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Proposal of New Stability-instability Criterion for Crack Extension Based on Crack Energy Density and Physical Systematization of Other Criteria*

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A new stability-instability criterion (named $T_\mathcal{E}(T_\mathcal{E}^*)$ criterion) for crack growth which is applicable to small scale yielding cracks and also to large scale yielding cracks is proposed based on crack energy density concept. $T_\mathcal{E}$ criterion is a criterion in which attention is paid to the rate of variation of crack energy density at a crack tip point of every moment, and $T_\mathcal{E}^*$ criterion is another version of $T_\mathcal{E}$ criterion and is a criterion in which attentton is paid to the rate of vatiation of crack energy density at a fixed point which will be a new crack tip point after extension.

The relations between $T_\mathcal{E}(T_\mathcal{E}^*)$ criterion and other known criteria (criterion based on $\mathcal{G}$-R-a curves; T_J, T_δ and T_W criteria) are also discussed and it is shown that the physical meanings of other criteria can be made clear and all criteria can be systematized based on the new criterion.

Key Words: Fracture, Stability-instability Criterion, Tearing Modulus, Crack Extension, Crack Energy Density

1. Introduction

As for a stable or unstable behavior of a growing crack, the energy balance approach by $\mathcal{G}$-R-a curves(1) has been considered effective to a crack accompanied with small scale yielding, but is has not been applied to a crack accompanied with large scale yielding successfully and various criteria based on tearing moduli (T_J(2)(3), T_δ(4), T_W(5)) have been proposed instead and received attention. However, in those criteria some problems remain, that is, as to T_J for instance, its physical meaning is not clear, it is applicable only under severely restricted conditions and, among other things, the relations between these criteria and the criterion based on energy balance are not clear. And it seems that a stability-instability criterion which permits a unified description of the phenomena from a brittle fracture to a ductile fracture has not been established yet.

By the way, in the previous papers one of the authors proposed a concept of crack energy density to take the place of energy release rate and showed by using the concept that various known criteria on the initiation condition of crack growth can be grasped systematically and their physical meanings can be made clear(6)(7). As this crack energy density is defined as a parameter which possesses a meaning under an arbitrary load history including crack extension, one can now expect to give a stability-instability criterion by crack energy density. From this point of view, in this paper the authors propose a new stability-instability criterion, named $T_\mathcal{E}(T_\mathcal{E}^*)$ criterion, based on crack energy density which is consistently applicable to a growing crack with small scale yielding and also to that with large scale yielding, and, after making clear its physical meaning, investigate the relations between this criterion and other criteria referred to in the above. It is shown that the positions and the role of other criteria can be understood systematically in the light of $T_\mathcal{E}(T_\mathcal{E}^*)$ criterion and the problems pointed out above can be solved.

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2. Proposal of a New Stability-instability Criterion Based on Crack Energy Density and Its Way of Thinking

Crack energy density $\mathcal{E}$ was defined as a value at a crack tip of every moment in the continuum model where the stress and strain become singular at a crack tip and in the model which reflects the discontinuities at a crack tip in an actual material (we call it the actual model hereafter) in the previous papers(6)(7). In this chapter, considering the work done per unit area not only on a crack tip but also on a plane containing a crack, we define the distribution of crack energy densities and propose a new stability-instability criterion named $T_\mathcal{E}$ criterion or another version of this $T_\mathcal{E}$ criterion, i.e., $T_\mathcal{E}^*$ criterion based on the distribution of crack energy densities.

2.1 Distribution of crack energy densities

We consider a crack in the actual model as schematically illustrated in Fig. 1. It is supposed here that a crack of which the length is a_0 in the initial state before loading (time $t=0$) as shown in Fig.

1(a) blunts and extends under loading and the length of extension becomes ($a-a_0$) at time $t=t$ as shown in Fig.1(b). In Fig.1 (a), Γ_0 is the shape of the crack tip in the initial state and a is the crack length at $t=t$ (the apparent growth length by blunting is not added to the crack length and a smooth crack extension is considered). Hereafter, a unit thickness is considered. X_1-X_2 coordinates are taken as shown in Fig.1(a) and a plane-like infinitesimal volume sandwiched between two broken lines in the figure is regarded as the actual cracked plane. Then the distribution of crack energy densities is defined, as the distribution of energies in the actual cracked plane, by

$$\mathcal{E}(t,X_1)=\lim_{\Delta X_1\to 0}\left\{\left(\int_{\Delta V}W dV\right)\Big/\Delta X_1\right\}$$
$$=\lim_{\Delta X_1\to 0}\left[\left\{\int_{X_1}^{X_1+\Delta X_1}\left(\int_{\Gamma(X_1)}W dX_2\right)dX_1\right\}\Big/\Delta X_1\right]$$
$$=\int_{\Gamma(X_1)}W dX_2 \quad \cdots\cdots(1)$$

where $\Gamma(X_1)$ is the path of which the shape is the same as Γ_0 at X_1 in the actual cracked plane in the initial state, ΔV is the volume in the actual cracked plane corresponding to ΔX_1 shown in Fig.1(a) and W is "the work done at each point in the initial state per unit volume through an actual deformation process", that is, the strain energy density in an extended sense. $\mathcal{E}(t,X_1)$ has a meaning of "the energy that the part of $\Gamma(X_1)$ in the initial state has absorbed actually up to now (time $t=t$ and crack length is a) per unit area in the actual cracked plane" and it is generally considered to make a discrete distribution along X_1 coordinate, but, in the following discussion, we regard it as a continuous distribution obtained by smoothing this discrete distribution, because such a smoothing is considered practical. Then the distribution is schematically represented by a chain line with a dot in Fig.1 (c) (a broken line shows $\mathcal{E}(t+\Delta t,X_1)$). Here, a chain line with two dots represents the crack energy density supplied to the part which the crack tip has passed before the crack tip arrives at each point.

While the distribution of crack energy densities is generally defined as the above, it may be easily defined considering a simple model such as Dugdale model as follows. That is, in this case, the above actual cracked plane becomes a complete plane and it is stretched out after loading as shown in Fig.2(a) by the shading part. And crack extension is realized by cutting off the plane and removing the traction force at the crack tip. Therefore, the distribution of the energies absorbed at the point $X_1(\geqq a)$ in the initial plane per unit area up to the present time t, that is, the distribution of crack energy densities $\mathcal{E}(t,X_1)$ is given, as the sum of products of adhesive force σ^* acting in the plane and corresponding increment of relative displacement $d\delta(t,X_1)$, by

$$\mathcal{E}(t,X_1)=\int_0^t \sigma^* d\delta(t,X_1) \quad \cdots\cdots(2)$$

and it is shown in Fig.2(b). Here a chain line with two dots in this figure has the same meaning as in Fig.1(c), and is evalu-

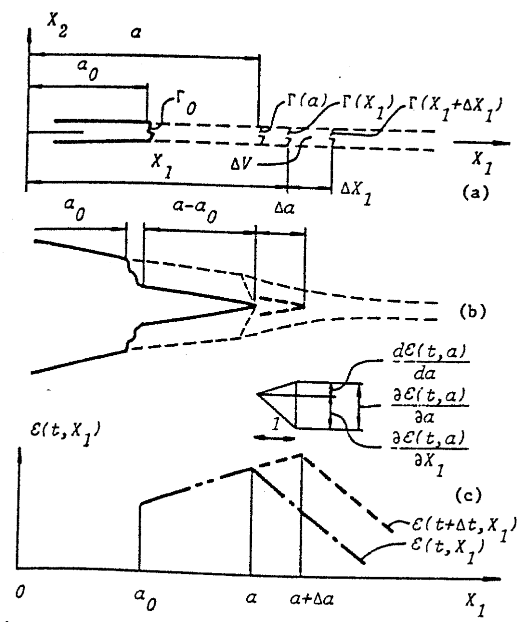

Fig.1 Crack extension in the actual model and distribution of crack energy densities

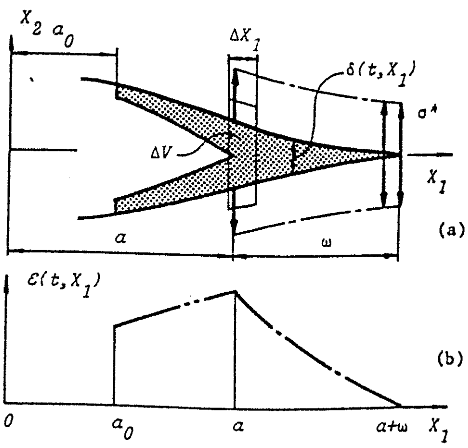

Fig.2 Crack extension in Dugdale model

ated by substituting zero for σ^* in Eq.(2) after the crack tip has passed the point X_1.

2.2 Proposal of $T_\mathcal{E}$ criterion

We represent the distribution of crack energy densities at time t of a smoothly and stably growing crack in an actual material by $\mathcal{E}_c(t,X_1)$ and call the crack energy density $\mathcal{E}_c(t,a)$ at the crack tip the critical crack energy density particularly. Here $\mathcal{E}_c(t,a)$ has a meaning of "the energy actually absorbed at the part of $\Gamma(a)$ in the actual cracked plane per unit area before the crack tip arrives at the position of $X_1=a$". On the other hand, we represent the distribution of crack energy densities determined by extending a crack statically under a given mechanical boundary condition (which may change with crack extension) by $\mathcal{E}_{ap}(t,X_1)$. Then, the condition of crack extension is given by

$$\mathcal{E}_{ap}(t,a)=\mathcal{E}_c(t,a) \quad \text{(extension)} \cdots\cdots(3)$$

This condition includes the initiation condition and we call the relation of Eq.(3)

$\mathcal{E}_c$ criterion for crack extension. Hereafter, the subscripts ap and c are assigned in the same way as used above also to other parameters, and a parameter with none of these subscripts is used to represent both parameters with these subscripts.

Now, when we grasp the phenomenon of crack extension as described above, the condition whether a crack that has started to grow or has grown stably behaves stably or unstably next moment is given by

$$\frac{d\mathcal{E}_{ap}(t,a)}{da}=\frac{\sigma_f^2}{E}T_{\mathcal{E}_{ap}}\geq\frac{d\mathcal{E}_c(t,a)}{da}=\frac{\sigma_f^2}{E}T_{\mathcal{E}_c}\text{(unstable)}$$

$$\frac{d\mathcal{E}_{ap}(t,a)}{da}<\frac{d\mathcal{E}_c(t,a)}{da}\text{ (stable)}\quad\cdots\cdots\cdots(4)$$

or

$$T_{\mathcal{E}_{ap}}\geq T_{\mathcal{E}_c}\text{ (unstable)},\quad T_{\mathcal{E}_{ap}}<T_{\mathcal{E}_c}\text{ (stable)}\quad\cdots\cdots\cdots(5)$$

where

$$\frac{d\mathcal{E}(t,a)}{da}=\lim_{\Delta a(\Delta t)\to 0}\frac{\mathcal{E}(t+\Delta t,a+\Delta a)-\mathcal{E}(t,a)}{\Delta t}\frac{\Delta t}{\Delta a}=\frac{d\mathcal{E}(t,a)}{dt}\frac{dt}{da}\quad\cdots\cdots\cdots(6)$$

E is Young's modulus and σ_f is an appropriate flow stress, and it should be noted that $\{d\mathcal{E}(t,a)\}/(da)$ is the gradient of a chain line with two dots at $X_1=a$ determined from the right side in Fig.1(c). We call Eq.(4) or (5) $T_{\mathcal{E}}$ criterion and newly propose this as a stability-instability criterion for crack extension. The applicabilities of $\mathcal{E}_c$ and $T_{\mathcal{E}}$ criteria are proved by seeing that the relations of $\mathcal{E}_c(t,a)$=constant and $T_{\mathcal{E}_c}=0$ hold when a crack grows under a uniform fracture mode.

2.3 Proposal of $T_{\mathcal{E}}^*$ criterion

Eq.(6) is transformed as follows;

$$\frac{d\mathcal{E}(t,a)}{da}=\lim_{\Delta a(\Delta t)\to 0}\left\{\frac{\mathcal{E}(t+\Delta t,a+\Delta a)-\mathcal{E}(t,a+\Delta a)}{\Delta t}\frac{\Delta t}{\Delta a}+\frac{\mathcal{E}(t,a+\Delta a)-\mathcal{E}(t,a)}{\Delta a}\right\}$$

$$=\lim_{\Delta a(\Delta t)\to 0}\frac{\mathcal{E}(t+\Delta t,a+\Delta a)-\mathcal{E}(t,a+\Delta a)}{\Delta t}\frac{\Delta t}{\Delta a}+\lim_{\Delta X_1\to 0}\frac{\mathcal{E}(t,a+\Delta X_1)-\mathcal{E}(t,a)}{\Delta X_1}\cdots(7)$$

Therefore, when we represent the first term and the second term in the right side of Eq.(7) by $\{\partial\mathcal{E}(t,a)\}/\partial a$ and $\{\partial\mathcal{E}(t,a)\}/\partial X_1$ respectively and consider that $\{\partial\mathcal{E}_{ap}(t,a)\}/\partial X_1=\{\partial\mathcal{E}_c(t,a)\}/\partial X_1$ holds at the time when a crack starts to grow or at the present time until when a crack has grown stably, the condition given by Eq.(4) or (5) can be expressed as

$$\frac{\partial\mathcal{E}_{ap}(t,a)}{\partial a}=\frac{\sigma_f^2}{E}T_{\mathcal{E}_{ap}}^*\geq\frac{\partial\mathcal{E}_c(t,a)}{\partial a}=\frac{\sigma_f^2}{E}T_{\mathcal{E}_c}^*\text{(unstable)}$$

$$\frac{\partial\mathcal{E}_{ap}(t,a)}{\partial a}<\frac{\partial\mathcal{E}_c(t,a)}{\partial a}\text{ (stable)}\quad\cdots\cdots\cdots(8)$$

or

$$T_{\mathcal{E}_{ap}}^*\geq T_{\mathcal{E}_c}^*\text{ (unstable)},\quad T_{\mathcal{E}_{ap}}^*<T_{\mathcal{E}_c}^*\text{ (stable)}\quad\cdots\cdots\cdots(9)$$

In this paper we call the condition represented by Eq.(8) or (9) $T_{\mathcal{E}}^*$ criterion. $T_{\mathcal{E}}$ and $T_{\mathcal{E}}^*$ criteria are equivalent though the parameters considered are different, and it can be said that $T_{\mathcal{E}}$ criterion is a criterion in which attention is paid to the rate of variation of crack energy density at the crack tip part of every moment and, on the other hand, $T_{\mathcal{E}}^*$ criterion is a criterion in which attention is paid to the rate of variation of crack energy density at the fixed part which will be a new crack tip part after extension. That is, $T_{\mathcal{E}}^*$ corresponds to the gradient at $X_1=a$ between the chain line with a dot and the chain line with two dots determined from the right side. Moreover, when a crack extends under a uniform fracture mode, $T_{\mathcal{E}_c}^*$ is presumed to be a constant if the gradient of the distribution of crack energy densities at $X_1=a$ determined from the right side does not change with an extension of crack, while $T_{\mathcal{E}_c}$ is presumed to be zero. The variation of the gradient is generally presumed to be small except in a special case of the ligament part of a specimen being small, so it is considered that $T_{\mathcal{E}_c}^*$ becomes constant in many cases when $T_{\mathcal{E}_c}$ is zero.

3. Relationship between Proposed Criterion and Other Criteria

In the past, the criterion based on $\mathcal{G}$-R-a curves for a crack with small scale yielding and T_J, T_δ and T_W criteria for a crack with large scale yielding have been accepted as available criteria. In this chapter we consider the relationships between those criteria and $T_{\mathcal{E}}(T_{\mathcal{E}}^*)$ criterion and show that those criteria can be grasped systematically in the light of $T_{\mathcal{E}}(T_{\mathcal{E}}^*)$ criterion.

3.1 Criterion based on $\mathcal{G}$-R-a curves

As for a stable or unstable behavior of a growing crack accompanied with small scale yielding, the energy balance approach by $\mathcal{G}$-R-a curves[1] has been considered an effective criterion. That is, it has been supposed that the behavior of crack extension is unstable when the released potential energy for imaginary crack extension exceeds the required energy to form new surfaces. However, it is clear that the quantity which has been considered the energy release rate of a crack with small scale yielding is not the energy release rate but merely the crack energy density[7]. Therefore, the criterion based on $\mathcal{G}$-R-a curves should be called the criterion based on $\mathcal{E}_{ap}$-$\mathcal{E}_c$-a curves and it is nothing but $T_{\mathcal{E}}$ criterion. It can be said that the availability of the criterion based on $\mathcal{G}$-R-a curves has proved that of $T_{\mathcal{E}}$ criterion.

3.2 T_J criterion

As a stability-instability criterion of a crack with large scale yielding, T_J criterion using the gradient of J-integral resistance curve is proposed by Paris et al.[2] and attracts our attention. By considering the relationship between parameter T_J in T_J criterion and parameter $T_{\mathcal{E}}^*$, it is shown here that T_J is equivalent to $T_{\mathcal{E}}^*$ under a certain condition and T_J criterion is positioned as a criterion corresponding to $T_{\mathcal{E}}^*$ criterion at the time when a crack starts to grow.

In T_J criterion, the condition of crack extension at time t is given, using J_{ap} determined from mechanical condition and J_c obtained from J-integral resistance curve, by

$$J_{ap}(t)=J_c(t) \quad \text{(extension)} \quad \cdots\cdots(10)$$

and the condition on stability-instability is given by

$$T_{Jap} \geqq T_{Jc} \text{ (unstable)}, \quad T_{Jap} < T_{Jc} \text{ (stable)} \quad \cdots\cdots(11)$$

where

$$T_J=\frac{E}{\sigma_f^2}\frac{dJ(t)}{da} \quad \cdots\cdots(12)$$

As evaluation of T_J has been carried out based on load-displacement curve or path integral, we discuss on T_J evaluated by each method in the following.

(1) T_J by load-displacement curve Here, as we take the relationship between T_J obtained from the simplified method by Rice[8] and T_C^*, we consider the load P-displacement u (or bending load M-angular displacement θ) curves of two cracked specimens as shown in Fig.3 of which the initial crack lengths are a_0 and $a_0+\Delta a_0$ respectively. In the figure, the state after the initiation of crack growth is also considered and two cracked specimens are loaded such that their displacements becomes equal to each other at every moment from time $t=0$ to t. Moreover, one-to-one correspondence between time t and $u(t)$ (or $\theta(t)$) is assumed. Cracks are assumed to start growing at point 1 ($t=t_1$) in case of initial crack length being a_0 and at point 2 ($t=t_2$) in case of initial crack length being $a_0+\Delta a_0$ in Fig.3, and we discuss the problem using Fig.3(a) when the crack of a short initial crack length starts to grow earlier than that of a long initial crack length and using Fig.3(b) when the sequence reverses. Besides, as to the cracked specimen with initial crack length a_0, the displacement and the load increase by Δu (or $\Delta\theta$) and ΔP respectively and the crack extends by Δa in the period of Δt $(=t_2-t_1)$, and the state after the extension of Δa ($t=t_2$) is shown by point 1_{ex} in Fig.3. The magnitudes of Δa, Δt, Δu and ΔP are determined dependent on that of Δa_0. In the following, when the rate of variation with time is discontinuous at the time when a crack starts to grow, the rates immediately before and after are represented by $(\ )^-$ and $(\ )^+$ respectively, and an infinitesimal of an increment represented by Δ is indicated by d.

Now, the value of J of a cracked specimen, $J(t_1)$, of initial crack length a_0 at $t=t_1$ is given generally, with load $P[a_0, u(t)]$ (or $P(a_0, t)$) as a function of initial crack length a_0 and displacement $u(t)$ (or time t) and area $\Delta A(t_1)$ which is area $\bar{2}01$ in Fig.3, by[9]

$$J(t_1)=-\frac{dA(t_1)}{da_0}=-\frac{1}{B}\int_0^{t_1}\left(\frac{\partial P}{\partial a_0}\right)_u du \quad \cdots(13)$$

where B is the thickness of a specimen. The simplified formulas by Rice are given based on the relation of Eq.(13) in particular cases of a cracked specimen loaded by bending moment or tensile force. In case of bending, as the relationship between angular displacement θ, bending moment $M(a_0, \theta)$ (or $M(a_0, t)$), ligament length b_0 and thickness B, that is

$$M=b_0^2Bf(\theta) \quad \text{or}$$

$$\left(\frac{\partial M}{\partial a_0}\right)_\theta=-\left(\frac{\partial M}{\partial b_0}\right)_\theta=-2b_0Bf(\theta)=-\frac{2}{b_0}M \quad \cdots\cdots(14)$$

holds approximately, its formula is given by

$$J(t_1)=-\frac{1}{B}\int_0^{t_1}\left(\frac{\partial M}{\partial a_0}\right)_\theta d\theta$$

$$=\frac{2}{Bb_0}\int_0^{t_1}Md\theta=\frac{2A_b(t_1)}{b_0} \quad \text{(bending)} \quad (15)$$

where $A_b(t_1)$ is area 101" in Fig.4. In case of a center-cracked (or both-edge cracked) specimen under tensile force, the relationship between displacement u_n corresponding to nonlinear behavior (cf. Fig.5), P, b_0 and B', that is

$$u_n=b_0g\left(\frac{P}{b_0B}\right) \quad \text{or} \quad \left(\frac{\partial u_n}{\partial a_0}\right)_P$$

$$=-\left(\frac{\partial u_n}{\partial b_0}\right)_P=\frac{P}{b_0}\left(\frac{\partial u_n}{\partial P}\right)_{b_0}-\frac{u_n}{b_0} \quad \cdots\cdots(16)$$

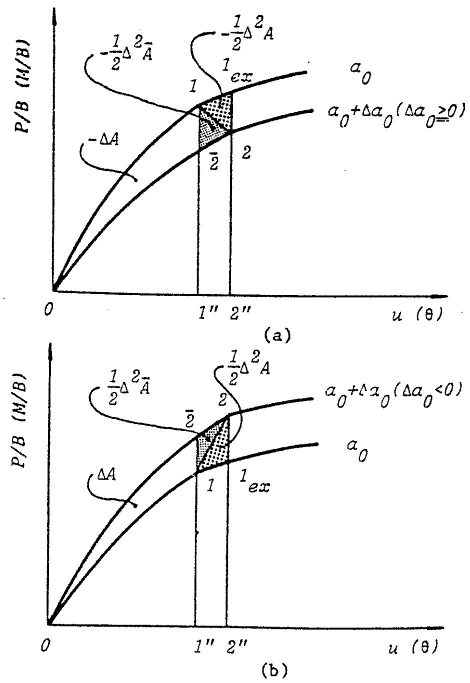

Fig.3 Two load-displacement curves of cracked specimens with different initial crack lengths

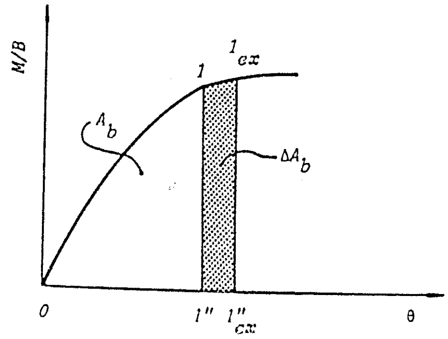

Fig.4 Bending moment-angular displacement curve

holds approximately, and the relation of $u(t)=u(a_0, P(a_0, t))=u_l+u_n$ holds because displacement u_l corresponding to linear behavior can be given as a function, $u_l(a_0, P)$, of a_0 and P. Therefore, the simplified formula by Rice is given, considering the relation between $u_l(a_0, P)$ and stress intensity factor $K_I(a_0, P)$ and $Pdu_l=u_l dP$, by

$$J(t_1)=\frac{1}{B}\int_0^{t_1}\left(\frac{\partial P}{\partial a_0}\right)_u du=\frac{1}{B}\int_0^{t_1}\left(\frac{\partial u}{\partial a_0}\right)_P dP$$
$$=\frac{1}{B}\int_0^{t_1}\left(\frac{\partial u_l}{\partial a_0}\right)_P dP+\frac{1}{B}\int_0^{t_1}\left(\frac{\partial u_n}{\partial a_0}\right)_P dP$$
$$=\frac{K_I^2}{E'}+\frac{1}{Bb_0}\int_0^{t_1}(Pdu_n-u_n dP)$$
$$=\frac{K_I^2}{E'}+\frac{2}{Bb_0}\left(\int_0^{t_1}Pdu-\frac{1}{2}P(a_0, t_1)u(t_1)\right)$$
$$=\frac{K_I^2(a_0, P(a_0, t_1))}{E'}+\frac{2A_t(t_1)}{b_0}\ \text{(tension)} \quad (17)$$

where $A_t(t_1)$ is area 101 in Fig.5, E' is E in case of plane stress and $E/(1-\nu^2)$ in case of plane strain and ν is Poisson's ratio. As T_J is a nondimensional value of the variation rate of J with crack extension $(dJ)/(da)$, we consider here the quantities given by the following equations as $(dJ)/(da)$.

$$\frac{dJ(t_1)}{da}=\frac{2}{b_0}\frac{dA_b(t_1)}{da}=\frac{2M(a_0, t_1)}{Bb_0}\frac{d\theta}{da}\ \text{(bending)} \quad (18)$$

$$\frac{dJ(t_1)}{da}=\frac{1}{E'}\left(\frac{\partial K_I^2(a_0, P(a_0, t_1))}{\partial P}\right)_{a_0}\frac{dP}{da}+\frac{2}{b_0}\frac{d\bar{A}_t(t_1)}{da}$$
$$=\frac{1}{B}\left[\left(\frac{\partial u_l(a_0, P(a_0, t_1))}{\partial a_0}\right)_P\frac{dP}{da}\right.$$
$$\left.+\frac{1}{b_0}\left\{P(a_0, t_1)\frac{du}{da}-u(t_1)\left(\frac{\partial P(a_0, t_1)}{\partial u}\right)^-_{a_0}\frac{du}{da}\right\}\right]$$
$$\text{(tension)} \quad (19)$$

Here, $\Delta A_b(t_1)$ is area $11''1''_{ex}1_{ex}$ in Fig.4 which expresses the variation corresponding to Δa (or Δt, Δu, ΔP) and $\Delta\bar{A}_t(t_1)$ is area $10\bar{1}_{ex}$ in Fig.5 which expresses the variation corresponding to Δa ($\bar{1}_{ex}$ is the point of $\{P(a_0, t_1)+[\{\partial P(a_0, t_1)\}/\partial u]^-\Delta u,\ u(t_1)+\Delta u\}$ on the load-displacement curve). The quantity evaluated by Eq.(18) corresponds to the gradient determined from the right side at $t=t_1$ on the resistance curve of a three-point bend specimen in accordance with the JSME S 001 J_{Ic} evaluation method. Here, substituting the relations of Eq.(14) and (16) at $t=t_1$ into Eq.(18) and (19) respectively, we obtain the relations

$$\frac{dJ(t_1)}{da}=-\frac{1}{B}\left(\frac{\partial M(a_0, t_1)}{\partial a_0}\right)_\theta\frac{d\theta}{da}$$
$$=\frac{1}{B}\left(\lim_{\Delta a_0\to 0}\frac{M(a_0, t_1)-M(a_0+\Delta a_0, t_1)}{\Delta a_0}\right)\frac{d\theta}{da}$$
$$=-\frac{d^2\bar{A}(t_1)}{da_0 da}\ \text{(bending)} \quad \cdots\cdots (20)$$

$$\frac{dJ(t_1)}{da}=\frac{1}{B}\left[\left(\frac{\partial u_l(a_0, P(a_0, t_1))}{\partial a_0}\right)_P\frac{dP}{da}\right.$$
$$+\frac{1}{b_0}\left\{P(a_0, t_1)\left(\frac{\partial u_n(a_0, P(a_0, t_1))}{\partial P}\right)_{b_0}^-\right.$$
$$\left.\left.-u_n(a_0, P(a_0, t_1))\right\}\left(\frac{\partial P(a_0, t_1)}{\partial u}\right)^-_{a_0}\frac{du}{da}\right]$$
$$=\frac{1}{B}\left\{\left(\frac{\partial u_l(a_0, P(a_0, t_1))}{\partial a_0}\right)_P\right.$$
$$\left.+\left(\frac{\partial u_n(a_0, P(a_0, t_1))}{\partial a_0}\right)_P\right\}\left(\frac{\partial P(a_0, t_1)}{\partial u}\right)^-_{a_0}\frac{du}{da}$$
$$=\frac{1}{B}\left(\frac{\partial u(a_0, P(a_0, t_1))}{\partial a_0}\right)_P\left(\frac{\partial P(a_0, t_1)}{\partial u}\right)^-_{a_0}\frac{du}{da}$$
$$=-\frac{1}{B}\left(\frac{\partial P(a_0, t_1)}{\partial a_0}\right)_u\frac{du}{da}$$

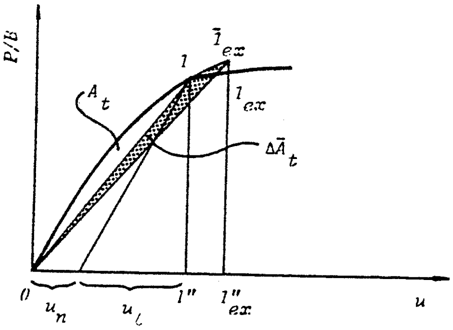

Fig.5 Tensile load-displacement curve

$$=\frac{1}{B}\left(\lim_{\Delta a_0\to 0}\frac{P(a_0, t_1)-P(a_0+\Delta a_0, t_1)}{\Delta a_0}\right)\frac{du}{da}$$
$$=-\frac{d^2\bar{A}(t_1)}{da_0 da}\ \text{(tension)} \quad \cdots\cdots (21)$$

where $-1/2\Delta^2\bar{A}(t_1)$ in Fig.3(a) (or $1/2\Delta^2\bar{A}(t_1)$ in Fig.3(b)) corresponds to area $12\bar{2}$. Equation (20) or (21) gives the relation between $\{dJ(t_1)/da\}$ evaluated by Eq.(18) or (19) and the area defined in Fig.3.

On the other hand, a relation between the additional rate of crack energy density, $\{\partial\mathcal{E}(t_1, a_0)\}/\partial a$, at the time when a crack starts to grow ($t=t_1$) and the area in Fig. 3 where two load-displacement curves are considered is given as follows. In Fig.6 (a), we consider that a crack of initial crack length $a_0+\Delta a_0'(|\Delta a_0'|\leq|\Delta a_0|)$ starts to grow at point 2' (the figure corresponds to Fig. 3(a), but the case corresponding to Fig.3 (b) can be also dealt with in the same way). As to a cracked specimen of initial crack length a_0, the displacement increases by $\Delta u'$ and the crack extends by $\Delta a'$ between $\Delta t'$ period from the time t_1 to the time when the crack of initial crack length $a_0+\Delta a_0'$ starts to grow, and the state after extension of $\Delta a'$ is expressed by point $1'_{ex}$ in Fig.3. $\Delta a', \Delta t'$ and $\Delta u'$ are determined corresponding to $\Delta a_0'$. Here, expressing the strain energy density W defined in section 2.1 as a function of initial crack length a_0, time t and place X_i (i = 1,2) by $W(a_0, t, X_1, X_2)$, we obtain the following equations of energy balance.

$$\frac{1}{B}\int_{t_1}^{t_1+\Delta t}P(a_0, t)du$$
$$=\int_{t_1}^{t_1+\Delta t}\left\{\int_{V_0-(\Delta V_0'-\Delta V')}\left(\frac{\partial W}{\partial t}\right)_{a_0=a_0,X_i}dV\right\}dt$$
$$+\int_{t_1}^{t_1+\Delta t}\left[\int_{a_0+\Delta a'}^{a_0+\Delta a_0}\left\{\int_{\Gamma(X_1)}\left(\frac{\partial W}{\partial t}\right)_{a_0=a_0,X_i}dX_2\right\}dX_1\right]dt \quad (22)$$

$$\frac{1}{B}\int_{t_1}^{t_1+\Delta t}P(a_0+\Delta a_0', t)du$$
$$=\int_{t_1}^{t_1+\Delta t}\left\{\int_{V_0-(\Delta V_0'-\Delta V')}\left(\frac{\partial W}{\partial t}\right)_{a_0=a_0+\Delta a_0',X_i}dV\right\}dt \quad \cdots (23)$$

Here, the left sides of equations are works done by external forces and V_0, $\Delta V_0'$ and $\Delta V'$ are the volumes in the initial state corresponding to the whole specimen, $\Delta a_0'$ and $\Delta a'$ as shown in Fig.6(b). Moreover, we can obtain the following relation on the part of $V_0-(\Delta V_0'-\Delta V')$.

$$\int_{t_1}^{t_1+\Delta t}\left\{\int_{V_0-(\Delta V_0'-\Delta V')}\left(\frac{\partial W}{\partial t}\right)_{a_0=a_0,X_i}dV\right\}dt$$
$$=\int_{t_1}^{t_1+\Delta t}\left\{\int_{V_0-(\Delta V_0'-\Delta V')}\left(\frac{\partial W}{\partial t}\right)_{a_0=a_0+\Delta a_0,X_i}dV\right\}dt$$

$$+\int_{\Delta S}\left(\int_0^{\Delta F_i,\,\Delta u'_{iF}} F_i^{*}\,du^{*}_{iF}\right)dS \qquad \cdots\cdots(24)$$

Here, ΔS is the surface before deformation excluding Γ_0 from the surface surrounding the volume ΔV_0 corresponding to Δa_0 (cf. Fig.6(b)), F_i^* is the traction force acting on the surface of a cracked specimen of initial crack length $a_0+\Delta a_0$ corresponding to ΔS so that the equation holds, du^*_{iF} is the infinitesimal increment of displacement corresponding to F_i^*, and ΔF_i and $\Delta u'_{iF}$ are the increments of traction force and the displacement in the above process respectively. On the other hand, we imagine the traction force T_i and the increment of traction force ΔT_i as working on ΔS of a cracked specimen of initial crack length a_0 corresponding to the actual increment of displacement Δu_i in the period of Δt so that the following relation holds.

$$\int_{t_1}^{t_1+\Delta t_1}\left[\int_{a_0+\Delta a'}^{a_0+\Delta a_0'}\left\{\int_{\Gamma(X_1)}\left(\frac{\partial W}{\partial t}\right)_{a_0=a_{0X_1}}dX_2\right\}dX_1\right]dt$$
$$=\int_{\Delta S}\left(\int_{T_i}^{T_i+\Delta T_i,\,\Delta u_i} T_i\,du_i\right)dS \qquad \cdots\cdots(25)$$

Now assume Δa_0 and Δa which are correlated to each other as infinitesimals. Then, $\Delta F_i, \Delta u'_{iF}$ and ΔT_i become infinitesimals of second order under the influences of both Δa_0 and Δa , T_i and Δu_i become infinitesimal of first order under the influence of Δa_0 and that of Δa respectively. Consequently, the second term of the right side of Eq.(24) can be neglected, since this term is a higher order infinitesimal than the right side of Eq.(25); and representing area $1 1_{ex} 2$ by $-1/2\Delta^2A(t_1)$ in Fig.3(a), we obtain the relation as

$$-\frac{d^2A(t_1)}{da_0da}=\frac{2}{B}\lim_{\Delta a(\Delta t)\to 0}\left\{\frac{1}{\Delta a_0\Delta a}\int_{t_1}^{t_1+\Delta t}(P(a_0,t)-P(a_0+\Delta a_0',t))du\right\}$$
$$=2\lim_{\Delta a(\Delta t)\to 0}\left[\frac{1}{\Delta a_0\Delta a}\int_{t_1}^{t_1+\Delta t}\left\{\int_{a_0+\Delta a'}^{a_0+\Delta a_0'}\left(\int_{\Gamma(X_1)}\left(\frac{\partial W}{\partial t}\right)_{a_0=a_{0X_1}}dX_2\right)dX_1\right\}dt\right]$$
$$=\lim_{\Delta a(\Delta t)\to 0}\left[\frac{\Delta a_0-\Delta a}{\Delta a_0}\left\{\int_{\Gamma(a_0)}\left(\frac{\partial W}{\partial t}\right)_{a_0=a_{0X_1}}dX_2\right\}\frac{\Delta t}{\Delta a}\right]$$
$$=\left(1-\frac{da}{da_0}\right)\frac{\partial\,\mathcal{E}(t_1,a_0)}{\partial a} \qquad \cdots\cdots(26)$$

The relation between the area of $\Delta^2A(t_1)$ defined from two load-displacement curves of cracked specimens with different initial crack lengths and the additional rate of crack energy density $\{\partial\mathcal{E}(t_1,a_0)\}/\partial a$ is given by Eq.(26). Therefore, the relation between $\{dJ(t_1)\}/da$ obtained from a load-displacement curve and $\{\partial\,\mathcal{E}(t_1,a_0)\}/\partial a$ can be obtained, based on the relation between $\Delta^2\bar{A}(t_1)$ and $\Delta^2A(t_1)$, as

$$\frac{\partial\,\mathcal{E}(t_1,a_0)}{\partial a}=C\frac{dJ(t_1)}{da} \qquad \cdots\cdots(27)$$

where

$$C=\left[1-\left\{\left(\frac{\partial P(a_0,t_1)}{\partial u}\right)^-_{a_0}-\left(\frac{\partial P(a_0,t_1)}{\partial u}\right)^+_{a_0}\right\}\Big/\left\{\left(\frac{\partial P(a_0,t_1)}{\partial a_0}\right)_u\frac{da_0}{da}\right\}\right]\Big/\left(1-\frac{da}{da_0}\right) \cdots\cdots(28)$$

Consequently, the relation between T_ε^* and T_J, that is

$$T_\varepsilon^*=CT_J \qquad \cdots\cdots(29)$$

holds at the time when a crack starts to grow. Thereupon it is known that T_J can be substituted for T_ε^* immediately after the time of initiation of crack extension under the condition that the value of C changes little, that is, C can be regarded as a constant approximately and T_J has a meaning of T_ε^* particularly under the condition of $C\doteqdot 1$. In addition, as we can observe the tendencies, even if the shapes of specimens are the same, that the value of $(da)/(da_0)$ becomes smaller than unity and the degree of discontinuity of load-displacement curve at the point of initiation of crack extension becomes smaller with an increase of ductility of material, we can consider that $C\approx 1$ is satisfied on a specific cracked specimen of ductile material.

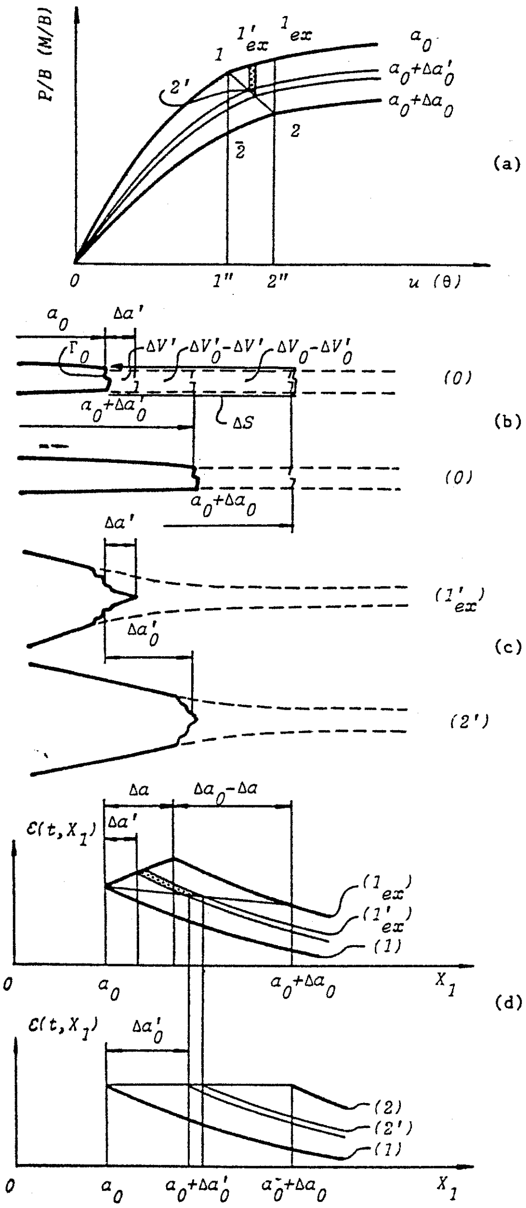

Fig.6 Crack extensions in cracked specimens with different initial crack lengths and distributions of crack energy densities

<u>(2) T_J by path integral</u> As to a growing crack, good agreement between J-values evaluated from the path integral along a path far from the crack tip by using a crack model (root radius ρ is zero) and by

using a notch model with a sufficiently small root radius can be expected and the possibility that a notch model with an appropriate root radius can be a continuum crack model which makes it possible to evaluate the crack energy density with its distribution is pointed out[10]. Therefore, we consider on an extending notch (the extension of crack is realized by assuming the rigidity of the volume in front of crack tip $\Gamma(a)$ as zero) shown in Fig.7 here. The distribution of crack energy densities $\mathcal{E}(t, X_1)$ of this model is evaluated by an integral which is given by

$$\mathcal{E}_J(t, X_1)=\int_\Gamma (Wn_1 - T_i u_{i,1})ds - \int_{\Gamma u-\Gamma(X_1)+\Gamma d}^{T_i u_{i,1}} ds - \int_A \int_0^t (\sigma_{ij,1} d\varepsilon_{ij} - d\sigma_{ij}, \varepsilon_{ij,1})dA \quad \cdots\cdots(30)$$

and is a path-independent without any restriction on constitutive law as $\mathcal{E}(t, X_1) = \mathcal{E}_J(t, X_1)$[10][11]. Therefore, considering $J(t) = \int_\Gamma (Wn_1 - T_i u_{i,1})ds$, the relation between $\{\partial \mathcal{E}(t, a)\}/(\partial a)$ and $\{dJ(t)\}/da$ is given by

$$\frac{\partial \mathcal{E}(t, a)}{\partial a} = \frac{\partial \mathcal{E}_J(t, a)}{\partial a} = \frac{dJ(t)}{da} - \int_{\Gamma(a)} \sigma_{ij} \varepsilon_{ij,1} dX_2 + \frac{dt}{da} \int_A (\varepsilon_{ij,1} \dot{\sigma}_{ij} - \sigma_{ij,1} \dot{\varepsilon}_{ij})dA \quad \cdots\cdots(31)$$

Here, W_t is strain energy density defined by $W = \int_0^t \sigma_{ij} d\varepsilon_{ij}$; σ_{ij}, ε_{ij}, T_i and u_i are stress, strain, traction force and displacement respectively; n_1 is X_1-direction component of outward normal unit vector; A is the area surrounded by $\Gamma + \Gamma_u - \Gamma(X_1) + \Gamma_d$; and $(\ )_{,i} = \partial(\ )/\partial X_i$, $(\dot{\ }) = \partial(\ )/\partial t$. The integration from the time 0 to t is carried out along an actual path of loading. Equation (31) is a general relation between T_J and $T_\mathcal{E}^*$ which holds always as well as immediately after the initiation of crack extension and both criteria can be transformed into each other based on this relation. It goes without saying that T_J criterion is equivalent to $T_\mathcal{E}^*$ criterion when

$$\frac{dJ(t)}{da} \gg - \int_{\Gamma(a)} \sigma_{ij} \varepsilon_{ij,1} dX_2 + \frac{dt}{da} \int_A (\varepsilon_{ij,1} \dot{\sigma}_{ij} - \sigma_{ij,1} \dot{\varepsilon}_{ij})dA \quad \cdots\cdots(32)$$

although more examination is required about the condition of Eq.(32).

3.3 T_δ criterion

T_δ criterion is proposed as a criterion which is applicable to a crack accompanied with large scale yielding which has extended large, and it is given, based on an analytical model, by[4]

$$T_{\delta ap} \geqq T_{\delta c} \text{ (unstable)}, \quad T_{\delta ap} < T_{\delta c} \text{ (stable)} \quad \cdots\cdots(33)$$

where T_δ is defined, representing the crack opening displacement of the place X_1 at the time t by $\delta(t, X_1)$ as shown in Fig.8, by

$$T_\delta = \frac{E}{\sigma_f} \lim_{\Delta a(\Delta t) \to 0} \frac{\delta(t+\Delta t, a) - \delta(t, a)}{\Delta t} \frac{\Delta t}{\Delta a} \quad \cdots\cdots(34)$$

Here, we consider T_δ as defined above by using Dugdale model in which its exact formulation is possible. In this case the relation

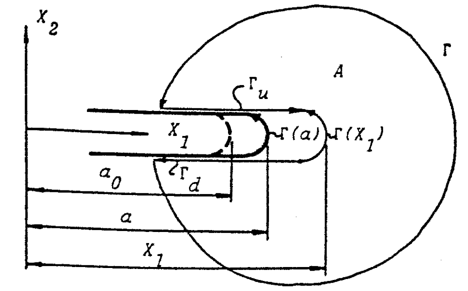

Fig.7 Integration path around a notch

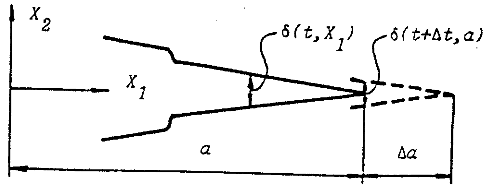

Fig.8 Increment of crack opening displacement accompanied with crack extension

$$T_\delta = \frac{E}{\sigma_f^2} \lim_{\Delta a(\Delta t) \to 0} \frac{\sigma_f \{\delta(t+\Delta t, a) - \delta(t, a)\}}{\Delta t} \frac{\Delta t}{\Delta a} = \frac{E}{\sigma_f^2} \lim_{\Delta a(\Delta t) \to 0} \frac{\sigma_f \{\delta(t+\Delta t, a+\Delta a) - \delta(t, a+\Delta a)\}}{\Delta t} \frac{\Delta t}{\Delta a} = \frac{E}{\sigma_f^2} \frac{\partial \mathcal{E}(t, a)}{\partial a} = T_\mathcal{E}^* \quad \cdots\cdots(35)$$

is obtained when we adopt flow stress σ_f as adhesive force σ^*. It is known from the equation that T_δ agrees with $T_\mathcal{E}^*$ numerically in the model, although there exists a difference that T_δ is defined for the changes at the crack tip before extension of Δa and, on the other hand, $T_\mathcal{E}^*$ is defined for the change at the new crack tip after extension of Δa. In general, T_δ disagrees with $T_\mathcal{E}^*$ numerically, but it can be considered that there exists one-to-one correspondence, between them, combined by a coefficient determined depending on the model adopted and how to define δ.

3.4 T_W criterion

T_W criterion is proposed as a criterion which is applicable to a crack accompanied with large scale yielding which has extended large and it is given, paying attention to the increment of plastic work done in the neighborhood of crack tip caused by crack extension, by

$$T_{W ap} \geqq T_{Wc} \text{ (unstable)}, \quad T_{W ap} < T_{Wc} \text{ (stable)} \quad \cdots\cdots(36)$$

where T_W is defined, using plastic strain energy $\tilde{W}_P(t, X_1)$ at the time t in the region closed by a circle of which the center is at the place X_1 and the radius is R as shown in Fig.9, by

$$T_W = \frac{E}{\sigma_f^2} \frac{1}{R} \lim_{\Delta a(\Delta t) \to 0} \frac{\tilde{W}_P(t+\Delta t, a) - \tilde{W}_P(t, a)}{\Delta t} \frac{\Delta t}{\Delta a} \quad (37)$$

When we use a Dugdale model as we did on T_δ, adopt ΔV and ΔX_1 as shown in Fig.2 as the closed region and R and put σ^* as σ_f, we obtain the follwoing relation

$$\lim_{\Delta X_1 \to 0} T_W = \frac{E}{\sigma_f^2} \lim_{\Delta X_1 \to 0} \frac{1}{\Delta X_1} \lim_{\Delta a(\Delta t) \to 0} \int_{\Delta X_1} \frac{\sigma_Y \{\delta(t+\Delta t, a) - \delta(t, a)\}}{\Delta t} \frac{\Delta t}{\Delta a} dX_1$$

$$=\frac{E}{\sigma_f^2}\lim_{\Delta a(\Delta t)\to 0}\frac{\sigma_Y\{\delta(t+\Delta t,a+\Delta a)-\delta(t,a+\Delta a)\}}{\Delta t}\frac{\Delta t}{\Delta a}$$
$$=\frac{E}{\sigma_f^2}\frac{\partial \mathcal{E}(t,a)}{\partial a}=T_{\mathcal{E}}^{*} \quad \cdots\cdots(38)$$

It can be considered from the relation of Eq.(38) that T_W can be positioned as a parameter which is conceptually the same as $T_{\mathcal{E}}^{*}$, although there is a difference that T_W is defined in the finite region surrounding a crack tip whereas $T_{\mathcal{E}}^{*}$ is defined at a crack tip.

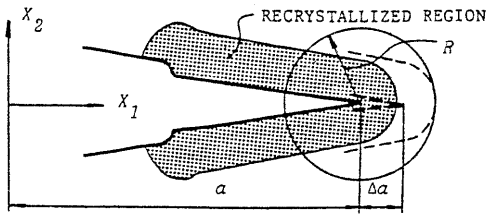

Fig.9 Closed, fixed region in the neighborhood of a crack tip

4. Conclusion

Concerning a stable or unstable behavior of a growing crack, a criterion named $T_{\mathcal{E}}(T_{\mathcal{E}}^{*})$ was newly proposed. Moreover, the relations between this criterion and other known criteria were discussed and it was shown that the unsolved problems on the stability-instability of a growing crack can be solved and various other criteria can ben grasped systematically in the light of $T_{\mathcal{E}}(T_{\mathcal{E}}^{*})$ criterion.

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