Bulletin of JSME, Vol. 29, No. 251 May 1986

Paper No. 251—4

Proposal of a New Crack Model Considering the Discontinuity in the Cracked Plane and Its Application to the Evaluation of Crack Parameter*

by Katsuhiko WATANABE** and Hideyuki AZEGAMI***

A new crack model which enables us to evaluate crack parameters such as crack energy density and its distribution, COD and COA under arbitrary load history is proposed and the availability of the proposed model is demonstrated through finite element analyses of elasto-plastic crack. The contents are as follows;

(1) A crack model considering the discontinuity in the cracked plane is introduced and the constitutive equation for a plane with discontinuity is formulated on an elasto-plastic case.

(2) The finite element formulation of the model is carried out by introducing a plane element.

(3) An elasto-plastic crack expressed by the proposed model under monotonic loading is analyzed by finite element method and the availability of the model is verified through the evaluation of crack energy density.

Key Words : Fracture, DWA Crack Model, Discontinuity in Cracked Plane, Crack Parameter, Crack Energy Density, Finite Element Analysis

1. Introduction

In the analysis of a crack problem based on usual continuum model, the singularities of stress and strain appear at a crack tip. At a crack tip in an actual material, however, the situation is different and necessary condition to be continuum is usually not satisfied because of discontinuous deformation. The Dugdale model[1] is regarded as a model of first-order approximation to express this discontinuous deformation, but it can not assert itself as a crack model with generality because, in the model, the non-linearity is limited to that of ideal plasticity and plastic deformation is admitted only in the plane containing a crack.

By the way, the authors discussed on the Dugdale model before and showed that the model can be rated as a crack model in which the plane containing a crack is deemed an ideally rigid-plastic body of which the constitutive law is given by a relation between out-of-plane stress component and corresponding out-of-plane relative displacement and the other part except the plane containing a crack is deemed a usual linear elastic body (in this paper, stress component acting on the upper and lower surfaces of the plane and corresponding relative displacement on the surfaces are called out-of-plane stress component and out-of-plane relative displacement respectively). And it was pointed out that, when the model is thus rated, it can be a model which enables us to follow the deformation every moment with the variation of applied load[2].

When the Dugdale model is grasped as described above, it is planned, as a natural extension of the model, to analyze a crack problem by applying the constitutive equations suitable to the problem given by a relation between out-of-plane stress component and out-of-plane relative displacement to the plane containing crack front (not restricted to the plane containing a crack) and by a relation between stress and strain to the part except the plane. In this paper, such a crack model is newly proposed as a general crack model which expresses the discontinuous deformation around a crack tip as first-order approximation and it is tried to apply this model to the evaluations of crack parameters. First, the authors present a new crack model and show how to give the constitutive equation and how to formulate the model for finite element analysis. Next, we carry out the finite element analysis of an elasto-plastic cracked panel under monotonously increasing load by the new model and show the availability of the model through the evaluations of crack energy density and J-integral.

2. Proposal of a Crack Model Considering the Discontinuity in the Plane Containing Crack Front

In this chapter, a new crack model is presented and it is shown how the crack parameters are defined in this model. The formulation of a constitutive equation especially for an elasto-plastic problem and

* Received 30th September, 1983.
** Associate Professor, Institute of Industrial Science, University of Tokyo, 7-22-1, Roppongi, Minato-ku, Tokyo.
*** Research Assistant, Toyohashi University of Technology, 1-1, Hibarigaoka, Tempaku-cho, Toyohashi-city.

the finite element formulation of the new model are also shown here.

2.1 DWA model and crack parameters

Around a crack tip tip in an actual material, usually the continuity of deformation does not hold because of concentrated slip deformations or other reasons. Then, we propose a general crack model shown in the following as the model which enables us to express the situation above around a crack tip as first-order approximation.

In Fig.1(a) which shows the state of a cracked body before deformation (time $\tau=0$), we divide the cracked body into two parts, that is, the plane containing the crack (shown by the solid line in front of crack tip in the figure) and the part except the plane above, and we consider that the latter behaves as a usual continuum and the former is stretched as shown in Figs.1(a) and (b) after deformation (time $\tau=t$) (Fig.1(b) shows the state where the plane is stretched and the crack extends until its length becomes a). When the constitutive equation for the latter is given by a relation between stress and strain and that for the former is given by a relation between out-of-plane stress component $\lfloor\sigma_n\rfloor$ and out-of-plane relative displacement $\lfloor\delta_n\rfloor$ defined by

$$\lfloor\sigma_n\rfloor = \lfloor\sigma_{22}, \sigma_{23}, \sigma_{12}\rfloor \quad \cdots\cdots(1)$$

$$\lfloor\delta_n\rfloor = \lfloor\delta_{22}, \delta_{23}, \delta_{12}\rfloor \quad \cdots\cdots(2)$$

referring to Fig.1(c), this model becomes a crack model which enables us to analyze a crack problem under an arbitrary applied load. It goes without saying that the constitutive equations suitable to the phenomenon should be adopted. Here, the Dugdale model is, as described before, a crack model in which the plane containing a crack is deemed an ideally rigid-plastic body and the part except the plane is deemed a linear elastic body[2], therefore, the proposed model can assert itself as the most generalized model of Dugdale model. Then, in this paper, we call this model DWA (Dugdale-Watanabe-Azegami) model for simplification of description. The plane containing a crack is considered here as the plane considering the discontinuity, but, as we can imagine an arbitrary plane containing crack front as the plane considering the discontinuity, we carry out the formulation in the following sections keeping in mind such a case.

The definitions of crack parameters in this crack model are given as shown in the following.

Crack energy density is defined as "the quantity expressed per unit area in the plane containing a crack of the work done during deformation at a point in the plane containing a crack before deformation"[(3)~(5)]. Therefore, when the strain energy plane density W_{plane} is defined by

$$W_{\text{plane}}=\int dW_{\text{plane}}, \quad dW_{\text{plane}}=\lfloor\sigma_n\rfloor\{d\delta_n\} \quad \cdots\cdots(3)$$

the crack energy density in the state where the crack extends until its length becomes

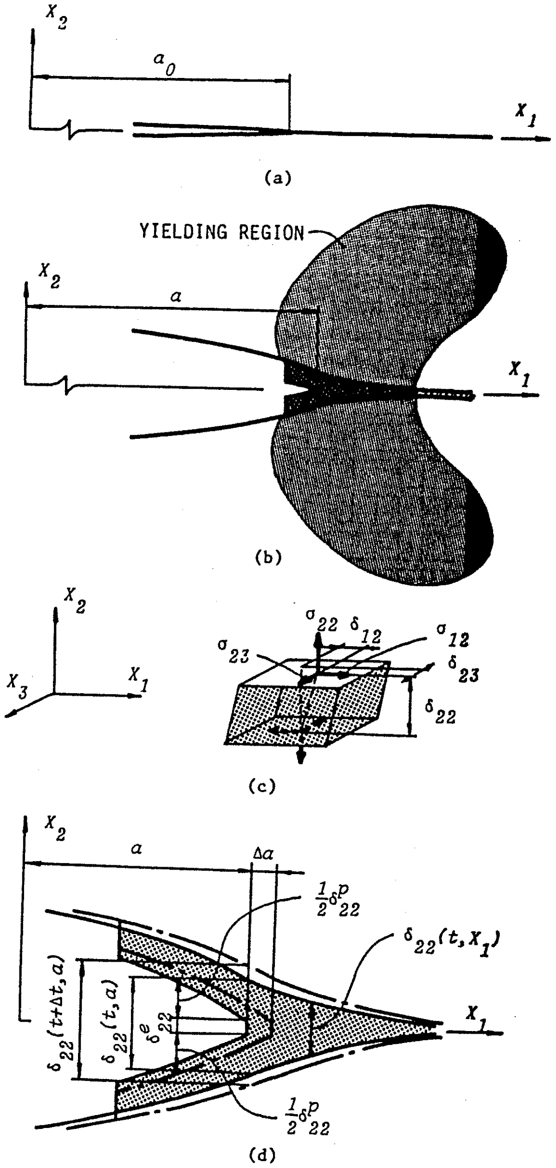


Fig.1 Crack model considering the discontinuity of displacement

a (Fig.1(d), time $\tau=t$) is given by

$$\mathcal{E}(t, X_1)=\int dW_{\text{plane}}(t, X_1) \quad \cdots\cdots(4)$$

and it distributes in X_1-direction. Especially, in case of an elasto-plastic problem, as the increment of out-of-plane relative displacement is divided into the elastic component $\{d\delta_n^e\}$ and the plastic one $\{d\delta_n^p\}$, it is expressed as

$$\{d\delta_n\}=\{d\delta_n^e\}+\{d\delta_n^p\} \quad \cdots\cdots(5)$$

and the elastic and plastic contributions to crack energy density are, respectively, given by

$$\mathcal{E}^e(t, X_1)=\int dW_{\text{plane}}^e, \quad dW_{\text{plane}}^e=\lfloor\sigma_n\rfloor\{d\delta_n^e\} \quad \cdots\cdots(6)$$

$$\mathcal{E}^p(t, X_1)=\int dW_{\text{plane}}^p, \quad dW_{\text{plane}}^p=\lfloor\sigma_n\rfloor\{d\delta_n^p\} \quad \cdots\cdots(7)$$

Crack opening displacement COD is defined as the out-of-plane relative displacement at crack tip, and its value at time $\tau=t$, using $\delta_{22}(t,X_1)$ distributing in the plane containing the crack, is given by

$$\mathrm{COD}(t)=\delta_{22}(t,a) \quad \cdots\cdots(8)$$

even after the crack has started to grow (Fig.1(d)). Crack opening angle COA, which is taken as a parameter for stable crack growth, is given, based on its definition[6], by

$$\mathrm{COA}(t)=\lim_{\Delta a\to 0}\frac{\delta_{22}(t+\Delta t,a)-\delta_{22}(t,a)}{\Delta t}\frac{\Delta t}{\Delta a} \quad \cdots\cdots(9)$$

at time $\tau=t$ (Fig.1(d)), where Δt and Δa are, respectively, the increments of time and crack length corresponding to each other.

2.2 Constitutive equation in an elasto-plastic problem

DWA model can be applied under an arbitrary constitutive equation, and the constitutive equation in an elasto-plastic problem is concretely formulated here.

To the part dealt with as a continuum, a usual constitutive equation based on the incremental theory can be applied, which is given by the relation between increments of stress and strain as

$$\left\{\frac{d\sigma_t}{d\sigma_n}\right\}=\left[\frac{D^e_{tt}}{D^e_{nt}}\middle|\frac{D^e_{tn}}{D^e_{nn}}\right]\left\{\frac{d\varepsilon_t}{d\varepsilon_n}\right\} \quad \text{(elastic)}$$

or $\{d\sigma_L\}=[D^e]\{d\varepsilon_L\}$ $\cdots\cdots(10)$

$$\left\{\frac{d\sigma_t}{d\sigma_n}\right\}=\left[\frac{D^p_{tt}}{D^p_{nt}}\middle|\frac{D^p_{tn}}{D^p_{nn}}\right]\left\{\frac{d\varepsilon_t}{d\varepsilon_n}\right\} \quad \text{(elasto-plastic)}$$

or $\{d\sigma_L\}=[D^p]\{d\varepsilon_L\}$ $\cdots\cdots(11)$

Here, $\llcorner d\sigma_t \lrcorner$ and $\llcorner d\varepsilon_t \lrcorner$ are the increments of stress and strain components in the plane parallel to the plane containing a crack, $\llcorner d\varepsilon_n \lrcorner$ is the increment of strain component in the plane perpendicular to the plane containing a crack and they are given by

$$\llcorner d\sigma_t \lrcorner = \llcorner d\sigma_{11}, d\sigma_{33}, d\sigma_{13} \lrcorner \quad \cdots\cdots(12)$$

$$\llcorner d\varepsilon_t \lrcorner = \llcorner d\varepsilon_{11}, d\varepsilon_{33}, d\gamma_{13} \lrcorner \quad \cdots\cdots(13)$$

$$\llcorner d\varepsilon_n \lrcorner = \llcorner d\varepsilon_{22}, d\gamma_{23}, d\gamma_{12} \lrcorner \quad \cdots\cdots(14)$$

Hereafter, we call $\llcorner d\sigma_t \lrcorner$, $\llcorner d\varepsilon_t \lrcorner$ and $\llcorner d\varepsilon_n \lrcorner$ in-plane stress, in-plane strain and out-of-plane strain respectively.

The constitutive equation for the plane admitting the discontinuity of displacement is given in the following manner. To elastic deformation, we apply Hooke's law in which a linear relation between out-of-plane stress and out-of-plane relative displacement is assumed. On the assumption that the material is isotropic in the plane, when one of two elastic constants is given by Poisson's ratio ν which is non-dimentional, we obtain the relation

$$\{d\sigma_n\}=[\bar{D}^e_{nn}]\{d\delta_n\} \quad \text{(elastic)}$$

$$[\bar{D}^e_{nn}]=\frac{\bar{E}}{1+\nu}\begin{bmatrix}\frac{1-\nu}{1-2\nu} & 0 & 0\\ 0 & \frac{1}{2} & 0\\ 0 & 0 & \frac{1}{2}\end{bmatrix} \quad \cdots\cdots(15)$$

where $[\bar{D}^e_{nn}]$ and $\bar{E}$ shall be called plane stiffness matrix to elastic deformation and plane Young's modulus. Here, the relation of Eq.(15) agrees with the constitutive equation of so-called joint element in finite element method[7]. By the way, in-plane stress $\llcorner \sigma_t \lrcorner$ and in-plane strain $\llcorner \varepsilon_t \lrcorner$ in the plane keep their physical meanings, since the continuity of displacement component in the plane is kept. It is only out-of-plane strain $\llcorner \varepsilon_n \lrcorner$ that has lost the meaning because of out-of-plane relative displacement. Therefore, eliminating $\llcorner d\varepsilon_n \lrcorner$ in Eqs.(10) and (15), the relation among all stress components, in-plane strain components and out-of-plane relative displacements is obtained by

$$\left\{\frac{d\sigma_t}{d\sigma_n}\right\}=\left[\frac{D^e_{tt}}{D^e_{nt}}\middle|\frac{D^e_{tn}}{D^e_{nn}}\right]\left[\frac{I}{\bar{H}^e_{nt}}\middle|\frac{0}{rI}\right]\left\{\frac{d\varepsilon_t}{d\delta_n}\right\}$$

$$[\bar{H}^e_{nt}]=-[D^e_{nn}]^{-1}[D^e_{nt}],\quad r=\frac{\bar{E}}{E} \quad \text{(elastic)}$$

or
$$\{d\sigma_L\}=[D^e][\bar{H}^e]\{d\varepsilon_\delta\}=[\bar{D}^e]\{d\varepsilon_\delta\} \quad \cdots\cdots(16)$$

where $[I]$ is third-degree unit matrix. It is noted that, as the strain energy related to in-plane stress and in-plane strain is zero, the stiffness corresponding to them becomes zero in Eq.(16). However, in-plane stress components have influence upon yield criterion, and, by considering them, it becomes possible to give yield criterion by a function of all stress components. In this paper, we assume that the yield criterion of the plane admitting the discontinuity is given by the same one as that of continuum part, that is, by

$$f(\sigma_{ij})=c \quad \cdots\cdots(17)$$

To elasto-plastic deformation, the constitutive equation is given as shown in the following. We consider that the increment of plastic out-of-plane relative displacement $\{d\delta^p_n\}$ is given, by using a plastic potential $f_n(\sigma_{ij})$, as

$$\{d\delta^p_n\}=\left\{\frac{\partial f_n}{\partial\sigma_n}\right\}\bar{c}\,df_n,\ f_n=\llcorner \sigma_n \lrcorner\left\{\frac{\partial f}{\partial\sigma_n}\right\} \quad \cdots(18)$$

where $\bar{c}$ is a proportional constant and $\llcorner \partial(\ )/\partial\sigma_n \lrcorner = \llcorner \partial(\ )/\partial\sigma_{22}, \partial(\ )/\partial\sigma_{23}, \partial(\ )/\partial\sigma_{12} \lrcorner$. Moreover, we define the increment of plastic strain energy plane density dW^p_{plane} by

$$dW^p_{\text{plane}}=\llcorner \sigma_n \lrcorner\{d\delta^p_n\} \quad \cdots\cdots(19)$$

and equivalent out-of-plane plane stress $\bar{\sigma}_n$ and its increment $d\bar{\sigma}_n$ by

$$\bar{\sigma}_n=\left\{\frac{\partial f_n}{\partial\sigma_n}\right\}\llcorner \sigma_n \lrcorner,\quad d\bar{\sigma}_n=\left\{\frac{\partial f_n}{\partial\sigma_n}\right\}\llcorner d\sigma_n \lrcorner \quad \cdots\cdots(20)$$

Here, assuming that f_n is a function of plastic strain energy plane density and taking an expression

$$\bar{c}\,df_n=d\bar{\delta}^p_n \quad \cdots\cdots(21)$$

into consideration, it can be said that the value of $\bar{H}'$ given by

$$\frac{d\bar{\sigma}_n}{d\bar{\delta}^p_n}=\bar{H}' \quad \cdots\cdots(22)$$

should exist uniquely from the relations of Eqs.(18)~(21), where $d\bar{\delta}^p_n$ and $\bar{H}'$ are the quantities which shall be called the increment of the equivalent plastic out-of-plane relative displacement and the relative displacement hardening rate respectively. On the other hand, using the relations of

Eqs.(5), (15) and (18), the relation

$$\{d\sigma_n\}=[\bar{D}^e_{nn}]\{d\delta_n\}-[\bar{D}^e_{nn}]\left\{\frac{\partial f_n}{\partial \sigma_n}\right\}\bar{c}df_n \quad \cdots\cdots\cdots(23)$$

is obtained and, by using the relations of Eqs.(20)∿(23), $\bar{c}df_n$ in Eq.(23) is given by

$$\bar{c}df_n=\frac{\dfrac{\partial f_n}{\llcorner \partial \sigma_n \lrcorner}[\bar{D}^e_{nn}]\{d\delta_n\}}{\bar{H}'+\dfrac{\partial f_n}{\llcorner \partial \sigma_n \lrcorner}[\bar{D}^e_{nn}]\left\{\dfrac{\partial f_n}{\partial \sigma_n}\right\}} \quad \cdots\cdots\cdots(24)$$

Moreover, substituting Eq.(24) into Eq.(23), we obtain

$$\{d\sigma_n\}=[\bar{D}^p_{nn}]\{d\delta_n\}$$

$$[\bar{D}^p_{nn}]=[\bar{D}^e_{nn}]-\frac{[\bar{D}^e_{nn}]\left\{\dfrac{\partial f_n}{\partial \sigma_n}\right\}\dfrac{\partial f_n}{\llcorner \partial \sigma_n \lrcorner}[\bar{D}^e_{nn}]}{\bar{H}'+\dfrac{\partial f_n}{\llcorner \partial \sigma_n \lrcorner}[\bar{D}^e_{nn}]\left\{\dfrac{\partial f_n}{\partial \sigma_n}\right\}}$$

(elasto-plastic) $\cdots\cdots\cdots\cdots(25)$

The relation among the increments of all stress components, in-plane stress components and out-of-plane relative displacements is given by

$$\left\{\frac{d\sigma_t}{d\sigma_n}\right\}=\left[\frac{D^p_{tt}}{D^p_{nt}}\middle|\frac{D^p_{tn}}{D^p_{nn}}\right]\left[\frac{I}{\bar{H}^p_{nt}}\middle|\frac{0}{\bar{H}^p_{nn}}\right]\left\{\frac{d\varepsilon_t}{d\delta_n}\right\}$$

$$[\bar{H}^p_{nt}]=-[D^p_{nn}]^{-1}[D^p_{nt}]$$

$$[\bar{H}^p_{nn}]=[D^p_{nn}]^{-1}[\bar{D}^p_{nn}] \qquad \text{(elasto-plastic)}$$

or $[d\sigma_L]=[D^p][\bar{H}^p]\{d\varepsilon_s\}$

$$=[\bar{D}^p]\{d\varepsilon_s\} \quad \cdots\cdots\cdots\cdots(26)$$

by eliminating $\{d\varepsilon_n\}$ from Eqs.(11) and (12) in the same manner as in case of elastic deformation.

When we formulate the constitutive equations as mentioned above, two material constants expressed by $\bar{E}$ and $\bar{H}'$ are newly required in addition to the material constants of continuum, and those correspond to Young's modulus E and the strain hardening rate H' of continuum and differ from $\bar{E}$ and $\bar{H}'$ only by the dimention of length. Hence, we shall call the model in which the relation

$$h=\frac{E}{\bar{E}}=\frac{H'}{\bar{H}'} \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(27)$$

holds the conformable model especially, where h becomes a material constant with the dimension of length. The value of h should be determined so as to be as suitable to actual behavior in the plane admitting the discontinuity as possible, and this will be discussed in another paper. Here, the usual continuum crack model corresponds to the DWA model in the limit when $h\to 0$, and in this case, the relative displacement $\delta_{ij}\to 0$, the stress at the crack tip $\sigma_{ij}\to\infty$ and the distribution of crack energy densities except at the crack tip becomes zero. Moreover, the Dugdale model corresponds to the nonconformable DWA model at the time when $E/\bar{E}\to 0$ and $H'/\bar{H}'\to\infty$ in the plane containing a crack and the continuum part is dealt with as a linear elastic body. Table 1 presents the results of $[\bar{H}^e_{nt}]$, $[\bar{H}^p_{nt}]$ and $[\bar{H}^p_{nn}]$ in Eq.(26) for the conformable DWA model under the condition that the yield criterion and the flow rule of the continuum part are given by Mises' one and associated one respectively. Here, $[\bar{H}^e_{nt}]$, $[\bar{H}^p_{nt}]$ and $[\bar{H}^p_{nn}]$ under plane strain state are given by adding the condition that $\sigma_{33}=\sigma_{13}=0$ and removing the rows and the columns concerned with $(\sigma_{33}, \sigma_{13}, \sigma_{23})$ and $(\varepsilon_{33}, \gamma_{13}, \delta_{23})$ in the expressions of three-dimensional stress case in Table 1, and the following definitions are adopted in two-dimensional problem.

$$\llcorner \sigma_t \lrcorner=\llcorner \sigma_{11} \lrcorner,\ \llcorner \sigma_n \lrcorner=\llcorner \sigma_{22}, \sigma_{12} \lrcorner \quad \cdots\cdots\cdots\cdots(28)$$

$$\llcorner \varepsilon_t \lrcorner=\llcorner \varepsilon_{11} \lrcorner,\ \llcorner \delta_n \lrcorner=\llcorner \delta_{22}, \delta_{12} \lrcorner \quad \cdots\cdots\cdots(29)$$

Table 1 $[\bar{H}]$ matrix for conformable DWA model in case of Mises' yield criterion and associated flow rule being applied to the continuum part

3-dimensional stress	$[\bar{H}^e_{nt}]=-\begin{bmatrix}\lambda_0&\lambda_0&0\\0&0&0\\0&0&0\end{bmatrix}$, $[\bar{H}^p_{nt}]=\frac{1}{\bar{S}_0}\left(-G\begin{bmatrix}\lambda_2&\lambda_2&0\\0&0&0\\0&0&0\end{bmatrix}+\frac{1}{S}\begin{bmatrix}S_1S_2+\lambda_2(S_4^2+S_6^2)&S_2S_3+\lambda_2(S_4^2+S_6^2)&S_2S_5\\\lambda_1S_1S_4-\lambda_2S_2S_4&\lambda_1S_3S_4-\lambda_2S_2S_4&\lambda_1S_4S_5\\\lambda_1S_1S_6-\lambda_2S_2S_6&\lambda_1S_3S_6-\lambda_2S_2S_6&\lambda_1S_5S_6\end{bmatrix}\right)$. $[\bar{H}^p_{nn}]=\frac{1}{h\bar{S}_0}\left(G\begin{bmatrix}\lambda_1&0&0\\0&\lambda_1&0\\0&0&\lambda_1\end{bmatrix}+\frac{1}{S}\begin{bmatrix}-\lambda_1(S_4^2+S_6^2)&S_2S_4&S_2S_6\\\lambda_1S_2S_4&-(S_2^2+\lambda_1S_6^2)&\lambda_1S_4S_6\\\lambda_1S_2S_6&\lambda_1S_4S_6&-(S_2^2+\lambda_1S_4^2)\end{bmatrix}-\frac{1}{\bar{S}}\begin{bmatrix}\bar{S}_2\bar{S}'_2&\bar{S}_2\bar{S}'_4&\bar{S}_2\bar{S}'_6\\\bar{S}_4\bar{S}'_2&\bar{S}_4\bar{S}'_4&\bar{S}_4\bar{S}'_6\\\bar{S}_6\bar{S}'_2&\bar{S}_6\bar{S}'_4&\bar{S}_6\bar{S}'_6\end{bmatrix}\right)$. $G=\frac{E}{2(1+\nu)}$, $\lambda_1=\frac{2(1-\nu)}{1-2\nu}$, $\lambda_2=\frac{2\nu}{1-2\nu}$, $\lambda_0=\frac{\nu}{1-\nu}$ $S_1=G(\lambda_1\sigma'_{11}+\lambda_2\sigma'_{22}+\lambda_2\sigma'_{33})$, $S_2=G(\lambda_2\sigma'_{11}+\lambda_1\sigma'_{22}+\lambda_2\sigma'_{33})$, $S_3=G(\lambda_2\sigma'_{11}+\lambda_2\sigma'_{22}+\lambda_1\sigma'_{33})$, $S_4=2G\sigma_{23}$, $S_5=2G\sigma_{13}$, $S_6=2G\sigma_{12}$, $S=\frac{4}{9}\bar{\sigma}^2H'+S_1\sigma'_{11}+S_2\sigma'_{22}+S_3\sigma'_{33}+2S_4\sigma_{23}+2S_5\sigma_{13}+2S_6\sigma_{12}$, $\bar{S}_2=\lambda_1G\sigma'_{22}+\frac{1}{S}\{-\lambda_1(S_4^2+S_6^2)\sigma'_{22}+2S_2S_4\sigma_{23}+2S_2S_6\sigma_{12}\}$, $\bar{S}_4=2\lambda_1G\sigma_{23}+\frac{1}{S}\{\lambda_1S_2S_4\sigma'_{22}-2(S_2^2+\lambda_1S_6^2)\sigma_{23}+2\lambda_1S_4S_6\sigma_{12}\}$, $\bar{S}_6=2\lambda_1G\sigma_{12}+\frac{1}{S}\{\lambda_1S_2S_6\sigma'_{22}+2\lambda_1S_4S_6\sigma_{23}-2(S_2^2+\lambda_1S_4^2)\sigma_{12}\}$, $\bar{S}'_2=\lambda_1G\sigma'_{22}$, $\bar{S}'_4=2G\sigma_{23}$, $\bar{S}'_6=2G\sigma_{12}$, $\bar{S}_0=\lambda_1G-\frac{1}{S}\{S_2^2+\lambda_1(S_4^2+S_6^2)\}$, $\bar{S}=\frac{4}{9}\bar{\sigma}^2h\bar{H}'+\bar{S}'_2\sigma'_{22}+2\bar{S}'_4\sigma_{23}+2\bar{S}'_6\sigma_{12}$,
plane stress	$\{\bar{H}^e_{nt}\}=-\begin{Bmatrix}\nu\\0\end{Bmatrix}$, $\{\bar{H}^p_{nt}\}=\frac{1}{\bar{S}_0}\left(-E'\begin{Bmatrix}\nu\lambda\\0\end{Bmatrix}+\frac{1}{S}\begin{Bmatrix}\nu S_6^2+\lambda S_1S_2\\S_1S_6-\nu S_2S_6\end{Bmatrix}\right)$, $[\bar{H}^p_{nn}]=\frac{1}{h\bar{S}_0}\left(E'\begin{bmatrix}\lambda&0\\0&\lambda\end{bmatrix}+\frac{1}{S}\begin{bmatrix}-S_6^2&\lambda S_2S_6\\S_2S_6&-\lambda S_2^2\end{bmatrix}-\frac{1}{\bar{S}}\begin{bmatrix}\bar{S}_2\bar{S}'_2&\bar{S}_2\bar{S}'_6\\\bar{S}_6\bar{S}'_2&\bar{S}_6\bar{S}'_6\end{bmatrix}\right)$, $E'=\frac{E}{1-\nu^2}$, $\lambda=\frac{1-\nu}{2}$, $S_1=E'(\sigma'_{11}+\nu\sigma'_{22})$, $S_2=E'(\nu\sigma'_{11}+\sigma'_{22})$, $S_6=2\lambda E'\sigma_{12}$, $S=\frac{4}{9}\bar{\sigma}^2H'+S_1\sigma'_{11}+S_2\sigma'_{22}+2S_6\sigma_{12}$, $\bar{S}_2=\lambda E'\sigma'_{22}+\frac{1}{S}(-S_6^2\sigma'_{22}+2\lambda S_2S_6\sigma_{12})$, $\bar{S}_6=2\lambda E'\sigma_{12}+\frac{1}{S}(S_2S_6\sigma'_{22}-2\lambda S_2^2\sigma_{12})$ $\bar{S}'_2=E'\sigma'_{22}$, $\bar{S}'_6=2\lambda E'\sigma_{12}$, $\bar{S}_0=\lambda E'-\frac{1}{S}(\lambda S_2^2+S_6^2)$, $\bar{S}=\frac{4}{9}\bar{\sigma}^2h\bar{H}'+\bar{S}'_2\sigma'_{22}+2\bar{S}'_6\sigma_{12}$

$\bar{\sigma}$: equivalent stress, σ'_{ij} : deviatoric stress, H' : strain-hardening rate

2.3 Finite element formulation of DWA model

We carry out the formulation for the finite element analysis of DWA model here, and it is shown through this formulation that the analysis of DWA model becomes possible by introducing plane elements into the plane admitting the discontinuity and extending an already-existing program of finite element method.

Consider the DWA crack model under body force and traction force. We represent the plane admitting the discontinuity, the volume of the continuum part and the surface traction force being applied by S_δ, V and S_σ respectively. The principle of virtual work to this DWA crack model is given, in the incremental form, by

$$\int_V \delta \llcorner d\varepsilon \lrcorner \{d\sigma\} dV + \int_{S_\delta} \delta \llcorner d\delta_n \lrcorner \{d\sigma_n\} dS = \int_V \delta \llcorner du \lrcorner \{dF\} dV + \int_{S_\sigma} \delta \llcorner du \lrcorner \{dp\} dS \quad \cdots (30)$$

where $\{d\varepsilon\}$ and $\{d\sigma\}$ are the increments of strain and stress in a global coordinate system, $\{d\delta_n\}$ and $\{d\sigma_n\}$ are the increments of out-of-plane relative displacement and out-of-plane stress for the plane S_δ and $\{du\}$, $\{dF\}$ and $\{dp\}$ are the increments of displacement, body force and traction force in a global coordinate system respectively. In addition, δ expresses the variation and $\delta\{du\}$ is an arbitrary increment of displacement which satisfies the geometrical boundary condition given. Here, since in-plane stress and in-plane strain are defined in the plane S_δ and the stiffness corresponding to them is zero, the second term in the left side of Eq.(30) is expressed by

$$\int_{S_\delta} \delta \llcorner d\delta_n \lrcorner \{d\sigma_n\} dS = \int_{S_\delta} \delta \llcorner d\delta_L \lrcorner \{d\sigma_L\} dS$$

$$\llcorner d\delta_L \lrcorner = \llcorner 0, 0, 0 | d\delta_n \lrcorner \quad \cdots (31)$$

where $\llcorner d\sigma_L \lrcorner$ is defined in Eq.(16). Now, we consider dividing the volume V into M elements of which the volumes are represented by $V^{(m)}$ and the plane S_δ into N plane elements of which the areas are represented by $S_\delta^{(n)}$. On the plane S_δ, we introduce the coordinate η normalized by out-of-plane relative displacement δ_{22} which is normal to the plane S_δ and of which the origin is the center of out-of-plane relative displacement (Fig.2). And we assume that the distribution of displacements is given by interpolating the displacements at $\eta=-1/2$ and $\eta=1/2$ linearly since the assumption does not restrict the generality of DWA model. The plane S_δ is divided into N plane elements as shown in Fig.2 for example. After having divided S_δ, we give the distribution of displacements in η-direction of the n-th element by interpolating the displacements in the n-th element linearly. On the divided DWA model, the relation of Eq.(30), using Eq.(31), becomes

$$\sum_{m=1}^{M} \int_{V(m)} \delta \llcorner d\varepsilon \lrcorner \{d\sigma\} dV + \sum_{n=1}^{N} \int_{S_\delta^{(n)}} \int_{\eta^{(n)}} \delta \llcorner d\delta_L \lrcorner \{d\sigma_L\} d\eta dS = \sum_{m=1}^{M} \int_{V(m)} \delta \llcorner du \lrcorner \{dF\} dV + \sum_{m=1}^{M} \int_{S_\sigma^{(m)}} \delta \llcorner du \lrcorner \{dp\} dS \quad \cdots (32)$$

where $S_\delta^{(n)}$ and $\eta^{(n)}$ are the area and the extent of η-coordinate corresponding to n-th plane element (Fig.2) and $S_\sigma^{(m)}$ is the surface traction force being applied of the m-th volume element.

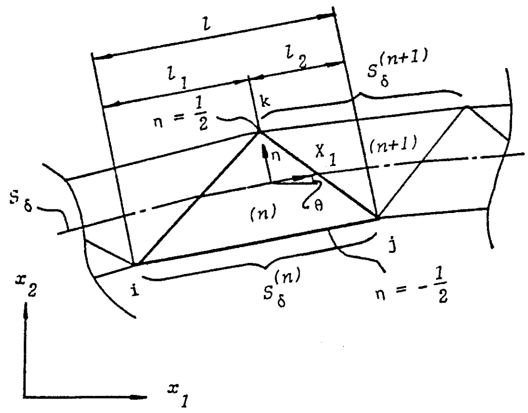


Fig.2 Introduction of plane element

To the volume element, the standard procedure for finite element formulation is applicable. That is, the increment of displacement $\{du\}$ in element is expressed with the increment of nodal displacement $\{d\bar{u}\}$ of element by

$$\{du\} = [N]\{d\bar{u}\} \quad \cdots (33)$$

where $[N]$ is called the shape function. Therefore, the increment of strain $\{d\varepsilon\}$ is given by

$$\{d\varepsilon\} = [\mathscr{D}]\{du\} = [\mathscr{D}][N]\{d\bar{u}\} = [B]\{d\bar{u}\} \quad \cdots (34)$$

where $[\mathscr{D}]$ is a matrix composed of differential operators which relate strain to displacement and $[B]$ is called a strain-nodal displacement matrix. Using the relations of Eqs.(10), (11) and (34), the increment of stres is given by

$$\{d\sigma\} = [D]\{d\varepsilon\} = [D][B]\{d\bar{u}\} \quad \cdots (35)$$

where $[D]$ is given by $[D^e]$ for elastic deformation and by $[D^p]$ for elasto-plastic deformation. On the other hand, for the plane element, a similar formulation can be made. That is, the increment of nodal displacement $\{d\bar{u}_L\}$ in a local coordinate system is expressed, by using that in a global coordinate system, as

$$\{d\bar{u}_L\} = [T]\{d\bar{u}\} \quad \cdots (36)$$

where $[T]$ is a matrix for coordinate transformation. The increment of displacement $\{du_L\}$ in an element in local coordinate system is expressed by

$$\{du_L\} = [\bar{N}]\{d\bar{u}_L\} \quad \cdots (37)$$

where $[\bar{N}]$ is the shape function of plane element, and is given as a function of η-coordinate and in-plane coordinate. The increment of in-plane strain and out-of-plane relative displacement $\{d\varepsilon_\delta\}$ and the increment of out-of-plane relative displacement $\{d\delta_L\}$ are expressed, by using the increment of displacement in an element and the increment of nodal displacement, as

$$\{d\varepsilon_\delta\} = \left\{\frac{d\varepsilon_t}{d\delta_n}\right\} = \left[\frac{\mathscr{D}_t}{\bar{\mathscr{D}}_n}\right]\{du_L\} = [\bar{\mathscr{D}}]\{du_L\} = [\bar{\mathscr{D}}][\bar{N}][T]\{d\bar{u}\} = [\bar{B}]\{d\bar{u}\} \quad \cdots (38)$$

$$\{d\delta_L\} = \left\{\frac{0}{d\delta_n}\right\} = \left[\frac{0}{\bar{\mathscr{D}}_n}\right]\{du_L\} = [\bar{\mathscr{D}}']\{du_L\}$$

$$=[\bar{\mathcal{D}}'][\bar{N}][T]\{d\bar{u}\}=[\bar{B}']\{d\bar{u}\} \qquad \cdots\cdots (39)$$

where $[\mathcal{D}_t]$ is a matrix composed of differential operators which relate strain to displacement and $[\bar{\mathcal{D}}_n]$ is a matrix which relates out-of-plane relative displacement to displacement in element. Using the relations of Eqs.(16), (26) and (38), the increment of stress is expressed by

$$\{d\sigma_L\}=[\bar{D}]\{d\varepsilon_s\}=[\bar{D}][\bar{B}]\{d\bar{u}\} \qquad \cdots\cdots (40)$$

where $[\bar{D}]$ is given by $[\bar{D}^e]$ for elastic deformation and by $[\bar{D}^p]$ for elasto-plastic deformation. Here, substituting Eqs.(33)∿(40) into Eq.(32) and considering that $\delta\llcorner d\bar{u}\lrcorner$ is an arbitrary displacement which satisfies the geometrical boundary condition, we obtain the stiffness equation for each element

$$[k]\{d\bar{u}\}=\{dP\}=\{dP_V\}+\{dP_S\}$$
(for volume element m) ······ (41)

$$[\bar{k}]\{d\bar{u}\}=\{0\}$$
(for plane element n) ······ (42)

where

$$[k]=\int_{V(m)}[B]^T[D][B]dV \qquad \cdots\cdots (43)$$

$$\{dP_V\}=\int_{V(m)}[N]^T\{dF\}dV,$$
$$\{dP_S\}=\int_{S_\sigma^{(m)}}[N]^T\{dp\}dS \qquad \cdots\cdots (44)$$

$$[\bar{k}]=\int_{S^{(n)}}\int_{\eta}^{(n)}[\bar{B}']^T[\bar{D}][\bar{B}]d\eta dS \qquad \cdots\cdots (45)$$

Here, $[k]$ is the stiffness matrix of volume element, $\{dP\}$ is the increment of equivalent nodal force vector and $[\bar{k}]$ is the stiffness matrix of plane element. Therefore, the stiffness equation of total system is obtained by assembling the stiffness equations to each element given by Eqs.(41) and (42). Although $[\bar{H}]$ matrix defined by Eq.(16) or Eq.(26) and $[\bar{D}]$ matrix are not symmetric, $[\bar{k}]$ matrix is symmetric when $[D]$ matrix is symmetric. Consequently, the matrix of total system is also symmetric under the same condition. When the stiffness equation is solved, the strain and stress of volume element are obtained by Eq.(34) and Eq.(35) and, moreover, the in-plane strain, out-of-plane relative displacement and stress of plane element are obtained by Eq.(38) and Eq.(40) respectively.

In case of a two-dimensional problem using a triangular element in which a linear displacement field is assumed, the concrete expression of plane element stiffness matrix is especially shown in the following. That is, if considering a triangular plane element of which the inclination from the global coordinate system is θ, the three nodes are indicated by i, j and k and the lengths of the three sides are by $l(i,j)$, $l_1(i,k)$ and $l_2(j,k)$ as shown in Fig.2, the plane element stiffness matrix will be given by

$$[\bar{k}]=\frac{l\bar{t}}{2}[\bar{B}']^T[\bar{D}][B] \qquad \cdots\cdots (46)$$

where, on the definition of $\llcorner d\bar{u}\lrcorner=\llcorner d\bar{u}_{i1}, d\bar{u}_{i2}, d\bar{u}_{j1}, d\bar{u}_{j2}, d\bar{u}_{k1}, d\bar{u}_{k2}\lrcorner$,

$$[\bar{B}]=\begin{bmatrix} -\xi\cos\theta & -\xi\sin\theta & \xi\cos\theta & \xi\sin\theta & 0 & 0 \\ \xi_2\sin\theta & -\xi_2\cos\theta & \xi_1\sin\theta & -\xi_1\cos\theta & -\sin\theta & \cos\theta \\ -\xi_2\cos\theta & -\xi_2\sin\theta & -\xi_1\cos\theta & -\xi_1\sin\theta & \cos\theta & \sin\theta \end{bmatrix} \qquad \cdots\cdots (47)$$

$$[\bar{B}']=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ \xi_2\sin\theta & -\xi_2\cos\theta & \xi_1\sin\theta & -\xi_1\cos\theta & -\sin\theta & \cos\theta \\ -\xi_2\cos\theta & -\xi_2\sin\theta & -\xi_1\cos\theta & -\xi_1\sin\theta & \cos\theta & \sin\theta \end{bmatrix} \qquad \cdots\cdots (48)$$

$$\xi=1/l, \quad \xi_1=l_1/l, \quad \xi_2=l_2/l \qquad \cdots\cdots (49)$$

and $\bar{t}$ is the thickness.

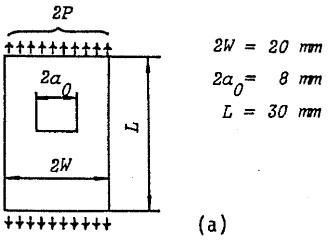
3. Elasto-plastic Finite Element Analysis of DWA Model

In this chapter, according to the formulation shown in Chapter 2, the DWA model analyses of an elasto-plastic crack and a crack corresponding to Dugdale model are carried out by means of finite element method, and the availability of DWA model as a crack model is fundamentally considered based on the results.

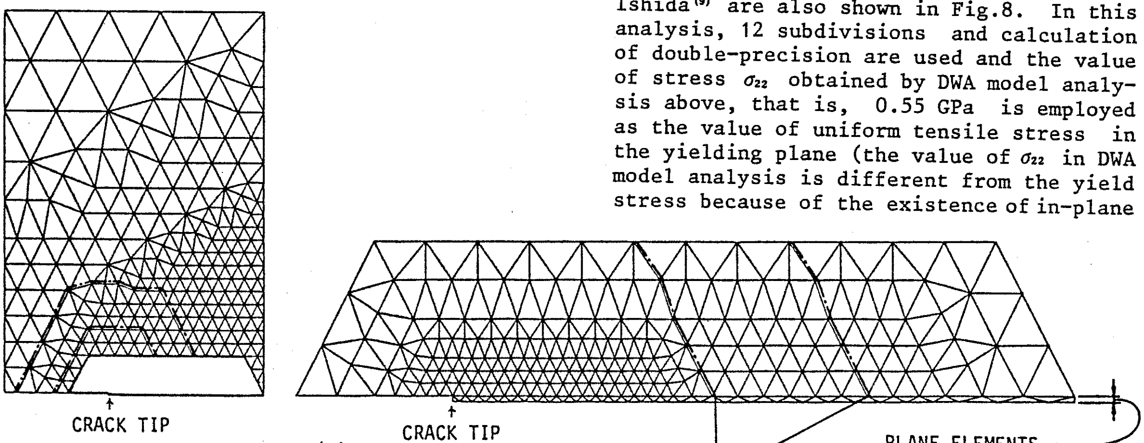
3.1 Elasto-plastic crack

A center cracked panel specimen, which is supposed to be made of A533B steel, under monotonously incresing uniform tension as shown in Fig.3(a) is analyzed. Because of symmetry, the mesh used here to a quarter part of specimen is shown in Fig.3(b), where the triangular plane elements of which the concrete expression is given in section 2.3 and the constant strain triangular volume elements are applied to the ligament plane and to the continuum part respectively. Sixty-three plane elements, 786 volume elements and 467 nodes are used. As the constitutive equations for plastic deformation, the associated flow rule with Mises' yield criterion is applied to volume elements and the flow rule given by $[\bar{H}]$ matrix in Table 1 to plane elements. As to the material constants, Young's modulus E = 205.8 GPa, Poisson's ratio ν = 0.3, yield stress σ_Y = 0.4802 GPa and strain hardening rate H' = $d\bar{\sigma}/d\bar{\varepsilon}^p$ = 2.058 GPa (the relation between equivalent stress and equivalent plastic strain is approximated by two straight lines). The value of h defined by Eq.(27) is assumed to be 0.02 mm (as the state, in which the plane elements in Fig.3(a) are doubly inserted into the ligament plane in order to keep symmetry, is supposed, the value of h for adopted plane elements becomes 0.01 mm). As for the method to increase the load, one by Yamada is adopted.

The results of the analysis are shown in Figs.4∿7. That is, Fig.4 shows the shape of plastic yielding region, Fig.5 does the relation between crack energy density at crack tip evaluated by Eqs.(3) and (4) $\mathcal{E}(t,a)$ and average displacement on loading surface u_m, Fig.6 does the relation between load $2P$ and average displacement on loading surface u_m and Fig.7 does the distribution of crack energy densities in the ligament plane. At the same time, the finite element analysis of usual continuum crack model is carried out by using the mesh obtained by removing the plane elements in Fig.3(a) and the results are also shown in Figs.4 and 5. The value of J-integral in Fig.5 is the mean value of those evaluated by using two integration paths shown in Fig.3(b) and it agrees with the value of crack energy density at crack tip well over the range of u_m. As it is shown that J-integral agrees with crack energy density at crack tip to usual continuum crack model when unloading and crack extension are not considered[8], it can be said that the DWA model is hopeful as a crack model to estimate the crack energy density.



(a)



(b)

Fig.3 Specimen and finite element mesh used in the analysis

3.2 Dugdale model

As described before, the Dugdale model corresponds to the DWA model at the time when $\bar{E}\to\infty$ and $\bar{H}'=0$ in the plane containing a crack and the continuum part is dealt with as a linear elastic body, so the analysis of Dugdale model by finite element method becomes possible by using completely the same procedure as in case of elasto-plastic crack.

The results of finite element analysis are shown in Fig.8. Fig.8(a) shows the relation between crack energy density at crack tip and displacement on loading surface, Fig.8(b) does the relation between load and displacement on loading surface and Fig.8(c) does the distribution of crack energy densities in the ligament plane. Here, specimen shape, loading condition and material constants E and ν are the same as in section 3.1, and $E/\bar{E}$ = 0.2 mm, $\bar{H}'$ = 0 and σ_Y = 0.4802 GPa are used in the ligament plane (the value of $E/\bar{E}$ to plane elements is 0.1 mm as we are in the same situation as in section 3.1). The results of so-called Dugdale model analysis by the boundary collocation method formulated by Ishida[9] are also shown in Fig.8. In this analysis, 12 subdivisions and calculation of double-precision are used and the value of stress σ_{22} obtained by DWA model analysis above, that is, 0.55 GPa is employed as the value of uniform tensile stress in the yielding plane (the value of σ_{22} in DWA model analysis is different from the yield stress because of the existence of in-plane

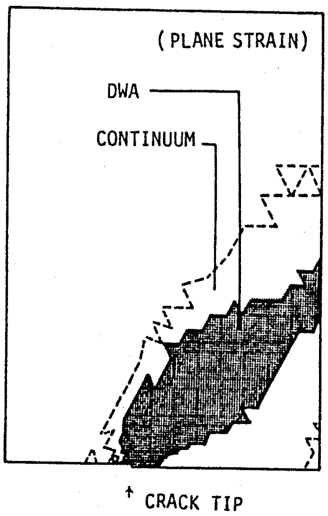


Fig.4 Shape of plastic yielding region (average displacement on loading surface u_m = 0.63 mm)

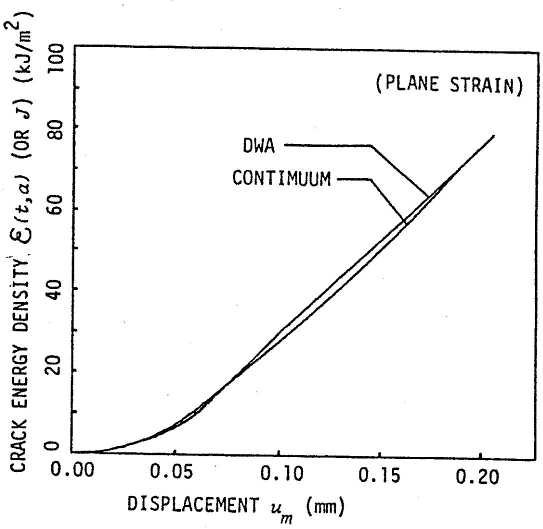


Fig.5 Crack energy density at crack tip (J-integral) versus average displacement on loading surface

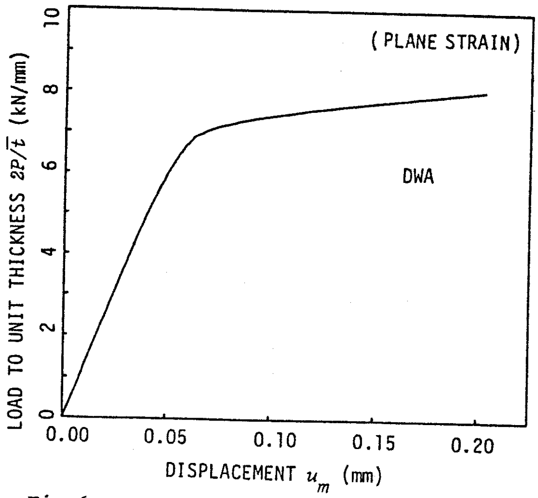


Fig.6 Load versus average displacement on loading surface

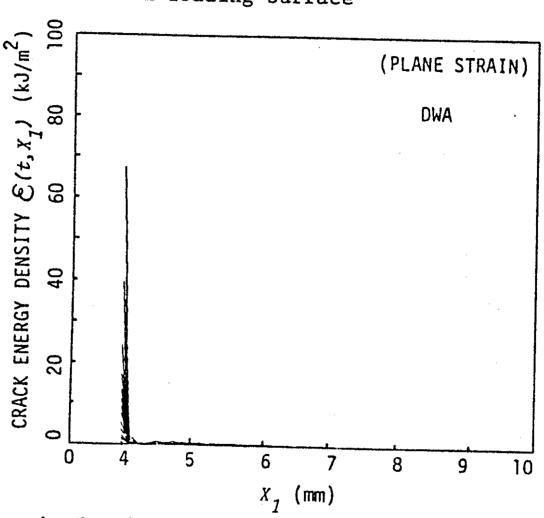


Fig.7 Distribution of crack energy densities in the ligament plane

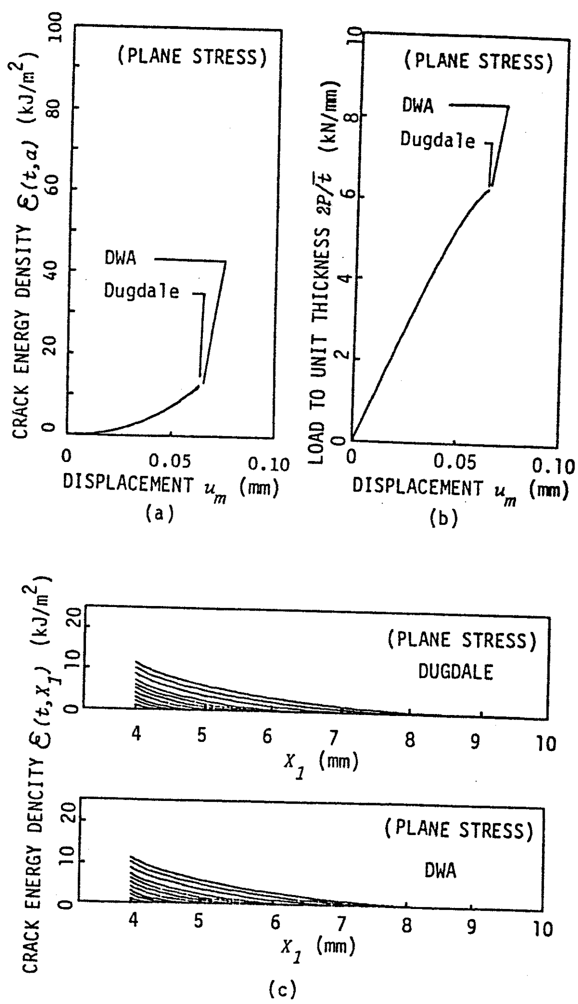


Fig.8 Dugdale model analyses by DWA model and boundary collocation method

stress). Extremely good agreement between the two results is observed in Fig.8, and it is confirmed from this fact that the DWA model and the method of analysis shown in this paper are really available for analysis of a crack problem.

4. Conclusions

A crack model named the DWA model which is expected to express the discontinuity around a crack tip in actual material as first-order approximation is proposed and the formulation to analyze an elasto-plastic crack problem by this model is shown concretely. Moreover, through the finite element analyses of an elasto-plastic crack and a crack corresponding to the Dugdale model under monotonously increasing load, it is shown that the proposed model is hopeful as a crack model to analyze the crack parameters.

References

(1) Dugdale, D. S., J. Mech. Phys. Solids, Vol.8 (1960), p.100.
(2) Watanabe, K., Azegami, H. & Kurashige, Y., Theoretical and Applied Mechanics, Vol.32, University of Tokyo Press (1984), p.427.
(3) Watanabe, K., Bulletin of the JSME, Vol.24, No.198 (1981), p.2059.
(4) Watanabe, K., Bulletin of the JSME, Vol.26, No.215 (1983), p.747.
(5) Watanabe, K. & Azegami, H., Bulletin of the JSME, Vol.28, No.246 (1985), p.2873.
(6) Shih, C. F. et al., ASTM STP 668 (1979), p.5.
(7) e. g. Yamada, Y., Introductory Solid Mechanics by Matrix Method (in Japanese), Baifukan (1980), p.209.
(8) Watanabe, K. & Kurashige, Y., Bulletin of the JSME, Vol.28, No.242 (1985), p.1582.
(9) Ishida, M., Mechanical Behavior of Materials, Proc.1971, ICM, 1 (1972), p.394.