Lesson 12
Application of Integrals (1)

12A
• Areas of plane regions
Area of the strip \[ \Delta S = f(x)\,dx \] where \( f(x) > 0 \)

Total area over \([a, b]\)

\[ S \approx \sum f(x)\Delta x \quad \rightarrow \quad S = \int_a^b f(x)\,dx \]
When \( f(x) \geq g(x) \) in \([a, b]\), area \( S \) is

\[
S = \int_{a}^{b} \{f(x) - g(x)\} \, dx
\]

[Proof]

Consider \( y = f(x) + k \) and \( y = g(x) + k \)

\[
S = S_1 - S_2 = \int_{a}^{b} \{f(x) + k\} \, dx - \int_{a}^{b} \{g(x) + k\} \, dx = \int_{a}^{b} \{f(x) - g(x)\} \, dx
\]
Case of a Negative Function

When \( f(x) < 0 \) in the domain \([a, b]\)

\[
S = \int_{a}^{b} \{0 - f(x)\} dx = -\int_{a}^{b} f(x) dx
\]

Ahh! That’s so easy!
Example 12-1  Find the area between the graph of $y = x^2 - x - 2$ and the $x$-axis, from $x = -2$ to $x = 3$.

Ans.

$$y = x^2 - x - 2 = (x - 2)(x + 1) = 0$$

Let the areas be $S_1$, $S_2$ and $S_3$

$$S_1 = \int_{-2}^{-1} (x^2 - x - 2)dx = \left[\frac{x^3}{3} - \frac{x^2}{2} - 2x\right]_{-2} = 1.833$$

$$S_2 = -\int_{-1}^{2} (x^2 - x - 2)dx = 4.5$$

$$S_3 = -\int_{2}^{3} (x^2 - x - 2)dx = 1.833$$

Therefore, $S = S_1 + S_2 + S_3 \approx 1.833 + 4.5 + 1.833 = 8.166$
Example 12-2  Determine the area of the region enclosed by the functions \( y = \sqrt{x} \) and \( y = x^2 \).

**Ans.**

The area is given by the blue region in the figure. Cross points:

\[
\sqrt{x} = x^2
\]

\[
x = x^4, \quad \therefore \quad x(x - 1)(x^2 + x + 1) = 0
\]

\((0,0), (1, 1)\)

\[
S = \int_{0}^{1} (\sqrt{x} - x^2) \, dx = \left[ \frac{2}{3} x^{\frac{3}{2}} - \frac{1}{3} x^3 \right]_{0}^{1} = \frac{1}{3}
\]
Area Between the Graph and the y-Axis

The area between the graph $x = g(y)$ and the $y$-axis is given by

$$ S = \int_c^d g(y) \, dy $$
Exercise

Ex.12-1  Find the area of the region enclosed by $y = \sin 2x$ and $y = \cos x$ in the domain $0 \leq x \leq \frac{\pi}{2}$.

Ans.

Pause the video and solve by yourself.
Ex. 12-1  Find the area of the region enclosed by \( y = \sin 2x \) and \( y = \cos x \) in the domain \( 0 \leq x \leq \frac{\pi}{2} \).

Ans.  We put \( \sin 2x = \cos x \)

\[ \therefore \cos x(2\sin x - 1) = 0, \]

\[ \therefore \cos x = 0 \quad \text{or} \quad \sin x = \frac{1}{2} \]

\[ \therefore x = \frac{\pi}{2}, \frac{\pi}{6} \]

\[ S = \int_{0}^{\frac{\pi}{6}} (\cos x - \sin 2x)dx + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (\sin 2x - \cos x)dx \]

\[ = \left[ \sin x - \frac{1}{2} \cos 2x \right]_{0}^{\frac{\pi}{6}} + \left[ -\frac{1}{2} \cos 2x - \sin x \right]_{\frac{\pi}{6}}^{\frac{\pi}{2}} = \frac{1}{2} \]
Lesson 12
Application of Integrals (1)

12B
• Volumes of Solids
Volumes of Solids

Volume of the slab \( \Delta V \approx S(x) \Delta x \)

Total volume of the solid between \( x=a \) and \( x=b \)

\[
V \approx \sum S(x) \Delta x \quad \rightarrow \quad V = \int_a^b S(x) \, dx
\]
[Example 12-2]
Find the volume of a cone with bottom radius \( r \) and height \( h \).

Ans.

Set the x-axis as shown in the figure.

Area of the bottom \( S(h) = \pi r^2 \)

From \( S(x) : S(h) = x^2 : h^2 \), we have

\[
S(x) = \frac{\pi r^2}{h^2} x^2
\]

Therefore

\[
V = \int_0^h \left( \frac{\pi r^2}{h^2} x^2 \right) dx = \frac{\pi r^2}{h^2} \left[ \frac{x^3}{3} \right]_0^h = \frac{\pi}{3} r^2 h
\]
Ex.12-2 Find the volume of the pyramid having a horizontal square cross section. The bottom side length is $a$ and the height is $h$. 

Ans.

Pause the video and solve by yourself.
Ex. 12-2  Find the volume of the pyramid having a horizontal square cross section. The bottom side length is \(a\) and the height is \(h\).

**Ans.**  We consider the \(z\)-axis vertically.

The horizontal cross section at \(z\) :

\[
S(z) : a^2 = (h - z)^2 : h^2
\]

\[
\therefore S(z) = \frac{a^2 (h - z)^2}{h^2}
\]

\[
V = \int_0^h \frac{a^2 (h - z)^2}{h^2} dz = \frac{a^2}{h^2} \int_0^h (h^2 - 2hz + z^2) dz
\]

\[
= \frac{a^2}{h^2} \left[ h^2z - hz^2 + \frac{1}{3}z^3 \right]_0^h = \frac{1}{3} a^2 h
\]
Volumes of Solids of Revolution

When the solid is generated by revolving a region about the x-axis

\[ V = \int_a^b S(x)\,dx = \int_a^b \pi r^2\,dx = \int_a^b \pi f(x)^2\,dx \]
[Examples 12-3]
Find the volume of a sphere with radius $r$:

Ans.

The upper half of the circle.

$$y = \sqrt{r^2 - x^2}$$

By rotating this blue area, we have a circle.

Volume of a circle

$$V = \int_{-r}^{r} \pi y^2 \, dx = \int_{-r}^{r} \pi (r^2 - x^2) \, dx$$

$$= \pi \left[ r^2 x - \frac{x^3}{3} \right]_{-r}^{r} = \frac{4}{3} \pi r^3$$

That makes sense!
Ex.12-3  Find the volume of the solid made by rotating the region surrounded by \( f(x) = \sqrt{x} \) and \( y=x \).

Ans.

Pause the video and solve by yourself.
**Exercise**

**Ex.12-3** Find the volume of the solid made by rotating the region surrounded by $y = \sqrt{x}$ and $y = x$.

**Ans.**

This volume can be obtained by subtracting $B$ from $A$, where:

$$V_A = \pi \int_0^1 (\sqrt{x})^2 \, dx = \pi \left[ \frac{1}{2} x^2 \right]_0^1 = \frac{1}{2} \pi$$

$$V_B = \pi \int_0^1 (x)^2 \, dx = \pi \left[ \frac{1}{3} x^3 \right]_0^1 = \frac{1}{3} \pi$$

$$V = V_B - V_A = \frac{1}{3} \pi - \frac{1}{2} \pi = -\frac{1}{6} \pi$$