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Solution to Shape Optimization Problem of Linear Elastic Continuum with Prescribed Vibrational Eigen-mode

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Key Words: Optimum Design, Computer Aided Design, Finite Element Method, Vibration of Continuous System, Modal Analysis, Traction Method

1. Introduction

Modal Analysis of natural vibration in linear elastic continua comes into wide use to investigate causes of vibration troubles in machines. To improve the dynamic property of the machines, it is effective to control the modal shapes of natural vibrations by changing geometrical boundary shapes of machine parts.

In the previous works, the authors presented a numerical solution for boundary shape optimization problems of linear elastic continua to maximize vibrational eigenvalues and to minimize frequency responses such as strain energy, kinetic energy and absolute mean compliance. However, control problem of natural vibration mode has not been presented.

This paper is devoted to presenting a solution to the boundary shape determination problems of linear elastic continua with a prescribed natural vibration mode. This problem can be formulated as minimization problem of an integral of squared error of natural vibration mode from prescribed mode on specified subboundary with respect to perturbation of the domain of linear elastic continuum.

2. Shape optimization problem with prescribed natural vibration mode

Let a continuum be defined on a domain $\Omega \subset \mathbb{R}^n$, $n = 2, 3,$ and a boundary Γ. The weak form of governing equation for r-th natural eigenvalue λ_r, that is the minus value of squared natural frequency, and natural vibration mode $u_r : \Omega \mapsto \mathbb{R}^n$ is represented by

$$a(u_r, v) = \lambda_r b(u_r, v) \quad u_r \in U \quad \forall v \in U \tag{1}$$

where the bilinear forms $a(\cdot,\cdot)$, $b(\cdot,\cdot)$ and admissible functional spaces U are defined by

$$a(u,v) = \int_\Omega C_{ijkl} u_{k,l} v_{i,j}\, dx \tag{2}$$

$$b(u,v) = \int_\Omega \rho u_i v_i\, dx \tag{3}$$

$$U = \left\{ v \in \left(H^1(\Omega)\right)^n \middle| v|_{\Gamma_0} = 0,\ \Gamma_0 \subset \Gamma \right\} \tag{4}$$

The symbols $(C_{ijkl})_{i,j,k,l=1,2,\cdots,n}$ and ρ are the Hooke stiffness and density respectively. In this paper, a partial differential notation with suffix $(\cdot)_{,i} = \partial(\cdot)/\partial x_i$ and the summation convention are used. The symbol $H^m(\Omega)$ denotes Sobolev space of m-times derivative and square integrable functions defined in Ω.

Domain perturbation of Ω can be represented by using a one-parameter family of one-to-one mappings

$$T_s \ :\ \Omega \ni X \mapsto x \in \Omega_s \subset \mathbb{R}^n \quad 0 \le s < \epsilon \tag{5}$$

$$T_s^{-1} \ :\ \Omega_s \ni x \mapsto X \in \Omega \tag{6}$$

where ϵ is a small positive number. The derivative of T_s with respect to s defined by

$$V(x) = \frac{\partial T_s}{\partial s}(T_s^{-1}(x)) \quad x \in \Omega_s \tag{7}$$

is called velocity.

Using the definitions above, the shape optimization problem with prescribed r-th natural vibration mode under domain measure constraint not more than M_0 can be formulated as minimization problem of a squared error integral $E(\alpha u_r - \bar{u}_r, \alpha u_r - \bar{u}_r)$ of natural vibration mode u_r from prescribed mode $\bar{u}_r$ on specified subboundary $\Gamma_D \subset \Gamma$ for all $\alpha \in \mathbb{R}$

$$\min_{\Omega \subset \mathbb{R}^n} E(\alpha u_r - \bar{u}_r, \alpha u_r - \bar{u}_r) \quad u_r \in U \quad \forall \alpha \in \mathbb{R}$$
$$\text{such that Eq. (1) and } \int_\Omega dx \le M_0 \tag{8}$$

where the bilinear form $E(\cdot,\cdot)$ is defined by

$$E(u,v) = \int_{\Gamma_D} u_i v_i\, dx \tag{9}$$

3. Shape gradient function

Applying the Lagrange multiplier method, or the adjoint variable method, to the optimization problem by Eq. (8), the Lagrange functional $L(u_r, v, \Lambda)$ is defined by

$$L = E(\alpha u_r - \bar{u}_r, \alpha u_r - \bar{u}_r) - a(u_r, v) + \lambda_r b(u_r, v) + \Lambda\left(\int_\Omega dx - M_0\right) \tag{10}$$

where $v \in U$ and Λ were introduced as the Lagrange multiplier function, or the adjoint function, with respect to the weak form and the Lagrange multiplier with respect to the domain measure constraint.

For the sake of simplicity, let the coefficient functions $\{C_{ijkl}\}_{i,j,k,l=1,2,\cdots,n}$ and ρ be fixed in $\mathbb{R}^n$ and the subboundary $\Gamma_0 \cup \Gamma_D$ is invariable during domain perturbations. Using the formulae of the material derivative, the shape derivative of the Lagrange functional is obtained by

$$\begin{aligned} \dot{L} =& 2\alpha E(\alpha u_r - \bar{u}_r, u_r') + 2\alpha' E(\alpha u_r - \bar{u}_r, u_r) \\ &- a(u_r', v) - a(u_r, v') + \lambda_r b(u_r', v) + \lambda_r b(u_r, v') \\ &+ \dot{\lambda}_r b(u_r, v) + \dot{\Lambda}\left(\int_\Omega dx - M_0\right) + \langle G\nu, V \rangle \end{aligned} \tag{11}$$

where the linear form $\langle G\nu, V\rangle$ with respect to the velocity V is defined by

$$\langle G\nu, V\rangle = \int_\Gamma G\nu_i V_i \, \mathrm{d}\Gamma \tag{12}$$

$$G = -\, C_{ijkl} u_{rk,l} v_{i,j} + \rho \lambda_r u_{ri} v_i + \Lambda \tag{13}$$

The notation $(\,\cdot\,)'$ represents the derivative with respect to domain perturbation of function fixed in the spatial coordinates. The symbol ν denotes the outer normal vector.

Considering the stationary conditions for all $u_r' \in U$, $v' \in U$, $\alpha' \in \mathbb{R}$ and $\dot{\Lambda} \in \mathbb{R}$, the Kuhn-Tucker conditons with repsect to u_r, v, α and Λ are obtained by

$$a(u_r, v') = \lambda_r b(u_r, v') \quad \forall v' \in U \tag{14}$$

$$a(v, u_r') = \lambda_r b(v, u_r') + 2\alpha E(\alpha u_r - \bar{u}_r, u_r') \quad \forall u_r' \in U \tag{15}$$

$$\alpha E(u_r, u_r) = E(\bar{u}_r, u_r) \tag{16}$$

$$\Lambda \geq 0, \quad \int_\Omega \mathrm{d}x \leq M_0, \quad \Lambda\left(\int_\Omega \mathrm{d}x - M_0\right) = 0 \tag{17}$$

Equation (14) represents the weak form by Eq.(1), so that Eq.(14) is satisfied by solving u_r with regular method. Equation (15) gives the governing equation of the adjoint mode $v \in U$, so it is called the adjoint equation. Solution to the adjoint equation will be presented in the next section. Equation (16) determines the modal scaling factor α. Inequality equations in Eq. (17) can be satisfied by increasing the Lagrange multiplier Λ when the problem is well-suited.

When u_r, v, α and Λ are determined as explained above, the derivative of the Lagrange functional agrees with that of the objective functional and the linear form $\langle G\nu, V\rangle$ with respect to V:

$$\dot{L}\Big|_{u_r, v, \alpha, \Lambda} = \langle G\nu, V\rangle \tag{18}$$

From the fact that the function $G\nu$ is a coefficient function with respect to velocity V that is the derivation of the design function T_s, $G\nu$ indicates a sensitivity function, which we call the shape gradient function, of this problem. The function G is called the shape gradient density function.

4. Solution for adjoint equation

Let us focus on the solution of Eq.(15). To avoid lack of uniqueness when the adjoint mode $v \in U$ includes the component of u_r, the admissible set for v must be $\{v \in U |\ b(u_r, v) = 0\}$.[1],[2] Using the Ritz vector representation, v can be approximated by

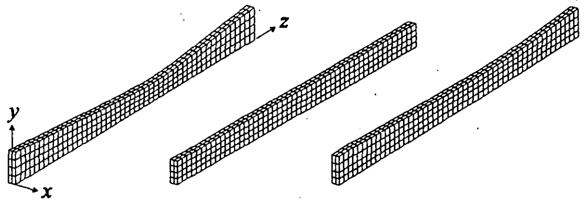
$$v = \sum_{p=1,2,\cdots,N,\ p\neq r} u_p \xi_p \tag{19}$$

Substituting Eq.(19) for Eq.(15), assuming $u_r' = u_p$ and considering p-th natural vibration equations, the adjoint modal variables $\{\xi_p\}_{p=1,2,\cdots,N,\ p\neq r}$ are calculated by

$$\xi_p = \frac{2\alpha E(\alpha u_r - \bar{u}_r, u_p)}{(\lambda_p - \lambda_r) b(u_p, u_p)} \tag{20}$$

Therefore, the adjoint mode v is evaluated with natural eigenvalues $\{\lambda_r\}_{p=1,2,\cdots,N,\ p\neq r}$ and natural vibration modes $\{\xi_p\}_{p=1,2,\cdots,N,\ p\neq r}$ by

$$v = \sum_{p=1,2,\cdots,N,\ p\neq r} \frac{2\alpha E(\alpha u_r - \bar{u}_r, u_p)}{(\lambda_p - \lambda_r) b(u_p, u_p)} u_p \tag{21}$$



(a) Reference (b) Initial (c) Optimized

Fig. 1 Shape optimization with prescribed first natural vibration mode of beam like continuum

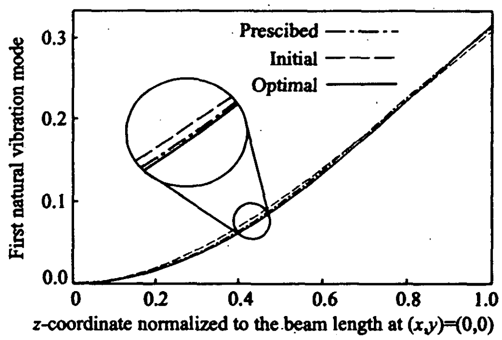


Fig. 2 Comparisons of first natural vibration modes

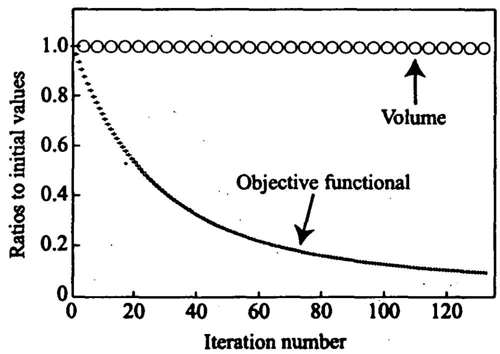


Fig. 3 Iteration history

5. Numerical analysis

To confirm the validity of the proposed method, numerical analysis of beam-like three-dimensional continuum clamped at one end was conducted using the traction method and the shape gradient function derived in the previous section. Figure 1 illustrates the shapes of a beam of which the first natural vibration mode on the bottom plane is used as the prescribed mode, initial beam and optimized beam. The bottom plane was completely fixed and the side planes were fixed in the normal direction during domain perturbations. Comparisons of the first natural vibration modes and the iteration history were shown in Figs. 2 and 3 respectively. The iteration history of the objective functional that successfully decreased demonstrates the validity of the presented theory.

References

(1) Haug, E. J., Choi, K. K. and Komkov, V. : *Design Sensitivity Analysis of Structural Systems*, Academic Press (1986).

(2) Nelson, R. B : *AIAA Journal*, Vol. 14 , 1201-1205 (1976).