Lesson 9
Rational Function and Irrational Function

9A
- Linear Rational Function
- Simple Rational Function
- Standard Form
- Inequality
Rational Function

\[ y = \frac{g(x)}{h(x)} \]

where \( g(x) \) and \( h(x) \) are polynomials

Linear Rational Function

\[ y = \frac{ax + b}{cx + d} \quad (ad - bc \neq 0, \ c \neq 0) \]

In the case of \( ad - bc = 0 \)

\[ \therefore \frac{a}{c} = \frac{b}{d} (\equiv k), \quad \therefore a = ck, \quad b = dk, \quad \therefore y = \frac{ckx + dk}{cx + d} = k \quad : \text{Constant} \]

In the case of \( c = 0 \)

\[ y = \frac{a}{d} x + \frac{b}{d} \quad : \text{Linear function} \]
The Simplest Rational Function

\[ y = \frac{k}{x} \]

- Two variables are inversely proportional.

- The horizontal asymptote is \( x = 0 \) / the vertical asymptote is \( y = 0 \)
- The graph is an **equilateral hyperbola**.

\[ k > 0 \quad k < 0 \]

asymptote
Change to the standard form

\[ y = \frac{ax + b}{cx + d} \]

\[ y = \frac{k}{x - p} + q \]

- The vertical asymptote is \( x = p \)
- the horizontal asymptote is \( y = q \).

- The domain is \( x \neq p \); the range is \( y \neq q \).
Example 1. Illustrate the rational function \( y = \frac{3x - 6}{x - 1} \), following each step:

1. **Step 1** Convert the function to the standard form.

2. **Step 2** Find the horizontal and vertical asymptotes.

3. **Step 3** Find the x-intercept and y-intercept.

4. **Step 4** Illustrate the graph.

**Ans.**

1. By rearranging, it becomes
   \[
   y = \frac{3x - 3 - 3}{x - 1} = 3 - \frac{3}{x - 1}
   \]

2. The horizontal asymptote is \( y = 3 \)
   The vertical asymptote is \( x = 1 \)

3. By putting \( y = 0 \), the x-intercept is \( x = 2 \)
   By putting \( x = 0 \), the y-intercept is \( y = 6 \)

4. See the right figure
Example 2. Solve the following inequality

\[
\frac{3x - 6}{x - 1} \geq -x + 6
\]

Ans.

We utilize the result of Example 1.

The cross points of \( y = \frac{3x - 6}{x - 1} \) and \( y = -x + 6 \) are

\[
\frac{3x - 6}{x - 1} = -x + 6 \quad \therefore \quad 3x - 6 = (x - 1)(-x + 6) \quad \therefore \quad x(x - 4) = 0 \quad \therefore \quad x = 0, x = 4
\]

From the figure and the cross points, we have \( 0 \leq x < 1, \quad 4 \leq x \)
Exercise 1. Solve the inequality \( \frac{x + 1}{x - 1} > x + 1 \)

Ans.

Pause the video and solve the problem.
Exercise 1. Solve the inequality \[ \frac{x + 1}{x - 1} > x + 1 \]

Ans.

- We consider two functions \[ y = \frac{x + 1}{x - 1} = \frac{2}{x - 1} + 1 \] and \[ y = x + 1 \]

- The horizontal asymptote and vertical asymptote of the former are \( y = 1 \) and \( x = 1 \), respectively.

- The \( x \)- and \( y \)-intercepts of the former are \( x = -1 \) and \( y = -1 \), respectively.

- The \( x \)- and the \( y \)-intercepts of the latter are \( x = -1 \) and \( y = 1 \), respectively.

- The cross-points are
  \[ \frac{x + 1}{x - 1} = x + 1 \quad \therefore (x + 1)(x - 2) = 0 \quad \therefore x = -1, \ 2 \]

- From the figure, the solution is \( x < -1, \ 1 < x < 2 \)
Lesson 9
Exponential Functions

9B

- Irrational Functions
- Standard Form
- Inequality
Irrational Function

• There is no rigorous definition of irrational function.
• We can say that irrational function is the one that cannot be written as the quotient of two polynomials (but this definition is not used.).
• Customary, a function which include variables in the root is called an irrational function.

Example

\[ y^2 = x \]

\[ y = \sqrt{x} \]

\[ y = -\sqrt{x} \]
Function

Fundamental Form

\[ y = \sqrt{ax} \]

**Domain**  \( x \leq 0 \)
**Range**  \( y \geq 0 \)

\( a < 0 \)

\( a > 0 \)

**Domain**  \( x \geq 0 \)
**Range**  \( y \geq 0 \)
Change to the standard form

\[ y = \sqrt{ax + b + q} \]

\[ y = \sqrt{a(x - p) + q} \]

That wasn’t so hard!
Example 3. Answer the following questions

1) Illustrate the graphs of the functions $y = \sqrt{2 - x}$ and $y = -x + \sqrt{2}$

2) Solve the equation $\sqrt{2 - x} = -x + \sqrt{2}$

3) Solve the inequality $\sqrt{2 - x} \geq -x + \sqrt{2}$

Ans. (1) See the right figure

$$\sqrt{2 - x} = -x + \sqrt{2} \quad \therefore \quad 2 - x = (-x + \sqrt{2})^2$$

$$\therefore \quad x(x - 2\sqrt{2} + 1) = 0 \quad \therefore \quad x = 0, \quad 2\sqrt{2} - 1$$

From the figure, the root is only $x = 0$

(3) From (1) and (2), we have $0 \leq x \leq 2$

[Note] We must check that the obtained roots really satisfy the equation.
Exercise 2. Solve the inequality $\sqrt{x + 2} > x$

Ans.

Pause the video and solve the problem.
Exercise 2. Solve the inequality $\sqrt{x + 2} > x$

Ans.

- We consider two functions $y = \sqrt{x + 2}$ and $y = x$

- The x- and the y-intercepts of the former are $x = -2$ and $y = \sqrt{2}$, respectively.

- The cross-points of these functions are

  \[
  \sqrt{x + 2} = x \quad \therefore \quad x + 2 = x^2 \\
  \therefore \quad (x + 1)(x - 2) = 0 \quad \therefore \quad x = -1, \; x = 2
  \]

Only $x = 2$ satisfies this condition.

- From the figure, the solution is $-2 \leq x < 2$

(Note the boundaries.)