# LECTURES ON RECONSTRUCTION ALGEBRAS I

MICHAEL WEMYSS

## 1. Introduction

Noncommutative algebra (=quivers) can be used to solve both explicit and non-explicit problems in algebraic geometry, and these lectures will try to explain some of the features of both approaches. I want to use these notes to give a gentle (!) introduction to the subject, and will try and make them as self-contained as possible. Since I want to eventually end up doing non-toric geometry, throughout I shall never adopt the language of toric geometry, even if the example I am considering is toric. First some motivation:

From a noncommutative perspective we would like to take a singularity $X = \text{Spec } R$ and produce a NC ring $A$ from which we can extract resolution(s) of $X$. We can then ask whether the NC ring has some geometrical meaning, and if so whether this gives information about $A$. We can also ask what $A$ says about $X$ and its resolutions.

From a more geometric perspective we may already have some resolution $Y$ of $X$ and would like produce other resolutions, for example by flopping certain curves. We may also want to describe the derived category of $Y$. This can sometimes be done using noncommutative algebra.

In practice however things are not quite as simple as this, since most of the time a specific problem will be a mixture of the two above problems. Sometimes it is easier to solve the problem using the geometry, sometimes it is easier using quivers. Thus geometry can give us results in noncommutative algebra and noncommutative algebra can give us results in geometry; it is the process of playing the two sides off each other which gives us the strongest results.

Today I'm going to define quivers and tell you how to think of them, then following King [King1] I'll talk about their moduli space(s) of finite dimensional representations. Time permitting I'll then show how to calculate the moduli spaces in some easy examples.

## 2. Quivers and Representations

Any algebra with a finite number of generators and a finite number of relations (i.e. almost all algebras you can think off) can be written as a quiver with relations[1]. You want to do this since the quiver gives you a way to visualize the algebra, and more importantly it gives you a way to visualize the finite dimensional modules (see later).

**Definition 2.1.** *A quiver $Q$ is just a finite directed graph.*

At this stage loops, double arrows,... are all allowed, and the directed graph need not be connected. For example

$$Q = \begin{tikzcd} \bullet \arrow[r, shift left] & \bullet \arrow[l, shift left] \arrow[loop above] \arrow[r] & \bullet & \bullet & \bullet \arrow[l, shift left] \arrow[l, shift right] \arrow[loop right] \end{tikzcd}$$

is an example of a quiver. A small technical point: for every vertex $i$ we actually also add in a trivial loop at that vertex and denote it by $e_i$, but we do not draw these loops. In the above example, the loops drawn are the non-trivial loops.

Denoting the vertices of $Q$ by $Q_0$ and the arrows by $Q_1$, you can view the directed graph $Q$ as simply a piece of combinatorial data $(Q_0, Q_1, h, t)$ where $h$ and $t$ are maps $Q_1 \to Q_0$. The map $h$ (the 'head') assigns to an arrow its head, and the map $t$ (the 'tail') assigns to an arrow its tail.

[1] This *cannot* be done in a unique way

**Definition 2.2.** *A non-trivial path of length $n$ in $Q$ is just a sequence of arrows $a_1 \cdots a_n$ in $Q$ with $h(a_i) = t(a_{i+1})$ for all $1 \leq i \leq n-1$. We call this path a cycle if $h(a_n) = t(a_1)$.*

We want to add more structure to the combinatorial data of a quiver by producing an algebra:

**Definition 2.3.** *For a given quiver $Q$, the path algebra $kQ$ is defined to be the $k$-algebra with basis given by the paths, with multiplication*

$$pq := \begin{cases} pq & h(p) = t(q) \\ 0 & \text{else} \end{cases} \qquad e_i p := \begin{cases} p & t(p) = i \\ 0 & \text{else} \end{cases} \qquad pe_i := \begin{cases} p & h(p) = i \\ 0 & \text{else} \end{cases}$$

*for any paths $p$ and $q$.*

This is an algebra, with identity $1_{kQ} = \sum_{i \in Q_0} e_i$. Note we are using the convention that $pq$ means $p$ then $q$; be aware that some savage barbarians[2] use the opposite convention. Note that by the definition of multiplication the path algebra is often noncommutative: for example if

$$Q = \bullet \xrightarrow{a} \bullet \xrightarrow{b} \bullet$$

then $ab \neq ba$ since $ba = 0$. In fact in this example $kQ$ is easy to describe: the basis of $kQ$ is $e_1$, $e_2$, $e_3$, $a$, $b$, $ab$. Its not hard to convince yourself that

$$kQ \cong \begin{pmatrix} k & k & k \\ 0 & k & k \\ 0 & 0 & k \end{pmatrix}.$$

**Exercise 2.4.** Let $Q$ be a quiver, then $kQ$ is finite dimensional if and only if $Q$ has no non-trivial cycles.

For quivers $Q$ without cycles, the resulting path algebras $kQ$ have been used in geometry although their use is generally limited to projective varieties; since in these talks we are going to be resolving singularities we need to make one more definition:

**Definition 2.5.** *For a given quiver $Q$, a relation is simply a $k$-linear combination of paths in $Q$. Given a finite number of relations, we can form their two sided ideal $R$ in the path algebra, and we thus define the algebra $kQ/R$ to be a quiver with relations.*

We can assume (by removing arrows if necessary) that the length of every path in every relation is greater than or equal to two. Note that with relations it is possible that $kQ/R$ can be finite dimensional even when $Q$ has cycles, though in these lectures most of the examples will involve infinite dimensional algebras.

In practice you should think of the relation $p - q$ as saying 'going along path $p$ is the same as going along path $q$', since $p = q$ in the quotient $kQ/R$.

Now as is standard in ring theory (and geometry), we tend to study a ring by instead studying its module category (=coherent sheaves), since this is an abelian category and so we have the machinery of homological algebra at our disposal. Representation theorists would tell us that we are we're studying the ring's representations - I'll now make this more precise.

**Definition 2.6.** *Let $kQ/R$ be a quiver with relations. A finite dimensional representation of $kQ/R$ is the assignment to every vertex $i$ of $Q$ a finite dimensional vector space $V_i$, and to every arrow $a$ a linear map $f_a : V_{t(a)} \to V_{h(a)}$, such that the relations $R$ between the linear maps hold. Denote $\alpha_i = \dim V_i$ and let $\alpha = (\alpha_i)$ be the collection of all the $\alpha_i$. We call $\alpha$ the dimension vector of the representation.*

[2] you know who you are

For example let $kQ/R$ be $\begin{tikzcd} \bullet \arrow[r, "a"] & \bullet \arrow[r, "b"] & \bullet \arrow[ll, bend left, "c"] \end{tikzcd}$ subject to $bc = 0$. Denoting

$$M := \begin{tikzcd} \mathbb{C} \arrow[r, "4"] & \mathbb{C} \arrow[r, "(1\ 0)"] & \mathbb{C}^2 \arrow[ll, bend left, "\binom{0}{1}"] \end{tikzcd} \qquad N := \begin{tikzcd} \mathbb{C} \arrow[r, "1"] & \mathbb{C} \arrow[r, "1"] & \mathbb{C} \arrow[ll, bend left, "3"] \end{tikzcd}$$

then $M$ is a representation of dimension vector $(1, 1, 2)$ whereas $N$ is *not* a representation of dimension vector $(1, 1, 1)$.

We also have the obvious notion of a morphism between two representations:

**Definition 2.7.** *Let $V = (V_i, f_a)$ and $W = (W_i, g_a)$ be finite dimensional representations of $kQ/R$. A morphism $\psi$ from $V$ to $W$ is given by specifying, for every vertex $i$, a linear map $\psi_i : V_i \to W_i$ such that for every arrow $a \in Q_1$,*

$$\begin{tikzcd} V_{t(a)} \arrow[r, "\psi_{t(a)}"] \arrow[d, "f_a"] & W_{t(a)} \arrow[d, "g_a"] \\ V_{h(a)} \arrow[r, "\psi_{h(a)}"] & W_{h(a)} \end{tikzcd}$$

*commutes.*

Note $\psi$ is an isomorphism if and only if each $\psi_i$ is a linear isomorphism. Also note we have the obvious notion of a subrepresentation. It is fairly clear that in this way the finite dimensional representations form a category, which we denote by $\texttt{fRep}(kQ, R)$

The whole point to all this is the following:

**Lemma 2.8.** *Let $A = kQ/R$ be a quiver with relations. Denote by $\texttt{fdmod}A$ the finite dimensional modules of $A$. Then there is a categorical equivalence*

$$\texttt{fRep}(kQ, R) \approx \texttt{fdmod}A$$

*Proof.* This is actually quite tautological. Given a representation $(V_i, f_a)$ then $\oplus_{i \in Q_0} V_i$ is the corresponding module. Conversely given any finite dimensional module $W$, setting $W_i = e_i W$ (where $e_i$ is the trivial path at vertex $i$) gives us the corresponding representation. $\square$

Thus we now see the benefit of writing an algebra $A$ as a quiver with relations, as by the above lemma we have a way to visualize the finite dimensional modules of $A$.

## 3. Moduli and GIT

In this section we consider a quiver with relations $A = kQ/R$ and define various moduli spaces of finite dimensional representations. In the process we have to take a very fast detour through the world of geometric invariant theory (GIT).

For a fixed dimension vector $\alpha$ we may consider all representations of $A = kQ/R$ with dimension vector $\alpha$:

$$\mathscr{R} := \mathrm{Rep}(A, \alpha) = \{\text{representations of } A \text{ of dimension } \alpha\}$$

This is an affine variety, so denote the co-ordinate ring by $k[\mathscr{R}]$. The variety (hence the co-ordinate ring) carries a natural action of $G := \prod_{i \in Q_0} \mathrm{GL}(\alpha_i)$ acting on an arrow $a$ as $g \cdot a = g_{t(a)}^{-1} a g_{h(a)}$. Actually its really an action of PGL since the diagonal one-parameter subgroup $\Delta = \{(\lambda 1, \cdots, \lambda 1) : \lambda \in k^*\}$ acts trivially, but this won't concern us much. Anyway, by linear algebra the *isomorphism classes* of representations of $A = kQ/R$ are in natural one-to-one correspondence with the orbits of this action.

To understand this space is normally an impossible problem (e.g. wild quiver type), so we want to throw away some representations and take what is known as a GIT quotient.

To make a GIT quotient we need to add the extra data of a character $\chi$ of $G$. Now the characters $\chi$ of $G = \prod_{i\in Q_0} \mathrm{GL}(\alpha_i)$ are given by powers of the determinants

$$\chi(g) = \prod_{i\in Q_0} det(g_i)^{\theta_i}$$

for some collection of integers $\theta_i \in \mathbb{Z}^{Q_0}$. Since such a $\chi$ determines and is determined by the $\theta_i$, we usually denote $\chi$ by $\chi_\theta$. Now consider the map

$$\begin{array}{rcl} \theta : \mathtt{fdmod}A & \to & \mathbb{Z} \\ M & \mapsto & \sum_{i\in Q_0} \theta_i \dim M_i \end{array}$$

This is additive on short exact sequences, so really its a map $K_0(\mathtt{fdmod}A) \to \mathbb{Z}$.

Now assume that our character satisfies $\chi_\theta(\Delta) = \{1\}$ (this is need to use Mumford's numerical criterion [King, 2.5]). It not too hard to see that this condition translates into $\sum_{i\in Q_0} \theta_i\alpha_i = 0$. Hence for these $\chi_\theta$, $\theta(M) = 0$ if $M$ has dimension vector $\alpha$.

We arrive at the key definition [King,1.1]

**Definition 3.1.** *Let $\mathscr{A}$ be an abelian category, and $\theta : K_0(\mathscr{A}) \to \mathbb{Z}$ an additive function. We call $\theta$ a character of $\mathscr{A}$. An object $M \in \mathscr{A}$ is called $\theta$-semistable if $\theta(M) = 0$ and every subobject $M' \subseteq M$ satisfies $\theta(M') \geq 0$. Such an object $M$ is called $\theta$-stable if the only subobjects $M'$ with $\theta(M') = 0$ are $M$ and $0$. We call $\theta$ generic if every $M$ which is $\theta$-semistable is actually $\theta$-stable.*

For $A = kQ/R$ as before, we are interested in the above definition for the case $\mathscr{A} = \mathtt{fdmod}A$. We shall see how this works in practice in the next section. The reason King gave the above definition is that it is equivalent to the other notion of stability from GIT, which we now describe:

$\mathscr{R}$ is an affine variety with an action of a linearly reductive group $G = \prod_{i\in Q_0} \mathrm{GL}(\alpha_i)$. Since $G$ is reductive, we have a quotient

$$\mathscr{R} \to \mathscr{R}/\!\!/G = \mathrm{Spec}k[\mathscr{R}]^G$$

which is dual to the inclusion $k[\mathscr{R}]^G \to k[\mathscr{R}]$. Its the reductiveness of the group which ensures that $k[\mathscr{R}]^G$ is a finitely generated $k$-algebra, and so $\mathrm{Spec}k[\mathscr{R}]^G$ is really a variety, not just a scheme. Virtually by definition the above is a categorical quotient (quite a weak condition); further its actually a good quotient (if you don't know what this means, don't worry)

To make a GIT quotient we have to add to this picture the extra data of $\chi$, some character of $G$.

**Definition 3.2.** *$f \in k[\mathscr{R}]$ is a semi-invariant of weight $\chi$ if $f(g\cdot x) = \chi(g)f(x)$ for all $g \in G$ and all $x \in \mathscr{R}$. We write the set of such $f$ as $\mathscr{R}^{G,\chi}$. We define*

$$\mathscr{R}/\!\!/_{\chi}G := \mathrm{Proj}\left(\bigoplus_{n\geq 0} k[\mathscr{R}]^{G,\chi^n}\right)$$

**Definition 3.3.** *$x \in \mathscr{R}$ is called $\chi$-semistable (in the sense of GIT) if there exists some semi-invariant $f$ of weight $\chi^n$ with $n > 0$ such that $f(x) \neq 0$, otherwise $x \in \mathscr{R}$ is called unstable.*

The set of semistable points $\mathscr{R}^{ss}$ forms an open subset of $\mathscr{R}$; in fact we have a morphism

$$q : \mathscr{R}^{ss} \to \mathscr{R}/\!\!/_{\chi}G$$

which is a good quotient. One more definition:

**Definition 3.4.** *$x \in \mathscr{R}$ is called $\chi$-stable (in the sense of GIT) if it is $\chi$-semistable, the $G$ orbit containing $x$ is closed in $\mathscr{R}^{ss}$ and further the stabilizer of $x$ is finite.*

In fact $q$ is a geometric quotient on the stable locus $\mathscr{R}^s$, meaning that $\mathscr{R}^s /\!\!/_{\chi} G$ really is an orbit space.

The point in the above discussion is the following result [King1, 3.1], which says the two notions are the same

**Proposition 3.5.** *Let $M \in \mathrm{Rep}(A, \alpha) = \mathscr{R}$, choose $\theta$ as in Definition 3.1. Then $M$ is $\theta$-semistable (in the sense of Definition 3.1) if and only if $M$ is $\chi_\theta$-semistable (in the sense of GIT). The same holds replacing semistability with stability.*

Thus we use the machinery from the GIT side to define for quivers the following:

**Definition 3.6.** *For $A = kQ/R$ choose dimension vector $\alpha$ and character $\theta$ satisfying $\sum_{i \in Q_0} \alpha_i \theta_i = 0$. Denote $\mathrm{Rep}(A, \alpha) = \mathscr{R}$ and $G = \mathrm{GL}(\alpha)$. We define*

$$\mathfrak{M}^{ss}_\theta(A, \alpha) := \mathscr{R} /\!\!/_{\chi_\theta} G := \mathrm{Proj}\left( \bigoplus_{n \geq 0} k[\mathscr{R}]^{G, \chi^n} \right)$$

*and call it the moduli space of $\theta$-semistable representations of dimension vector $\alpha$.*

This is by definition projective over the ordinary quotient $\mathscr{R} /\!\!/ G = \mathrm{Spec} k[\mathscr{R}]^G$. We make some remarks

(i) If $k[\mathscr{R}]^G = k$ then $\mathfrak{M}^{ss}_\theta(A, \alpha)$ is a projective variety.
(ii) In the resolution of singularities we ideally would like the zeroth piece $\mathrm{Spec} k[\mathscr{R}]^G$ to be the singularity since then the moduli space is projective over it! However, even in cases where we use NC rings to resolve singularities, $\mathrm{Spec} k[\mathscr{R}]^G$ might not be the thing we want; see Example 4.6 later.
(iii) Note that $\mathfrak{M}^{ss}_\theta(A, \alpha)$ may be empty.
(iv) One way to compute this space is to compute semi-invariants, but this in general is quite hard.

One small point before we continue: we can't just call $\mathfrak{M}^{ss}_\theta(A, \alpha)$ a moduli space, we really have to justify that it *is* a moduli space, i.e. why it parameterizes certain objects. We shall describe this more precisely in a future section. For now though we shall concern ourselves with showing how to calculate the moduli space in some examples:

## 4. Examples

The last section was quite abstract, here we show how it works in practice. For $A = kQ/R$, we may want to construct a space $X$ from $A$ as a moduli space of $\theta$-stable $A$-modules. What this means [King2]:

> "To specify such a moduli space we must give a dimension vector $\alpha$ and a weight vector (or 'character') $\theta$ satisfying $\sum_{i \in Q_0} \theta_i \alpha_i = 0$. The moduli space of $\theta$-stable $A$-modules of dimension vector $\alpha$ is then the parameter space for those $A$-modules which have no proper submodules with any dimension vector $\beta$ for which $\sum_{i \in Q_0} \theta_i \beta_i \leq 0$."

For computational ease I will only compute moduli with dimension vector $(1, \ldots, 1)$ in this section; I will return and do a computation of some other dimension vectors in a future section. There are many different (and better) ways to view the following example, but here I give the easiest:

**Example 4.1.** Consider the quiver

$$\bullet \rightrightarrows \bullet$$

with no relations. Choose $\alpha = (1, 1)$ and $\theta = (-1, 1)$. With these choices, since $\sum \theta_i \alpha_i = 0$ we can form the moduli space. Now a representation of dimension vector $\alpha = (1, 1)$ is $\theta$-semistable by definition if $\theta(M') \geq 0$ for all subobjects $M'$. But the only possible subobjects in this example are of dimension vector $(0, 0)$, $(0, 1)$ and $(1, 0)$, and $\theta$ is $\geq 0$ on all but the last (in fact its easy to see that $\theta$ is generic in this example). Thus a representation of dimension vector $(1,1)$ is $\theta$-semistable if and only if it has no submodules of dimension vector $(1,0)$. Now take an arbitrary representation $M$ of dimension vector $(1,1)$

$$M = \mathbb{C} \underset{b}{\overset{a}{\rightrightarrows}} \mathbb{C}\ .$$

Notice that $M$ has a submodule of dimension vector $(1,0)$ if and only if $a = b = 0$, since the diagram

$$\begin{tikzcd} \mathbb{C} \arrow[r, "\cong"] \arrow[d, shift right, "0"'] \arrow[d, shift left, "0"] & \mathbb{C} \arrow[d, shift right, "b"'] \arrow[d, shift left, "a"] \\ 0 \arrow[r, "0"] & \mathbb{C} \end{tikzcd}$$

must commute. Thus by our choice of stability $\theta$,

$M$ is $\theta$-semistable $\iff$ $M$ has no submodule of dim vector $(1,0)$ $\iff$ $a \neq 0$ or $b \neq 0$.

and so we see that the semistable objects parametrize $\mathbb{P}^1$ via the ratio $(a:b)$, so the moduli space is just $\mathbb{P}^1$. Another way to see this: we have two open sets, one corresponding to $a \neq 0$ and the other to $b \neq 0$. After changing basis we can set them to be the identity, and so we have

$$U_0 = \{ \ \mathbb{C} \underset{b}{\overset{1}{\rightrightarrows}} \mathbb{C} \quad : b \in \mathbb{C}\} \quad U_1 = \{ \ \mathbb{C} \underset{1}{\overset{a}{\rightrightarrows}} \mathbb{C} \quad : a \in \mathbb{C}\}$$

Now the gluing is given by, whenever $U_0 \ni b \neq 0$

$$U_0 \ni b = \quad \mathbb{C} \underset{b}{\overset{1}{\rightrightarrows}} \mathbb{C} \quad = \quad \mathbb{C} \underset{1}{\overset{b^{-1}}{\rightrightarrows}} \mathbb{C} \quad = b^{-1} \in U_1$$

which is evidently just $\mathbb{P}^1$.

This lecture series is devoted to resolving singularities, so we warm up by blowing up the origin in $\mathbb{C}^2$:

**Example 4.2.** Consider the quiver with relations

$$\bullet \ \underset{\xleftarrow{t}}{\overset{a,\,b}{\rightrightarrows}} \ \bullet \qquad atb = bta$$

and again choose dimension vector $(1,1)$ and stability $\theta_0 = (-1,1)$. Exactly as above if

$$M = \quad \mathbb{C} \ \underset{\xleftarrow{t}}{\overset{a,\,b}{\rightrightarrows}} \ \mathbb{C}$$

then

$M$ is $\theta$-semistable $\iff$ $M$ has no submodule of dim vector $(1,0)$ $\iff$ $a \neq 0$ or $b \neq 0$.

For the first open set in the moduli $U_0$ (when $a \neq 0$): after changing basis so that $a = 1$ we see that the open set is parameterized by the two scalars $b$ and $t$ subject to the single relation (substituting $a = 1$ into the quiver relations) $tb = bt$. But this always holds so it isn't really a relation, thus the open set $U_0$ is just $\mathbb{C}^2$ with co-ordinates $b, t$. We write this as $\mathbb{C}^2_{b,t}$. Similarly for the other open set:

$$\begin{array}{cc} \mathbb{C} \ \underset{\xleftarrow{t}}{\overset{1,\,b}{\rightrightarrows}} \ \mathbb{C} & \qquad \mathbb{C} \ \underset{\xleftarrow{t}}{\overset{a,\,1}{\rightrightarrows}} \ \mathbb{C} \\ & \\ U_0 = \mathbb{C}^2_{b,t} & \qquad U_1 = \mathbb{C}^2_{a,t}. \end{array}$$

Now the gluing is given by, whenever $b \neq 0$

$$U_0 \ni (b,t) = \quad \mathbb{C} \ \underset{\xleftarrow{t}}{\overset{1,\,b}{\rightrightarrows}} \ \mathbb{C} \quad = \quad \mathbb{C} \ \underset{\xleftarrow{bt}}{\overset{b^{-1},\,1}{\rightrightarrows}} \ \mathbb{C} \quad = (b^{-1}, bt) \in U_1$$

and so we see that this is just the blowup of the origin of $\mathbb{C}^2$.

**Exercise 4.3.** What does the stability $\theta_1 = (1,-1)$ give us in the above example?

**Example 4.4.** Consider the group $\frac{1}{3}(1,1) := \langle \left(\begin{smallmatrix} \varepsilon_3 & 0 \\ 0 & \varepsilon_3 \end{smallmatrix}\right) \rangle$ where $\varepsilon_3$ is a primitive third root of unity. This acts on $\mathbb{C}^2$ giving us a quotient singularity $\mathbb{C}[x,y]^{\frac{1}{3}(1,1)}$. Consider the quiver with relations (the reconstruction algebra)

$$\bullet \ \underset{\xleftarrow{a_1,\, a_2,\, k_1}}{\overset{c_1,\, c_2}{\rightrightarrows}} \ \bullet \qquad \begin{array}{ll} c_1a_2 = c_2a_1 & a_2c_1 = a_1c_2 \\ c_1k_1 = c_2a_2 & k_1c_1 = a_2c_2 \end{array}$$

Choose dimension vector (1,1). We are going to calculate the moduli space for stability $\theta_0 = (-1, 1)$, then calculate the moduli space for stability $\theta_1 = (1, -1)$.
(i) Take $\theta_0 = (-1, 1)$. As in the examples above, for a module

$$M = \quad \mathbb{C} \begin{array}{c} \xrightarrow{c_1} \\ \xrightarrow{c_2} \\ \xleftarrow{a_1} \\ \xleftarrow{a_2} \\ \xleftarrow{k_1} \end{array} \mathbb{C}$$

to be semistable requires $c_1 \neq 0$ or $c_2 \neq 0$ and so we have two open sets $U_0 = (c_1 \neq 0)$ and $U_1 = (c_2 \neq 0)$. Changing basis so that these are 1, by the relations we have

$$\begin{array}{cc} \mathbb{C} \begin{array}{c} \xrightarrow{1} \\ \xrightarrow{c_2} \\ \xleftarrow{a_1} \\ \xleftarrow{c_2 a_1} \\ \xleftarrow{c_2^2 a_1} \end{array} \mathbb{C} & \qquad \mathbb{C} \begin{array}{c} \xrightarrow{c_1} \\ \xrightarrow{1} \\ \xleftarrow{c_1^2 k_1} \\ \xleftarrow{c_1 k_1} \\ \xleftarrow{k_1} \end{array} \mathbb{C} \\ \\ U_0 = \mathbb{C}^2_{c_2, a_1} & \qquad U_1 = \mathbb{C}^2_{c_1, k_1}. \end{array}$$

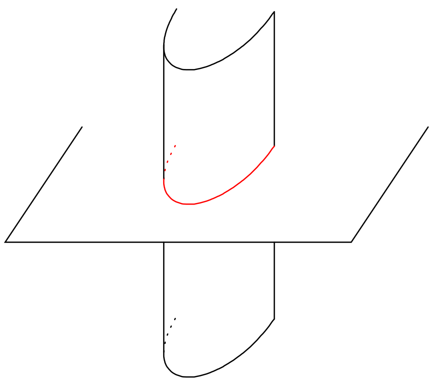
Now the gluing is given by, whenever $c_2 \neq 0$

$$U_0 \ni (c_2, a_1) = \quad \mathbb{C} \begin{array}{c} \xrightarrow{1} \\ \xrightarrow{c_2} \\ \xleftarrow{a_1} \\ \xleftarrow{c_2 a_1} \\ \xleftarrow{c_2^2 a_1} \end{array} \mathbb{C} \quad = \quad \mathbb{C} \begin{array}{c} \xrightarrow{c_2^{-1}} \\ \xrightarrow{1} \\ \xleftarrow{c_2 a_1} \\ \xleftarrow{c_2^2 a_1} \\ \xleftarrow{c_2^3 a_1} \end{array} \mathbb{C} \quad = (c_2^{-1}, c_2^3 a_1) \in U_1$$

since we read off the co-ordinates in $U_1$ in the $c_1$ and $k_1$ positions. Thus by inspection we see that our space is $\mathscr{O}_{\mathbb{P}^1}(-3)$, the minimal resolution.
(ii) Take $\theta_1 = (1, -1)$. Its clear that we now have 3 open sets $U_0 = (a_1 \neq 0)$, $U_1 = (a_2 \neq 0)$, $U_2 = (k_1 \neq 0)$. Consider first $U_0$: after changing basis so that $a_1 = 1$, we have

$$\mathbb{C} \begin{array}{c} \xrightarrow{c_1} \\ \xrightarrow{c_2} \\ \xleftarrow{1} \\ \xleftarrow{a_2} \\ \xleftarrow{k_1} \end{array} \mathbb{C} \qquad \begin{array}{cc} c_1 a_2 = c_2 & a_2 c_1 = c_2 \\ c_1 k_1 = c_2 a_2 & k_1 c_1 = a_2 c_2 \end{array}$$

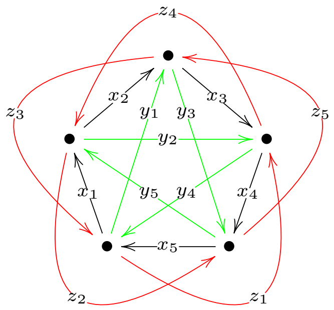
which is parameterized by the three variables $c_1, a_2, k_1$ subject to the one relation $c_1 k_1 = c_1 a_2^2$ i.e. $c_1(k_1 - a_2^2) = 0$. This is singular in dimension 1! If we draw $U_0$, it looks something like



It has two components, namely the $c_1 = 0$ component and the $k_1 = a_2^2$ component. The $k_1 = a_2^2$ component is the one that we want, since it ends up giving us (part of) the minimal resolution.

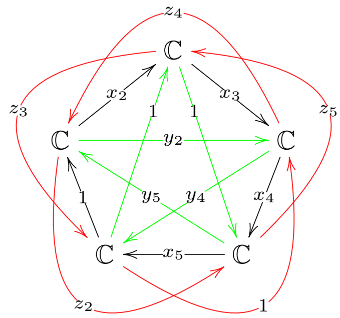
From the above example we see that a moduli space may not be smooth and might have components. Note that in the above example there is one component which is particularly nice, however the next example shows that a moduli space may be both irreducible and singular.

**Example 4.5.** Consider the group $\frac{1}{5}(1,2,3) := \left\langle \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & \varepsilon^2 & 0 \\ 0 & 0 & \varepsilon^3 \end{pmatrix} : \varepsilon^5 = 1 \right\rangle$ giving a three dimensional quotient singularity. The algebra to consider is




$$\begin{array}{lll} x_1y_2 = y_1x_3 & x_1z_2 = z_1x_4 & y_1z_3 = z_1y_4 \\ x_2y_3 = y_2x_4 & x_2z_3 = z_2x_5 & y_2z_4 = z_2y_5 \\ x_3y_4 = y_3x_5 & x_3z_4 = z_3x_1 & y_3z_5 = z_3y_1 \\ x_4y_5 = y_4x_1 & x_4z_5 = z_4x_2 & y_4z_1 = z_4y_2 \\ x_5y_1 = y_5x_2 & x_5z_1 = z_5x_3 & y_5z_2 = z_5y_3 \end{array}$$

Consider $\alpha = (1,1,1,1,1)$ with stability $\theta = (-4,1,1,1,1)$. Consider the open set given by $x_1 \neq 0$, $y_1 \neq 0$, $y_3 \neq 0$ and $z_1 \neq 0$. After changing basis so that these are the identity we have



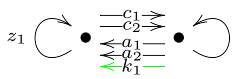

$$\begin{array}{ccc} y_2 = x_3 & z_2 = x_4 & z_3 = y_4 \\ x_2 = y_2x_4 & x_2z_3 = z_2x_5 & y_2z_4 = z_2y_5 \\ x_3y_4 = x_5 & x_3z_4 = z_3 & z_5 = z_3 \\ x_4y_5 = y_4 & x_4z_5 = z_4x_2 & y_4 = z_4y_2 \\ x_5 = y_5x_2 & x_5 = z_5x_3 & y_5z_2 = z_5 \end{array}$$

from which elimination of variables gives that this open set is parameterized by $a = y_5$, $b = x_3$, $c = z_4$ and $d = x_4$ subject to the one relation $ad = bc$. This is singular at the origin and so consequently the moduli space is singular. In fact in this example it is also irreducible.

**Example 4.6.** Consider the group $\frac{1}{3}(1,1,0)$ giving the three dimensional singularity

$$\mathbb{C}[x,y,z]^{\frac{1}{3}(1,1,0)} = \mathbb{C}[x,y]^{\frac{1}{3}(1,1)} \otimes_{\mathbb{C}} \mathbb{C}[z]$$

i.e. really just a surface crossed with $\mathbb{C}$. In this case the algebra to consider is the higher-dimensional reconstruction algebra




$$\begin{array}{llll} c_1a_2 = c_2a_1 & a_2c_1 = a_1c_2 & z_1c_1 = c_1z_2 & z_2a_1 = a_1z_1 \\ c_1k_1 = c_2a_2 & k_1c_1 = a_2c_2 & z_1c_2 = c_2z_2 & z_2a_2 = a_2z_1 \\ & & & z_2k_1 = k_1z_1 \end{array}$$

An easy calculation shows that for $\alpha = (1,1)$ and $\theta_0 = (-1,1)$ we resolve the singularity; unsurprisingly its just the minimal resolution crossed with $\mathbb{C}$. Again the same is true for $\theta_1 = (1,-1)$ but again we have to pass to components. The point in this example is that although for $\theta_0 = (-1,1)$ the moduli space is projective over $\mathbb{C}[x,y,z]^{\frac{1}{3}(1,1,0)}$, the zeroth part of the graded ring which we take the Proj of (i.e. the invariants $k[\mathscr{R}]^G$) is *not* $\mathbb{C}[x,y,z]^{\frac{1}{3}(1,1,0)}$, so a little care should be taken. The reason for this is that both $z_1$ and $z_2$ belong to $k[\mathscr{R}]^G$, and there is no relation which tells us they are the same (they are however the same as soon as $c_1 \neq 0$ or $c_2 \neq 0$). Thus $k[\mathscr{R}]^G$ has an 'extra' $z$.

One of the advantages of quivers is that they allow you to resolve singularities explictly in examples you wouldn't be able to do otherwise, especially in the case of quotients by a non-abelian group: we will illustrate this principle in more complicated examples in a future lecture.