Lesson 3
Linear and Quadratic Inequalities

3A
• Inequalities of numbers
• Linear inequalities
### Inequality signs

\[
\begin{align*}
    a < b & \quad \text{is less than } b \\
    a \leq b & \quad \text{is less than or equal to } b \\
    a > b & \quad \text{is greater than } b \\
    a \geq b & \quad \text{is greater than or equal to } b
\end{align*}
\]

### Intervals

A (real) interval is a set of real number that lies between two numbers.

**Closed interval**  \([a, b]\)  \(\{x \in \mathbb{R} : a \leq x \leq b\}\)

**Open interval**  \((a, b)\)  \(\{x \in \mathbb{R} : a < x < b\}\)

**Half-open interval**  \([a, b)\)  \(\{x \in \mathbb{R} : a \leq x < b\}\)

**Half-open interval**  \((a, b]\)  \(\{x \in \mathbb{R} : a < x \leq b\}\)
Some Properties of Inequalities

1. Transitivity
   If \(a > b\) and \(b > c\), then \(a > c\).

2. Addition
   If \(a > b\), then \(a + c > b + c\).

3. Subtraction
   If \(a > b\), then \(a - c > b - c\).

4. Multiplication and Division
   If \(a > b\) and \(c > 0\), then \(ac > bc\) and \(\frac{a}{c} > \frac{b}{c}\).
   
   If \(a > b\) and \(c < 0\), then \(ac < bc\) and \(\frac{a}{c} < \frac{b}{c}\).

From the third property, we can derive the following by putting \(b = c\).

If \(a > c\), then \(a - c > 0\).
Example 1. Prove the following inequality \( \frac{a + b}{2} \geq \sqrt{ab} \quad (a \geq 0, \ b \geq 0) \)

Ans. \[
\frac{a + b}{2} - \sqrt{ab} = \frac{a + b - 2\sqrt{ab}}{2} = \frac{\sqrt{a^2 + b^2 - 2ab}}{2} = \frac{(\sqrt{a} - \sqrt{b})^2}{2} \geq 0
\]

Therefore \( \frac{a + b}{2} \geq \sqrt{ab} \)

Equality holds when \( a = b \).

[ Note ]
\( \frac{a + b}{2} \): Arithmetic mean
\( \sqrt{ab} \): Geometric mean

Example
\[
\frac{12 + 3}{2} = 7.5 \quad \sqrt{12 \times 3} = 6
\]
\[
\frac{8 + 7}{2} = 7.5 \quad \sqrt{8 \times 7} = 7.48
\]
\[
\frac{7.5 + 7.5}{2} = 7.5 \quad \sqrt{7.5 \times 7.5} = 7.5
\]
Linear Inequality

One balance weight has 100g. Let the weight of the apple be $x$. Then we have

$$x + 100 > 3 \times 100 \quad \therefore x > 200$$

Linear inequality \[ ax + b > cx + d \]

$\rightarrow$ \[ ax - cx > d - b \]

$\rightarrow$ Divide by $(a - c)$ but be careful of its sign.

Example 1. Solve the following inequality $4x - 2 > 10$

Ans. $4x - 2 > 10$

$\therefore 4x > 12$ \quad \leftarrow$ add 2 to both sides

$\therefore x > 3$ \quad \leftarrow$ divide by 4
The inequality in Example 1

\[ 4x - 12 > 0 \]

Corresponding to this, we consider

\[ y = 4x - 12 \]

and illustrate this in the \( x-y \) plane.

The \( x \)-intercept is \( x = 3 \).

The domain corresponding to \( y > 0 \) is \( x > 3 \).

Therefore,

the solution. \( x > 3 \)
Exercise 1  Solve the following double inequality:

\[
\begin{align*}
7x - 1 & \geq 4x - 7 \\
x + 5 & > 3(1 + x)
\end{align*}
\]

Ans.

Pause the video and solve the problem.
Exercise 1  Solve the following double inequality

\[ 7x - 1 \geq 4x - 7 \]
\[ x + 5 > 3(1 + x) \]

Ans.  The first inequality

\[ 7x - 1 \geq 4x - 7 \quad \therefore \quad 3x \geq -6 \quad \therefore \quad x \geq -2 \quad (1) \]

The second equation

\[ x + 5 > 3(1 + x) \quad \therefore \quad -2x > -2 \quad \therefore \quad x < 1 \quad (2) \]

The intersection of the two solutions

\[ -2 \leq x < 1 \]
Lesson 3
Linear and Quadratic Inequalities

3B
• Quadratic Functions and Roots
• Quadratic Inequalities
Equations and Graphs of Functions

Quadratic Inequality

After rearrangement, quadratic inequality has the following standard form

$$ax^2 + bx + c > 0$$

[ Review ] Quadratic Functions and Roots

$$D = b^2 - 4ac$$

- Two real roots \( D = b^2 - 4ac > 0 \)
- Double root \( D = b^2 - 4ac = 0 \)
- No real root \( D = b^2 - 4ac < 0 \)

Case of \( a > 0 \)

Case of \( a < 0 \)
Steps to Solve Quadratic Inequalities

Step 1. Rearrange the inequality to the standard form

\[ ax^2 + bx + c > 0 \]

Step 2. Illustrate the corresponding quadratic function

\[ y = ax^2 + bx + c = a(x - p)^2 + q \]

Step 3. Solve the quadratic equation \( ax^2 + bx + c = 0 \) and find its roots \( \alpha \) and \( \beta \).

Step 4. Find the sign of \( y \) in each interval divided by \( \alpha \) and \( \beta \), and select the intervals which satisfy the inequality \( ax^2 + bx + c > 0 \).
Case of \( a > 0 \) and \( D = b^2 - 4ac > 0 \)

**Example 2** Solve the inequality \( x^2 - 4x + 3 > 0 \)

**Ans.**
The standard form \( y = (x - 2)^2 - 1 \)

By factoring, we have \( y = (x - 1)(x - 3) \)

therefore, the roots are \( x = 1, \ x = 3 \)

The inequality is satisfied in the shaded domain.

The solution is \( x < 1, \ x > 3 \)
Example 3   Solve the inequality  \[ x^2 - 4x + 4 > 0 \]

Ans.
The standard form  \[ y = (x - 2)^2 \]

The graph has one contact point at \( x = 2 \).

Therefore, the answer is all real numbers except \( x = 2 \).
Case of \( a > 0 \) and \( D = b^2 - 4ac < 0 \)

Example 4 Solve the inequality \( x^2 - 4x + 5 > 0 \)

Ans.

The standard form \( y = (x - 2)^2 + 1 \)

The graph has no contact point

The solution of this inequality is all real numbers.
Exercise 2. Solve the following inequalities.

(1) \(-x^2 - 2x + 2 < 0\) \hspace{1cm} (2) \(x^2 + x + 2 < 0\)

Pause the video and solve the problem.
Exercise 2. Solve the following inequalities.

(1) \(-x^2 - 2x + 2 < 0\)  (2) \(x^2 + x + 2 < 0\)

Ans. (1) The corresponding quadratic equation:

\(-x^2 - 2x + 2 = 0\)

The roots:

\[x = \frac{2 \pm \sqrt{(-2)^2 - 4(-1)(2)}}{2(-1)} = -1 \pm \sqrt{3}\]

From the figure:

\[x < -1 - \sqrt{3}, \quad x > -1 + \sqrt{3}\]

(2) \(D = 1^2 - 4 \times 1 \times 2 = -7 < 0\)

The graph \(y = x^2 + x + 2\) does not cross with the \(x\)-axis.

A parabola opening upward.

Therefore, there is no solution.
Exercise 3. Solve the following simultaneous inequalities.

\[ x^2 + 4x + 3 > 0 \]
\[ 2x^2 + x - 6 \leq x^2 + 2x \]

Pause the video and solve the problem.
**Exercise 3.** Solve the following simultaneous inequalities.

\[
x^2 + 4x + 3 > 0
\]
\[
2x^2 + x - 6 \leq x^2 + 2x
\]

**Ans.** The first equation is

\[
x^2 + 4x + 3 = (x + 3)(x + 1) > 0
\]

The solutions are \( x < -3, \quad x > -1 \)

The second equation is

\[
(2x^2 + x - 6) - (x^2 + 2x) = x^2 - x - 6 = (x + 2)(x - 3) \leq 0
\]

Whose solution lies in the interval \( -2 \leq x \leq 3 \)

From the figure, we have

\[
-1 < x \leq 3
\]