Lesson 8
Approximation of a Function

8A
• Linear Approximation
Linear Approximation

Definition of the Derivative

\[
\lim_{h \to 0} \frac{f(a + h) - f(a)}{h} = f'(a)
\]

When \(|h|\) is small

\[
\frac{f(a + h) - f(a)}{h} \approx f'(a)
\]

Rearrange

\[
f(a + h) \approx f(a) + f'(a)h
\]

If we put \(a + h = x\)

\[
f(x) \approx f(a) + f'(a)(x - a)
\]
Example

**Example 8-1** Derive the approximate expression of \( f(x) = \sqrt{x} \) in the neighborhood of \( a = 1 \).

**Ans.**

\[
f'(x) = 1 \cdot \frac{1}{2} x^{-\frac{1}{2}}
\]

At \( a = 1 \)

\[
f(1) = \sqrt{1} = 1 \quad \text{and} \quad f'(1) = \frac{1}{2}
\]

Therefore

\[
\sqrt{x} \approx f(1) + f'(1)(x - 1) = 1 + \frac{1}{2}(x - 1) = \frac{1}{2}x + \frac{1}{2}
\]

For example

\( x = 1.1 \quad \text{Calculator} \quad \sqrt{1.1} = 1.0488... \)

\( \text{This approximation} \quad \sqrt{1.1} = \frac{1}{2} \times 1.1 + \frac{1}{2} = 1.05 \)

\{ \text{Error 1.1%} \}
Example 8-2 (1) When \( h \) is small, find the linear approximation of \( \sin(a + h) \).

(2) Find the approximate value of \( \sin 31^\circ \).

Ans. (1) \( f(x) = \sin x \) \quad \therefore \quad f'(x) = \cos x

\[
\sin(a + h) = f(a + h) \approx f(a) + f'(a)h = \sin a + h \cos a
\]

(2)

\[
31^\circ = \frac{\pi}{6} + \frac{\pi}{180}
\]

We put \( h = \frac{\pi}{180} \)

\[
\sin\left(\frac{\pi}{6} + \frac{\pi}{180}\right) \approx \sin\frac{\pi}{6} + \frac{\pi}{180} \cos\frac{\pi}{6} = \frac{1}{2} + \frac{\pi}{180} \cdot \frac{\sqrt{3}}{2}
\]

Using \( \pi = 3.142, \sqrt{3} = 1.732 \)

We have

\[
\sin 31^\circ \approx \frac{1}{2} + \frac{3.142}{180} \cdot \frac{1.732}{2} \approx 0.5151
\]
Case that \( |x| \) is small

\[ a + h = x \]
\[ f(x) \approx f(a) + f'(a)(x - a) \]

When \( x \) is small

We put \( a = 0 \).

\[ f(x) \approx f(0) + f'(0)x \]

Examples 8-3

When \( |x| \) is small enough, derive the approximate expression of \( \sqrt{1 + x} \)

Ans.

\[ (\sqrt{1 + x})' = \frac{1}{2\sqrt{1 + x}} \]

Therefore

\[ f(x) \approx f(0) + f'(0)x = 1 + \frac{1}{2}x \]
Exercise

Ex.8-1 Find the approximate value of $\cos 29^\circ$ in the following two ways.

1. Use the formula $f(x) = f(a + h) \approx f(a) + f'(a)h$

2. Use the formula $f(x) \approx f(0) + f'(0)x$

Ans.

Pause the video and solve the problem by yourself.
Ex.8-1 Find the approximate value of \( \cos 29^\circ \) in the following two ways.

(1) Use the formula \( f(x) = f(a + h) \approx f(a) + f'(a)h \)

(2) Use the formula \( f(x) \approx f(0) + f'(0)x \)

Ans. (1) We put \( f(x) = \cos x \)

Then \( f'(x) = -\sin x \)

\[
\cos 29^\circ = \cos\left(\frac{\pi}{6} - \frac{\pi}{180}\right) \approx \cos\left(\frac{\pi}{6}\right) - \frac{\pi}{180} \sin\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} - \frac{\pi}{180} \cdot \frac{1}{2}
\]

\[
\approx \frac{1.732}{2} - \frac{3.142}{180} \cdot \frac{1}{2} = 0.875
\]

(2) We put \( f(x) = \cos\left(\frac{\pi}{6} + x\right) \)

Then \( f'(x) = -\sin\left(\frac{\pi}{6} + x\right) \)

\[
\cos 29^\circ = f\left(-\frac{\pi}{180}\right) \approx \cos\left(\frac{\pi}{6}\right) - \sin\left(\frac{\pi}{6}\right)\left(-\frac{\pi}{180}\right)
\]

\[
= \frac{\sqrt{3}}{2} + \frac{\pi}{180} \cdot \frac{1}{2} \approx \frac{1.732}{2} + \frac{3.142}{180} \cdot \frac{1}{2} = 0.875
\]
Lesson 8
Application of a Function

8B
- Tayler Expansion
Suppose that $f(x)$ is expanded in power series

$$f(x) = \sum_{n=0}^{\infty} a_n (x - c)^n = a_0 + a_1 (x - c) + a_2 (x - c)^2 + a_3 (x - c)^3 + ...$$

By differentiating term by term, we have

$$f'(x) = a_1 + 2a_2 (x - c) + 3a_3 (x - c)^2 + 4a_4 (x - c)^3 + ...$$

$$f''(x) = 2a_2 + 2 \cdot 3a_3 (x - c) + 3 \cdot 4a_4 (x - c)^2 + ...$$

......

Setting $x = c$, we have

$$f(c) = a_0, \quad f'(c) = a_1, \quad f''(c) = 2a_2, \quad ... \quad f^{(k)}(c) = k! a_k, \quad ...$$

Namely, the coefficients are given by

$$a_k = \frac{f^{(k)}(c)}{k!}$$
Taylor series

For a function $f(x)$ that has continuous derivatives in the neighborhood of $c$, the Taylor series expansion of $f(x)$ is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (x - c)^n$$

$$= f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \frac{f'''(c)}{3!}(x - c)^3 + \ldots$$

For $c = 0$

Maclaurin series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + f'(0)x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + \ldots$$
Example 8-3 Derive the approximate expression for $\sin x$ near $x = 0$ up to the seventh order.

Ans. We put $f(x) = \sin x$

Then $f'(x) = \cos x, f''(x) = -\sin x$

In general $f^{(2n)}(x) = (-1)^n \sin x, f^{(2n+1)}(x) = (-1)^n \cos x$

For $x = 0$ : $f^{(2n)}(0) = 0$

Using these values

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}.$$
Ex.8-2 Find Maclaurin series for $f(x) = e^x$

Ans.

Pause the video and solve the problem by yourself.
Ex. 8-2 Find Maclaurin series for \( f(x) = e^x \)

Ans.

The \( n \)-th derivative is \( f^{(n)}(x) = e^x \) for all \( n \).

Thus, these values for \( x = 0 \) are

\[
f(0) = f'(0) = f''(0) = \ldots = e^0 = 1
\]

The coefficients are

\[
a_k = \frac{f^{(k)}(0)}{k!} = \frac{1}{k!}
\]

Therefore

\[
e^x = \sum_{k=1}^{\infty} \frac{x^k}{k!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots
\]