Lesson 9
Antiderivatives

9A
• Antiderivative
Inverse Problem

To find the function when its derivative is given.

[Ex.] velocity \( v(t) \left( = \frac{ds(t)}{dt} \right) \) \( \rightarrow \) position \( s(t) \)

Antiderivative

A function \( F(x) \) is an antiderivative (or primitive integral) of \( f(x) \) if \( F'(x) = f(x) \).

[Ex.] \( F(x) = x^3 \) is an antiderivative of \( f(x) = 3x^2 \) because \( \frac{d}{dx} \left( x^3 \right) = 3x^2 \).
Example:
\[ x^3, \ x^3 - 2, \ x^3 + 2 \ : \ \text{antiderivatives of} \ 3x^2 \]

**Theorem**

When \( F'(x) = f(x) \), every other antiderivative is of the form
\[ F(x) + C \quad (C \ : \ \text{constant}) \]

**Indefinite Integral**

The collection of all antiderivatives of a function is called the **general antiderivative** or **indefinite integral** of \( f(x) \) and denoted by \( \int f(x)\,dx \)

Namely,
\[ \int f(x)\,dx = F(x) + C \]
Fundamental Indefinite Integrals

\[ x^3 + C \overset{\text{Differentiation}}{\rightarrow} 3x^2 \overset{\text{Integration}}{\leftarrow} \]

Algebraic Functions

\[ \text{< Differentiation >} \quad \frac{d}{dx} x^{n+1} = (n + 1)x^n \quad \text{< Integration >} \quad \int x^n \, dx = \frac{x^{n+1}}{n+1} + C \quad \text{for} \quad n \neq -1 \]

Example

\[ \int x^5 \, dx = \frac{1}{6} x^6 + C \]

Note

For \( n = -1 \)

\[ \int x^{-1} \, dx = \frac{x^{-1+1}}{-1+1} + C = \frac{x^0}{0} + C \quad \text{Meaningless} \]
Graphical Interpretation.

**Differentiation**

Graph: \[ y = f(x) \]

Slope: \[ \frac{df(x)}{dx} \]

**Integration**

Slope: \[ \frac{df(x)}{dx} \]

Graph: \[ y = f(x) \]

<Another interpretation>

Graph: \[ y = f(x) \] \(\Rightarrow\) Area

(We will study this later.)
Graphical Interpretation of Differentiation

How to find the derivative \( y = \frac{1}{x} \) graphically from \( y = \ln x \)

\[ y = \ln x \]

\[ y = \frac{d}{dx} \ln x = \frac{1}{x} \]
Graphical Interpretation of Integration

How to find the indefinite function graphically from \( y = \frac{1}{x} \)

\[ y = \ln|x| + C \]
Algebraic derivation

For \( x > 0 \) (Defined domain of \( y = \ln x \) )

\[
\frac{d}{dx} \ln x = \frac{1}{x} \quad \iff \quad \int \frac{1}{x} \, dx = \ln x + C
\]

For \( x < 0 \)

From the graphical discussion, we assume \( y = \ln|x| \)

Put \( x = -u \) (\( u > 0 \))

From the chain rule

\[
\frac{dy}{dx} = \frac{d(\ln|x|)}{dx} = \frac{d(\ln u)}{du} \cdot \frac{du}{dx} = \frac{1}{u} \cdot (-1) = \frac{1}{x}
\]

For all \( x \neq 0 \)

\[
\int \frac{1}{x} \, dx = \ln|x| + C
\]
Trigonometric Integrals

\[ \frac{d}{dx} \sin x = \cos x \quad \Rightarrow \quad \int \cos x \, dx = \sin x + C \]

\[ \frac{d}{dx} \cos x = -\sin x \quad \Rightarrow \quad \int \sin x \, dx = -\cos x + C \]

\[ \frac{d}{dx} \tan x = \frac{1}{\cos^2 x} \quad \Rightarrow \quad \int \frac{1}{\cos^2 x} \, dx = \tan x + C \]
Exponential Integrals

\[
\frac{d}{dx} e^x = e^x \quad \Rightarrow \quad \int e^x \, dx = e^x + C
\]

\[
\frac{d}{dx} a^x = a^x \ln a \quad \Rightarrow \quad \int a^x \, dx = \frac{a^x}{\ln a} + C
\]
Basic Rules of Integration

Sum Rule

\[ \int \{f(x) + g(x)\} \, dx = \int f(x) \, dx + \int g(x) \, dx \]

Constant Multiple Rule

\[ \int cf(x) \, dx = c \int f(x) \, dx \]
Examples 9-2  Find the indefinite integrals of the following function.

(1)  \((x + 1)(x - 2)\)

(2)  \(\sin x + 2 \cos x\)

Ans.

(1)  \[\int (x + 1)(x - 2)\,dx = \int (x^2 - x - 2)\,dx = \int x^2\,dx - \int x\,dx - 2\int dx\]

\[= \frac{x^3}{3} - \frac{x^2}{2} - 2x + C\]

(2)  \[\int (\sin x + 2 \cos x)\,dx = \int \sin x + 2\int \cos x\,dx\]

\[= -\cos x + 2\sin x + C\]
Exercise. 9-1  The slope of the curve $y = f(x)$ is given by $3x^2$ and this curve passes the point $(1,2)$. Determine the equation of this curve.

Ans.

Pause the video and solve by yourself.
Exercise 12-1] The slope of the curve $y = f(x)$ is given by $3x^2$ and this curve passes the point (1,2). Determine the equation of this curve.

Ans.

\[ f'(x) = 3x^2 \]

\[ f(x) = \int 3x^2 \, dx = x^3 + C \]

Since this line passes the point (1,2),

\[ f(1) = 1 + C = 2 \]

\[ C = 1 \]

\[ y = x^3 + 1 \]
Lesson 9
Antiderivatives

9B
- Substitution Method
- Integration by Parts (IBP)
Substitution Method (1)

Applicable when the integrand is of the form \( f(u(x))u'(x) \)

Example: \((x^2 + 1)^5 x \quad \sin^5 x \cos x\)

\[
\begin{align*}
\text{Let } & F'(u) = f(u) \quad \text{or} \quad F(u) = \int f(u)du \quad (1) \\
\text{From the Chain Rule } & \frac{d}{dx} F(u(x)) = F'(u)u'(x) = f(u)u'(x) \\
\text{By integration, we have } & F(u(x)) = \int f(u)u'(x)dx \quad (2) \\
\text{From (1) and (2), we have } & \begin{array}{c}
\int f(u(x))u'(x)dx = \int f(u)du
\end{array}
\end{align*}
\]
Substitution Using Differentials

\[ u = u(x) \quad \rightarrow \quad \frac{du}{dx} = u'(x) \quad \rightarrow \quad du = u'(x) \, dx \]

< Example >  If \( u = x^3 \), then \( du = 3x^2 \, dx \)

\[
\int f(u)u'(x) \, dx = \int f(u) \, du \\
\]

\[ du = \frac{du}{dx} \, dx \]
Examples 9-4 Find the indefinite integral of the following functions.

(1) \[ \int (x^2 + 1)^5 \, dx \]

(2) \[ \int \sin^5 x \cos x \, dx \]

Ans. (1) We put \( x^2 + 1 = u \)

Then \( 2x \frac{dx}{du} = 1 \), \( \therefore \) \( xdx = \frac{1}{2} \, du \)

Substitution

\[ \int (x^2 + 1)^5 \, dx = \int u^5 \frac{1}{2} \, du = \frac{1}{12} u^6 + C = \frac{1}{12} (x^2 + 1)^6 + C \]

(2) We put \( \sin x = u \)

Then \( \cos x \frac{dx}{du} = 1 \), \( \therefore \) \( \cos x \, dx = \frac{1}{\cos x} \, du \)

Substitution

\[ \int \sin^5 x \cos x \, dx = \int u^5 \, du = \frac{1}{6} u^6 + C = \frac{1}{6} \sin^6 x + C \]
Substitution Method

\[ \int f(x)dx = \int f(x(t))x'(t)dt \]

This is also written as follows.

\[ \int f(x)dx = \int f(x(t)) \frac{dx}{dt} dt \]
Examples 9-5  Find the indefinite integral of the following functions.

\[ \int x\sqrt{1-x} \, dx \]

**Ans.** We put \( \sqrt{1-x} = t \). Then \( x = 1 - t^2 \) and \( \frac{dx}{dt} = -2t \).

Therefore

\[ \int x\sqrt{1-x} \, dx = \int (1-t^2)t(-2t) \, dt = \int (1-t^2)t(-2t) \, dt \]

\[ = 2 \int (t^4 - t^2) \, dt = 2 \left( \frac{t^5}{5} - \frac{t^3}{3} \right) + C = -\frac{2}{15} (2 + 3x)(1-x)\sqrt{1-x} + C \]
Integration by Parts

We studied the derivative of product

\[ \{f(x)g(x)\}' = f'(x)g(x) + f(x)g'(x) \]

After rearrangement, we have

\[ f(x)g'(x) = \{f(x)g(x)\}' - f'(x)g(x) \]

Therefore, we obtain the following formula

\[
\int f(x)g(x)'\,dx = f(x)g(x) - \int f(x)'g(x)\,dx
\]
Examples 9-5 Evaluate the following integral.

$$\int x \cos x \, dx$$

Ans.

$$\int x \cos x \, dx = x \sin x - \int 1 \cdot \sin x \, dx = x \sin x + \cos x + C$$
Exercise 9-2 Evaluate the following.

(1) \[ \int \cos^2 x \sin x \, dx \]

(2) \[ \int xe^{6x} \, dx \]

Ans.

Pause the video and solve by yourself.
Exercise 9-2  Evaluate the followings.

(1) \[ \int \cos^2 x \sin x \, dx \]  
(2) \[ \int xe^{6x} \, dx \]

Ans.  
(1) Since \((\cos x)' = -\sin x\), we put \(\cos x = u\)

\[ \therefore -\sin x = \frac{du}{dx} \]

Therefore

\[ \int \cos^2 x \sin x \, dx = \int u^2 (-du) = -\frac{1}{3} u^3 + C = -\frac{1}{3} \cos^3 x + C \]

(2) 

\[ \int xe^{6x} \, dx = x \left( \frac{e^{6x}}{6} \right) - \int 1 \cdot \left( \frac{e^{6x}}{6} \right) \, dx = \frac{x}{6} e^{6x} - \frac{x}{36} e^{6x} + C \]