Lesson 3
Differentiation Formulas

3A
- Power Rule
- Linearity Rule
- Product Rule
- Quotient Rule
The Power Rule

\[ \frac{d}{dx} x^n = nx^{n-1} \]

• This rule holds for any real number.

[Examples 3-1] Differentiate the following functions.

(1) \[ f(x) = \frac{1}{x^2} \]

(2) \[ f(x) = \sqrt{x^3} \]

(1) \[ f'(x) = \frac{d}{dt} \left( x^{-2} \right) = -2x^{-2-1} = -2x^{-3} = -\frac{2}{x^3} \]

(2) \[ f'(x) = \frac{d}{dt} \left( \sqrt{x^3} \right) = \frac{d}{dt} \left( x^{\frac{3}{2}} \right) = \frac{3}{2} \cdot \frac{1}{x^{\frac{1}{2}}} = \frac{3}{2} \sqrt{x} \]
Linearity Rules

Assume that $f(x)$ and $g(x)$ are differentiable functions.

- **Constant Multiple rule**: $(cf)' = cf''$
- **Sum Rule**: $(f + g)' = f' + g'$
- **Difference Rule**: $(f - g)' = f' - g'$

[ Proof of Sum Rule ]

By definition

$$\{f(x) + g(x)\}' = \lim_{h \to 0} \frac{(f(x+h) + g(x+h)) - (f(x) + g(x))}{h}$$

After rearrangement, we have

$$\{f(x) + g(x)\}' = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = f'(x) + g'(x)$$
Product Rule

Assume that \( f(x) \) and \( g(x) \) are differentiable functions.

\[
(fg)' = f'g + fg'
\]

\[
-f(x)g(x+h)+f(x)g(x+h) = 0 \quad (ii)
\]

\[
\left\{ f(x)g(x) \right\}' = \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}
\]

\[
= \lim_{h \to 0} \left\{ \frac{f(x+h) - f(x)}{h} \cdot g(x+h) + f(x) \cdot \frac{g(x+h) - g(x)}{h} \right\} \quad (iii)
\]

\[
= f'(x)g(x) + f(x)g'(x)
\]

[Examples 3-3] Find the derivative function of \( h(x) = 3x^2(5x + 1) \)

Ans. \( h'(x) = 6x(5x + 1) + 3x^2(5) = 45x^2 + 6x \)
The Quotient Rule

\[ \left( \frac{f}{g} \right)' = \frac{f'g - fg'}{g^2} \]

In particular \[ \left( \frac{1}{g} \right)' = -\frac{g'}{g^2} \]

The second one is proved as

\[ \left( \frac{1}{g(x)} \right)' = \lim_{h \to 0} \frac{1}{h} \cdot \left\{ \frac{1}{g(x + h)} - \frac{1}{g(x)} \right\} \]

\[ = \lim_{h \to 0} \left\{ \frac{-g(x + h) - g(x)}{h} \cdot \frac{1}{g(x + h)g(x)} \right\} = -\frac{g'(x)}{g(x)^2} \]

Using this, we have

\[ \left( \frac{f(x)}{g(x)} \right)' = \left( f(x) \cdot \frac{1}{g(x)} \right)' = \frac{f'(x)}{g(x)} + f(x) \cdot \frac{-g'(x)}{\{g(x)\}^2} = \frac{f'(x)g(x) - f(x)g'(x)}{\{g(x)\}^2} \]

[Examples 3-4] Compute the derivative function of \[ f(x) = \frac{x}{x + 1} \]

Ans.
\[ f'(x) = \frac{1(x + 1) - x(1)}{(x + 1)^2} = \frac{1}{(x + 1)^2} \]
[ Exercise 3-1 ] Calculate the derivatives of the following functions in two ways. First use the Quotient Rule, then rewrite the function algebraically and apply the Power Rule directly.

(1) \( g(x) = \frac{x^3 + 2x^2 + 3x^{-1}}{x} \)  

(2) \( h(t) = \frac{t^2 - 1}{t - 1} \)

Pause the video and solve the problem by yourself.


**Exercise**

[Ex.3-1] Calculate the derivatives of the following functions in two ways. First use the Quotient Rule, then rewrite the function algebraically and apply the Power Rule directly.

(1) \( g(x) = \frac{x^3 + 2x^2 + 3x^{-1}}{x} \)

(2) \( h(t) = \frac{t^2 - 1}{t - 1} \)

**Ans.**

(1) Quotient Rule

\[
g'(x) = \frac{(3x^2 + 4x - 3x^{-2})x - (x^3 + 2x^2 + 3x^{-1}) \cdot 1}{x^2} = \frac{2x^3 + 2x^2 - 6x^{-1}}{x^2} = 2x + 2 - 6x^{-3}
\]

Power Rule

\[
g'(x) = (x^2 + 2x + 3x^{-2})' = 2x + 2 - 6x^{-3}
\]

(2) Quotient Rule

\[
h'(t) = \frac{2t(t - 1) - (t^2 - 1) \cdot 1}{(t - 1)^2} = \frac{t^2 - 2t + 1}{(t - 1)^2} = \frac{(t - 1)^2}{(t - 1)^2} = 1
\]

Power Rule

\[
h'(t) = (t + 1)' = 1
\]
Lesson 3
Differentiation Formulas

3B
• Chain Rule
• Derivative of Implicit Functions
The Chain Rule

Composite function \( y = f(g(x)) \)

\[ y = f(u) \text{ and } u = g(x) \]

The Chain Rule

If \( f(x) \) and \( g(x) \) are differentiable, the next relationship holds.

\[
\frac{df(g(x))}{dx} = \frac{df(g)}{dg} \frac{dg(x)}{dx}
\]

Setting \( u = g(x) \), we may also write this as

\[
\frac{dy}{dx} = f'(u) \frac{du}{dx} \quad \text{or} \quad \frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}
\]
Example

[Examples 3-5] Calculate the derivative of \( y = \sqrt{x^3 + 1} \)

Ans.

This is a composite function in the form

\[
f(u) = \sqrt{u} \quad \text{and} \quad u = g(x) = x^3 + 1
\]

Since \( f'(u) = \frac{1}{2} u^{-1/2} \) and \( g'(x) = 3x^2 \), we have

\[
\frac{d}{dx} \sqrt{x^3 + 1} = \frac{1}{2} u^{-1/2} (3x^2) = \frac{1}{2} (x^3 + 1)^{-1/2} (3x^2) = \frac{3x^2}{2 \sqrt{x^3 + 1}}
\]
Derivative of Implicit Functions (1)

Two kinds of function

- Explicit function: \( y = f(x) \)  
  \[ \text{[Ex.]} \quad y = x^2 \]

- Implicit function: \( f(x, y) = 0 \)  
  \[ \text{[Ex.]} \quad x^2 + y^2 = 1 \]

Two ways to calculate the derivative.

Ex: circle \( x^2 + y^2 = 4 \)

(1) Solve for \( y \) and then differentiate.

\[
y = \pm \sqrt{4 - x^2}
\]

\[
\frac{dy}{dx} = \left( \pm \frac{x}{\sqrt{4 - x^2}} \right) = -\frac{x}{y}
\]
(2) Take derivative of each term and apply the chain rule.

[Ex.] A circle \( x^2 + y^2 = 1 \)

Take the derivative of both sides

\[
\frac{d}{dx} (x^2 + y^2) = \frac{d}{dx} (1)
\]

\[
\therefore 2x + \frac{d}{dx} (y^2) = 0
\]

Applying the chain rule

\[
2x + \frac{d}{dy} (y^2) \frac{dy}{dx} = 0
\]

\[
\therefore 2x + 2y \frac{dy}{dx} = 0
\]

Then

\[
\frac{dy}{dx} = -\frac{x}{y}
\]
[Exercise 3-2] Calculate the derivatives of the following functions

(1) \( y = (x^2 + 2x - 1)^2 \)

(2) \( y = \frac{1}{(3x - 2)^2} \)

Pause the video and solve the problem by yourself.
[ Ex.3-3 ] Calculate the derivatives of the following functions

1. \[ y = (x^2 + 2x - 1)^2 \]

2. \[ y = \frac{1}{(3x - 2)^2} \]

(1) \[ y' = 2(x^2 + 2x - 1)(2x + 2) = 4(x^2 + 2x - 1)(x + 1) \]

(2) \[ y' = \left\{(3x - 2)^{-2}\right\}' = -2(3x - 2) \cdot 3 = -6(3x - 2) \]
[ Ex.3-4 ] Find the derivative $\frac{dy}{dx}$ of the function $\frac{x^2}{9} + \frac{y^2}{4} = 1$
[ Ex.3-4 ] Find the derivative \( \frac{dy}{dx} \) of the function \( \frac{x^2}{9} + \frac{y^2}{4} = 1 \)

Taking the derivative of both sides with respect to \( x \), we have

\[
\frac{2x}{9} + \frac{2y}{4} \frac{dy}{dx} = 0
\]

Therefore, when \( y \neq 0 \), we have

\[
\frac{dy}{dx} = -\frac{4x}{9y}
\]