Lesson 2
Derivative and Graphs

2A
• Tangent Lines to a Graph of a Function
The straight line through \( P(a, b) \) with slope \( m \)

\[
m = \frac{y-b}{x-a}
\]

Therefore

\[
y - b = m(x - a)
\]

The tangent line to the graph of \( y = f(x) \) at point \( (a, f(a)) \) is

\[
y - f(a) = f'(a)(x - a)
\]
[Examples 2-1] Find an equation of the tangent line to the graph 
\[ y = f(x) = x^2 + 1 \] at \( x = 1 \).

**Ans.**

The \( y \)-coordinate at \( x = 1 \) is \( f(1) = 2 \)

Since \( f'(x) = 2x \) we have \( f'(1) = 2 \)

Therefore,
\[ y - 2 = 2(x - 1) \]
\[ \therefore \quad y = 2x \]
[Examples 2-2] Find the equations and contact points of the tangent lines which contact with \( y = x^2 + 4 \) and pass \((1, 1)\).

**Ans.**

Let the contact point be \((a, a^2 + 4)\).

The derivative function \( f'(x) = 2x \).

The tangent line is

\[
y - (a^2 + 4) = 2a(x - a)
\]

Since this line passes \((1, 1)\)

\[
1 - (a^2 + 4) = 2a(1 - a)
\]

\[
\therefore a^2 - 2a - 3 = 0, \quad \therefore a = -1, 3
\]

When \( a = -1 \)

contact point \((-1, 5)\), tangent line \( y = -2x + 3 \)

When \( a = 3 \)

contact point \((3, 13)\), tangent line \( y = -6x + 5 \)
Behavior of graphs based on its derivatives

In open domains A and C:

\[ f'(x) < 0 \rightarrow f(x) \text{ is increasing.} \]

In open domain B:

\[ f'(x) > 0 \rightarrow f(x) \text{ is decreasing.} \]
[Examples 2-3] Investigate the change of the function \( y = x^3 + 3x^2 - 2 \) and illustrate the graph.

Derivative function \( y' = 3x^2 + 6x \)

Horizontal tangent line
\[
\begin{align*}
y' &= 3x(x + 2) = 0 \quad x = -2, \ 0 \\
y' > 0 & \text{ in } x < -2, \ x > 0 \\
y' < 0 & \text{ in } -2 < x < 0
\end{align*}
\]

<table>
<thead>
<tr>
<th>( x )</th>
<th>......</th>
<th>-2</th>
<th>......</th>
<th>0</th>
<th>......</th>
</tr>
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<tbody>
<tr>
<td>( y' )</td>
<td>+</td>
<td>0</td>
<td>-</td>
<td>0</td>
<td>+</td>
</tr>
<tr>
<td>( y )</td>
<td>( \nearrow )</td>
<td>2</td>
<td>( \searrow )</td>
<td>-2</td>
<td>( \nearrow )</td>
</tr>
</tbody>
</table>

y-intercept : \((0, 2)\)

x-intercepts:
\[
y = x^3 + 3x^2 - 2 = (x + 1)(x^2 + 2x - 2) = 0
\]
\[
(-1, 0), \quad (-1 + \sqrt{3}, 0) \quad (-1 - \sqrt{3}, 0)
\]
[Exercise 2-1] About the graph \( y = x^3 - 2x^2 - 3x \), answer the following questions.

(1) Find the equation of the tangent line at \((-1, 0)\).

(2) Find the cross-point of the graph and this tangent line.

(Hint: The cross-point of \( y = f(x) \) and \( y = ax + b \) is given by the roots of \( f(x) = ax + b \)).

Ans.

Pause the video and solve the problem by yourself.
[Exercise 2-1] About the graph \( y = x^3 - 2x^2 - 3x \), answer the following questions.

(1) Find the equation of the tangent line at \((-1, 0)\).
(2) Find the cross-point of the graph and this tangent line.

\[ y' = 3x^2 - 4x - 3 \]

\[ \therefore f''(-1) = 4 \]

\(1\) Tangent line
\[ y - 0 = 4(x + 1) \]
\[ \therefore y = 4x + 4 \]

\(2\) Cross point of the tangent line and the curve
\[ x^3 - 2x^2 - 3x = 4x + 4 \]

By factoring
\[ (x + 1)^2(x - 4) = 0 \]

Therefore,
\[ x = 4 \]

The cross-point is \((4, 20)\).
Lesson 2
Derivative and Graphs

2B
• Local Extrema
• Global Extrema
• Second derivative test
Local Extrema

- The function has a local maximum (local minimum) at \( x = c \)
  
  if \( f(c) \) is the maximum (minimum) value in a neighborhood around \( c \).

- If \( f(c) \) is a local extrema then \( f'(c) = 0 \)
Global Extrema

- **Absolute maximum (or global maximum)** is the maximum value in the whole domain $[a, b]$.

- **Absolute minimum (or global minimum)** is the minimum value in the whole domain $[a, b]$. 
(1) That the slope is zero $f'(c) = 0$ does not necessarily mean that the point is a local max./min. point.

(2) A number $c$ in the domain of $f(x)$ is called a **critical point** if either $f'(c) = 0$ or $f'(c)$ does not exist.
**Example**

**Examples 2-4** Find the maximum and minimum values of 
\[ f(x) = 2x^3 + 3x^2 - 12x + 1 \] in the domain \(-1 \leq x \leq 2\).

**Ans.**

1. Find extrema: 
   \[ f'(x) = 6x^2 + 6x - 12 = 6(x + 2)(x - 1) \]
   \[ x = -2, 1 \]
   \[ f(1) = -6 \]

2. Find the values at the boundaries: 
   \[ f(-1) = 14 \quad f(2) = 5 \]

3. Find \(y\) - intercept: 
   \[ y = f(0) = 1 \]

4. Make a table

<table>
<thead>
<tr>
<th>(x)</th>
<th>(-1)</th>
<th>(\ldots)</th>
<th>1</th>
<th>(\ldots)</th>
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<tbody>
<tr>
<td>(f'(x))</td>
<td>-</td>
<td>-</td>
<td>0</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>(f(x))</td>
<td>14</td>
<td>Local Min.</td>
<td>-6</td>
<td>Max.</td>
<td>5</td>
</tr>
</tbody>
</table>

5. Illustrate the graph

6. Find the max. and min. values:
   Max. value 14 (at \(x=-1\)), Min. value -6 (at \(x=1\))
Shape of a graph and Second Derivative

Shape of a graph

Local maximum

Concave down

\[ f''(x) < 0 \]

At \( x = a \), slope decreases

\[ f''(a) < 0 \]

Local minimum

Concave up

\[ f''(x) > 0 \]

At \( x = a \), slope increases

\[ f''(b) > 0 \]
Second Derivative Test

Second derivative test

Suppose that \( f'(c) = 0 \) at \( x = c \).

- If \( f''(c) > 0 \) then \( f(x) \) is a local minimum at \( x = c \).
- If \( f''(c) < 0 \) then \( f(x) \) is a local maximum at \( x = c \).
- If \( f''(x) = 0 \) then the situation is inconclusive.

Inflection Point

The point where the concavity of the graph changes from up to down or vice versa is called an inflection point.
[Exercise 3.2] A square cup was made by cutting the four corners of a thick square paper as shown in the Figure. When does the volume of this cup becomes maximum?

Ans.

Pause the video and solve the problem by yourself.
[Exercise 3.2] A square cup was made by cutting the four corners of a thick square paper as shown in the Figure. When does the volume of this cup become maximum?

**Ans.** The side length of the squares cut off from the corner: \( x \)

Then,
- the volume
- Condition \( 0 < x < 6 \)
- The derivative \( V' = 12(x - 2)(x - 6) \)

<table>
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<th>( \ldots )</th>
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<th>( \ldots )</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
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<td>0</td>
<td>-</td>
<td></td>
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<tr>
<td>( V )</td>
<td></td>
<td>128</td>
<td></td>
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</table>

\[ V = (12 - 2x)^2 x = 4(x^3 - 12x^2 + 36x) \]