shorter than h – 1.

Case (≤(Π)) Δ ⊢ v : σ[α[μ.σ[σ’.σ]], with τ = σ[α[μ.σ[σ’.σ]], By I.H. 1, Δ ⊢ v : τ has a proof shorter than h – 1.

Case (≤(Π)) Δ ⊢ v : τ₁ ≤ τ. By I.H. 3, Δ ⊢ v : τ has a proof shorter than h – 1.

Case (≤(Π)) Δ ⊢ v : τ₁ ≤ τ. By I.H. 4, Δ ⊢ v : τ has a proof shorter than h – 1.

In either case above we could reduce the statements to
where the I.H.s can be applied.

- The following subcases apply for statement 1:
  Case \(( suburban)\) \(\Delta' + v : \tau_1 \cap \tau_1\) with \(\tau_1 = \tau_2\). Trivial.
  Case \(( suburban)\) \(\Delta' + v : \tau_1 \cup \tau_2\) with \(\tau_1 \cap \tau_1\) and \(\tau_2 \leq \tau_2\). By I.H. 1, \(\Delta + v : \tau_1\) with a proof shorter than \(h - 1\), for all \(i \in \{1,2\}\). Applying \(( suburban)\) we get a proof of \(\Delta + v : \tau_1\), which is shorter than \(h\).

- The following subcases apply for statement 2:
  Case \(( suburban)\) \(\Delta + v : \tau_1 \cup \tau_2\). Trivial.
  Case \(( suburban)\) \(\Delta + v : \tau_1 \cap \tau_2\) with \(\tau_1 \cap \tau_1\) and \(\tau_2 \leq \tau_2\). By I.H. 2, \(\Delta + v : \tau_1\) has a proof shorter than \(h - 1\) for some \(i \in \{1,2\}\). So applying \(( suburban)\) gives \(\Delta + v : \tau_1\) a proof shorter than \(h\).

- The following subcases apply for statement 3:
  Case \(( suburban)\) \(\Delta + v : \sigma' \cup \sigma_1\) with \(\sigma_1 = \rho \cap \sigma_1\) and \(\tau_1 = \rho \cap \sigma_2\). By I.H. 1, we have a proof shorter than \(h - 1\) for both
  \[\Delta + v : \sigma_1\] for some \(i \in \{1,2\}\) (A.3)

Applying \(( suburban)\) with (A.1) and (A.3), we get a proof shorter than \(h\) of \(\Delta + v : \tau_1\).

- The following subcases apply for statement 4:
  Case \(( suburban)\) \(\Delta + v : \sigma' \cup \sigma_1\) with \(\sigma_1 = \rho \cap \sigma_1\) and \(\tau_1 = \rho \cap \sigma_2\). By Lemma 11 we have
  \[\Gamma + \sigma'[\alpha' \mapsto \mu']\leq \sigma[a \mapsto \mu_{\sigma'}]\] (A.4)

On the other hand, by I.H. 3 \(\Delta + v : \sigma'[\alpha' \mapsto \mu']\) has a proof shorter than \(h - 1\). Applying \(( suburban)\) with (A.4), we get a proof of \(\Delta + v : \sigma[a \mapsto \mu_{\sigma'}]\), which is shorter than \(h\).

- No other subcases of subtyping apply because of the form \(\tau\).

Others. Not applicable because of the form of \(v\) or \(\tau\).

Lemma 13 (Substitution) Let \(x \notin \text{Dom}(\Delta)\). If \(\Delta, x : \sigma + t : \tau, \Delta + v : \sigma\) then \(\Delta + [x \mapsto v] : \tau\).

Proof By induction on the proof of \(\Delta, x : \sigma + t : \tau\).

Case (ass). If \(\Delta = \lambda x. t\), then \(\sigma = \sigma'\), and \(\Delta + v : \sigma\) is the goal. Otherwise \((t : \tau) \in \Delta\) and \(\Delta[x \mapsto e] = t\), so by (ass) we get \(\Delta + [x \mapsto e] = \tau\).

Case (base), (prim), (E). Trivial.

Case \(( suburban)\) \(\Delta, x : \sigma, x' : \sigma' + t' : \tau'\) with \(\Delta = \lambda x'. t'\) and \(\sigma = \sigma' \mapsto \tau'\). Here we assumed that \(x'\) is properly renamed to avoid capture. By the I.H. we have \(\Delta, x : \sigma' + t'[x \mapsto e] : \tau'\), and applying \(( suburban)\), we get \(\Delta + \lambda x'. t'[x \mapsto e] : \sigma' \mapsto \tau'\).

Case \(( suburban)\). There are two subcases to examine.

Case \(\Delta, x : \sigma_1, x' : \sigma_2 + t : \tau\) with \(\sigma = \sigma_1 \cup \sigma_2\). This case is obvious from the I.H.

Case \(\Delta, x : \sigma_1 + t : \tau, \Delta, x : \sigma_2 + t : \tau\) with \(\sigma = \sigma_1 \cup \sigma_2\). By Lemma 12 if \(\Delta + v : \sigma_1 \cup \sigma_2\), we have either \(\Delta + v : \sigma_1\) or \(\Delta + v : \sigma_2\). In either case, by the I.H., we get \(\Delta + [x \mapsto e] : \tau\).

Others. Obvious from the I.H.

Lemma 14 (Abstraction) If \(\Delta + \lambda x. t : \sigma = \tau\) then \(\Delta, x : \sigma + t : \tau\).

Proof By induction on \(\Delta + \lambda x. t : \sigma = \tau\).

Case \(( suburban)\). Trivial.

Cases \(( suburban)\) or \(( \sigma L)\) or \(( \sigma R)\). Obvious from the I.H.

Case \(( suburban)\). Because of the form \(\sigma = \tau\), the following subcases are possible:

Case \(\Delta : v : \sigma = \tau\) with \(\sigma \leq \sigma'\) and \(\tau \leq \tau'\). By the I.H. we have \(\Delta, x : \sigma + t : \tau\). Applying \(( suburban)\) and \(( \sigma L)\) we get \(\Delta, x : \sigma + t : \tau\).

Cases \(( suburban)\) or \(( \sigma R)\) or \(( \sigma L)\) or \(( \sigma E)\). In either case, Lemma 12 gives a shorter proof for \(\Delta + \lambda x. t : \sigma = \tau\). So by the I.H. we get \(\Delta, x : \sigma + t : \tau\).

Case \(\Delta + \lambda x. t : \sigma = \tau\) with \(\sigma = \sigma_1 \cup \sigma_2\).Lemma 12 gives a shorter proof of \(\Delta + \lambda x. t : \sigma = \tau\) for both \(i \in \{1,2\}\). So by the I.H. we have \(\Delta, x : \sigma + t : \tau\).

Applying \(\text{dom}(\Delta)\) we get \(\Delta, x : \sigma + t : \tau\).

Case \(( suburban)\) or \(( \sigma L)\) or \(( \sigma R)\) or \(( \sigma E)\). Analogously we have \(\Delta, x : \sigma + t : \tau\). Applying \(( \sigma L)\) or \(\text{dom}(\Delta)\) we get \(\Delta, x : \sigma + t : \tau\).

Others. Not applicable because of the form \(\lambda x. t \sigma = \tau\).

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