The free energies of six-vertex models and the *n*-equivalence relation

Kazuhiko Minami[a)]
Graduate School of Mathematics, Nagoya University, Furou-cho, Chikusa-ku, Nagoya, Aichi 464-8602, Japan

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The free energies of six-vertex models on a general domain D with various boundary conditions are investigated with the use of the n-equivalence relation, which help classify the thermodynamic limit properties. It is derived that the free energy of the six-vertex model on the rectangle is unique in the limit (height, width) $\rightarrow(\infty,\infty)$. It is derived that the free energies of the model on the domain D are classified through the densities of left/down arrows on the boundary. Specifically, the free energy is identical to that obtained by Lieb [Phys. Rev. Lett. **18**, 1046 (1967); **19**, 108 (1967); Phys. Rev. **162**, 162 (1967)] and Sutherland [Phys. Rev. Lett **19**, 103 (1967)] with the cyclic boundary condition when the densities are both equal to $\frac{1}{2}$. This fact explains several results already obtained through the transfer matrix calculation. The relation to the domino tiling (or dimer, or matching) problems is also noted.
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I. INTRODUCTION

The six-vertex model is a solvable lattice model, usually introduced on the rectangle and considered with the use of the Bethe ansatz method or the Yang–Baxter relation. The model was first solved by Lieb[1–3] and generally by Sutherland,[4] both assuming the cyclic boundary condition in the horizontal and vertical directions. In this paper, f_{LS} denotes the free energy obtained by Sutherland.

We have, on the other hand, an example where the free energy is exactly obtained with another specific boundary condition called the domain wall boundary condition. This boundary condition is introduced in the paper calculating the norms of the Bethe wave functions.[5] The partition function of this case is expressed in terms of a determinant,[6,7] the free energy is obtained,[8] expressed in terms of a matrix integral,[9] the large N asymptotics is derived,[10] and the phase separation is numerically investigated.[11] In this case, the free energy is expressed in terms of the elementary functions, and thus it is apparently different from f_{LS}. In the six-vertex model, the boundary condition is relevant even in the thermodynamic limit. This fact seems to be unusual compared to other lattice models such as the Ising model.

In this paper, we study the free energies of the six-vertex models introduced generally on a domain D with continuous boundary and with various boundary conditions. The free energies are investigated and classified, with the use of the n-equivalence relation, and our study includes the cases where the transfer matrix method cannot be directly applied. The main result of this paper is Proposition 3, which states that the density of down arrows and that of left arrows on the boundary determine the free energy of the system on D. The free energy is identical to f_{LS} when the two densities are both equal to $\frac{1}{2}$. This result also means that the free energy is still intensive even if the boundary effect remains relevant in the thermodynamic limit.

[a)]Electronic mail: minami@math.nagoya-u.ac.jp.

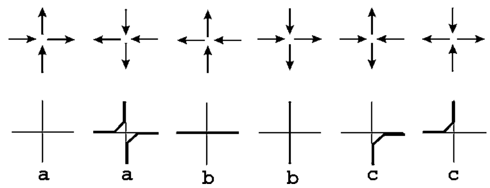

FIG. 1. Six vertices, corresponding line configurations, and their Boltzmann weights.

Section II A is a short summary on the six-vertex model and the corresponding transfer matrix treatment. In Sec. II B, we introduce the domain D and, in Sec. II C, introduce an equivalence relation of boundary conditions which is called the n-equivalence relation.[12] Two boundary conditions yield the identical free energy if they are n-equivalent. This n-equivalence is a generalization of the concept of the boundary condition, helps to classify the infinite limit properties, and also corresponds to the irreducibility of the transfer matrix. In Sec. II D, it is derived, with the use of the n-equivalence, that the free energy of the six-vertex model on the rectangle R with w columns and h rows is unique in the thermodynamic limit $(w,h)\to(\infty,\infty)$, specifically independent of the ratio w/h, and independent of the order of two limits $w\to\infty$ and $h\to\infty$. The six-vertex models on a cylinder and on a rectangle are considered and, finally, we obtain Proposition 3.

There exist several exact calculations which yield f_{LS} with various boundary conditions. Our results explain why these free energies are equal to f_{LS} and also can determine the exact free energies of six-vertex models which have not yet been solved. The results can also be written in terms of the domino tiling language and also we can certify that Proposition 3 is consistent with the results in this area. All of these are summarized in Sec. III.

The free energy of the six-vertex model on a rectangle converges in the limit $h\to\infty$, where h is the height of the rectangle. In this limit, one obtains $w^{-1}\log\lambda_1(w)$, where $\lambda_1(w)$ is the maximum eigenvalue of the transfer matrix. We assume the convergence of the next limit $w\to\infty$. From the symmetry of the system, the free energy is also convergent in the sequential limits $w\to\infty$ and $h\to\infty$.

The existence of this limit has been derived for special cases with assumptions, through the Bethe ansatz method or the Yang–Baxter relation. If Δ satisfies $\Delta<1$, the limit is finally obtained as a solution of a Fredholm integral equation of the second kind with a symmetric difference kernel [see, for example, (8.8.6) of Ref. 13]. The solution of this kind of integral equation is already classified (see, for example, Ref. 14), and it is known that the existence and uniqueness are determined by the coupling parameter which is, in our case, $1/2\pi$ and is independent of ρ. On the other hand, the solution is directly obtained and revealed to be unique for $\Delta<1$ and $\rho=\frac{1}{2}$. This fact justifies the assumption that the limit $w\to\infty$ exists, at least in a finite interval of ρ around $\rho=\frac{1}{2}$.

The existence of the limit $(h,w)\to(\infty,\infty)$ yields the existence of the van Hove limit (see, for example, Ref. 15).

II. SIX-VERTEX MODELS ON DOMAIN *D*

A. The six-vertex model

Let us consider the square lattice and assign an arrow on each bond. The arrows are arranged such that two arrows come in and the other two go out at each site (the ice rule). Then, there exist six types of possible local arrow arrangements, as shown in Fig. 1.

In this paper, we are going to use the term "vertex" as a site and four bonds around it. Each vertex is assumed to have a finite energy. The energy is assumed to be unchanged by reversing all the arrows on the four bonds. Then, we have three energy parameters and hence three types of Boltzmann weights a, b, and c assigned to the vertices (see again Fig. 1). We also introduce the Boltzmann constant k_{B}, the temperature T, $\beta=1/k_{\mathrm{B}}T$, and the total number of sites N. The partition function is

$$Z = \sum_{\text{config.}} \prod_{i=1}^{N} e^{-\beta\epsilon_i}, \tag{1}$$

where ϵ_i is the energy of the ith vertex and the sum $\Sigma_{\text{config.}}$ will be taken over all the possible arrow configurations. The free energy f is obtained through $-\beta f = \lim_{N\to\infty} N^{-1} \log Z$.

Let us introduce a line assigned on each arrow pointing down or pointing to the left (Fig. 1). Then, each global arrow configuration corresponds to a line configuration on the lattice. In this case, the ice rule is the restriction that each line begins from a bond on the boundary, continue until it ends at another bond on the boundary, and that the lines do not intersect each other.

Let us consider a rectangle R with w columns and h rows. Here, we assume that h and w are both even. Let $\eta \equiv \{x_1, \ldots, x_m\}$ be a line configuration on a row of vertical bonds in R, and let $\eta' \equiv \{x'_1, \ldots, x'_m\}$ be that on the row below. The symbol $\{x_1, x_2, \ldots, x_m\}$ denotes that there is a line on the x_kth bond $(k=1, \ldots, m)$ and there is no line on the other bonds. The (η', η)-element of the transfer matrix V is introduced as

$$V_{\eta'\eta} \equiv \langle x'_1, \ldots, x'_m | V | x_1, \ldots, x_m \rangle = \sum_{\text{config.}} \prod_{k=1}^{w} e^{-\beta\epsilon_k}, \tag{2}$$

where $|x_1, \ldots, x_m\rangle$ is the state corresponding to $\eta \equiv \{x_1, \ldots, x_m\}$, ϵ_k $(k=1, \ldots, w)$ is the energy of the kth vertex on the row between the two rows of vertical bonds, and here the sum $\Sigma_{\text{config.}}$ is taken over all the possible line configurations with fixed η and η'.

When we introduce the cyclic boundary condition in the horizontal direction, the transfer matrices are identical for all rows. When we assume that the boundary condition is also cyclic in the vertical direction, the partition function Z is written as

$$Z = \operatorname{tr} V^h = \sum_i \lambda_i^h \sim \lambda_1^h \quad (h \to \infty), \tag{3}$$

where λ_i's are the eigenvalues of V and $\lambda_1 \geqslant |\lambda_i|$ for all i.

Following the notations in Ref. 13, let us introduce $\Delta = (a^2+b^2-c^2)/2ab$. The transfer matrix is block diagonalized according to the number of lines m. The maximum eigenvalue of the transfer matrix lies in the block element with $m=0$ or w $(\Delta > 1)$, with $m = w/2$ $(\Delta < 1)$. In the case that $\Delta > 1$, we have two types of frozen phase where specific line configurations are dominant. In this case, the free energy is a constant and all the arguments in this paper become trivial. We thus concentrate on the case where $\Delta < 1$.

B. Domain *D*

Here, we introduce the domain D. Let us consider a continuous and closed line $\gamma(t) = (x(t), y(t))$ $(0 \leqslant t \leqslant 1)$, which satisfies $\gamma(t_1) \neq \gamma(t_2)$ if $t_1 \neq t_2$ except $\gamma(0) = \gamma(1)$. Let us assume that the sites are on the points $(n_1 a_1, n_2 a_2)$, where n_1 and n_2 are integers, and a_1 and a_2 are the lattice spacings.

The sites inside γ belong to D. The sites on the line γ can be suitably defined to belong or not to belong to D. The vertices belong to D when the corresponding sites belong to D. Each row and column in D is assumed to be simply connected.

The bonds/vertices in D are called the boundary bonds/vertices when they have nonzero intersection with γ. The sites of the boundary vertices are called the boundary sites. It is assumed that the number of vertices on the boundary divided by the total number of sites vanishes in the thermodynamic limit.

We will take the following limit: fix the line γ and take the thermodynamic limit $a_1, a_2 \to 0$. This corresponds to taking the limit $w, h \to \infty$, where w and h are the number of columns and rows in D.

C. The *n*-equivalence

Now, we introduce the n-equivalence relation.[12] Let us assume that each site and bond can take one of a finite number of discrete states. Let us introduce the n-boundary sites, which are the sites n step inside the boundary, i.e., the sites which satisfy the condition that the minimum number of steps (number of bonds) to be reached from boundary sites is n. The set of bonds between $(n-1)$- and n-boundary sites are called the n-boundary bonds. The n-boundary sites together with the n-boundary bonds are called the n-boundary. Configurations on the n-boundary are called n-boundary configurations. Let $\{\Gamma_i\}$ be the set of all the possible configurations on the n-boundary, under the restriction that the boundary condition is Γ on the actual boundary of D. Two boundary conditions Γ and Γ' are called n-equivalent when $\{\Gamma_i\}=\{\Gamma'_i\}$ as a set of n-boundary configurations. The following proposition says that the free energies with n-equivalent boundary conditions are identical in the thermodynamic limit, if they exist.

Proposition 1: Fix a sequence of the domain $\{D_N\}$, where N is the number of sites. Let Γ_N and Γ'_N be boundary conditions for D_N. Let $Z_N(\Gamma_N)$ and $Z_N(\Gamma'_N)$ be the partition functions on D_N with the boundary conditions Γ_N and Γ'_N, respectively. Suppose that Γ_N and Γ'_N are n-equivalent for each N with $n=o(N/N')$, where N' is the number of boundary sites, then

$$\lim_{N\to\infty}\frac{1}{N}\log\frac{Z_N(\Gamma_N)}{Z_N(\Gamma'_N)}=0 \tag{4}$$

Proof: Let $Z_N(\Gamma_N)=\Sigma_i B_{N,i}(\Gamma_N)Z_{N,i}$ be the partition function of the system on D_N. The factor $Z_{N,i}$ is the partition function from the variables inside the n-boundary with a fixed n-boundary configuration $\Gamma_{N,i}$. Here, n is a function of N, $n=n(N)$, and the index i runs from 1 to $i_{\max}$, where $i_{\max}$ is the number of allowed configurations on the n-boundary and satisfies $i_{\max}\leqslant O(r^{N'})$ with a positive constant r. The factor $B_{N,i}(\Gamma_N)$ is the contribution from the other variables with the boundary condition Γ_N and fixed $\Gamma_{N,i}$. The factor $B_{N,i}(\Gamma_N)Z_{N,i}$ is thus the partition function with Γ_N and fixed $\Gamma_{N,i}$.

All the factors $B_{N,i}(\Gamma_N)$ and $Z_{N,i}$ are positive. There exist constants b and C, which satisfy $0<b<B_{N,i}(\Gamma_N)<\exp(CnN')$ for all N. Then, it follows that

$$0<\frac{b}{\exp(CnN')}<\frac{\Sigma_i B_{N,i}(\Gamma_N)Z_{N,i}}{\Sigma_i B_{N,i}(\Gamma'_N)Z_{N,i}}<\frac{\exp(CnN')}{b}, \tag{5}$$

where $n=o(N/N')$. This yields (4). ■

It should be noted that we do not need to count the number of ways to realize each n-boundary configuration, i.e., do not need to count the "degeneracy" of each configuration. The existence of one needed n-boundary configuration is sufficient to yield the corresponding free energy.

All the boundary conditions are 1-equivalent, for example, in the case of Ising models with finite interactions. It follows that the free energies of the Ising models do not depend on the boundary condition.

This n-equivalence is a generalization of the concept of boundary condition, can be considered in other lattice models such as the 19-vertex model, and can be introduced in stochastic processes.[12]

The equivalence is introduced sometimes in a generalized form. One can introduce a boundary condition Γ fixed on a subset of the boundary and introduce the corresponding n-boundary and the corresponding n-equivalences. In the following argument, we sometimes concentrate on one or two of the four edges of a rectangle R and consider the corresponding n-equivalences.

D. Results

Lemma 1: Consider a rectangle R and assume the cyclic boundary condition in the horizontal direction. Then, all the line configurations with m lines on the upper edge (the first row of vertical bonds) of R are 2m-equivalent to each other.

Proof: Let $\{x_1,x_2,\ldots,x_m\}$, $x_i<x_{i+1}$, be a line configuration on the first row of R. Beginning

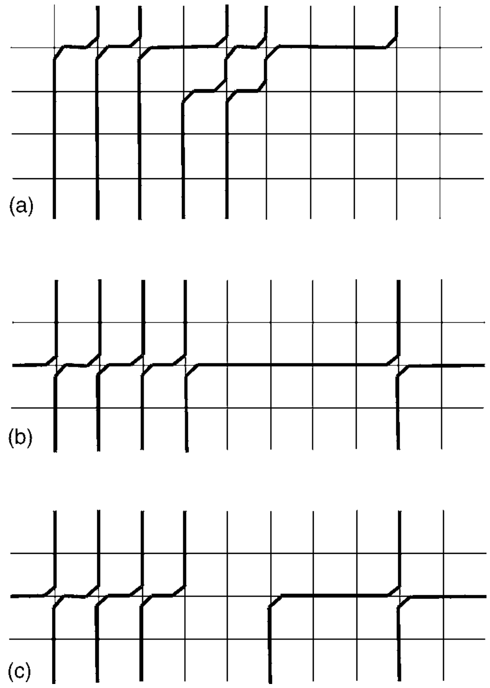

FIG. 2. Shift of lines introduced in the proof of Lemma 1 and Lemma 2.

from a line arrangement $\{x_1,x_2,x_3,x_4,x_5\}=\{2,3,5,6,9\}$, as shown in Fig. 2(a), for example, one can introduce the shift of lines $\{2,3,5,6,9\}\to\{1,2,3,5,6\}\to\{1,2,3,4,5\}$. When one generally obtains the arrangement $\{1,2,\ldots,i,x_j,x_{j+1},\ldots,x_{m-k}\}$, $i+1<x_j$, after kth step, one can introduce the next arrangement as $\{1,2,\ldots,i,i+1,x_j,\ldots,x_{m-(k+1)}\}$. We need $m-l$ steps to realize the shift $\{x_1,x_2,\ldots,x_m\}\to\{1,2,\ldots,m\}$ if $x_l\leqslant m<x_{l+1}$, and this means that $\{x_1,x_2,\ldots,x_m\}$ is m-equivalent to $\{1,2,\ldots,m\}$. Then, all the line configurations $\{x'_1,x'_2,\ldots,x'_m\}$ are again possible on the $2m$-boundary. ∎

Let us introduce, on the left and right edges, the line configurations with m_2 lines per the first h_p horizontal bonds on the boundary, and m_2 lines per the next h_p horizontal bonds, and so on. In this case, one can introduce the line density $\rho_2=m_2/h_p$ on the boundary.

Lemma 2: Suppose that the boundary line configurations on the right and left edges of R are identical and fixed, with the density $\rho_2=m_2/h_p$, where $0<\rho_2<1$. Then, all the line configurations with m lines on the upper edge of R are $\bar{m}$-equivalent where $\bar{m}=\alpha m+\alpha'$ (α, α' are constant).

Proof: Let us introduce the shift of lines $\{1,2,\ldots,i,x_j,x_{j+1},\ldots,x_{m-k}\}\to\{1,2,\ldots,i,i+1,x_j,\ldots,x_{m-(k+1)}\}$, $i+1<x_j$, on each row in which there are no lines on the right and left boundary bonds, and $\{x'_1,x'_2,\ldots,x'_m\}\to\{x'_m,x'_1,\ldots,x'_{m-1}\}$ on each row in which there are lines on the boundary bonds, as shown in Fig. 2(b). The line configuration will be arranged as $\{x_1,x_2,\ldots,x_m\}\to\{1,2,\ldots,m\}$ within m' steps, where $m'=[m/(h_p-m_2)]h_p+h_p$.

Next, we consider an arbitrary line configuration $\{x''_1,x''_2,\ldots,x''_m\}$, $x''_i<x''_{i+1}$, and the shift $\{1,2,\ldots,m\}\to\{x''_1,x''_2,\ldots,x''_m\}$. Let us introduce $\{x'_1,x'_2,\ldots,x'_m\}\to\{x'_1,x'_2,\ldots,x'_m\}$ on each row in which there are no lines on the right and left boundary bonds, and $\{1,2,\ldots,i,x''_{i+1},\ldots,x''_m\}\to\{x''_m,1,2,\ldots,i-1,x''_i,x''_{i+1},\ldots,x''_{m-1}\}$, as shown in Fig. 2(c), on each row in which there are lines on the boundary bonds. Finally, the configuration $\{x''_1,x''_2,\ldots,x''_m\}$ is possible on the $\bar{m}$-boundary, where $\bar{m}\leqslant m'+m''$ and $m''=[m/m_2]h_p+h_p$. ∎

Boundary line configurations where no configuration is allowed on the lattice should be excluded from our argument because in this case, the system cannot be a six-vertex model.

Proposition 2: Consider the six-vertex model on a rectangle R. Assume the cyclic boundary condition in the horizontal direction or otherwise assume that the boundary configurations on the *right and left edges of R are identical to each other, periodic in the vertical direction with a fixed period h_p, with a fixed line density $\rho_2 = m_2/h_p$ satisfying $0 < \rho_2 < 1$. Assume that h is a multiple of h_p. Then,*

(i) the transfer matrix of each row or the product of the transfer matrices of sequential h_p rows, respectively, is block diagonalized according to the number of lines, and each block element is irreducible;
(ii) the free energy is unique in the limit $(w,h)\to(\infty,\infty)$; specifically, the limit is independent of the order of two limits $w\to\infty$ and $h\to\infty$, independent of the ratio w/h when one takes $h\to\infty$ with fixed w/h.

Proof: First, let us consider the case where the boundary condition is cyclic in the horizontal direction. The case with the fixed boundary condition can be treated similarly and will be considered at the last of this proof. Let m be the number of lines on the upper edge (the first row of vertical bonds) of the rectangle R. Then, there are m lines on every row of R because the number of lines is invariant. Thus, the transfer matrix V is block diagonalized according to m. Let V_m be the block element of V with a fixed line number m. All the elements of V_m are non-negative because they are sums of Boltzmann weights. From Lemma 1, all the line configurations with m lines are $2m$-equivalent. Hence, there exists at least one allowed line arrangement on the lattice, for any line configuration with m lines on the n-boundary under any fixed line configuration with m lines on the upper boundary, provided that $n \geqslant 2m+2m$. Thus, we find that all the elements of V_m^{2m+2m} are positive. (All the elements of V_m^{2m} are already positive, which is obvious from the proof of Lemma 1.) Hence, we find that V_m^{2m+2m} is irreducible because, generally, the matrix should be irreducible if all of its elements are positive. It follows that V_m is irreducible because V_m^{4m} cannot be irreducible if V_m is not; this proves (i). Then, the Perron–Frobenius theorem works and we know the following. There exists an nondegenerate eigenvalue $\lambda_1(w) > 0$ such that $\lambda_1(w) \geqslant |\lambda_i(w)|$, where $\lambda_i(w)$ $(i \geqslant 2)$ are the other eigenvalues of V_m. We also know that all the elements of an eigenvector associated with $\lambda_1(w)$ are positive, i.e., the projections satisfy $\langle x_1, \ldots, x_m | \max\rangle > 0$, where $|\max\rangle$ is the eigenstate associated with the maximum eigenvalue of V_m. These results are valid for every finite m.

The partition function Z with finite w and h is written as

$$Z = \langle x_1', \ldots, x_m' | V^h | x_1, \ldots, x_m\rangle = c_1(w)\lambda_1(w)^h + \sum_{i=2}^{i_w} c_i(w)\lambda_i(w)^h = c_1\lambda_1^h\left[1 + \sum_{i=2}^{i_w}\frac{c_i}{c_1}\left(\frac{\lambda_i}{\lambda_1}\right)^h\right], \tag{6}$$

where $\{x_1, \ldots, x_m\}$ and $\{x_1', \ldots, x_m'\}$ are the line configurations on the upper and the lower edges of R, respectively, i_w is finite,

$$c_1 = \langle x_1', \ldots, x_m' | \max\rangle\langle \max | x_1, \ldots, x_m\rangle \tag{7}$$

is positive and independent of h, and here we assumed that $|\max\rangle$ is normalized. The coefficients c_i are independent of h, and we will consider the limit of

$$-\beta f_{h,w} = \frac{1}{hw}\log Z = \frac{1}{w}\log \lambda_1(w) + \frac{1}{h} z'(h,w), \tag{8}$$

$$z'(h,w) = \frac{1}{w}\log c_1\left[1 + \sum_{i=2}^{i_w}\frac{c_i\lambda_i^h}{c_1\lambda_1^h}\right]. \tag{9}$$

[The matrix V is assumed to be diagonalizable in (6). In the case that V is not diagonalizable, V can be expressed in a Jordan form, and the terms $c_i\lambda_i^h$ $(i \geqslant 2)$ in (9) are replaced by

$$c_{ih}\lambda_i^h + c_{ih-1}\lambda_i^{h-1} + \cdots + c_{i1}\lambda_i + c_{i0}, \tag{10}$$

where c_{ik} ($k=h,h-1,\ldots,0$) are independent of h, and the following argument remains valid.]

The factor $z'(h,w)$ is finite when h and w are finite. This factor $z'(h,w)$ depends on h only through the terms $(\lambda_i/\lambda_1)^h$, where $|\lambda_i/\lambda_1|\leqslant 1$, and hence remains finite in the limit $h\to\infty$ with fixed w.

Therefore, taking the limit $h\to\infty$ in (8), the factor $h^{-1}z'(h,w)$ converges to zero. The term $w^{-1}\log\lambda_1(w)$ is independent of h. Hence, the term $(hw)^{-1}\log Z$ is convergent in this limit. Next, taking $w\to\infty$ in (8), one finds that $w^{-1}\log\lambda_1(w)$ is convergent in the limit $w\to\infty$ because $(hw)^{-1}\log Z$ is assumed to be convergent in this limit.

On the other hand, let us first take the limit $w\to\infty$ in (8). We now know that $w^{-1}\log\lambda_1(w)$ is convergent in this limit, $(hw)^{-1}\log Z$ is assumed to be convergent in this limit, and therefore the factor $z'(h,w)$ is convergent in the limit $w\to\infty$.

Convergent series are generally bounded, and therefore $z'(h,w)$ is bounded as a series of w with fixed h, i.e., $|z'(h,w)|\leqslant C(h)$, where $C(h)$ is independent of w. Here, let us note that Z, c_1, and $\lambda_1(w)$ are real and positive in (6), and hence the third factor in (6) is real and positive. With the use of the fact that $\log(1+x)<1+x$ for all positive $1+x$, we obtain

$$|z'(h,w)| < \frac{1}{w}\left[|\log c_1| + 1 + \sum_{i=2}^{i_w}\left|\frac{c_i}{c_1}\left(\frac{\lambda_i}{\lambda_1}\right)^h\right|\right]. \tag{11}$$

The right hand side depends on h only through the terms $(\lambda_i/\lambda_1)^h$ which are nonincreasing series of h. Therefore, the factor $C(h)$ can be chosen as a constant C which is independent of both w and h.

Hence, from (8), we find that

$$\left|-\beta f_{h,w} - \frac{1}{w}\log\lambda_1(w)\right| < \frac{C}{h}, \tag{12}$$

which means that the convergence of $f_{h,w}$ in the limit $h\to\infty$ is uniform throughout w. This proves (ii) when we consider the known fact that the limit of double series is generally unique provided that the convergence is uniform.

This last point on the uniqueness of double series is well known but we are going to show a proof of it. We derived that the convergence in the limit $h\to\infty$ is uniform: there exists a number $f_{\infty,w}$, which satisfies that for every $\epsilon>0$, there is an integer $h_0(\epsilon)$ which is independent of w, such that $|f_{h,w}-f_{\infty,w}|<\epsilon$ for all $h\geqslant h_0(\epsilon)$. The free energy is convergent in the next limit $w\to\infty$: there exists a number f, which satisfies that for every $\epsilon>0$, there is an integer $w_0(\epsilon)$, such that $|f_{\infty,w}-f|<\epsilon$ for all $w\geqslant w_0(\epsilon)$. Then, for all $h,w\geqslant\max\{h_0(\epsilon),w_0(\epsilon)\}$, we obtain

$$|f_{h,w}-f| \leqslant |f_{h,w}-f_{\infty,w}| + |f_{\infty,w}-f| < 2\epsilon, \tag{13}$$

which means

$$\lim_{(h,w)\to(\infty,\infty)} f_{h,w} = f \tag{14}$$

as a double series. Taking $w\to\infty$ in (13), one obtains $|f_{h,\infty}-f|\leqslant 2\epsilon$ which means

$$\lim_{h\to\infty}\lim_{w\to\infty} f_{h,w} = f. \tag{15}$$

Taking $h\to\infty$ in (13), one obtains $|f_{\infty,w}-f|\leqslant 2\epsilon$ which means

$$\lim_{w\to\infty}\lim_{h\to\infty} f_{h,w} = f. \tag{16}$$

At last, let us consider the case where the boundary line configurations on the right and left edges of R are identical, fixed, and the lines are located periodically with the period h_p, with the line density $\rho_2 = m_2/h_p$ satisfying $0 < \rho_2 < 1$. Then, one can introduce a transfer matrix $V = V_1 V_2 \cdots V_{h_p}$, where $V_1 \cdots V_{h_p}$ are the transfer matrices of sequential h_p rows of R, respectively. The partition function is expressed as a linear combination of $\lambda_i(w)^{h/h_p}$, where $\lambda_i(w)$ are now the eigenvalues of $V = V_1 V_2 \cdots V_{h_p}$. Then, Lemma 2 works and we obtain the same result. ■

It should be noted that the $2m$-equivalence of line configurations corresponds to the irreducibility of the block element V_m^{4m} and hence to the irreducibility of V_m. The matrix V is irreducible if the line configurations used as a bases for the matrix representation of V are n-equivalent to each other for some finite n.

It is derived in (ii) that the limit is unique in all the limiting procedures $h \to \infty$ and $w \to \infty$. We will iteratively use this fact in the following argument.

Here, we explicitly show what can be obtained from our formula, using only the n-equivalence and the uniform convergence of the free energy.

Lemma 3: Consider the six-vertex model on a rectangle R. Assume the cyclic boundary condition in the horizontal direction and assume that the number of lines on the upper and the lower edges are $m = m(w)$. Then, all the boundary line configurations with the same $m(w)$ yield the identical free energy in the limit $(w,h) \to (\infty,\infty)$. Specifically, we have $f = f_{\rm LS}$ when $m = w/2$.

Proof: Because of Lemma 1, the line configuration on the upper edge is $2m$-equivalent to arbitrary line configurations with m lines and hence $2m$-equivalent to the cyclic boundary condition with m lines. We take the limit $h \to \infty$ with fixed w and obtain the free energy with $(w,h) \to (w,\infty)$. The resulted functions are the same for all of these fixed and the cyclic boundary conditions on the upper and the lower edges with m lines. Next, taking the limit $w \to \infty$, the limits of the functions are of course identical to each other. The limit is unique for all the limiting procedures because of Proposition 2. In particular, we obtain the maximum eigenvalue of V and the known free energy $f_{\rm LS}$ when $m = w/2$. ■

In the case that the system is cyclic in two directions, we know that the maximum eigenvalue of the row to row transfer matrix lies in the block element with $w/2$ lines on each row. Because of the symmetry of the system, the maximum eigenvalue of the column to column transfer matrix also lies in the block element with $h/2$ lines on each column, otherwise we have contradictions. Furthermore, we can derive the following results.

Lemma 4: In Lemma 3, $f_{\rm LS}$ appears with the restriction that the boundary condition is cyclic in the horizontal direction and that the number of lines is $h/2$ on each column.

Proof: In the case that the boundary is cyclic in the horizontal direction with $h/2$ lines on each column, the "alternate" line configuration with $\rho_2 = 1/2$ with the period $h_p = 2$ is possible on the left and right edges. Consider the case that the line configurations on the left and right edges are alternative, from the first low to the $4m$th low and from the $(h-4m+1)$th low to the last low. Then, in the proof of Lemma 3, one can use Lemma 2 instead of Lemma 1, and then the configurations are $4m$-equivalent instead of $2m$. Taking the limit $h \to \infty$ with the fixed m, all the fixed and cyclic boundary line configurations on the upper and the lower edges with m lines yield the identical free energy. In the case that $m(w) = w/2$, the system is equivalent to the system cyclic in two directions with $w/2$ lines on each row and $h/2$ lines on each column, and we obtain $f_{\rm LS}$. ■

It should be noted in this proof that the depth $4m$ is finite through the limit $h \to \infty$ and hence irrelevant to the free energy. Inside the $4m$-boundary, we have $h - 2 \times 4m$ rows, and there the number of lines is $(h - 2 \times 4m)/2$ on each column.

Let us consider a fixed boundary line configuration on the boundary of R. We assume that w and h are multiples of w_p and h_p, respectively. The boundary line configuration on the upper edge is identical to that on the lower edge, and the configuration on the left edge is identical to that on the right edge. The line configurations are periodic on the boundaries, the line density is $\rho_1 = m_1/w_p$ on the upper and lower edges, and $\rho_2 = m_2/h_p$ on the left and right edges. The limit will be taken with fixed w_p and h_p. In the proof of the next proposition, we do not need the explicit form of $\lambda_1(w)$.

Lemma 5: With these conditions, the line densities ρ_1 and ρ_2 determine the free energy of the six-vertex model on the rectangle R: $f = f(\rho_1, \rho_2)$. Specifically, $f(\frac{1}{2}, \frac{1}{2}) = f_{\rm LS}$.

Proof: First, let us assume $0 < \rho_1 < 1$ and $0 < \rho_2 < 1$. Lemma 2 yields that all the line configurations on the upper edge with the density ρ_1 are n-equivalent with some n, which is independent of h, and all the line configurations on the lower edge with the density ρ_1 are also n-equivalent with the same n. The boundary effect is relevant to the limit only through ρ_1 when we take $h \to \infty$ with fixed w. Next, taking $w \to \infty$, we obtain the thermodynamic limit. This argument is also valid for the left and right edges with the density ρ_2 taking the limit $w \to \infty$ with fixed h at first, and next $h \to \infty$. The limit is unique because of Proposition 2.

Consider the case with $\rho_1 = \rho_2 = \frac{1}{2}$. The boundary configurations on the upper and lower edges are n-equivalent to the cyclic boundary with $w/2$ lines with some n, which is independent of h. Taking $h \to \infty$ with fixed w, we obtain the limit identical to that obtained under the cyclic boundary condition in the vertical direction with $w/2$ lines, and under the boundary configuration with $\rho_2 = \frac{1}{2}$ on the left and right edges. Next, taking $w \to \infty$, we obtain the thermodynamic limit and, from Lemma 4, the limit is identical to f_{LS}.

In the case $\rho_2 = 0$, the free energy is $(1-\rho_1)\epsilon_1 + \rho_1 \epsilon_2$ and, in the case $\rho_1 = 0$, it is $(1-\rho_2)\epsilon_1 + \rho_2 \epsilon_2$. As for the cases with the densities equal to 1, one can use the fact that $f(\rho_1, \rho_2) = f(1-\rho_1, 1-\rho_2)$, which comes from the symmetry of the vertex energies. ■

Let us consider the sequence of vertical boundary bonds on the line γ. Let us assume a fixed and periodic boundary line configuration on this sequence with the line density $\rho_1 = n_1/w_p$, i.e., n_1 lines on w_p vertical bonds on γ. Also, let us assume a fixed and periodic boundary line configuration on the sequence of horizontal bonds on γ with the line density $\rho_2 = n_2/h_p$, i.e., n_2 lines on h_p horizontal bonds on γ. Assume that there exists a line on a horizontal bond on the left edge of the domain D if and only if there is a line on the horizontal bond at the right edge of D on the same row. Assume also that there exists a line on a vertical bond on the lower edge of D if and only if there is a line on the vertical bond at the upper edge of D on the same column. The limit will be taken with fixed w_p and h_p. With these conditions, one can derive the following.

Proposition 3: The line densities ρ_1 and ρ_2 determine the free energy of the six-vertex model on the domain D: $f = f(\rho_1, \rho_2)$. Specifically, $f(\frac{1}{2}, \frac{1}{2}) = f_{\mathrm{LS}}$.

Proof: Let D_0 be a rectangle, which is sufficiently large satisfying $D \subset D_0$. The width and height of D_0 are Δx and Δy, respectively. Let $D = \cup_i R_i$, where R_i are successive small rectangles. The width of R_i is Δx, the height of R_i is $\Delta y_i = \Delta h$, and the lower edge of R_i coincides with the upper edge of R_{i+1}. Let $D' = \cup_i R_i'$, where R_i' is a rectangle satisfying $R_i' \subset R_i$. The width of R_i' is Δx_i, the height of R_i' is Δy_i, and the widths Δx_i are assumed to take their maximum value with the restriction that $R_i' \subset R_i$, $R_i' \subset D$ and that the number of columns in each R_i' is a multiple of w_p. The sites in R_i' or those on the edges of R_i' belong to R_i', where the exception is that the sites on $R_i' \cap R_{i+1}'$ belong to R_i'. Assume that the boundary line configuration on the edges of D' is identical to the corresponding boundary line configuration of D, i.e., there exists a line on a bond at the right edge of D' if and only if there is a line on the bond at the right edge of D on the same row, and so on in the case of the other edges.

We assumed in each R_i' that the number of columns is a multiple of w_p. In addition to it, let us assume in each R_i' that the number of rows is also a multiple of h_p. The latter choice corresponds to introduce a sequence $a_2 \to 0$, where each a_2 satisfies $a_2 \times n' h_p = \Delta h$ $(n' = 1, 2, \ldots)$. This sequence is sufficient to consider the thermodynamic limit because the difference from the remaining cases vanishes as $N \to \infty$ together with the ratio N'/N, where N' is the number of boundary sites.

Because of the line conservation property, the sum of the line numbers on the upper and right edges of each small rectangle R_i' is equal to the sum of the line numbers on the lower and left edges of R_i'. Because of the restricted boundary condition, the line configuration on the right edge of R_i' and that on the left edge are identical to each other, and hence the number of lines on the upper edge of R_i' is equal to that on the lower edge. All the possible configurations on the upper edge are n-equivalent with some n independent of a_2, and all the configurations on the lower edge are also n-equivalent; hence, they yield the identical free energy in each R_i' in the limit $a_2 \to 0$. Next, taking $a_1 \to 0$, we find that each rectangle R_i' yields $f(\rho_1, \rho_2)$, which is the free energy obtained in Lemma 5. Thus, we obtain $f_{D'} = f(\rho_1, \rho_2)$, where $f_{D'}$ is the free energy of the six-vertex model on D'.

The result is independent of the ratio $\Delta h / \Delta x_i$ and the ratio can be taken sufficiently small. The ratio of the energy contribution, the contribution from $D \backslash D'$ over that from D, goes to zero in the limit $\Delta h \to 0$ because the line γ, which determines the boundary of D, is continuous. Therefore, we obtain $|f_D - f_{D'}| < \epsilon$ for arbitrary positive ϵ. ■

The result means that the free energy is still additive even in the situation where the boundary condition remains relevant in the thermodynamic limit.

These results are applications of Ref. 12 to the case of the six-vertex model and give sufficient conditions to obtain f_{LS}. It should be noted that we did not need to diagonalize sequential transfer matrices but it was sufficient for us to check that the configurations are n-equivalent, for the purpose to classify the thermodynamic limit properties.

III. CONCLUSION

Our propositions explain several results already obtained and are also able to determine the exact free energies of the six-vertex models which have not been solved.

One can introduce boundary conditions, such as the cyclic boundary condition, in which various boundary configurations are allowed. If the vertex energies and the temperature satisfy $\Delta < 1$, and if we can find configurations being n-equivalent to those with $\rho_1 = \rho_2 = \frac{1}{2}$, then the free energy is f_{LS}. If we are in the parameter region with $\Delta > 1$, and if one of the boundary configurations with $(\rho_1, \rho_2) = (0,0)$ or $(1, 1)$ is allowed, the system falls in trivial frozen phases.

Owczarek and Baxter[16] solved (by the Bethe ansatz method) the six-vertex model with the cyclic boundary and a "free" boundary condition, respectively, in two directions. Batchelor *et al.*[17] solved (by the Yang–Baxter relation) the six-vertex model on a rectangle R with a specific boundary condition in which they assumed that the arrow at one end of a row points to the right (left) if that on the other end of the same row points to the left (right), and assumed that the boundary condition is cyclic in the vertical direction. One can find that the boundary configuration with the line density equal to $\frac{1}{2}$ is realized under the restrictions in both of their studies,[16,17] and hence it can be directly derived from our results that the free energies of these systems are f_{LS}. Furthermore, one can easily construct a bunch of n-equivalent cases which have not yet been solved and are extremely difficult to solve directly by the Bethe ansatz method or the Yang–Baxter relation, but we now know that the free energies of all of these cases should be f_{LS}.

It should be noted that equivalences of boundary conditions in the six-vertex model is also investigated in Ref. 18 yielding f_{LS}.

One of the most interesting problems related to our results is the domino tiling (refer to Refs. 19 and 20). The problem is to find the number of possible ways to completely cover a region using dominos (1×2 rectangles). The problem is naturally equivalent to find the number of dimer coverings on a given lattice.

Each domino configuration is expressed in terms of a height function $h(x)$. The correspondence is unique except for an overall constant. More precisely, $h(x)$ is introduced as follows: coloring the squares in a checker-board pattern, $h(x)$ increases by 1 in each unit moving anticlockwise around black squares on the boundary of dominos and decreases by 1 around white squares, as shown in Fig. 3(a). Not all the regions can be tiled by dominos. The necessary and sufficient condition for tilability is written in terms of the height function $|h(x) - h(y)| \leq d(x,y)$ for all x and y on the boundary of the region, where $d(x,y)$ is the minimal number of steps to move from x to y with only black squares on its left.

Kasteleyn[21] and independently Temperley and Fisher[22] derived the number of tilings on the $m \times n$ rectangle and obtained that in the thermodynamic limit, it behaves $\exp(mnG/\pi)$, where G denotes Catalan's constant $G = 1/1^2 - 1/3^2 + 1/5^2 - 1/7^2 + \cdots$. The number of tilings on the Aztec diamond (the "square" rotated by $\pi/4$ and tiled by horizontal and vertical dominos) is also obtained[23] exactly as $2^{n(n+1)/2}$, where n is half the diameter of the region. This is completely different from that of the $m \times n$ rectangle. We recognize that the number of possible ways of tiling strongly depends on the shape of the boundary.

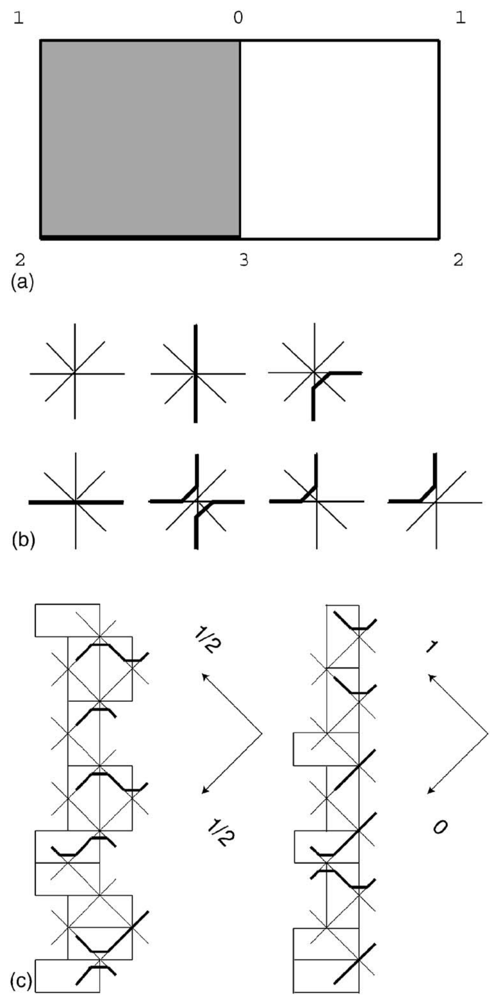

FIG. 3. (a) The height function for domino configurations. (b) Correspondence between vertices and domino tilings. The horizontal and the vertical lines are the lattice for the six-vertex model, while the lines rotated $\pm\pi/4$ are the edges of dominos. (c) Two boundaries with constant tilt (0,0) but different line densities.

Cohn *et al.*[24] showed a variational principle for the number of tilings. Their result is as follows: assuming that the region is tilable and sufficiently "fat," and assuming that the slope of the height function $(s,t)=(\partial h/\partial x, \partial h/\partial y)$ is asymptotically constant on the boundary, then the asymptotic number of tilings per domino is a function of (s,t).

It is known[8,25] that the number of possible domino tilings per domino is equal to the value of the partition function of the six-vertex model with $a=b=1$ and $c=\sqrt{2}$, i.e., $\Delta=0$. The equivalence is obtained from the correspondence of configurations of dominos and those of vertices shown in Fig. 3(b).

The boundary of the $m\times n$ rectangle has a constant slope (0, 0) and, as a six-vertex model, line densities are $\rho_1=1$ and $\rho_2=0$ on the upper edge. Introducing the boundary shown in Fig. 3(c) periodically, we find that the line densities vary as $\rho_1=1-\epsilon$ and $\rho_2=0+\epsilon$ while the slope remains (0, 0). Taking $\epsilon=\frac{1}{2}$, we have $\rho_1=\rho_2=\frac{1}{2}$. This modification can also be done for other three edges and we find, from Proposition 3, that the number of tilings can be obtained from the partition function by Lieb and Sutherland with the parameters $a=b=1$, $c=\sqrt{2}$, i.e., $\Delta=0$. The free energy is (see, for example, Ref. 13)

$$f_{\mathrm{LS}} = -k_{\mathrm{B}}T\int_{-\infty}^{+\infty}\frac{\sinh^2\frac{\pi}{2}x}{2x\sinh\pi x\cosh\frac{\pi}{2}x}dx. \tag{17}$$

Counting the residues at $z=i,3i,5i,\ldots$ on the imaginary axis, we find Catalan's constant G and obtain that $-\beta f=2G/\pi$. The factor 2 corresponds to the fact that the number of vertices is equal to the number of dominos and twice the number of squares, and the result from our Proposition 3 is consistent with that previously obtained by Kasteleyn and by Temperley–Fisher. The limits $m\to\infty$ and $n\to\infty$ in the domino case are unique. This fact is also consistent with our Proposition 2.

The six-vertex model with the domain wall boundary condition[8] corresponds to the domino tiling of the Aztec diamond. The boundary line densities on the upper and lower edges of this boundary condition are not identical and therefore do not satisfy the condition of Proposition 3. This boundary is not n-equivalent to other known cases where the model is solved exactly and, accordingly, the free energy with the domain wall boundary condition is distinct from others.

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