Lesson 15
Graphs and Equations (II)

15A
• Circle
• Circle and a Straight Line
Condition

\[ \sqrt{(x - a)^2 + (y - b)^2} = r \quad (\text{where } r > 0) \]

Standard expression

\[ (x - a)^2 + (y - b)^2 = r^2 \]

By expanding this, we obtain

\[ x^2 - 2ax + y^2 - 2by + a^2 + b^2 = r^2 \]

General expression

\[ x^2 + y^2 + lx + my + n = 0 \quad (\text{where } l^2 + m^2 > 4n) \]
Terminologies

**Lines**
- Chord
- Arc
- Diameter
- Radius
- Tangent

**Slices**
- Sector
- Segment

**Special types of sector**
- Quadrant
- Semicircle
Example

[Example 15.1] Answer the following questions.
(1) Find the circle which passes points A(-1, 7), B(2, -2) and C(6, 0).
(2) Find the center of this circle.

Ans.

\[x^2 + y^2 + lx + my + n = 0\]

(1) Because the circle passes points A, B, C
\[-l + 7m + n = -50\]
\[2l - 2m + n = -8\]
\[6l + n = -36\]

Therefore
\[l = -4, \ m = -6, \ n = -12\]
and
\[x^2 + y^2 - 4x - 6y - 12 = 0\]

(2) This equation is rearranges to the standard form as follows.
\[(x^2 - 4x + 4) + (y^2 - 6y + 9) - 4 - 9 - 12 = 0\]
\[\therefore \ (x - 2)^2 + (x - 3)^2 = 5^2\]

Therefore, the center is (2, 3)
Relationship between a Circle and a Straight Line

Three cases

\[ d < r \]
\[ d = r \]
\[ d > r \]

(a) Cross                (b) Contact              (c) No contact

Simultaneous Equation Problem

Circle \[ x^2 + y^2 + lx + my + n = 0 \]
Line \[ y = px + q \]

Discriminant \[ D = b^2 - 4ac \]

(a) Cross : \[ D > 0 \]  (b) Contact : \[ D = 0 \]  (c) No contact : \[ D < 0 \]
[Example 15.2] Find the coordinates of the common points of a circle and a straight line in the following.

(1) Circle \( x^2 + y^2 = 5 \) (i) and line \( y = x - 1 \) (ii).

(2) Circle \( x^2 + y^2 = 5 \) (iii) and line \( 2x - y + 5 = 0 \) (iv).

Ans.

(1) Substituting Eq. (ii) into Eq. (i), we have

\[
x^2 - x - 2 = (x + 1)(x - 2) = 0
\]

\[
\therefore x = -1, \ 2 \quad \therefore y = -2, \ 1
\]

The common points are \((-1, -2)\) and \((2, 1)\).

(2) Substituting Eq. (iv) into Eq. (iii), we have

\[
x^2 + 4x + 4 = (x + 2)^2 = 0
\]

\[
\therefore x = -2, \quad \therefore y = 1
\]

Therefore, the contact point is \((-2, 1)\).
[Exercise 15.1] The straight line $y = -2x + 1$ crosses with the circle $x^2 + y^2 = 1$. Find the chord length.

Pause the video and solve the problem by yourself.
Answer to the Exercise

[Exercise 15.1] The straight line $y = -2x + 1$ crosses with the circle $x^2 + y^2 = 1$. Find the chord length.

Ans.

By eliminating $y$, we have

$$x^2 + (-2x + 1)^2 = 1$$

$$5x^2 - 4x = 5x(x - \frac{4}{5}) = 0$$

$\therefore x = 0, \frac{4}{5}$

Then, the two cross points are

$$\left(0, \ 1\right), \ \left(\frac{4}{5}, \ -\frac{3}{5}\right)$$

The chord length is

$$\sqrt{\left(0 - \frac{4}{5}\right)^2 + \left(1 + \frac{3}{5}\right)^2} = \frac{4\sqrt{5}}{5}$$
Lesson 15
Graphs and Equations (II)

15B
- Tangent line to a Circle
- Line through a cross point
- Circle through cross points
- Region defined by inequalities
Tangent Line to a Circle

**Tangent Line**

- A tangent line to a circle intersects at a single point $T$.
- The radius of a circle is perpendicular to the tangent line.

- The equation of the tangent line is given by

\[ y - y_1 = -\frac{x_1}{y_1} (x - x_1) \]

That is,

\[ x_1 x + y_1 y = r^2 \]

Uhh…. What?
[Example 15.3] Find the tangent lines to circle $x^2 + y^2 = 2$, which passes point (3, 1).

**Ans.** Put tangent line $x_1 x + y_1 y = 2$

Since his line passes (3, 1)

$$3x_1 + y_1 = 2 \quad (i)$$

Since $(x_1, y_1)$ is on the given circle

$$x_1^2 + y_1^2 = 2 \quad (ii)$$

From (i) and (ii), we have

$$5x_1^2 - 6x_1 + 1 = (5x_1 - 1)(x_1 - 1) = 0$$

$$x_1 = \frac{1}{5}, \ 1 \quad \text{Therefore} \quad (x_1, y_1) = \left(\frac{1}{5}, \frac{7}{5}\right), \ (1, -1)$$

The tangent lines are

$$\frac{1}{5} x + \frac{7}{5} y = 2 \quad x - y = 2$$
Line and Circle Passing Through Cross Points

**Line**

Two lines: \( a_1x + b_1y + c_1 = 0 \) \( \quad \boxed{1} \)  
and \( a_2x + b_2y + c_2 = 0 \) \( \quad \boxed{2} \)

The line which passes the cross point is

\[ k \cdot (a_1x + b_1y + c_1) + a_2x + b_2y + c_2 = 0 \]

**Circle**

Two circles:

\( f(x, y) = x^2 + y^2 + l_1x + m_1y + n_1 = 0 \) \( \quad \boxed{1} \)

\( g(x, y) = x^2 + y^2 + l_2x + m_2y + n_2 = 0 \)

\( \quad \boxed{2} \) The circle which passes the cross points

\[ k \cdot f(x, y) + g(x, y) = 0 \]
[Example 15.4] Find the expression of a circle which passes the cross points of two circles \( x^2 + y^2 - 2x - 4y + 4 = 0 \) and \( x^2 + y^2 - 4x - 2y = 0 \), and also passes point \((1, 2)\).

Ans.

The line which passes the cross points.

\[
k(x^2 + y^2 - 2x - 4y + 4) + x^2 + y^2 - 4x - 2y = 0
\]

Substituting the coordinate of the point \((1, 2)\), we have

\[
-k - 3 = 0 \quad \therefore \quad k = -3
\]

Then

\[
-3(x^2 + y^2 - 2x - 4y + 4) + x^2 + y^2 - 4x - 2y = 0
\]

The answer is

\[
(x^2 + y^2) - x - 5y + 6 = 0
\]
Region Defined by Inequalities

**Line**

Straight line \( l : \ y = mx + n \)

**Circle**

Circle \( : (x - a)^2 + (y - b)^2 = r^2 \)
[Example 15.4] Illustrate the region given by the following expression.

\[ x^2 - 4x + y^2 - 6y + 9 < 0 \]

Ans. First, we rearrange the expression to the following standard form.

\[ (x - 2)^2 + (y - 3)^2 < 2^2 \]

Therefore, the boundary is given by a circle with radius \(2\) and with center \((2, 3)\).

The answer is given by the shaded region.
[Exercise 15.2] Illustrate the region which satisfies the following inequality.

\[(y - x + 3)(x^2 + y^2 - 12x - 6y + 20) < 0\]

Ans.

Pause the video and solve the problem by yourself.
[Exercise 15.2] Illustrate the region which satisfies the following inequality.

\((y - x + 3)(x^2 + y^2 - 12x - 6y + 20) < 0\)

**Ans.**

\[ AB < 0 \iff A > 0, \quad B < 0 \quad \text{or} \quad A < 0, \quad B > 0 \]

We add the following two regions. After rearrangement, we have

1. \( y > x - 3 \) and \( (x - 6)^2 + (y - 3)^2 < 5^2 \)
2. \( y < x - 3 \) and \( (x - 6)^2 + (y - 3)^2 > 5^2 \)

Let the boundary \( y = x - 3 \) be \( l \)
and the boundary
\[ (x - 6)^2 + (y - 3)^2 = 5^2 \]

be \( C \).

Then,
the range (1) is above \( l \) and inside of \( C \).
the range (2) is below \( l \) and outside of \( C \).