Lesson 4
Derivatives of Trigonometric Functions

4A
• Derivative of Sine Function
• Limit of \( \frac{\sin x}{x} \)
• Derivatives of Basic Trigonometric Function
Variation of slopes

Derivative by definition

\[
\begin{align*}
\frac{df}{dx} &= \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{\sin (x + h) - \sin x}{h} \\
&= \lim_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\
&= \lim_{h \to 0} \left( \cos x \frac{\sin h}{h} - \sin x \frac{1 - \cos h}{h} \right)
\end{align*}
\]
Consider a sector with central angle \( x \)

Compare the areas of \( \triangle OAB \), sector OAB, and \( \triangle OAT \)

\[
\frac{1}{2} \cdot 1 \cdot \sin x < \left( \pi \cdot 1^2 \right) \cdot \frac{x}{2\pi} < \frac{1}{2} \cdot 1 \cdot \tan x
\]

\[
\therefore \sin x < x < \tan x
\]

Divide by \( \sin x(>0) \)

\[
\therefore 1 < \frac{x}{\sin x} < \frac{1}{\cos x}
\]

\[
\therefore 1 > \frac{\sin x}{x} > \cos x
\]

As \( x \to 0 \)

\[
\lim_{x \to 0} \frac{\sin x}{x} = 1
\]
Derivative of Sine Function—Cont.

\[ f'(x) = \lim_{h \to 0} \left( \cos x \frac{\sin h}{h} - \sin x \frac{1 - \cos h}{h} \right) \]

\[ = \frac{(1 - \cos h)(1 + \cos h)}{h(1 + \cos h)} = \frac{1 - \cos^2 h}{h(1 + \cos h)} = \frac{\sinh h}{h} \frac{\sinh h}{1 + \cos h} \]

Therefore

\[ (\sin x)' = \cos x \]

That makes sense!
[Example 4-1] Derive the derivative of \( \cos x \) and \( \tan x \).

Ans.

(1) From the triangle in the right side
\[
\cos x = \sin \left( \frac{\pi}{2} - x \right)
\]
Therefore
\[
\left( \cos x \right)' = \left( \sin \left( \frac{\pi}{2} - x \right) \right)' = \cos \left( \frac{\pi}{2} - x \right) \cdot \left( \frac{\pi}{2} - x \right)
= \sin x \times (-1) = -\sin x
\]

(2) From the quotient rule
\[
\left( \tan x \right)' = \left( \frac{\sin x}{\cos x} \right)' = \frac{\cos x \cdot \cos x - \sin x (-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x}
= \frac{1}{\cos^2 x} = \sec^2 x
\]
Summary of Derivatives of Tri. Functions

(1) The basic trigonometric derivatives (Memorize!)

\[
\frac{d}{dx} \sin x = \cos x, \quad \frac{d}{dx} \cos x = -\sin x
\]

(2) Other standard relationships

( Derive from (1) if necessary)

\[
\frac{d}{dx} \tan x = \sec^2 x, \quad \frac{d}{dx} \sec x = \sec x \tan x
\]

\[
\frac{d}{dx} \cot x = -\csc^2 x, \quad \frac{d}{dx} \csc x = -\csc x \cot x
\]

[ note ]
These formula are valid only when the angle $x$ is measured in radians.
[Example 4.2] Find the derivatives of the following functions:

(1) \( y = \cos^2 x \)  \hspace{1cm} (2) \( y = x \sin x + \cos x \)

Ans.

(1) Chain rule

\[
y' = 2 \cos x \cdot \left(\cos x\right)' = 2 \cos x \cdot (- \sin x) = -2 \sin x \cos x = -\sin 2x
\]

(2) Product rule

\[
y' = (x \cdot \sin x)' + (\cos x)' = (1 \cdot \sin x + x \cos x) - \sin x = x \cos x
\]
Exercise

[Ex.4.1] Find the derivatives of the following functions:

(1) \[ y = \sin ax^2 \]
(2) \[ y = \frac{1}{\tan x} \]
(3) \[ y = \cos \left( \frac{x}{2} + \frac{\pi}{6} \right) \]

Pause the video and solve the problem by yourself.
[Ex.4.1] Find the derivatives of the following functions:

1. \( y = \sin ax^2 \)
2. \( y = \frac{1}{\tan x} \)
3. \( y = \cos \left( \frac{x}{2} + \frac{\pi}{6} \right) \)

(1) \[ \frac{d}{dx} \sin(ax^2) = \frac{d}{du} \sin u \frac{d}{dx} (ax^2) = \cos u \cdot (2ax) = 2ax \cos(ax^2) \]

(2) \[ y' = \left( \frac{\cos x}{\sin x} \right)' = \frac{(-\sin x) \sin x - \cos x (\cos x)}{\sin^2 x} = -\frac{\sin^2 x + \cos^2 x}{\sin^2 x} = -\frac{1}{\sin^2 x} \]

(3) \[ y' = \frac{d}{du} \cos u \frac{d}{dx} \left( \frac{x}{2} + \frac{\pi}{6} \right) = -\frac{1}{2} \sin \left( \frac{x}{2} + \frac{\pi}{6} \right) \]
Lesson 4
Derivatives of Trigonometric Functions

4B
- Derivatives and Motions
- Position, Velocity and Acceleration
- Simple Harmonic Motion
Velocity and Acceleration

Point P is moving on the straight line:

\[ x = f(t) \]

Its position is given by \( x = f(t) \)

The average velocity between \( t_1 \) and \( t_2 \):

\[ \bar{v} = \frac{\Delta x}{\Delta t} = \frac{x_2 - x_1}{t_2 - t_1} \]

The instantaneous velocity at \( t = t_1 \):

\[ v(t_1) = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{ds}{dt} = f'(t) \]

The instantaneous acceleration at \( t = t_1 \):

\[ \alpha = \frac{dv}{dt} = f''(t) \]
Example

[ Example 4-3 ] The position of the mass moving on the $x$-axis is given by $s(t) = t^3 - 3t^2 - 9t + 10$.

(1) Find the velocity and the acceleration at $t = 2$.

(2) Investigate the motion during $-2 \leq t \leq 4$.

\[ v = \frac{ds}{dt} = 3t^2 - 6t - 9 = 3(t + 1)(t - 3) \quad \therefore \quad v(2) = -9 \]

\[ a = \frac{dv}{dt} = 6(t - 1) \quad \therefore \quad a(2) = 6 \]

(2)

\begin{center}
\begin{tabular}{|c|c|c|c|c|c|}
\hline
$t$ & $-2$ & \ldots & $-1$ & \ldots & $3$ & \ldots & $4$ \\
\hline
$v$ & $+$ & $0$ & $-$ & $0$ & $+$ \\
\hline
$s$ & 8 & $\nearrow$ & 15 & $\searrow$ & $-17$ & $\nearrow$ & $-10$ \\
\hline
\end{tabular}
\end{center}
Vertical position: \[ y = A \sin(\omega t + \alpha) \]

Velocity: \[ v = \frac{dy}{dt} = A \omega \cos(\omega t + \alpha) \]

Acceleration: \[ a = \frac{dv}{dt} = \frac{d^2v}{dt^2} = -A \omega^2 \sin(\omega t + \alpha) \]
[Exercise.4.2] Point P is moving on the x-axis. Its position is given by
\[ x = 2t + \cos t \]. Find the time when the point has the maximum velocity and its maximum velocity.

Pause the video and solve the problem by yourself.
[Exercise.4.2] Point P is moving on the x-axis. Its position is given by
\[ x = 2t + \cos t \]. Find the time when the point has the maximum velocity and its maximum velocity.

Ans.

Velocity \( v = \frac{dx}{dt} = 2 - \sin t \)

Maximum velocity occurs at \( \sin t = -1 \)

Therefore \( t = \frac{3}{2} \pi + 2n \pi \)

Maximum velocity is 3.