Course II : Calculus
**What is Calculus?**

**Calculus** is a branch of mathematics.
- functions, • limits, • derivatives, • integrals, • power series

**Calculus** is the study of change.
- cf. • *Geometry* is the study of shape.
  - • *Algebra* is the study of operation.

**Calculus** is a gateway to advanced mathematics.
- We must study and understand completely.

**Calculus** has wide applications in
- science, • engineering, • economics, • biology

**Calculus** has two branches
- • differential calculus, • integral calculus,
Lesson 01  Limit of Functions and Derivatives
Lesson 02  Derivative and Graphs
Lesson 03  Differentiation Formulas
Lesson 04  Derivatives of Trigonometric Functions
Lesson 05  Derivatives of Logarithmic Functions and Exponential Functions
Lesson 06  Applications of Derivatives to Equations and Inequality
Lesson 07  Application to Physics
Lesson 08  Approximation of a Function
Lesson 09  Antiderivatives
Lesson 10  Definite Integrals
Lesson 11  Estimating Area by Rectangles
Lesson 12  Application of Integrals (1)
Lesson 13  Application of Integrals (2)
Lesson 14  Differential Equations (1)
Lesson 15  Differential Equations (2)
Lesson 1
Limit of Functions and Derivatives

1A
• Limit of a function
Definition of a Limit

If a function \( f(x) \) can be made to be as close to \( L \) as desired by making \( x \) sufficiently close to \( a \), we say that "the limit of \( f(x) \), as \( x \) approaches \( a \), is \( L \)" and we write as follows

\[
\lim_{{x \to a}} f(x) = L
\]

We can also write \( f(x) \to L \) as \( x \to a \) and read "\( f(x) \) approaches \( L \) as \( x \) approaches \( a \)."
[Example] \( f(x) = x^2 \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>1.000000</td>
<td>3.0</td>
<td>9.000000</td>
</tr>
<tr>
<td>1.5</td>
<td>2.250000</td>
<td>2.5</td>
<td>6.250000</td>
</tr>
<tr>
<td>1.8</td>
<td>3.240000</td>
<td>2.2</td>
<td>4.840000</td>
</tr>
<tr>
<td>1.9</td>
<td>3.610000</td>
<td>2.1</td>
<td>4.410000</td>
</tr>
<tr>
<td>1.99</td>
<td>3.960100</td>
<td>2.01</td>
<td>4.040100</td>
</tr>
<tr>
<td>1.999</td>
<td>3.996001</td>
<td>2.001</td>
<td>4.004001</td>
</tr>
</tbody>
</table>

The limit of \( f(x) = x^2 \) as \( x \) approaches 2 is 4
Several Comments about the Limit

○ For the limit of a function to exist, the left-hand and right-hand limits must be equal, that is

\[
\lim_{{x \to a^-}} f(x) = L \quad \text{and} \quad \lim_{{x \to a^+}} f(x) = L
\]

○ \( \lim_{{x \to a}} f(x) = L \) is not always equal to \( f(a) \)
Example

Example 1.1 Find the limit value of the following function.

\[ \lim_{x \to 1} \frac{x^2 - x}{x - 1} \]

Ans.

\[ \lim_{x \to 1} \frac{x^2 - x}{x - 1} = \lim_{x \to 1} \frac{x(x - 1)}{x - 1} = \lim_{x \to 1} x = 1 \]

○ Even if the function has not a value at \( x = a \), the limit may exist.

Indeterminate Form

The forms \( 0^0, \frac{0}{0}, 1^\infty, \infty - \infty, \infty/\infty, 0 \times \infty, \infty^0 \), etc. are called indeterminate forms because they do not give enough information to determine values.
Example

Example 1.2 Find the limit value of the following function.

\[
\lim_{x \to 1} \frac{\sqrt{x - 1} - 1}{x}
\]

Ans.

\[
\lim_{x \to 1} \frac{\sqrt{x + 1} - 1}{x} = \lim_{x \to 1} \frac{\left(\sqrt{x + 1} - 1\right)\left(\sqrt{x + 1} + 1\right)}{x\left(\sqrt{x + 1} + 1\right)} = \lim_{x \to 1} \frac{(x + 1) - 1}{x\left(\sqrt{x + 1} + 1\right)}
\]

\[
= \lim_{x \to 1} \frac{1}{\left(\sqrt{x + 1} + 1\right)} = \frac{1}{2}
\]

Means to find a limit of an Indeterminate Form 0/0

(1) Case of Polynomial → Factor them

(2) Case of Irrational Function → Multiply the conjugate
Example 1.3 Determine the values of $a$ and $b$ so that the following expression holds.

\[
\lim_{x \to 1} \frac{x^2 + x - 2}{x^2 + ax + b} = 1
\]

Ans.

When $x \to 1$, then $x^2 + x - 2 \to 0$ and $x^2 + ax + b \to 1 + a + b$

In order for limit to exist, $1 + a + b$ must be zero. \[\therefore b = -a - 1\]

Substituting this, we have

\[
\lim_{x \to 1} \frac{x^2 + x - 2}{x^2 + ax + b} = \lim_{x \to 1} \frac{(x - 1)(x + 2)}{(x - 1)(x + a + 1)} = \lim_{x \to 1} \frac{x + 2}{x + a + 1} = \frac{3}{a + 2}
\]

Therefore,

\[
\frac{3}{a + 2} = 1 \quad \therefore \quad a = 1 \text{ and } b = -2
\]
Exercise

[ Exercise 1.1 ] Determine the values of $a$ and $b$ so that the following relationship holds.

$$\lim_{x \to 1} \frac{ax^2 + bx + 1}{x - 1} = 3$$

Ans.

Pause the video and try to solve by yourself
[Exercise 1.1] Determine the values of $a$ and $b$ so that the following relationship holds.

$$\lim_{x \to 1} \frac{ax^2 + bx + 1}{x - 1} = 3$$

**Ans.**

When $x \to 1$, then $x - 1 \to 0$ and $ax^2 + bx + 1 \to a + b + 1$

In order to exist a limit value $1$, $a + b + 1 = 0 \quad \therefore \quad b = -a - 1$

Substituting this, we have

$$\lim_{x \to 1} \frac{ax^2 + bx + 1}{x - 1} = \lim_{x \to 1} \frac{(x - 1)(ax - 1)}{x - 1} = \lim_{x \to 1} (ax - 1) = a - 1$$

Therefore,

$$a - 1 = 3 \quad \therefore \quad a = 4 \quad \text{and} \quad b = -5$$
Lesson 1
Limit of Functions and Derivatives

1B
• Derivatives of Functions
Average Rate of Change

The slope

\[ \Delta y = y_2 - y_1 \]

\[ \Delta x = x_2 - x_1 \]

The slope \( \frac{\Delta y}{\Delta x} \)

\[ \frac{\Delta y}{\Delta x} \quad \text{or} \quad \frac{f(b) - f(a)}{b - a} \]

Average Rate of Change

Increments

\[ \Delta x = x_2 - x_1 \]

\[ \Delta y = y_2 - y_1 \]
Definition of a Derivative

Derivative

The slope at point A (the tangent line $T$) can be obtained by making point B approach point A.

\[
\lim_{b \to a} \frac{f(b) - f(a)}{b - a} = f'(a)
\]

This is called the derivative of $f(x)$ at $a$.

Putting $b = a + h$, we also have

\[
\lim_{h \to 0} \frac{f(a + h) - f(a)}{h} = f'(a)
\]

That makes sense!
[Example 1-4] About the function \( f(x) = x^2 \)

(1) Find the average rate of change between \( x = 1 \) and \( x = 2 \).

(2) Find the instantaneous rate of change at \( x = a \).

(3) Find the point where the instantaneous rate of change is equal to the average rate of change between \( x = 1 \) and \( x = 2 \).

**Ans.**

(1) \( \frac{f(2) - f(1)}{2 - 1} = 4 - 1 = 3 \)

(2) \( f'(a) = \lim_{h \to 0} \frac{(a + h)^2 - a^2}{h} = \lim_{h \to 0} (2a + h) \)

\[ = 2a \]

(3) Using the results of (1) and (2), we put \( 2a = 3 \)

\[ \therefore a = \frac{3}{2} \]
Derivative as a Function

Let the number $a$ varies and replace it by $x$.

\[ f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \]

$f'(x)$ is called the derivative of $f(x)$ or the derivative function of $f(x)$ (because it has been “derived” from $f(x)$.)

Alternative notation

\[ f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx} f(x) = Df(x) = D_x f(x) \]

[Note]

The definition $\frac{dy}{dx}$ is read as: “the derivative with respect to $x$”, “by $y$”, $dx$ over “$dx$ simply “$dy$“$dx$
[Example 1-4] Find the derivative function of

\( f(x) = x \) \hspace{1cm} \( f(x) = x^2 \) \hspace{1cm} \( f(x) = x^3 \).

**Ans.** Definition

\[
  f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}
\]

(1) \( f'(x) = \lim_{h \to 0} \frac{(x + h) - x}{h} = \lim_{h \to 0} (1) = 1 \)

(2) \( f'(x) = \lim_{h \to 0} \frac{(x + h)^2 - x^2}{h} = \lim_{h \to 0} (2x + h) = 2x \)

(3) \( f'(x) = \lim_{h \to 0} \frac{(x + h)^3 - x^3}{h} = \lim_{h \to 0} (3x^2 + 3xh + h^2) = 3x^2 \)

**Formula**

\[
  \frac{d}{dx} (x^n) = nx^{n-1}
\]
Higher Derivatives

Since \( f'(x) \) is a function, it also has its own derivative which is denoted by
\[
\frac{d^2 y}{dx^2} = f''(x) = f^{(2)}(x)
\]
: The second derivative function

We can continue
\[
\frac{d^3 y}{dx^3} = f'''(x) = f^{(3)}(x)
\]
: The third derivative function

The process of finding a derivative function is called differentiation.
Example

[Example 1-5 ] If \( f(x) = x^3 - x \), find and interpret \( f''(x) \)

Using the formula \( \frac{d}{dx} (x^n) = nx^{n-1} \), we get

\[
\begin{align*}
  f'(x) &= 3x^2 - 1 \\
  f''(x) &= 6x
\end{align*}
\]

These derivatives are illustrated in the Right-hand side.

- \( f''(x) \) is the slope of the curve \( y = f'(x) \)
- \( f''(x) \) is the rate of change of \( y = f'(x) \)
[ Exercise 1.2 ] Function \( f(x) = x^3 + ax^2 + bx + c \)
Satisfy the conditions \( f(1) = 3 \), \( f(0) = 1 \) and \( f'(-1) = 16 \). Find the constants \( a \), \( b \) and \( c \).

Ans.

Pause the video and try to solve by yourself
[Exercise 1.2] Function \( f(x) = x^3 + ax^2 + bx + c \)
Satisfy the conditions \( f(1) = 3 \), \( f(0) = 1 \) and \( f'(-1) = 16 \). Find the constants \( a \), \( b \) and \( c \).

**Ans.**

The derivative function is
\[
f'(x) = 3x^2 + 2ax + b
\]

Given condition
\[
f(1) = 1 + a + b + c = 3
\]
\[
f(0) = c = 1
\]
\[
f'(-1) = 3 - 2a + b = 16
\]

From these equations
\[
a = -4, \quad b = 5, \quad c = 1
\]