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An Evaluation of the Fracture Resistance of a Stably Growing Crack by Crack Energy Density (1st Report, Derivation of Fundamental Relations and Proposal of Evaluation Method)*

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The objective of this study is to propose a practical method to evaluate the fracture resistance of a stably growing crack by crack energy density and to verify it through its applications to actual stable crack growth problems. The contents of this report are as follows:
(1) More refined investigation of the relationship obtained before between initial crack length, load-displacement curves and the crack energy density which holds until a crack starts to grow is made by using two crack models.
(2) A relationship between initial crack length, present crack length, load-displacement curves and the additional rate of crack energy density caused by crack extension which holds generally for a growing crack is derived by using the same crack models as used in (1).
(3) A method to evaluate the fracture resistance of a stably growing crack from load-displacement curves which can be easily obtained by experiments is proposed, based on the relations above.

Key Words : Fracture, Stable Crack Growth, Fracture Resistance, Crack Energy Density, Additional Rate of Crack Energy Density

1. Introduction

It is important to evaluate the character of a stably growing crack in a ductile material in order to estimate the fracture mode of a structure including a crack and, moreover, avoid a design bringing about a dangerous fracture mode. In connection with this problem, several methods were proposed based on the energy release rate (or the stress intensity factor)[1], the J-integral[2][3], the crack tip opening angle[4] and the plastic work done in the recrystallized region around a crack tip[5]. However, when we try to evaluate the fracture resistance as a material characteristic by these methods, there are some problems; that is, the former two methods need to satisfy the strict conditions of the small-scale-yielding and the J-controlled crack growth respectively though they are applied easily through the measurements of load and load-point-displacement, and the latter two methods require accurate observation around a crack tip though they are applicable with no restriction.

By the way, in the previous work[6], the authors proposed an idea to evaluate the character of a stably growing crack based on the crack energy density[7][8] and systematized the above methods through considerations on relations between the above methods and the proposed idea. In the present serial study, they go one step farther, propose a practical method to evaluate the fracture resistance of a material by the crack energy density and establish it as a fracture resistance evaluation method through the concrete verification of its applicability. In this first report, fundamental relationships between the crack energy density and the mechanical parameters like load, load-point-displacement and so on are derived, and an evaluation method of fracture resistance by the crack energy density is proposed based on the above relations. This method makes it possible to evaluate the fracture resistance, without any restriction on its applicability, only from the mechanical parameters reflecting the overall behavior of a specimen such as load, load-point-displacement and the like.

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2. Concept and Definition of Crack Energy Density

A crack tip in an actual material is, except a completely elastic crack, in a state of complicated deformation caused by such uneven plastic deformation as a slip and a twinning, so a complete description of its state is difficult. A crack parameter should express totally and representatively the intensity of a change around a crack tip and the crack tip opening displacement is regarded as a parameter which expresses the intensity of the change in

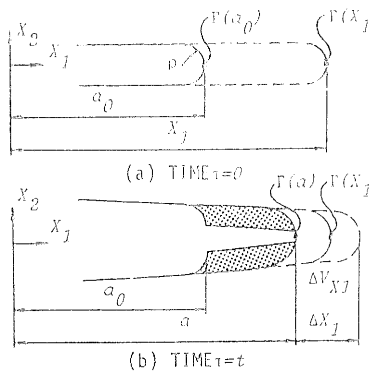

Fig.1 Notch model

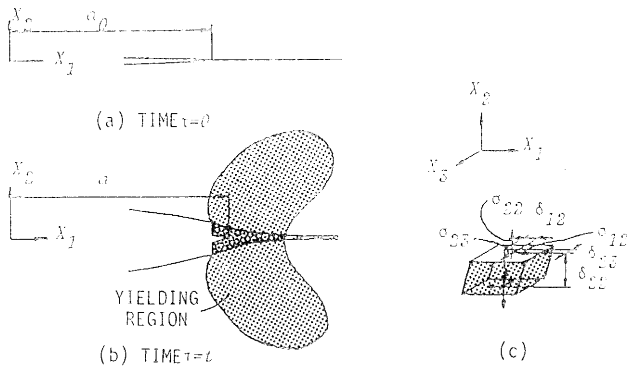

Fig.2 Discontinuous model

the dimension of displacement. On the other hand, the crack energy density can assert itself as a crack parameter to express the intensity of the change in the dimension of energy per unit area, and it is defined generally as "the work done per unit area in the plane containing a crack tip line during deformation at each position in the plane containing a crack tip line before deformation".

The crack energy density distributes in the plane containing a crack tip line[6][9], and, in this paper, discussion is made by employing a continuum notch model (called notch model hereafter)[10] and a crack model considering the discontinuity in the plane containing a crack tip line formulated by the authors (called discontinuous model hereafter)[11] because the distribution of crack energy densities can be defined concretely in these models. For the simplification of the following argument, a two-dimensional cracked plate with thickness B under crack opening mode (Mode I) is considered hereafter.

2.1 In case of notch model

The notch model is one in which a cracked material is substituted by a notched continuum with a sufficiently small radius of notch curvature, and the notched part is shown in Fig.1 with a system of coordinates. Here, $\Gamma(a_0)$ is the path along the bottom of the notch in the initial state at the time $\tau=0$, and $\Gamma(X_1)$ is the path of which the shape is the same as $\Gamma(a_0)$ at an arbitrary position in front of the notch tip. In the figure, it is supposed that a crack of which the initial length was a_0 in the initial state at the time $\tau=0$ (Fig.1(a)) blunted until the time $\tau=t_0$, extended thereafter under loading and its length has become a at the time $\tau=t$ (Fig.1(b)). Crack extension is realized by cutting off the bottom of the notch and putting the stresses acting on the new surfaces into zero. The crack energy density in the state of Fig.1(b) $\mathcal{E}(t, X_1)$, based on the definition above, is given by

$$\mathcal{E}(t, X_1)=\int_{\Gamma(X_1)} W dX_2 \qquad \cdots\cdots(1)$$

Here, W is strain energy density defined by

$$W=\int_0^t \sigma_{ij}\dot{\varepsilon}_{ij}d\tau \qquad \cdots\cdots(2)$$

where σ_{ij} and ε_{ij} are stress and strain tensors respectively and $(\dot{\ })=\partial(\ )/\partial\tau$ (when an extending crack is considered, $(\dot{\ })$ is defined for the change from $\tau-d\tau$ to τ. $(\dot{\ })$ for the changes from $\tau-d\tau$ to τ and from τ to $\tau+d\tau$ are equal and their distinction is not required in the region except the crack tip, but their distinction is necessary in order to let the quantity at the crack tip at the time τ have the meaning).

2.2 In case of discontinuous model

The discontinuous model is composed of two parts, that is, the plane containing a crack tip line which is stretched by applied stress and the usual continuum part except the above plane; and the neighborhood of the crack tip is shown in Fig.2. In this model, as the out-of-plane relative displacement $\llcorner\delta_n\lrcorner$ at a place of $(X_1, 0)$ before deformation at the time $\tau=t$ and the out-of-plane stress $\llcorner\sigma_n\lrcorner$ corresponding to the relative displacement are (referring to Fig.2(c)) given by

$$\llcorner\delta_n\lrcorner=\llcorner\delta_{22}, \delta_{23}, \delta_{12}\lrcorner \qquad \cdots\cdots(3)$$

$$\llcorner\sigma_n\lrcorner=\llcorner\sigma_{22}, \sigma_{23}, \sigma_{12}\lrcorner \qquad \cdots\cdots(4)$$

the strain energy plane density as the energy per unit area is defined by

$$W_{plane}=\int_0^t \llcorner\sigma_n\lrcorner\{\dot{\delta}_n\}d\tau \qquad \cdots\cdots(5)$$

and the crack energy density, based on the definition, is given by

$$\mathcal{E}(t, X_1)=W_{plane} \qquad \cdots\cdots(6)$$

3. Relation between Crack Energy Density at Crack Tip and Load-displacement Curves before Onset of Crack Extension

In this chapter, a general relation between load, load-point-displacement, initial crack length and crack energy density at crack tip before onset of crack extension is derived by using two models above. Here, the relation for the notch model was previously discussed[8] but that is also presented in a more refined manner corresponding to the description in this paper in contrast with the discontinuous model case.

3.1 Relation on notch model

We consider two notch models as shown in Fig.3 which are in the states before onsets of crack extension under loading ($\tau=t\leqq t_0$ and $t_0+\Delta t_0$, where $t_0+\Delta t_0$ is the

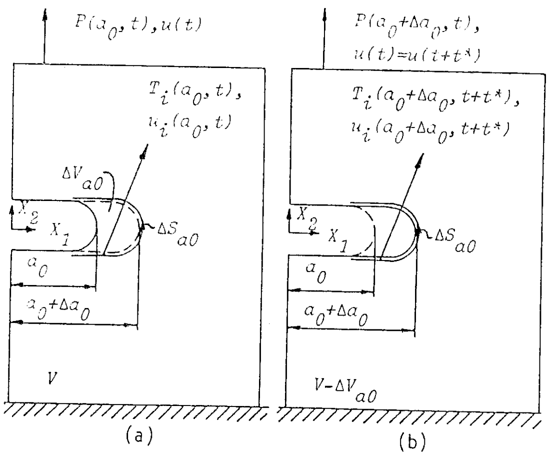

Fig.3 Two notch models differing in initial crack length before onset of crack growth

time when the crack of the specimen in Fig. 3(b) begins to extend). The two notch models are idential except that their initial crack lengths a_0 and $a_0+\Delta a_0$ are slightly different, and they are loaded in such a way that their load-point-displacements $u(\tau)$ become equal at the time τ and increase monotonously (one-to-one correspondence between u and τ is assumed).

Here, we represent load, whole strain energy, strain energy density, traction force and displacement by $P(a_0,\tau)$, $U(a_0,\tau)$, $W(a_0,\tau)$, $T_i(a_0,\tau)$ and $u_i(a_0,\tau)$, respectively, as functions of the initial crack length a_0 and the time τ. Then, the whole strain energies of the two notch models shown in Fig.3, $U(a_0,t)$ and $U(a_0+\Delta a_0,t)$, at the time $\tau=t$ are expressed by

$$U(a_0,t)=\int_V W(a_0,t)dV=\int_0^t P(a_0,\tau)\dot{u}d\tau \quad \cdots(7)$$

$$U(a_0+\Delta a_0,t)=\int_{V-\Delta V_{a_0}} W(a_0+\Delta a_0,t)dV$$
$$=\int_0^t P(a_0+\Delta a_0,\tau)\dot{u}d\tau \quad \cdots\cdots\cdots(8)$$

where V and $V-\Delta V_{a_0}$ are the volumes corresponding to all the parts of the notch models with initial crack lengths a_0 and $a_0+\Delta a_0$ respectively (refer to Fig.3).

On the other hand, we can consider the state at the time $\tau=t+t^*$ in which the following relation on the parts of $V-\Delta V_{a_0}$ of the two notch models holds.

$$U(a_0,t)-B\int_{a_0}^{a_0+\Delta a_0}\int_{\Gamma(X_1)} W(a_0,t)dX_2dX_1$$
$$=U(a_0+\Delta a_0,t)$$
$$+\int_{\Delta S_{a_0}}\int_t^{t+t^*} T_i(a_0+\Delta a_0,\tau)\dot{u}_i(a_0+\Delta a_0,\tau)d\tau dS$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots(9)$$

Here, ΔS_{a_0} is the surface which appears when ΔV_{a_0} is removed, and $T_i(a_0+\Delta a_0,\tau)$ and $u_i(a_0+\Delta a_0,\tau)$ at the time $\tau\geqq t$ are the traction force applied on ΔS_{a_0} of the notch model with initial crack length $a_0+\Delta a_0$ under the condition of holding the load-point-displacement to be $u(\tau)=u(t)$ and the corresponding displacement respectively (refer to Fig.3(b)). Especially, $T_i(a_0+\Delta a_0,t)=0$, $T_i(a_0+\Delta a_0,t+t^*)$ is the applied traction force and $u_i(a_0+\Delta a_0,t+t^*)-u_i(a_0+\Delta a_0,t)$ is the displacement caused by $T_i(a_0+\Delta a_0,t+t^*)$. In the case of linear elastic continuum, Eq.(9) holds when substituting the traction force $T_i(a_0,t)$ on ΔS_{a_0} of the notch model with initial crack length a_0 for $T_i(a_0+\Delta a_0,t+t^*)$.

Moreover, the strain energy on ΔV_{a_0} of the notch model with initial crack length a_0 is expressed by

$$B\int_{a_0}^{a_0+\Delta a_0}\int_{\Gamma(X_1)} W(a_0,t)dX_2dX_1$$
$$=-\int_{\Delta S_{a_0}}\int_0^t T_i(a_0,\tau)\dot{u}_i(a_0,\tau)d\tau dS \quad \cdots(10)$$

where $T_i(a_0,\tau)$ and $u_i(a_0,\tau)$ are the traction force and the displacement, respectively, on ΔS_{a_0} of the notch model with initial crack length a_0 at the time $\tau(\leqq t)$.

Here, paying attention to the difference of the whole strain energy $\Delta U_{a_0}(a_0,t)$ caused by the difference of initial crack length Δa_0, we obtain from Eqs.(9) and (10)

$$-\Delta U_{a_0}(a_0,t)$$
$$=U(a_0,t)-U(a_0+\Delta a_0,t)$$
$$=-\int_{\Delta S_{a_0}}\int_0^t T_i(a_0,\tau)\dot{u}_i(a_0,\tau)d\tau dS$$
$$+\int_{\Delta S_{a_0}}\int_t^{t+t^*} T_i(a_0+\Delta a_0,\tau)\dot{u}_i(a_0+\Delta a_0,\tau)d\tau dS$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots(11)$$

In this relation, we consider the case of $\Delta a_0\to 0$. In this case, $-\Delta U_{a_0}(a_0,t)\to 0$, and, as the first and the second term in the right hand are positive, both become zero. However, their infinitesimal orders are different. When $\Delta a_0\to 0$, that is, as ΔS_{a_0} approaches the free surface $\Gamma(a_0)$, we get $T_i(a_0,t)\to 0$ but $u_i(a_0,t)$ does not vary and is finite. Therefore, the first term becomes the first order of inifinitesimal. On the other hand, the second term becomes the second order of infinitesimal, because the displacement of $u_i(a_0+\Delta a_0,t+t^*)-u_i(a_0+\Delta a_0,t)$ and the traction force $T_i(a_0+\Delta a_0,t+t^*)$ are caused by each other and therefore become zero together. Accordingly, when we consider the variation of whole strain energy with an infinitesimal increase of initial crack length a_0, we can neglect the second term in the right hand and obtain the following relation by substituting Eq.(10) into Eq.(11).

$$-\frac{\partial U}{\partial a_0}(a_0,t)=\lim_{\Delta a_0\to 0}-\frac{\Delta U_{a_0}(a_0,t)}{\Delta a_0}$$
$$=\lim_{\Delta a_0\to 0}\left[\left\{B\int_{a_0}^{a_0+\Delta a_0}\int_{\Gamma(X_1)} W(a_0,t)dX_2dX_1\right\}\Big/\Delta a_0\right]$$
$$=B\int_{\Gamma(a_0)} W(a_0,t)dX_2 \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots(12)$$

Here, $\partial Z/\partial X(X,Y)$ represents a partial derivative of a function Z of X and Y with X. The term of $\int_{\Gamma(a_0)} W(a_0,t)dX_2$ in Eq.(12) is the crack energy density at the crack tip $\mathcal{E}(t,a_0)\{=\mathcal{E}(t,X_1)|_{X_1=a_0}\}$ defined in Eq.(1). Therefore, the following relation is obtained by substituting Eqs.(7) and (8) into Eq.(12).

$$\mathcal{E}(t,a_0)=-\frac{1}{B}\frac{\partial U}{\partial a_0}(a_0,t)$$
$$=-\frac{1}{B}\int_0^t\frac{\partial P}{\partial a_0}(a_0,\tau)\dot{u}d\tau \quad \cdots\cdots\cdots\cdots\cdots(13)$$

Moreover, from the assumption of one-to-one correspondence between load-displacement u and time τ, Eq.(13) is also expressed by

$$\mathcal{E}(t,a_0)=-\frac{1}{B}\int_0^{u(t)}\frac{\partial P}{\partial a_0}(a_0,u)du \quad \cdots\cdots\cdots(14)$$

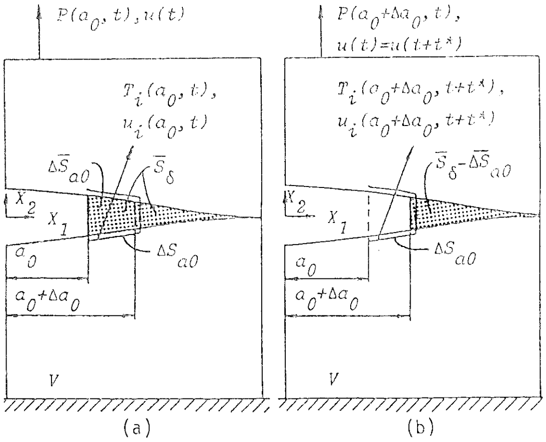

Fig.4 Two discontinuous models differing in initial crack length before onset of crack growth

3.2 Relation on discontinuous model

We consider the discontinuous models with initial crack lengths a_0 and $a_0+\Delta a_0$ as shown in Fig.4 which are in the states before onsets of crack extension ($\tau \leqq t_0$ and $t_0+\Delta t_0$) under loading. Also on the discontinuous models, when the strain energy plane density is represented as a function of the initial crack length a_0 and the time τ by $W_{\text{plane}}(a_0, \tau)$, the whole strain energies of the two discontinuous models are expressed by

$$U(a_0, t)=\int_V W(a_0, t)dV+\int_{\bar{S}_\delta} W_{\text{plane}}(a_0, t)dS$$
$$=\int_0^t P(a_0, \tau)\dot{u}_i d\tau \cdots\cdots (15)$$

$$U(a_0+\Delta a_0, t)=\int_V W(a_0+\Delta a_0, t)dV$$
$$+\int_{\bar{S}_\delta-\Delta\bar{S}_{a_0}} W_{\text{plane}}(a_0+\Delta a_0, t)dS$$
$$=\int_0^t P(a_0+\Delta a_0, \tau)\dot{u}d\tau \cdots\cdots (16)$$

where V is the volume corresponding to all parts of the continuum, and $\bar{S}_\delta$ and $\bar{S}_\delta-\Delta\bar{S}_{a_0}$ are the planes considering the discontinuities of the discontinuous models with initial crack lengths a_0 and $a_0+\Delta a_0$ respectively (refer to Fig.4). On the other hand, we can consider the state at the time $\tau=t+t^*$ in which the following relation on the parts of $V+\bar{S}_\delta-\Delta\bar{S}_{a_0}$ of the two discontinuous models holds.

$$U(a_0, t)-B\int_{a_0}^{a_0+\Delta a_0} W_{\text{plane}}(a_0, t)dX_1$$
$$=U(a_0+\Delta a_0, t)$$
$$+\int_{\Delta S_{a_0}}\int_t^{t+t^*} T_i(a_0+\Delta a_0, \tau)\dot{u}_i(a_0+\Delta a_0, \tau)d\tau dS$$
$$\cdots\cdots (17)$$

Here, ΔS_{a_0} is the surface of the continuum part being in contact with $\Delta\bar{S}_{a_0}$, and $T_i(a_0+\Delta a_0, \tau)$ and $u_i(a_0+\Delta a_0, \tau)$ at the time $\tau \geqq t$ are the traction force acting on ΔS_{a_0} of the discontinuous model with initial crack length $a_0+\Delta a_0$ under the condition of holding the load-point-displacement to be $u(\tau)=u(t)$ and the corresponding displacement respectively. In the case of the Dugdale model, Eq.(17) holds substituting yield stress σ_y for $T_i(a_0+\Delta a_0, t+t^*)$.

Moreover, the strain energy on $\Delta\bar{S}_{a_0}$ of the disontinuous model with initial crack length a_0 is expressed by

$$B\int_{a_0}^{a_0+\Delta a_0} W_{\text{plane}}(a_0, t)dX_1$$
$$=-\int_{\Delta S_{a_0}}\int_0^t T_i(a_0, \tau)\dot{u}_i(a_0, \tau)d\tau dS \cdots (18)$$

Here, paying attention to the difference of the whole strain energies $\Delta U_{a_0}(a_0, t)$ caused by the difference of initial crack length Δa_0 , we obtain formally the same equation as Eq.(11) from Eqs.(17) and (18). In this equation, we consider the case of $\Delta a_0\to 0$. The traction forces $T_i(a_0+\Delta a_0, t+t^*)$ and $T_i(a_0, t)$ in the first and the second term in the right hand side are finite under the condition of $\Delta a_0\to 0$. However, the area of the surface ΔS_{a_0} on which $T_i(a_0+\Delta a_0, t+t^*)$ acts becomes the first order of inifinitesimal. Moreover, the displacement $u_i(a_0, t)$ in the first term of the right hand side is finite because the displacement becomes equal to half the relative displacement at the crack tip of the discontinuous model with initial crack length a_0 at the time $\tau=t$. However, the displacement of $u_i(a_0+\Delta a_0, t+t^*)-u_i(a_0+\Delta a_0, t)$ in the second term of the right hand side becomes the first order of infinitesimal because the area of ΔS_{a_0} becomes zero when $\Delta a_0\to 0$. Consequently, the first and the second term in the right hand side of Eq. (11) become the first and the second order inifinitesimal respectively. Hence, the second term can be neglected when we obtain the differential coefficient of whole strain energy with initial crack length $\partial U/\partial a_0(a_0, t)$, and the following relation is obtained by substituting Eq.(18) into Eq. (11).

$$-\frac{\partial U}{\partial a_0}(a_0, t)=\lim_{\Delta a_0\to 0}-\frac{\Delta U_{a_0}(a_0, t)}{\Delta a_0}$$
$$=\lim_{\Delta a_0\to 0}\left[\left\{B\int_{a_0}^{a_0+\Delta a_0} W_{\text{plane}}(a_0, t)dX_1\right\}\Big/\Delta a_0\right]$$
$$=BW_{\text{plane}}(a_0, t)|_{X_1=a_0} \cdots\cdots (19)$$

Here, $W_{\text{plane}}(a_0, t)|_{X_1=a_0}$ is the crack energy density at the crack tip $\mathcal{E}(t, a_0)$ defined by Eq.(6). Therefore, the same relations as Eqs.(13) and (14) are obtained by substituting Eqs.(15) and (16) into Eq.(19).

4. Relation between Additional Rate of Crack Energy Density and Load-displacement Curve on Growing Crack

Subsequently, a relation between load, growing crack length, load-point-displacement, initial crack length and additional rate of crack energy density after onset of stable crack growth is derived for the two models above.

4.1 Relation on notch model

Figure 5 shows the states after onset of stable crack growth at the time $\tau=t(>t_0$ and $t_0+\Delta t_0)$ for the notch models shown in Fig.3 under load corresponding to a monotonously increasing $u(\tau)$. Here, the crack growth is assumed to be smooth. Also in this chapter, we follow the same manner of description as in the previous chapter, and the growing crack length, which is the sum of the initial crack length and the crack growth length, is represented as a

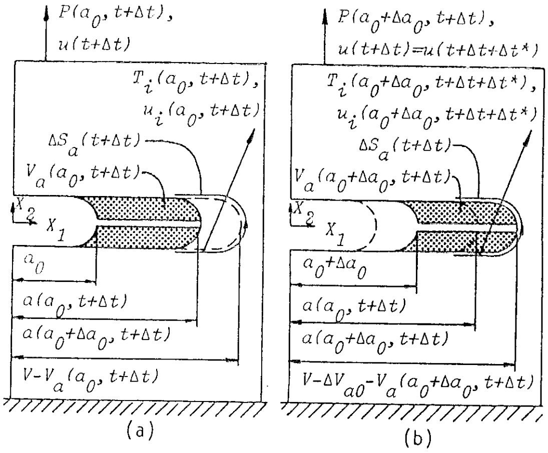

Fig.5 Two notch models differing in initial crack length after onset of crack growth

function of the initial crack length a_0 and the time τ by $a(a_0, \tau)$.

Following the manner of description, the increments of the whole strain energies of the two notch models shown in Fig. 5, $\Delta U_a(a_0, t)$ and $\Delta U_a(a_0+\Delta a_0, t)$, during the period of Δt from $\tau=t$ to $\tau=t+\Delta t$ are expressed by

$$\begin{aligned}\Delta U_a(a_0, t) &= U(a_0, t+\Delta t)-U(a_0, t)\\ &= \int_t^{t+\Delta t}\int_{V-V_a(a_0,\tau)}\dot{W}(a_0, \tau)dVd\tau\\ &= \int_t^{t+\Delta t}P(a_0, \tau)\dot{u}d\tau \quad\cdots\cdots(20)\end{aligned}$$

$$\begin{aligned}&\Delta U_a(a_0+\Delta a_0, t)\\ &= U(a_0+\Delta a_0, t+\Delta t)-U(a_0+\Delta a_0, t)\\ &= \int_t^{t+\Delta t}\int_{V-\Delta V_{a0}-V_a(a_0+\Delta a_0,\tau)}\dot{W}(a_0+\Delta a_0, \tau)dVd\tau\\ &= \int_t^{t+\Delta t}P(a_0+\Delta a_0, \tau)\dot{u}d\tau \quad\cdots\cdots(21)\end{aligned}$$

where $V_a(a_0, \tau)$ and $V_a(a_0+\Delta a_0, \tau)$ are the volumes corresponding to the parts through which the crack tips of the notch models with initial crack lengths a_0 and $a_0+\Delta a_0$ have passed before the time τ (refer to Fig.5).

On the other hand, we can consider the state at the time $\tau=t+\Delta t+\Delta t^*$ in which the following relation on the parts of $V-\Delta V_{a_0}-V_a(a_0+\Delta a_0, \tau)$ at the time of $t\leqq\tau\leqq t+\Delta t$ in the two notch models holds.

$$\begin{aligned}&\Delta U_a(a_0, t)-B\int_t^{t+\Delta t}\int_{a(a_0,\tau)}^{a(a_0+\Delta a_0,\tau)}\int_{\Gamma(X_1)}\dot{W}(a_0, \tau)dX_2dX_1d\tau\\ &= \Delta U_a(a_0+\Delta a_0, t)\\ &\quad+\int_{t+\Delta t}^{t+\Delta t+\Delta t^*}\int_{\Delta S_a(t+\Delta t)}T_i(a_0+\Delta a_0, \tau)\dot{u}_i(a_0+\Delta a_0, \tau)dSd\tau \quad\cdots\cdots(22)\end{aligned}$$

Here, $\Delta S_a(t+\Delta t)$ is the surface obtained by excluding $\Gamma\{a(a_0, t+\Delta t)\}$ from the surface which appears after removing $\Delta V_{a_0}+V_a(a_0+\Delta a_0, t+\Delta t)$, and $T_i(a_0+\Delta a_0, \tau)$ and $u_i(a_0+\Delta a_0, \tau)$ at the time $\tau\geqq t+\Delta t$ are the traction force acting on $\Delta S_a(t+\Delta t)$ of the notch model with initial crack length $a_0+\Delta a_0$ under the condition of holding the load-point-displacement to be $u(\tau)=u(t+\Delta t)$ and the corresponding displacement (refer to Fig.5).

Moreover, on the part of $\Delta V_{a_0}+V_a(a_0+\Delta a_0, \tau)-V_a(a_0, \tau)$ at the time of $t\leqq\tau\leqq t+\Delta t$ in the notch model with initial crack length a_0, the relation

$$\begin{aligned}&B\int_t^{t+\Delta t}\int_{a(a_0,\tau)}^{a(a_0+\Delta a_0,\tau)}\int_{\Gamma(X_1)}\dot{W}(a_0, \tau)dX_2dX_1d\tau\\ &= -\int_t^{t+\Delta t}\int_{\Delta S_a(\tau)}T_i(a_0, \tau)\dot{u}_i(a_0, \tau)dSd\tau \cdots\cdots(23)\end{aligned}$$

holds where $\Delta S_a(\tau)$ is the surface obtained by excluding $\Gamma\{a(a_0, \tau)\}$ from the surface which appears after removing $\Delta V_{a_0}+V_a(a_0+\Delta a_0, \tau)-V_a(a_0, \tau)$, and $T_i(a_0, \tau)$ and $\dot{u}_i(a_0, \tau)$ are the traction force and the displacement on $\Delta S_a(\tau)$ respectively.

Here, paying attention to the difference of the increments of whole strain energies $\Delta^2U(a_0, t)$ during the period of Δt from the time $\tau=t$ caused by the difference of initial crack length Δa_0, we obtain from Eqs.(22) and (23)

$$\begin{aligned}&-\Delta^2U(a_0, t)\\ &= \Delta U_a(a_0, t)-\Delta U_a(a_0+\Delta a_0, t)\\ &= -\int_t^{t+\Delta t}\int_{\Delta S_a(\tau)}T_i(a_0, \tau)\dot{u}_i(a_0, \tau)dSd\tau\\ &\quad+\int_{t+\Delta t}^{t+\Delta t+\Delta t^*}\int_{\Delta S_a(t+\Delta t)}T_i(a_0+\Delta a_0, \tau)\dot{u}_i(a_0+\Delta a_0, \tau)dSd\tau\\ &\qquad\cdots\cdots(24)\end{aligned}$$

In this relation, we consider the case of $\Delta a_0\to 0$ and $\Delta t\to 0$. First, we consider the case of $\Delta a_0\to 0$. The traction forces of $T_i(a_0, \tau)$ and $T_i(a_0+\Delta a_0, t+\Delta t+\Delta t^*)$ in the first and the second term of the right hand side become the first order infinitesimals together because $\Delta S_a(\tau)$ and $\Delta S_a(t+\Delta t)$ approach the free surfaces $\Gamma\{a(a_0, \tau)\}$ and $\Gamma\{a(a_0, t+\Delta t)\}$, respectively, with $\Delta a_0\to 0$. Further, the displacement of $u_i(a_0, t+\Delta t)-u_i(a_0, t)$ in the first term of the right hand side is finite because it becomes the increment of displacement at $\Gamma\{a(a_0, t)\}$ between t and $t+\Delta t$ of notch model with initial crack length a_0 when $\Delta a_0\to 0$, but the displacement of $u_i(a_0+\Delta a_0, t+\Delta t+\Delta t^*)-u_i(a_0+\Delta a_0, t+\Delta t)$ in the second term of the right hand side becomes the first order of infinitesimal because it corresponds to $T_i(a_0+\Delta a_0, t+\Delta t+\Delta t^*)\to 0$. Second, we consider the case of $\Delta t\to 0$. The traction force of $T_i(a_0, t+\Delta t)$ in the first term of the right hand side is finite because it becomes $T_i(a_0, t)$ with $\Delta t\to 0$ and $\Delta S_a(t)$ does not become a free surface, but the traction force of $T_i(a_0+\Delta a_0, t+\Delta t+\Delta t^*)$ in the second term of the right hand side becomes the first order of infinitesimal because every term in Eq.(22) defining the second term of the right hand side becomes zero and the second term in the right hand side of Eq.(22) is determined by $T_i(a_0+\Delta a_0, t+\Delta t+\Delta t^*)$. Further, the displacements $u_i(a_0, t+\Delta t)-u_i(a_0, t)$ and $u_i(a_0+\Delta a_0, t+\Delta t+\Delta t^*)-u_i(a_0+\Delta a_0, t+\Delta t)$ in the first and the second term of the right hand side become the first order infinitesimals together because $u_i(a_0, t+\Delta t)\to u_i(a_0, t)$ and $T_i(a_0+\Delta a_0, t+\Delta t+\Delta t^*)\to 0$, respectively, with $\Delta t\to 0$. Based on the above discussion, the first and the second term in the right hand side of Eq.(24) become the first and the second order infinitesimal respectively when Δa_0 becomes the first order infinitesimal, and they become the same order infinitesimals when Δt becomes the first order of infinitesimal. Hence, when we obtain the variational rate of whole strain energy change on initial crack length and growing crack length $\partial^2U/(\partial a_0\partial a)(a_0, t)$, the second term in the right hand side can be neglected, and the following relation is obtained by substituting Eq. (23) into Eq.(22).

$$\frac{\partial^2 U}{\partial a_0 \partial a}(a_0, t)$$
$$= -\frac{1}{\dot{a}(a_0, t)} \frac{\partial^2 U}{\partial a_0 \partial t}(a_0, t)$$
$$= \frac{1}{\dot{a}(a_0, t)} \lim_{\Delta a_0, \Delta t \to 0} \frac{\Delta^2 U(a_0, t)}{\Delta a_0 \Delta t}$$
$$= \frac{1}{\dot{a}(a_0, t)} \lim_{\Delta a_0, \Delta t \to 0} \frac{B}{\Delta a_0 \Delta t} \int_t^{t+\Delta t} \int_{a(a_0,\tau)}^{a(a_0+\Delta a_0,\tau)} \int_{\Gamma(X_1)} \dot{W}(a_0, \tau) dX_2 dX_1 d\tau$$
$$= \frac{B}{\dot{a}(a_0, t)} \frac{\partial a}{\partial a_0}(a_0, \tau) \int_{\Gamma\{a(a_0, t)\}} \dot{W}(a_0, t) dX_2 \quad \cdots\cdots (25)$$

Here, we define the additional rate of crack energy density change (called additional rate hereafter) $\partial \mathcal{E}/\partial a(t, X_1)$, for the region of $X_1 \geqq a$, by

$$\frac{\partial \mathcal{E}}{\partial a}(t, X_1) = \frac{1}{\dot{a}} \frac{\partial \mathcal{E}}{\partial \tau}(t, X_1) \quad \cdots\cdots (26)$$

then, $\int_{\Gamma\{a(a_0,t)\}} \dot{W}(a_0, t) dX_2$ in Eq.(25) is the additional rate at the crack tip $\partial \mathcal{E}/\partial a(t, a)$ $\{= \partial \mathcal{E}_1/\partial a(t, X_1)|_{X_1=a}\}$. Therefore, using the relations of Eqs.(20) and (21), the relation

$$\frac{\partial \mathcal{E}}{\partial a}(t, a)$$
$$= -\frac{1}{B} \frac{\partial P}{\partial a_0}(a_0, t) \frac{\dot{u}(t)}{\dot{a}(a_0, t)} \Big/ \frac{\partial a}{\partial a_0}(a_0, t) \quad \cdots\cdots (27)$$

is obtained. Moreover, from the assumption of one-to-one correspondence between load-point-displacement u and time τ, Eq.(27) is expressed by

$$\frac{\partial \mathcal{E}}{\partial a}(t, a)$$
$$= -\frac{1}{B} \frac{\partial P}{\partial a_0}(a_0, u) \Big/ \left\{ \frac{\partial a}{\partial u}(a_0, u) \cdot \frac{\partial a}{\partial a_0}(a_0, u) \right\} \quad (28)$$

4.2 Relation on discontinuous model

Figure 6 shows the states after onset of stable crack growth at the time $\tau = t (> t_0$ and $t_0 + \Delta t_0)$ for the discontinuous models shown in Fig.4 under a load corresponding to a monotonously increaseg $u(\tau)$.

Also in case of the discontinuous models, the increments of the whole strain energies $\Delta U_a(a_0, t)$ and $\Delta U_a(a_0+\Delta a_0, t)$ between t and $t+\Delta t$ are expressed by

$$\Delta U_a(a_0, t)$$
$$= U(a_0, t+\Delta t) - U(a_0, t)$$
$$= \int_t^{t+\Delta t} \int_V \dot{W}(a_0, \tau) dV d\tau + \int_t^{t+\Delta t} \int_{\bar{S}_s - \bar{S}_a(a_0,\tau)} \dot{W}_{\mathrm{plane}}(a_0, \tau) dS d\tau$$
$$= \int_t^{t+\Delta t} P(a_0, \tau) \dot{u} d\tau \quad \cdots\cdots (29)$$

$$\Delta U_a(a_0+\Delta a_0, t)$$
$$= U(a_0+\Delta a_0, t+\Delta t) - U(a_0+\Delta a_0, t)$$
$$= \int_t^{t+\Delta t} \int_V \dot{W}(a_0+\Delta a_0, \tau) dV d\tau + \int_t^{t+\Delta t} \int_{\bar{S}_s - \Delta\bar{S}_{a_0} - \bar{S}_a(a_0+\Delta a_0,\tau)} \dot{W}_{\mathrm{plane}}(a_0+\Delta a_0, \tau) dS d\tau$$
$$= \int_t^{t+\Delta t} P(a_0+\Delta a_0, \tau) \dot{u} d\tau \quad \cdots\cdots (30)$$

where $\bar{S}_a(a_0, \tau)$ and $\bar{S}_a(a_0+\Delta a_0, \tau)$ are the planes corresponding to the parts through which the crack tips of the discontinous models with initial crack lengths a_0 and $a_0+\Delta a_0$ have passed before the time τ (refer to Fig.5).

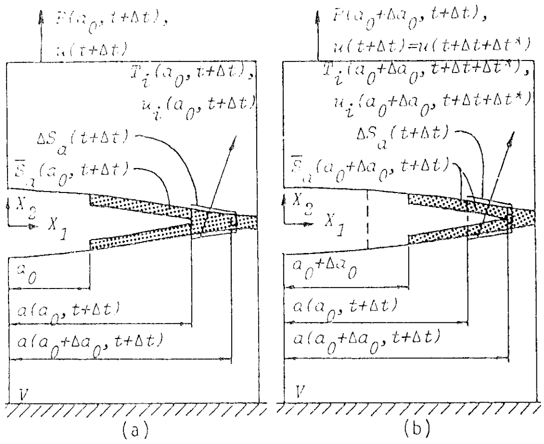

Fig.6 Two discontinuous models differing in initial crack length after onset of crack growth

On the other hand, we can consider the state at the time $\tau = t + \Delta t + \Delta t^*$ in which the following relation on the part of $V + \bar{S}_s - \Delta\bar{S}_{a_0} - \bar{S}_a(a_0+\Delta a_0, \tau)$ at the time $t \leqq \tau \leqq t+\Delta t$ in the two discontinuous models holds.

$$\Delta U_a(a_0, t) - B\int_t^{t+\Delta t} \int_{a(a_0,\tau)}^{a(a_0+\Delta a_0,\tau)} \dot{W}_{\mathrm{plane}}(a_0, \tau) dX_1 d\tau$$
$$= \Delta U_a(a_0+\Delta a_0, t) + \int_{t+\Delta t}^{t+\Delta t+\Delta t^*} \int_{\Delta S_a(t+\Delta t)} T_i(a_0+\Delta a_0, \tau) \dot{u}_i(a_0+\Delta a_0, \tau) dS d\tau \quad \cdots\cdots (31)$$

Here, $\Delta S_a(t+\Delta t)$ is the surface of the continuum part in contact with $\Delta\bar{S}_{a_0} + \bar{S}_a(a_0+\Delta a_0, t+\Delta t) - \bar{S}_a(a_0, t+\Delta t)$, and $T_i(a_0+\Delta a_0, \tau)$ and $u_i(a_0+\Delta a_0, \tau)$ at the time $\tau \geqq t+\Delta t$ are the traction force acting on $\Delta S_a(t+\Delta t)$ of the discontinuous model with initial crack length $a_0+\Delta a_0$ under the condition of holding the load-point-displacement to be $u(\tau) = u(t+\Delta t)$ and the corresponding displacement (refer to Fig.6). In the case of the Dugdale model, Eq.(31) holds when putting $T_i(a_0+\Delta a_0, t+\Delta t+\Delta t^*) = \sigma_y$.

Moreover, on $\Delta\bar{S}_{a_0} + \bar{S}_a(a_0+\Delta a_0, \tau) - \bar{S}_a(a_0, \tau)$ at the time of $t \leqq \tau \leqq t+\Delta t$ in the discontinuous model with initial crack length a_0, the relation

$$B\int_t^{t+\Delta t} \int_{a(a_0,\tau)}^{a(a_0+\Delta a_0,\tau)} \dot{W}_{\mathrm{plane}}(a_0, \tau) dX_1 d\tau$$
$$= -\int_t^{t+\Delta t} \int_{\Delta S_a(\tau)} T_i(a_0, \tau) \dot{u}_i(a_0, \tau) dS d\tau \quad \cdots\cdots (32)$$

holds, where $\Delta S_a(\tau)$ is the surface of the continuum which appears after removing $\Delta\bar{S}_{a_0} + \bar{S}_a(a_0+\Delta a_0, \tau) - \bar{S}_a(a_0, \tau)$.

Here, paying attention to the difference of the increments of whole strain energies $\Delta^2 U(a_0, t)$ between t and $t+\Delta t$ caused by the difference of initial crack length Δa_0, we obtain formally the same relation as Eq.(24) from Eqs.(31) and (32), and, in this relation, we consider the case of $\Delta a_0 \to 0$ and $\Delta t \to 0$. First, we consider the case of $\Delta a_0 \to 0$. The traction forces of $T_i(a_0, t+\Delta t)$ and $T_i(a_0+\Delta a_0, t+\Delta t+\Delta t^*)$ in the first and the second term of the right hand side are finite, but the areas of the surfaces $\Delta S_a(\tau)$ and $\Delta S_a(t+\Delta t)$ on which the traction forces act become the first order infinitesimals together when $\Delta a_0 \to 0$. Further, the displacement of $u_i(a_0, t+\Delta t)-u_i(a_0, t)$ in the first term of the right hand side is finite because the displacement becomes equal to half the increment of the relative displacement at the crack tip of the discontinuous model with initial crack length a_0 corresponding to time increment Δt from the time $\tau=t$ with $\Delta a_0\to 0$, but the displacement of $u_i(a_0+\Delta a_0, t+\Delta t+\Delta t^*)-u_i(a_0+\Delta a_0, t+\Delta t)$ in the second term of the right hand side becomes the first order infinitesimal because $\Delta S_a(t+\Delta t)\to 0$ with $\Delta a_0\to 0$. Second, we consider the case of $\Delta t\to 0$. The traction force of $T_i(a_0, t+\Delta t)$ in the first term of the right hand side is finite because it becomes $T_i(a_0, t)$ with $\Delta t\to 0$, but the traction force of $T_i(a_0+\Delta a_0, t+\Delta t+\Delta t^*)$ in the second term of the right hand side becomes the first order infiniterimal because every term in Eq.(31) defining the second term of the right hand side becomes zero and the second term in the right hand side of Eq.(31) is determined by $T_i(a_0+\Delta a_0, t+\Delta t+\Delta t^*)$. Further, the displacements of $u_i(a_0, t+\Delta t)-u_i(a_0, t)$ and $u_i(a_0+\Delta a_0, t+\Delta t+\Delta t^*)-u_i(a_0+\Delta a_0, t+\Delta t)$ in the first and the second term of the right hand side become the first order infinitesimals together because $u_i(a_0, t+\Delta t)\to u_i(a_0, t)$ and $T_i(a_0+\Delta a_0, t+\Delta t+\Delta t^*)\to 0$ respectively with $\Delta t\to 0$. From the results on infinitesimals, the first and the second term in the right hand side of Eq.(24) become the first and the second order infinitesimal when Δa_0 becomes the first order infinitesimal, and they become the same order infinitesimals also when Δt becomes the first order infinitesimal. Hence, when we obtain the variational rate of increase of whole strain energy on initial crack length and growing crack length $\partial^2 U/(\partial a_0\partial a)(a_0, t)$, the second term in the right hand can be neglected, and the following relation is obtained by substituting Eq.(32) into Eq.(31).

$$-\frac{\partial^2 U}{\partial a_0\partial a}(a_0, t) = -\frac{1}{\dot a(a_0, t)}\frac{\partial^2 U}{\partial a_0\partial t}(a_0, t) = \frac{1}{\dot a(a_0, t)}\lim_{\Delta a_0, \Delta t\to 0} -\frac{\Delta^2 U(a_0, t)}{\Delta a_0\Delta t}$$
$$= \frac{1}{\dot a(a_0, t)}\lim_{\Delta a_0, \Delta t\to 0}\frac{B}{\Delta a_0\Delta t}\int_t^{t+\Delta t}\int_{a(a_0,\tau)}^{a(a_0+\Delta a_0,\tau)}\dot W_{\mathrm{plane}}(a_0, \tau)dX_1 d\tau$$
$$= \frac{1}{\dot a(a_0, t)}\frac{\partial a}{\partial a_0}(a_0, t)\dot W_{\mathrm{plane}}(a_0, t)|_{X_1=a(a_0,t)} \quad \cdots(33)$$

Here, $\dot W_{\mathrm{plane}}(a_0, t)|_{X_1=a(a_0,t)}$ in Eq.(33) is the additional rate at the crack tip $\partial\mathcal{E}/\partial a(t, a)$ defined by Eqs.(5) and (26). Therefore, using the relations of Eqs.(29) and (30), the same relations as those in Eqs.(27) and (28) for the notch model are obtained.

5. Proposal of Evaluation Method of Fracture Resistance

The variation of crack energy density distribution on a growing crack can be shown shematically in Fig.7. Here, the solid line represents the distribution at the time of onset of crack growth ($\tau=t_0$), the chain line with a dot does the change of crack energy density at the crack tip between $\tau=t_0$ and $\tau=t-dt$ when the growing crack length becomes $a-da$, the chain line with two dots does the distribution at the time $\tau=t-dt$ and the broken line represents the distribution at the time of $\tau=t$. In the previous paper[6], the authors proposed, based on the fact that the crack energy density distribution can be defined momentarily, to express the criterion for stable crack growth by

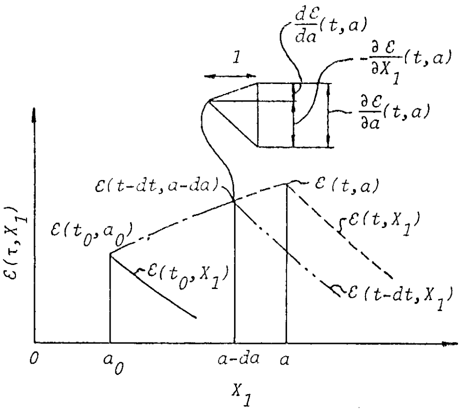

Fig.7 Crack energy density with crack growth

$$\mathcal{E}(t, a)=\mathcal{E}_c(t, a) \quad \text{(growth)} \quad \cdots\cdots(34)$$

and the criterion for crack growth becoming unstable at the time τ by

$$\frac{d\mathcal{E}}{da}(t, a)>\frac{d\mathcal{E}_c}{da}(t, a) \quad \text{(unstable)} \quad \cdots(35)$$

or, as another equivalent expression, by

$$\frac{\partial\mathcal{E}}{\partial a}(t, a)>\frac{\partial\mathcal{E}_c}{\partial a}(t, a) \quad \text{(unstable)} \quad \cdots(36)$$

Here, the left hand sides of the three equations above represent the quantities determined by prescribed mechanical condition, and the right hand sides with the subscript c represent the fracture resistances determined through an actual process of stable crack growth. Further, $d\mathcal{E}/da(t, a)$ at $\tau=t$ is defined by

$$\frac{d\mathcal{E}}{da}(\tau, a)=\frac{1}{\dot a}\frac{\partial\mathcal{E}}{\partial\tau}(\tau, a) = \frac{1}{\dot a}\frac{\partial\mathcal{E}}{\partial\tau}(\tau, a)+\frac{\partial\mathcal{E}}{\partial X_1}(\tau, a) = \frac{\partial\mathcal{E}}{\partial a}(\tau, a)+\frac{\partial\mathcal{E}}{\partial X_1}(\tau, a) \quad \cdots\cdots(37)$$

(refer to Fig.7), and the equivalency of Eqs.(35) and (36) is derived from the fact that $\partial\mathcal{E}/\partial X_1(t, a)=\partial\mathcal{E}_c/\partial X_1(t, a)$ at the place of crack tip $X_1=a-da$ at the time $\tau=t-dt$.

By the way, as to the criterion for the crack growth, considering that the greater-than-signs in Eqs.(35) and (36) become equal-signs in the process of stable crack growth and that the relation

$$\mathcal{E}(t, a)=\mathcal{E}(t_0, a_0)+\int_{t_0}^{t}\frac{d\mathcal{E}}{da}(\tau, a)\dot a d\tau \quad \cdots(38)$$

holds, another criterion which is equivalent to Eq.(34) is given by the combination of the relation of $\mathcal{E}(t_0, a_0)=\mathcal{E}_c(t_0, a_0)$ and the relation given by replacing the greater-than-sign with the equal-sign in Eq.(35) or (36), for instance

$$\frac{\partial\mathcal{E}}{\partial a}(t, a)=\frac{\partial\mathcal{E}_c}{\partial a}(t, a) \quad \text{(growth)} \quad \cdots(39)$$

The relations derived in the previous chapters show that it is possible to evaluate directly the fracture resistance of $\mathcal{E}_c(t_0, a_0)$ and $\partial \mathcal{E}_c/\partial a(t, a)$ by Eqs.(14) and (28), respectively, from the measurements of the parameters, which represent the overall behavior of a specimen such as load, load-point-displacement and so on, through the experiments of stable crack growths for several specimens with various initial crack lengths. Moreover, if a suitable value of $\partial \mathcal{E}_c/\partial X_1(\tau, a)$ is chosen, it becomes possible to evaluate the fracture resistance of $\mathcal{E}_c(t, a)$ by Eqs.(37) and (38) using $\mathcal{E}_c(t_0, a_0)$ and $\partial \mathcal{E}_c/\partial a(t, a)$ evaluated above. Thereupon, in this paper, we propose to evaluate the fracture resistance, in the manner above, by the combination of $\mathcal{E}_c(t_0, a_0)$ and $\partial \mathcal{E}_c/\partial a(t, a)$ or by $\mathcal{E}_c(t, a)$. Here, it is considered, as the simplest estimation method of $\partial \mathcal{E}_c/\partial X_1(t, a)$, to evaluate it, from the constant value $(\partial \mathcal{E}_c/\partial a)_{uni}$ of $\partial \mathcal{E}_c/\partial a(\tau, a)$ under the condition of crack extension accompanied with a uniform fracture mode, by

$$\frac{\partial \mathcal{E}_c}{\partial X_1}(\tau, a) = -\left(\frac{\partial \mathcal{E}_c}{\partial a}\right)_{uni} \qquad \cdots\cdots (40)$$

because the relation $d\mathcal{E}_c/da(\tau, a)=0$ holds in Eq.(37) under a uniform fracture mode.

6. Conclusion

From a standpoint of crack energy density, we derived the relation between the crack energy density, the load-displacement curve and so on considering stable crack growth behaviors in the mechanical models, and proposed an evaluation method of fracture resistance for stable crack growth based on the relation above. In the next reports, we are going to apply the proposed method to actual problems and examine the applicability of the method.

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