KÄHLER GEOMETRY OF LOOP SPACES

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Contents

Foreword

This book deals with infinite-dimensional Kähler manifolds, more precisely, with three particular examples of such manifolds — loop spaces of compact Lie groups, Teichmüller spaces of complex structures on loop spaces, and Grassmannians of Hilbert spaces. There is an opinion that there could not be a comprehensive theory of Kähler manifolds in the infinite-dimensional setting. Such an opinion is based on the belief that infinite-dimensional Kähler manifolds are too rich and too different from each other so that any of them deserves its own theory. It's hard to say now whether a general theory of infinite-dimensional Kähler manifolds may or may not exist but it is certainly true that each of our three examples deserves a separate study. Any of these manifolds can be considered as a universal object in a certain category, containing all its finite-dimensional counterparts. In particular, main ingredients of Kähler geometry of these finite-dimensional spaces may be recovered from the corresponding ingredients, attached to the universal object, by restriction. Therefore, one can expect that it may be more natural and sometimes easier to study these ingredients for the universal object, rather than for its finite-dimensional counterparts. We'll give several examples of this sort in our book, and I'm sure that many more are to be found in future.

The choice of the three infinite-dimensional Kähler spaces for our study is, by no means, accidental. It is motivated by the relation of these spaces to various problems in modern mathematical physics. We do not consider these intriguing relations in our book in order to save its volume with only one exception. Since our first interest in infinite-dimensional Kähler manifolds emerged from the geometric quantization of loop spaces (related to string theory), we could not refuse ourselves in supplying the book with a second part, devoted to this subject (together with a brief survey of the geometric quantization of finite-dimensional Kähler manifolds).

My interest in the geometric quantization of infinite-dimensional phase manifolds arose from reading the papers by Bowick–Rajeev [14] and Kirillov–Yuriev [44]. (It was my colleague A.Popov from Dubna Institute of Nuclear Research, who draw my attention to these papers.) I began to study the Pressley–Segal treatise on loop spaces [65], which became my handbook on this subject and infinite-dimensional Kähler manifolds, in general. The current edition may be considered as an attempt, inspired by [65], to expose in a concise form geometric ideas, lying behind the loop space theory. It should be also mentioned here a stimulating paper by Nag–Sullivan [58], which has revealed the role of the universal Teichmüller space and the Sobolev space of half-differentiable functions on the circle for the geometric quantization of loop spaces and string theory.

Let us present now our main heros in more detail. The first one is the loop space ΩG of a compact Lie group G. It is a Kähler Frechet manifold, which can be considered as a universal flag manifold of the group G in the sense that it contains all flag manifolds of G as complex Kähler submanifolds. There is an essentially unique natural symplectic form on this manifold. On the other hand, ΩG has a lot of different complex structures, compatible with this symplectic form. The admissible complex structures on ΩG are parameterized by points of the space $\mathrm{Diff}_+(S^1)/\mathrm{Möb}(S^1)$ of orientation-preserving diffeomorphisms of the circle, normalized modulo Möbius transformations.

The space $\mathcal{S} = \mathrm{Diff}_+(S^1)/\mathrm{Möb}(S^1)$ is our second hero. It is also a Kähler Frechet manifold, which has a unique natural complex structure and a 1-parameter family of compatible symplectic forms. These forms coincide with realizations of the canonical Kirillov form on different coadjoint orbits of the Virasoro group (being a central extension of $\mathrm{Diff}_+(S^1)$), identified with $\mathcal{S}$. The space $\mathcal{S}$ can be also regarded as a "smooth" part of the universal Teichmüller space $\mathcal{T}$. This space, introduced and studied by L.Ahlfors and L.Bers, consists of quasisymmetric homeomorphisms of the circle (i.e. orientation-preserving homeomorphisms of S^1, extending to quasiconformal homeomorphisms of the disc), normalized modulo Möbius transformations. The universal Teichmüller space $\mathcal{T}$ is a complex Banach manifold, which can be provided with a natural Kähler pseudometric (which is only densely defined on $\mathcal{T}$). This pseudometric restricts to a Kähler metric on $\mathcal{S} \subset \mathcal{T}$. As it can be guessed from its name, the universal Teichmüller space $\mathcal{T}$ contains all classical Teichmüller spaces (of compact Riemann surfaces of finite genus) as complex submanifolds. Moreover, the Kähler pseudometric of $\mathcal{T}$ restrics to the Weil–Petersson Kähler metric on each of these classical Teichmüller spaces.

The group of quasisymmetric homeomorphisms of the circle acts naturally on the Sobolev space V of half-differentiable functions on the circle, preserving its natural symplectic form. This action defines an embedding of the universal Teichmüller space $\mathcal{T}$ into an infinite-dimensional Grassmannian $\mathrm{Gr}(V)$ of V. The constructed map generates also an embedding of the "smooth" part $\mathcal{S} \subset \mathcal{T}$ into a "smooth" part of $\mathrm{Gr}(V)$, represented by the Hilbert–Schmidt Grassmannian $\mathrm{Gr}_{\mathrm{HS}}(V) \subset \mathrm{Gr}(V)$. The Hilbert–Schmidt Grassmannian $\mathrm{Gr}_{\mathrm{HS}}(V)$, which is our third hero, is a Kähler Hilbert manifold. It can be considered as a universal Grassmann manifold, since all finite-dimensional Grassmannians are contained in $\mathrm{Gr}_{\mathrm{HS}}(V)$ as complex submanifolds. Moreover, the loop space ΩG can be also embedded into $\mathrm{Gr}_{\mathrm{HS}}(V)$, more precisely, into the Hilbert–Schmidt Siegel disc $\mathcal{D}_{\mathrm{HS}}$, identified with the "lower hemisphere" of $\mathrm{Gr}_{\mathrm{HS}}(V)$.

These are the three main heros of our book, which may be considered as an accessible introduction to the Kähler geometry of these remarkable spaces and a starting point to their detailed study. Basic properties of the three spaces are summarized in the table at the end of the foreword.

Briefly on the content of the book.

Book I: Kähler geometry of loop spaces. To facilitate the reading, we have collected in Part I all necessary background, which may be considered as external with respect to the main stream of the book.

We start from Chapters 1 and 2, devoted to Frechet manifolds and Frechet Lie groups. A key reference for these Chapters is a fundamental paper by Hamilton

[32], which was our main guide to Frechet manifolds.

Chapter 3 contains necessary basic facts on flag manifolds and irreducible representations of semisimple Lie groups. This is a standard material, which can be found in general books on Lie groups, Lie algebras and representation theory.

Chapter 4 is devoted to central extensions of Lie groups and algebras — the concept, crucial for the representation theory of infinite-dimensional groups and algebras. A comprehensive presentation of this subject is given in Pressley–Segal book [65]. This also applies to the next Chapter 5, where we study Grassmannians of a Hilbert space.

Chapter 6 deals with quasiconformal maps. It is a classical notion, covered in many books, in particular, in a beautiful (and short) book by Ahlfors [1].

Part II is devoted to the loop spaces ΩG of compact Lie groups G.

In Chapter 7 we describe the Kähler geometry of the loop space ΩG and a canonical embedding of flag manifolds of a Lie group G into ΩG.

In Chapter 8, devoted to the central extensions of loop groups and algebras, we follow mostly Pressley–Segal book [65]. The same applies to the next Chapter 9, where the Grassmann realization of the loop spaces is constructed.

Part III is devoted to various spaces of complex structures on loop spaces ΩG.

We start in Chapter 10 with the description of the coadjoint action of the Virasoro group and its orbits, due mainly to Kirillov. Among these orbits only two kinds admit a Kähler structure, namely, the "smooth" part $\mathcal{S} = \mathrm{Diff}_+(S^1)/\mathrm{Möb}(S^1)$ of the universal Teichmüller space $\mathcal{T}$ and the homogeneous space $\mathcal{R} = \mathrm{Diff}_+(S^1)/S^1$.

In Chapter 11 we introduce the universal Teichmüller space $\mathcal{T}$ and define a pseudoKähler structure on it, using its embedding into the complex Banach space of holomorphic quadratic differentials in the disc. The classical Teichmüller spaces $T(G)$, where G is a Fuchsian group, are identified with the subspaces of $\mathcal{T}$, consisting of G-invariant quasisymmetric homeomorphisms of S^1. The Kähler pseudometric on $\mathcal{T}$ restricts to a natural Kähler metric on the "smooth" part $\mathcal{S} \subset \mathcal{T}$ and to the Weil–Petersson metric on $T(G)$. A Grassmann realization of $\mathcal{T}$ was constructed by Nag–Sullivan in [58]. This realization agrees with a natural Grassmann realization of the "smooth" part $\mathcal{S}$.

Book II: Geometric quantization of loop spaces. Part IV is a brief introduction to the geometric quantization of finite-dimensional Kähler manifolds. More detailed presentations of this theory may be found in various books on the subject, e.g., in [29] and [70].

In Chapter 12 we define the Dirac quantization of classical systems. The Kostant–Souriau prequantization of symplectic manifolds with integral symplectic forms is constructed in Chapter 13.

Chapter 14 is devoted to the Blattner–Kostant–Sternberg (BKS) quantization. A more detailed exposition of this subject may be found in [29], [70]. We introduce Fock spaces of half-forms on a Kähler phase manifold and define a BKS-pairing between them. Using this pairing, one can construct a quantization of the original phase manifold in a Fock space of half-forms.

The geometric quantization of loop spaces is considered in Part V. We start in Chapter 15 with the geometric quantization of the loop space of a d-dimensional vector space. Its quantization is based on a twistor-like construction of a Fock bundle of half-forms over the space of complex structures on the Sobolev space V

of half-differentiable functions on S^1. There is a projective action of the Hilbert–Schmidt symplectic group of V on this bundle, and its infinitesimal version yields a quantization of the original loop space. At the end of this Chapter we discuss the geometric quantization of the universal Teichmüller space $\mathcal{T}$. The standard Dirac quantization does not apply to the whole of $\mathcal{T}$, and it seems more natural in this case to use an approach, based on the "quantized calculus" of Connes and Sullivan. (We are grateful to Alain Connes for drawing our attention to this approach, presented in [16].)

In Chapter 16 we construct a geometric quantization of the loop space ΩG of a compact Lie group G. It is based on the Borel–Weil theorem for the loop groups, given in Pressley–Segal book [65]. We follow the same scheme, as in Chapter 15, using the projective action of the diffeomorphism group on the Fock bundle, defined by Goodman–Wallach [26],[27].

Concluding this foreword, I want to thank all my colleagues, who made it possible this book to appear. Book I of the present edition is an extended version of the book, published in Russian in 2001 by Moscow Center of Continuous Mathematical Education. Book II may be considered as an extended version of a joint paper with Johann Davidov [17], published in Steklov Institute Proceedings in 1999. That paper was based on my previous collaboration with Alexander Popov.

This book is based on the lecture course on the Kähler geometry of loop spaces and their geometric quantization, which I gave in Nagoya University in 2003 by the invitation of Professor Ryoichi Kobayashi. I am deeply grateful to him and Nagoya University for the invitation to give this lecture course and warm hospitality during my stay in Nagoya.

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	Loop Spaces	Grassmann Manifolds	Spaces of Complex Structures
Universal Spaces	$HG = H^{1/2}(S^1, G)/G$	$\mathrm{Gr}_b(H)$	$\mathcal{T} = QS(S^1)/\operatorname{Möb}(S^1)$
Smooth Parts	$\Omega G = LG/G$	$\mathrm{Gr}_{HS}(H)$	$\mathcal{S} = \mathrm{Diff}_+(S^1)/\operatorname{Möb}(S^1)$
Finite-dimensional Parts	$F(G) = G/K$	$\mathrm{Gr}(\mathbb{C}^d)$	$T(G)$
Bundles over	$\Omega_T G = LG/T$	$\mathrm{Det}(H)$	$\mathrm{Diff}_+(S^1)/S^1$

Part I

PRELIMINARY CONCEPTS

Chapter 1

Frechet manifolds

This Chapter is devoted to the Frechet manifolds, having Frechet vector spaces as their local models. We start our exposition by recalling basic facts on Frechet spaces in Sec. 1.1. In Sec. 1.2 we introduce Frechet manifolds and define various geometric structures on them, including vector bundles and connections, differential forms, symplectic and complex structures.

1.1 Frechet vector spaces

1.1.1 Basic definitions

In contrast with Banach spaces, whose topology is defined by a norm, the topology of a Frechet vector space is determined by a system of seminorms. Recall that

Definition 1. A *seminorm* on a vector space F is a real-valued function $p:\ F \to \mathbb{R}$, which satisfies the following conditions:

1. $p(f) \geq 0 \quad \text{for any } f \in F$;
2. $p(f+g) \leq p(f) + p(g) \quad \text{for any } f, g \in F$;
3. $p(cf) = |c|p(f) \quad$ for any $f \in F$ and any element c of the basic number field k (we restrict to $k = \mathbb{R}$ and $k = \mathbb{C}$ in the sequel).

As one can see from this definition, the only difference between seminorms and norms is that a seminorm p is not required to satisfy the property: $p(f) = 0 \iff f = 0$.

A system of seminorms $\{p_n\}_{n\in\mathbb{N}}$ determines on the vector space F a unique *topology*, for which

$$f_j \to f \iff p_n(f_j - f) \to 0 \quad \text{for any } n \in \mathbb{N} \ .$$

This topology is Hausdorff, if the following condition is fulfilled:

$$f = 0 \Longleftrightarrow p_n(f) = 0 \quad \text{for all } n \in \mathbb{N} \ .$$

A sequence $\{f_j\}$ of elements of F is called a *Cauchy sequence* with respect to this topology if $p_n(f_j - f_k) \to 0$ for $j, k \to \infty$ for any $n \in \mathbb{N}$. The space F is *complete*, if any Cauchy sequence in F has a limit in F.

Definition 2. A Hausdorff topological vector space F with the topology, defined by a countable system of seminorms, is called a *Frechet space* iff it is complete.

Example 1. Any *Banach space* is a Frechet space with a system of seminorms, represented by a single norm.

Example 2. The vector space $C^\infty[a,b]$, consisting of C^∞-smooth real-valued functions f on an interval $[a,b]$, is a Frechet space with a system of seminorms

$$p_n(f) = \sum_{j=0}^{n} \sup_{[a,b]} \left| f^{(j)}(x) \right| .$$

Example 3. The vector space $C^\infty(X)$, consisting of C^∞-smooth real-valued functions f on a compact manifold X, is a Frechet space with a system of seminorms

$$p_n(f) = \sum_{|j|=0}^{n} \sup_{X} \left| d^j f(x) \right| .$$

Example 4. Let $V \to X$ be a vector bundle over a compact Riemannian manifold X, provided with a Riemannian metric and connection. Then the vector space $C^\infty(X,V)$, consisting of C^∞-smooth sections f of $V \to X$, is a Frechet space with a system of seminorms

$$p_n(f) = \sum_{|j|=0}^{n} \sup_{X} \left| D^j f(x) \right| ,$$

where $D^j f$ is the jth covariant derivative of a section f, and the "length" $|h|$ of a section h is computed, using the metrics on X and V.

A closed subspace of a Frechet space is also a Frechet space and the same is true for the quotient of a Frechet space by its closed subspace.

Example 5. The vector space $C^\infty_{2\pi}$, consisting of C^∞-smooth real-valued 2π-periodic functions on the real line $\mathbb{R}$, may be identified with the closed subspace in the Frechet space $C^\infty[0,2\pi]$, consisting of functions $f \in C^\infty[0,2\pi]$ such that all their derivatives $f^{(j)}$ match together at the end points: $f^{(j)}(0) = f^{(j)}(2\pi)$. It implies that $C^\infty_{2\pi}$ is also a Frechet space.

Many well-known properties of Banach spaces, such as the Hahn-Banach theorem and the closed graph theorem, are fulfilled in Frechet spaces as well.

However, there is a number of properties of Banach spaces, which do not transfer to the Frechet case. For example, the theorem of existence and uniqueness of solutions of ordinary differential equations for Banach spaces do not extend to general Frechet spaces. Another example: the dual of a Frechet space, which is <u>not</u> a Banach space, <u>cannot</u> be a Frechet space. In particular, the dual of the Frechet space $C^\infty(X)$ of C^∞-smooth real-valued functions on a compact manifold X, which is the space $\mathcal{D}'(X)$ of distributions on X, is not a Frechet space. Note also that the space $L(F,G)$ of linear operators, acting from a Frechet space F to another Frechet space G, is not, generally speaking, a Frechet space.

1.1.2 Derivative

Definition 3. Let F and G be Frechet spaces and $A : F \to G$ be a continuous map. The *derivative* of A *at a point* $f \in F$ *in a direction* $h \in F$ is the limit

$$D_f A(h) = \lim_{t \to 0} \frac{A(f + th) - A(f)}{t} \in G \; .$$

The map A is *differentiable at* f *in the direction* h, if this limit exists. The map A is *continuously differentiable* (or belongs to the class $C^1(U)$) on an open subset $U \subset F$, if this limit exists for any $f \in U$ and all $h \in F$ and the map

$$DA : U \times F \longrightarrow G$$

is continuous.

Example 6. Let $f : [a, b] \to F$ be a path in a Frechet space F, i.e. a continuous map from an interval $[a, b]$ to F. Denote by $\mathbf{1}$ the unit vector in $\mathbb{R}$, then the derivative $f'(t)$ (if it exists) coincides with $D_{f(t)}(\mathbf{1})$.

Example 7. A continuous linear map $L : F \to G$ of Frechet spaces belongs to the class C^1 and $D_f L(h) = Lh$ since

$$D_f L(h) = \lim_{t \to 0} \frac{L(f + th) - Lf}{t} = \lim_{t \to 0} \frac{tLh}{t} = Lh \; .$$

Example 8. Let U be a relatively open subset of a band $[a, b] \times \mathbb{R} \subset \mathbb{R}^2_{(x,y)}$ and $F = F(x, y)$ be a smooth function on U. Denote by $\mathcal{U}$ an open subset in $C^\infty[a, b]$, consisting of functions $y = f(x)$, having their graphs inside U. Consider a map $A : \mathcal{U} \longrightarrow C^\infty[a, b]$, given by the formula

$$A(f)(x) = F(x, f(x)) \; .$$

Then A belongs to the class C^1 and

$$D_f A(h)(x) = d_y F(x, f(x))h(x) \; .$$

Example 9. More generally, let X be a compact manifold and $V \to X$, $W \to X$ be two vector bundles over X. Given an open subset U in V, denote by $\mathcal{U}$ the open subset in $C^\infty(X, V)$, consisting of sections f of $V \to X$, having their image in U: $f(X) \subset U$. Let $F : U \to W$ be an arbitrary smooth *bundle map*, sending any fibre V_p, $p \in X$, into the fibre W_p over the same point p.

Define a *fibrewise operator* $A : \mathcal{U} \longrightarrow C^\infty(X, W)$, acting by the formula

$$A(f) = F \circ f \; .$$

Denote by x a local coordinate on X in a neighborhood of a given point p and by y and z coordinates in the fibres V_p and W_p respectively. Then the map F is given locally by a function $z = F(x, y)$. A section f has a local representation $y = f(x)$, and the bundle operator A is given locally by the formula $A(f)(x) = F(x, f(x))$.

The derivative of A in the chosen local coordinates has the form

$$D_f A(h)(x) = d_y F(x, f(x))h(x) \; ,$$

where $d_y F$ is the matrix of partial derivatives in y, applied to a vector-valued function h, representing locally a section $h \in C^\infty(X, V)$.

If A is a C^1-map $F \to G$, then

$$D_f A(h_1 + h_2) = D_f A(h_1) + D_f A(h_2) \ .$$

In other words, a continuously differentiable map A is necessarily linear in h. This important property shows that the derivative "behaves" like a differential with respect to the variable h.

Moreover, a map $A : U \subset F \to G$ is continuously differentiable on a convex open subset $U \subset F$ if and only if there exists a continuous map

$$L : U \times U \times F \longrightarrow G, \quad L = L(f_1, f_2)h \ ,$$

which is linear in h and for any $f_1, f_2 \in U$ satisfies the relation

$$A(f_1) - A(f_2) = L(f_1, f_2)(f_1 - f_2) \ .$$

In this case $D_f A(h) = L(f, f)h$.

If two maps $A : F \to G$ and $B : G \to H$ are continuously differentiable, then their composition $B \circ A : F \to H$ is also continuously differentiable and the chain rule for the derivatives is fulfilled

$$D_f[B \circ A](h) = D_{A(f)} B(D_f A(h)) \ .$$

In particular, if $f(t)$ is a C^1-path in F and $A : F \to G$ is a C^1-map, then $A(f(t))$ is a C^1-path in G and

$$A(f(t))' = D_{f(t)} A(f'(t)) \ .$$

Suppose now that the basic number field $k = \mathbb{C}$ and $A : U \subset F \to G$ is a map between complex Frechet spaces. We shall call this map *holomorphic* if it belongs to the class $C^1(U)$ and its derivative $DA : F \times F \to G$ is complex linear in $h \in F$.

By iterating the definition of the derivative, one can define higher order derivatives of maps between Frechet spaces. In particular, the *second derivative* of a map $A : F \to G$ is defined by the formula

$$D_f^2 A(h, k) = \lim_{t \to 0} \frac{D_{f+tk} A(h) - D_f A(h)}{t} \ .$$

A map $A : U \to G$ belongs to the class $C^2(U)$ on an open subset $U \subset F$ if DA belongs to $C^1(U)$, which is equivalent to the existence and continuity of the second derivative as a map $D^2 A : U \times F \times F \to G$.

Similarly to the first derivative, the second derivative $D_f^2 A(h, k)$ is linear separately in h and k if A is of class C^2. Moreover, in this case it can be given by the limit of the second finite difference

$$D_f^2 A(h, k) = \lim_{t,s \to 0} \frac{A(f + th + sk) - A(f + th) - A(f + sk) + A(f)}{ts}$$

and is symmetric in h, k.

By induction, one can define the *nth order derivative* $D_f^n A(h_1, \ldots, h_n)$ as the partial derivative of the $(n-1)$th derivative $D_f^{n-1} A(h_1, \ldots, h_{n-1})$ with respect to f in the direction of h_n, more precisely:

$$D_f^n A(h_1, \ldots, h_n) = \lim_{t \to 0} \frac{D_{f+th_n}^{n-1} A(h_1, \ldots, h_{n-1}) - D_f^{n-1} A(h_1, \ldots, h_{n-1})}{t} \ .$$

Again, a map $A : U \to G$ belongs to the class $C^n(U)$ on an open subset $U \subset F$ if $D^n_f A(h_1, \dots, h_n)$ exists and is continuous as a map $D^n A : U \times F \cdots \times F \to G$. In this case $D^n_f A(h_1, \dots, h_n)$ is symmetric and linear in $h_1, \dots, h_n$. We say that a map $A : U \to G$ belongs to the class $C^\infty(U)$ on an open subset $U \subset F$ if it belongs to all classes $C^n(U)$ for $n \in \mathbb{N}$.

1.2 Frechet manifolds

1.2.1 Basic definitions

Definition 4. A *Frechet manifold* is a Hausdorff topological space $\mathcal{X}$, provided with an *atlas*, i.e. a covering of $\mathcal{X}$ by open subsets (*coordinate neighborhoods*) $\{U_\alpha\}$, and a collection of *charts*, i.e. homeomorphisms (*coordinate maps*)

$$\varphi_\alpha : U_\alpha \xrightarrow{\approx} u_\alpha \subset F_\alpha$$

onto open subsets u_α in *model* Frechet spaces F_α. The *transition functions*

$$\varphi_{\beta\alpha} : \varphi_\alpha(U_\alpha \cap U_\beta) \xrightarrow{\varphi_\alpha^{-1}} U_\alpha \cap U_\beta \xrightarrow{\varphi_\beta} \varphi_\beta(U_\alpha \cap U_\beta)$$

are smooth (i.e. of class C^∞) maps of Frechet spaces.

If all Frechet spaces F_α in this definition coincide with some Banach spaces E_α, we call such an $\mathcal{X}$ a *Banach manifold*. Respectively, when all F_α coincide with a separable Hilbert space $H = l_2$, we call it a *Hilbert manifold*.

There is one more specification of the above definition in the case when the basic field $k = \mathbb{C}$.

Definition 5. A *complex Frechet manifold* is a Frechet manifold $\mathcal{X}$, for which all model Frechet spaces F_α are complex, and the transition functions $\varphi_{\beta\alpha}$ are holomorphic.

We add the definition of a (closed) Frechet submanifold for the future use.

Definition 6. A closed subset $\mathcal{Y}$ in a Frechet manifold $\mathcal{X}$ is called a *submanifold* of $\mathcal{X}$ if for any point of $\mathcal{Y}$ there exists a coordinate neighborhood U of $\mathcal{X}$ with a coordinate chart, mapping U onto a neighborhood u in the product of Frechet spaces $F \times G$, which identifies $U \cap \mathcal{Y}$ with the subset $u \cap F \times \{0\}$.

Example 10. Let X be a (finite-dimensional) smooth manifold. Then the set of all smooth submanifolds in X, denoted by $\mathcal{S}(X)$, is a Frechet manifold. Indeed, consider a submanifold $S \in \mathcal{S}(X)$, having the normal bundle $NS = (TX|S)/TS$.

Then there exists a local exponential diffeomorphism

$$\exp : v \longrightarrow V ,$$

mapping a neighborhood v of the zero section in NS onto a tubular neighborhood V of S in X. This diffeomorphism generates a local coordinate chart φ with

$$\varphi^{-1} : \mathfrak{v} \longrightarrow \mathfrak{V} ,$$

mapping the neighborhood $\mathfrak{v}$ of zero in the Frechet space $C^\infty(S, NS)$, consisting of sections of NS with their image in v, onto the neighborhood $\mathfrak{V}$ of the submanifold S in $\mathcal{S}(X)$, consisting of submanifolds in X, lying in V.

Example 11. Let X be a compact smooth manifold and $\pi : E \to X$ is a *smooth bundle*, i.e. E is a smooth manifold, π is a smooth map, whose tangent π_* is everywhere surjective. Then the space of smooth sections of the bundle E, denoted by $C^\infty(X, E)$, is a Frechet manifold.

In order to construct coordinate charts on $C^\infty(X, E)$, we define for a given section f a *vertical vector bundle* $T^v_f E \to X$, *associated with* f, with the fibre at $p \in X$, equal to the kernel of π_*, restricted to $T_{f(p)}E$. Choose a neighborhood u of the zero section of $T^v_f E \to X$ together with a fibrewise diffeomorphism of u onto a tubular neighborhood U of the image $f(X)$ in E. This diffeomorphism generates a local coordinate chart φ with

$$\varphi^{-1} : \mathfrak{u} \longrightarrow \mathfrak{U} ,$$

mapping the neighborhood $\mathfrak{u}$ of the zero section in the Frechet space $C^\infty(X, T^v_f E)$, consisting of sections of $T^v_f E \to X$ with their image in u, onto the neighborhood $\mathfrak{U}$ of f in $C^\infty(X, E)$, consisting of sections of $E \to X$ with their image in U. The transition functions are given by fibrewise operators, as in Ex. 9 from Sec. 1.1.

Example 12. The manifold $C^\infty(X, Y)$ of smooth maps from a smooth compact manifold X into a smooth manifold Y is a particular case of the above construction, when the bundle $E = X \times Y \to X$ is trivial. The group $\mathrm{Diff}(X)$ of diffeomorphisms of X onto itself is an open subspace in $C^\infty(X, Y)$ and so inherits its structure of a Frechet manifold.

Example 13. The latter example is especially interesting for us when X is a circle, which we identify with $S^1 = \{|z| = 1 : z \in \mathbb{C}\}$. In this case the manifold $C^\infty(S^1, Y)$ is called the *space of (free) loops* in the manifold Y.

Consider the simplest example of that sort when Y is also a circle S^1. The manifold $C^\infty(S^1, S^1)$ consists of a countable number of connected components, denoted by $C^\infty_k(S^1, S^1)$ with $k \in \mathbb{Z}$, which are numerated by the index (rotation number) of a map $S^1 \to S^1$. By pulling up to the universal coverings, we can associate with a map $f : S^1 \to S^1$ the map $\tilde{f} : \mathbb{R}^1 \longrightarrow \mathbb{R}^1$, defined up to an additive constant of the form $2\pi n$, $n \in \mathbb{Z}$. In particular, the maps $f \in C^\infty_0(S^1, S^1)$ of index 0 have the pullbacks $\tilde{f}$, which are smooth 2π-periodic functions, i.e. belong to the Frechet space $C^\infty_{2\pi}$ (cf. Ex. 5 in Sec. 1.1). So we have a global coordinate chart for the whole component $C^\infty_0(S^1, S^1)$:

$$\varphi : C^\infty_0(S^1, S^1) \xrightarrow{\approx} C^\infty_{2\pi}/2\pi\mathbb{Z} , \quad f \longmapsto [\tilde{f}] .$$

In the same way, the maps $f \in C^\infty_k(S^1, S^1)$ of index k have the pullbacks $\tilde{f}$, which satisfy the relation: $\tilde{f}(x + 2\pi) = \tilde{f}(x) + 2\pi k$. Translating such a function by kx, i.e. replacing $\tilde{f}(x)$ by $\tilde{f}_1(x) := \tilde{f}(x) - kx$, we obtain a 2π-periodic function $\tilde{f}_1$. Hence, we have again a global coordinate chart on $C^\infty_k(S^1, S^1)$:

$$\varphi : C^\infty_k(S^1, S^1) \xrightarrow{\approx} C^\infty_{2\pi}/2\pi\mathbb{Z} .$$

For the whole manifold $C^\infty(S^1, S^1)$ we get a diffeomorphism

$$C^\infty(S^1, S^1) \xrightarrow{\approx} \mathbb{Z} \times C^\infty_{2\pi}/2\pi\mathbb{Z} .$$

Example 14. Consider an open submanifold $\mathrm{Diff}(S^1)$ in $C^\infty(S^1, S^1)$, consisting of all diffeomorphisms of the circle S^1. It has two connected components: the identity component $\mathrm{Diff}_+(S^1)$, consisting of diffeomorphisms of S^1, preserving its orientation (this component belongs to the subspace $C_1^\infty(S^1, S^1)$), and $\mathrm{Diff}_-(S^1)$, consisting of diffeomorphisms of S^1, reversing its orientation (this component belongs to the subspace $C_{-1}^\infty(S^1, S^1)$).

The maps $f \in \mathrm{Diff}_+(S^1)$ pull back to functions $\tilde{f}$, satisfying the relation

$$\tilde{f}(x + 2\pi) = \tilde{f}(x) + 2\pi \ .$$

They have 2π-periodic derivatives $\tilde{f}'(x)$, which are everywhere positive, since diffeomorphisms f preserve the orientation. We also have:

$$\frac{1}{2\pi} \int_0^{2\pi} \tilde{f}'(x)\, dx = \frac{\tilde{f}(2\pi) - \tilde{f}(0)}{2\pi} = 1 \ ,$$

i.e. the average of $\tilde{f}'(x)$ over the period is equal to 1. Denote by C the subset of $C_{2\pi}^\infty$, consisting of smooth 2π-periodic strictly positive functions on the real line with the average, equal to 1. It is an open convex subset in an affine subspace of codimension 1 in $C_{2\pi}^\infty$, hence a Frechet submanifold. The above argument implies that our manifold $\mathrm{Diff}_+(S^1)$ is diffeomorphic to $S^1 \times C$. Indeed, the function $\tilde{f}$ is defined by $\tilde{f}'$ up to an additive constant $\tilde{f}(0) \in \mathbb{R}$, but the function $\tilde{f}$ itself is defined by $f : S^1 \to S^1$ up to an additive constant $2\pi n \in 2\pi\mathbb{Z}$. Hence, $\tilde{f}'$ determines f up to an element of $S^1 = \mathbb{R}/2\pi\mathbb{Z}$. Since C is contractible, we see that $\mathrm{Diff}_+(S^1)$ *is homotopy equivalent to* S^1.

1.2.2 Frechet vector bundles

Let $\mathcal{X}$, $\mathcal{V}$ be two Frechet manifolds and $\pi : \mathcal{V} \to \mathcal{X}$ be a smooth surjection such that each fibre $\pi^{-1}(x)$, $x \in \mathcal{X}$, of π has the structure of a Frechet vector space.

Definition 7. A Frechet manifold $\mathcal{V}$ is called a *Frechet vector bundle* over $\mathcal{X}$ if the following conditions are satisfied. There exists an atlas $\{U_\alpha\}$ of coordinate neighborhoods in $\mathcal{X}$ such that for any α the preimage $V_\alpha = \pi^{-1}(U_\alpha)$ of the coordinate neighborhood $\{U_\alpha\}$ belongs to a coordinate neighborhood in $\mathcal{V}$. The corresponding coordinate charts have the form

$$\varphi_\alpha : U_\alpha \longrightarrow u_\alpha = \varphi(U_\alpha) \subset F_\alpha \ , \tag{1.1}$$

$$\psi_\alpha : V_\alpha \longrightarrow v_\alpha = \psi_\alpha(V_\alpha) = u_\alpha \times G_\alpha \tag{1.2}$$

and are compatible in the sense that the following diagram is commutative

$$\begin{tikzcd}
V_\alpha = \pi^{-1}(U_\alpha) \arrow[r, "\psi_\alpha"'] \arrow[d, "\pi"'] & v_\alpha = u_\alpha \times G_\alpha \subset F_\alpha \times G_\alpha \arrow[d, "\text{projection}"] \\
U_\alpha \arrow[r, "\varphi_\alpha"] & u_\alpha \subset F_\alpha
\end{tikzcd}$$

The structure of a vector space on π-fibres, induced from the right vertical arrow, coincides with the original one and the transition functions

$$\psi_{\beta\alpha} := \psi_\beta \circ \psi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \times G_\alpha \longrightarrow \varphi_\beta(U_\alpha \cap U_\beta) \times G_\beta$$

are linear in the second variable.

This definition applies with evident modifications to Banach and Hilbert vector bundles. If all Frechet spaces in the above definition, as well as π-fibres, are complex and the transition functions are holomorphic, we obtain the definition of a holomorphic Frechet vector bundle.

Example 15. The *tangent bundle* $T\mathcal{X}$ of a Frechet manifold $\mathcal{X}$ is a Frechet vector bundle. The fibre of $T\mathcal{X}$ at $x \in \mathcal{X}$ is formed by vectors $x'(t)|_{t=0}$, where $x(t)$ is a smooth path in $\mathcal{X}$, emanating from x. The coordinate transition function for $T\mathcal{X}$ are given by the derivatives of coordinate transition functions for $\mathcal{X}$.

Example 16. If, in particular, $\mathcal{X} = C^\infty(X,Y)$, then a path $f : [0,1] \to C^\infty(X,Y)$ is given by a map $f : [0,1] \times X \to Y$, i.e. by a 1-parameter family of maps $f_t : X \to Y$, $t \in [0,1]$. For any $x \in X$ the image $f_t(x)$ for $0 \le t \le 1$ constitutes a path in Y, whose tangent vector at $f_t(x)$ coincides with the derivative $f_t'(x) \in T_{f_t(x)}Y = f_t^*(TY)_x$. Hence, f_t' is a section of the inverse image $f_t^*TY \to X$ of the tangent bundle TY under the map f_t and

$$T_f C^\infty(X,Y) = C^\infty(X, f^*TY) .$$

Example 17. Let X be a (finite-dimensional) smooth manifold and $\mathcal{S}(X)$ be the Frechet manifold of its smooth compact submanifolds (cf. Ex. 10). Then its tangent bundle $T\mathcal{S}(X)$ has the fibre at $S \in \mathcal{S}(X)$, equal to the Frechet space of sections $C^\infty(S, NS)$ of the normal bundle NS.

We shall need later another Frechet vector bundle, related to the Frechet manifold $\mathcal{S}(X)$. Namely, denote by $C^\infty(S)$ the Frechet space of smooth functions on S. Then the union of the spaces $C^\infty(S)$ over all $S \in \mathcal{S}(X)$ is a Frechet vector bundle $C^\infty\mathcal{S}(X) \to \mathcal{S}(X)$. Indeed, a coordinate chart φ on $\mathcal{S}(X)$ in a neighborhood of the submanifold $S \in \mathcal{S}(X)$ maps this neighborhood into the Frechet space $C^\infty(S, NS)$ of smooth sections of the normal bundle NS. Using this map, we can identify diffeomorphically submanifolds S', close to S, with the submanifold S, which corresponds to the zero section of NS. Accordingly, smooth functions on S' will be identified with smooth functions on S, which defines a coordinate chart ψ on $C^\infty\mathcal{S}(X)$ in a neighborhood of S with values in $C^\infty(S, NS) \times C^\infty(S)$, compatible with the coordinate chart φ on $\mathcal{S}(X)$.

Definition 8. A map $\mathcal{A} : \mathcal{X} \to \mathcal{Y}$ between Frechet manifolds is called *smooth* if for any point $x \in \mathcal{X}$ we can find coordinate charts φ in a neighborhood of this point and ψ in a neighborhood of its image $y = \mathcal{A}(x)$ such that the composition $\psi \circ \mathcal{A} \circ \varphi^{-1}$, called otherwise a *local representative* of $\mathcal{A}$, is a smooth map of Frechet spaces.

We say that a smooth map $\mathcal{A} : \mathcal{X} \to \mathcal{Y}$ is an *immersion* (resp. *submersion*) if for any point $x \in \mathcal{X}$ we can find coordinate charts near x and its image $y = \mathcal{A}(x)$ so that the local representative of $\mathcal{A}$ is an immersion (resp. submersion) of Frechet spaces, i.e. it is an inclusion of a summand (resp. projection onto a summand) in a direct sum of Frechet spaces.

Example 18. A smooth map $\mathcal{A} : \mathcal{X} \to \mathcal{Y}$ between Frechet manifolds generates a *tangent map* $T(\mathcal{A}) : T\mathcal{X} \to T\mathcal{Y}$ of their tangent bundles. This map sends any fibre $T_x\mathcal{X}$ at $x \in \mathcal{X}$ to the fibre $T_y\mathcal{Y}$ at the image point $y = \mathcal{A}(x) \in \mathcal{Y}$. In a coordinate chart it is given by the derivative of the corresponding local representative. The linear map $D\mathcal{A} : T_x\mathcal{X} \to T_y\mathcal{Y}$, induced by $T(\mathcal{A})$ on the tangent space $T_x\mathcal{X}$, is the *derivative* of $\mathcal{A}$ at x, which agrees with the definition, given in Subsec. 1.1.2, in the case when $\mathcal{X}$ and $\mathcal{Y}$ are Frechet spaces.

Definition 9. A smooth map $\pi : \mathcal{E} \to \mathcal{X}$ between Frechet manifolds is called a *Frechet fibre bundle*, if it is a submersion and for any point $x \in \mathcal{X}$ we can find an open neighborhood U of this point such that there exists a Frechet manifold $\mathcal{F}$ and a diffeomorphism $\psi : \pi^{-1}(U) \to \mathcal{F}$ such that the following diagram is commutative:

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\psi"'] \arrow[d, "\pi"'] & U \times \mathcal{F} \arrow[d, "\text{projection}"] \\ U \arrow[r, "\text{id}"] & U \end{tikzcd}$$

As in the finite-dimensional situation, a smooth map $\mathcal{A} : \mathcal{E}_1 \to \mathcal{E}_2$ of a fibre bundle $\pi_1 : \mathcal{E}_1 \to \mathcal{X}$ to a fibre bundle $\pi_2 : \mathcal{E}_2 \to \mathcal{X}$ is called a *fibre bundle map* if it sends fibres to fibres, i.e. for any $x \in \mathcal{X}$ it sends the fibre $\pi_1^{-1}(x)$ to the fibre $\pi_2^{-1}(x)$.

Example 19. Let $\pi_1 : \mathcal{E}_1 \to \mathcal{X}$ and $\pi_2 : \mathcal{E}_2 \to \mathcal{X}$ be two fibre bundles of Frechet manifolds. Then we can form a new fibre bundle over $\mathcal{X}$, called the *fibre product* of these two bundles, which a closed submanifold in $\mathcal{E}_1 \times \mathcal{E}_2$. Namely, we set

$$\mathcal{E}_1 \times_{\mathcal{X}} \mathcal{E}_2 = \{(e_1, e_2) \in \mathcal{E}_1 \times \mathcal{E}_2 : \ \pi_1(e_1) = \pi_2(e_2)\} \ .$$

It is a closed subset in $\mathcal{E}_1 \times \mathcal{E}_2$, since $\mathcal{E}_1 \times_{\mathcal{X}} \mathcal{E}_2$ coincides with the preimage of the diagonal Δ in $\mathcal{E}_1 \times \mathcal{E}_2$ under the product map $\pi_1 \times \pi_2 : \mathcal{E}_1 \times \mathcal{E}_2 \to \mathcal{X} \times \mathcal{X}$. To prove that it is a fibre bundle over $\mathcal{X}$ and a submanifold in $\mathcal{E}_1 \times \mathcal{E}_2$, take an arbitrary point $x \in \mathcal{X}$ and choose an open neighborhood U so that $\pi_1 : \pi_1^{-1}(U) \to U$ and $\pi_2 : \pi_2^{-1}(U) \to U$ are compatible with the projections $U \times \mathcal{F}_1 \to U$ and $U \times \mathcal{F}_2 \to U$ respectively in the sense of Def. 9. This generates a diffeomorphism $\pi_1^{-1}(U) \times \pi_2^{-1}(U) \subset \mathcal{E}_1 \times \mathcal{E}_2$ into $U \times \mathcal{F}_1 \times U \times \mathcal{F}_2$. Restricting this diffeomorphism to the diagonal Δ in $U \times U$, we obtain for $\mathcal{E}_1 \times_{\mathcal{X}} \mathcal{E}_2$ a local diffeomorphism ψ, required in the Def. 9. The same argument shows that $\mathcal{E}_1 \times_{\mathcal{X}} \mathcal{E}_2$ is a closed submanifold in $\mathcal{E}_1 \times \mathcal{E}_2$.

1.2.3 Connections

Let $\pi : \mathcal{V} \to \mathcal{X}$ be a Frechet vector bundle over a Frechet manifold $\mathcal{X}$. Given a point $v \in \mathcal{V}$ denote by $V_v = \operatorname{Ker} D\pi$ the subspace in $T_v\mathcal{V}$, formed by vectors, annihilated by the derivative $D\pi : T_v\mathcal{V} \to T_{\pi(v)}\mathcal{X}$. By mimicking the finite-dimensional definition, we want to define a connection $\mathcal{H}$ on $\pi : \mathcal{V} \to \mathcal{X}$ as a rule, assigning to any point $v \in \mathcal{V}$ a subspace H_v in $T_v\mathcal{V}$, complementary to V_v.

The tangent bundle $T\mathcal{V}$ can be considered as a Frechet vector bundle $\pi_{\mathcal{V}} : T\mathcal{V} \to \mathcal{V}$ over $\mathcal{V}$ and also as a Frechet vector bundle $T\pi : T\mathcal{V} \to T\mathcal{X}$ over $T\mathcal{X}$. So we have a natural projection

$$(\pi_{\mathcal{V}}, T\pi) : T\mathcal{V} \longrightarrow \mathcal{V} \oplus T\mathcal{X} \ , \qquad \delta v \longmapsto (\pi_{\mathcal{V}}(\delta v), T\pi(\delta v))$$

for $\delta v \in T\mathcal{V}$. Note that the composite map $\pi \circ \pi_{\mathcal{V}} : T\mathcal{V} \to \mathcal{X}$ provides $T\mathcal{V}$ with a structure of a fibre bundle over $\mathcal{X}$.

Definition 10. A *connection* on a Frechet vector bundle $\pi : \mathcal{V} \to \mathcal{X}$ is a smooth fibre bundle map

$$\mathcal{H} : \mathcal{V} \oplus T\mathcal{X} \longrightarrow T\mathcal{V}$$

of fibre bundles over $\mathcal{X}$ such that

$$(\pi_{\mathcal{V}}, T\pi) \circ \mathcal{H} = \text{id} \quad \text{on} \quad \mathcal{V} \oplus T\mathcal{X}$$

and is bilinear. The latter means that for any $x \in \mathcal{X}$ the restriction of $\mathcal{H}$ to the fibre over x is a map $\mathcal{H}_x : \mathcal{V}_x \oplus T_x\mathcal{X} \to T_x\mathcal{V}$, which is linear in both arguments.

To understand what this definition means in local terms, consider a coordinate neighborhood U in $\mathcal{X}$, over which we have the following identifications

$$TU \longleftrightarrow U \times F \ , \quad \pi^{-1}(U) \longleftrightarrow U \times G \ , \ T(U \times G) \longleftrightarrow (U \times G) \times (F \times G) \ .$$

In these terms our connection $\mathcal{H}$ has the following representation

$$\mathcal{H}(x, v, \xi) = (x, v, H_1(x, v, \xi), H_2(x, v, \xi))$$

where $x \in U$, $v \in G$, $\xi \in F$. Since $(\pi_{\mathcal{V}}, T\pi) \circ \mathcal{H} = \text{id}$ on $\mathcal{V} \oplus T\mathcal{X}$, we have $H_1(x, v, \xi) = \xi$ and the bilinearity condition implies that $H_2(x, v, \xi)$ is bilinear in (v, ξ). We shall denote this map, called the *Christoffel symbol* of the connection $\mathcal{H}$, by

$$\Gamma : U \times G \times F \longrightarrow G \ , \quad \Gamma_x(v, \xi) := H_2(x, v, \xi) \ .$$

Denote, as above, by V the subbundle in $T\mathcal{V}$, given by the kernel $\operatorname{Ker} T\pi$ of the tangent map $T\pi : T\mathcal{V} \to T\mathcal{X}$. We call V the *vertical subbundle* of $T\mathcal{V}$. The complementary subbundle H in $T\mathcal{V}$, given by the image $\operatorname{Im}\mathcal{H}$ of the map $\mathcal{H} : \mathcal{V} \oplus T\mathcal{X} \to T\mathcal{V}$, is called the *horizontal subbundle* of $T\mathcal{V}$. Note that, while the vertical subbundle V is canonically defined by $\pi : \mathcal{V} \to \mathcal{X}$, the horizontal subbundle H is determined by the connection $\mathcal{H}$.

There is another way to view the connection, based on the notion of *covariant derivative*. The covariant derivative is defined in terms of connection $\mathcal{H}$ as follows. Consider a path $v(t)$ in $\mathcal{V}$, represented in local coordinates as $v(t) = (x(t), g(t))$ with $x(t) \in U$, $g(t) \in G$. Then its covariant derivative $\nabla v(t)$ is equal to

$$\nabla v(t) = (\xi(t), \Xi(t)) \ ,$$

where

$$\xi(t) = x'(t) \ , \quad \Xi(t) = g'(t) - \Gamma_{x(t)}(g(t), \xi(t)) \ .$$

The path $v(t)$ in $\mathcal{V}$, covering the path $x(t)$ in $\mathcal{X}$, is *horizontal* iff $\nabla v(t) = 0$.

For Banach manifolds we can always find for a given path $x(t)$ in $\mathcal{X}$ with the initial value $x(0)$ a uniquely determined horizontal lift $v(t)$ in $\mathcal{V}$, covering $x(t)$. On the contrary, for Frechet manifolds the horizontal lift may not exist and, even if it exists, it may be not unique. This is due to the absence of the existence and uniqueness theorem for the ordinary differential equations in Frechet spaces.

By definition, a *connection on a Frechet manifold* $\mathcal{X}$ is a connection on its tangent bundle $T\mathcal{X}$. If $x(t)$ is a path in $\mathcal{X}$, then its derivative $v(t) := x'(t)$ is a path in $T\mathcal{X}$. Its covariant derivative $\nabla v(t)$ is called otherwise the *acceleration* of $x(t)$. A path $x(t)$ is a *geodesic* of $\mathcal{X}$ iff its acceleration is zero. We say that a connection $\mathcal{H}$ on $T\mathcal{X}$ is *symmetric* if its local representatives $\Gamma_x(\xi, \eta)$ are symmetric in $(\xi, \eta) \in T_x\mathcal{X} \times T_x\mathcal{X}$.

Definition 11. The *curvature* $\mathcal{R}$ of a connection $\mathcal{H}$ on a Frechet vector bundle $\pi : \mathcal{V} \to \mathcal{X}$ is a trilinear map

$$\mathcal{R} : \mathcal{V} \times T\mathcal{X} \times T\mathcal{X} \longrightarrow \mathcal{V} ,$$

given in terms of local representatives by the formula

$$\mathcal{R}_x(v, \xi, \eta) := D\Gamma_x(v, \xi, \eta) - D\Gamma_x(v, \eta, \xi) - \Gamma_x(\Gamma_x(v, \xi), \eta) + \Gamma_x(\Gamma_x(v, \eta), \xi) ,$$

where $\Gamma_x(v, \xi)$ is a local representative of the connection $\mathcal{H}$. This definition does not depend upon the choice of a local chart.

Example 20. Consider the Frechet manifold $C^\infty(X, Y)$ of smooth maps from a compact manifold X into a manifold Y. Suppose that Y has a connection, represented locally by the Christoffel symbol $\Gamma_y(\xi, \eta)$. Then we can define a connection on $C^\infty(X, Y)$ locally by the Christoffel symbol

$$\left(\Gamma_f(\tilde{\xi}, \tilde{\eta})\right)(x) = \Gamma_{f(x)}(\tilde{\xi}(x), \tilde{\eta}(x)) \quad \text{for } x \in X ,$$

where $f \in C^\infty(X, Y)$, $\tilde{\xi}, \tilde{\eta} \in T_f C^\infty(X, Y) = C^\infty(X, f^*TY)$ (cf. Ex.16 in Subsec. 1.2.2). Note that $\tilde{\xi}(x), \tilde{\eta}(x) \in T_{f(x)}Y$.

A path $f(t)$ in $C^\infty(X, Y)$, evaluated at $x \in X$, yields a path $f_t(x)$ in Y. The path $f(t)$ is a geodesic in $C^\infty(X, Y)$ if and only if the path $f_t(x)$ is a geodesic in Y for any $x \in X$. The curvature $\mathcal{R}$ of the introduced connection on $C^\infty(X, Y)$ is given in terms of the curvature R of the connection on Y by the formula

$$\mathcal{R}_f(\tilde{\xi}, \tilde{\eta}, \tilde{\zeta})(x) = R_{f(x)}(\tilde{\xi}(x), \tilde{\eta}(x), \tilde{\zeta}(x)) ,$$

i.e. is computed from R pointwise.

Example 21. Consider the Frechet manifold $\mathcal{S}(X)$ of smooth compact submanifolds S in a Riemannian manifold X (cf. Ex. 10 in Subsec. 1.2.1 and Ex. 17 in Subsec. 1.2.2). For any $S \in \mathcal{S}(X)$ and $f \in C^\infty(S)$ we can define vector bundles Tf and Nf over S by setting

$$Tf := \text{graph of } Df = \{(v, D_v f) : v \in TS\} \subset TX \times \mathbb{R}$$

and $Nf = TX \times \mathbb{R}/Tf$.

Then we have the following natural isomorphisms

$$T_S\mathcal{S}(X) = C^\infty(S, NS) , \quad T_{(S,f)}C^\infty\mathcal{S}(X) = C^\infty(S, Nf) .$$

The vector bundle Nf may be included into the following exact sequence of vector bundle maps over S

$$0 \longrightarrow \mathbb{R} \longrightarrow Nf \longrightarrow NS \longrightarrow 0 ,$$

which induces an exact sequence of maps of Frechet vector spaces

$$0 \longrightarrow C^\infty(S) \longrightarrow C^\infty(S, Nf) \longrightarrow C^\infty(S, NS) \longrightarrow 0 .$$

By above isomorphisms, it coincides with the exact sequence

$$0 \longrightarrow C^\infty(S) \longrightarrow T_{(S,f)}C^\infty\mathcal{S}(X) \longrightarrow T_S\mathcal{S}(X) \longrightarrow 0 .$$

The third arrow in this sequence is the tangent map of the vector bundle projection $C^\infty\mathcal{S}(X) \to \mathcal{S}(X)$, while the second arrow realizes $C^\infty(S)$ as the vertical subspace of this bundle at $f \in C^\infty(S)$.

To define a complementary subspace, we need a connection on $C^\infty\mathcal{S}(X)$, which is generated by the Riemannian connection on X. This connection $\mathcal{H}$ may be described as follows. For $S \in \mathcal{S}(X)$ we can identify its normal bundle NS with the subbundle of $TX|S$, consisting of vectors, orthogonal to TS with respect to the Riemannian metric of X. Then $NS \times \mathbb{R}$ would be a complementary subbundle to Tf in $TX \times \mathbb{R}$, so we can identify $Nf = TX \times \mathbb{R}/Tf$ with $NS \times \mathbb{R}$. We set $Hf = NS \times \{0\}$ to be the horizontal subbundle, complementary to the vertical subspace $\{0\} \times \mathbb{R}$. Then $C^\infty(S, Hf)$, which is complementary to the vertical subspace $C^\infty(S)$, will be the horizontal subspace of our connection $\mathcal{H}$. Note that it projects isomorphically onto the space $C^\infty(S, NS) = T_S\mathcal{S}(X)$, since $Hf = NS \times \{0\} \sim NS$.

Let us compute the curvature of this connection. Using the Riemannian connection ∇ on X, we can define covariant derivatives ∇f of $f \in C^\infty(S)$ and $\nabla\xi$ of $\xi \in C^\infty(S, NS)$ and compute their inner product $\nabla f \cdot \nabla\xi$ in $TX|S$. The curvature $\mathcal{R}$ of the connection $\mathcal{H}$ is a trilinear map

$$\mathcal{R} : C^\infty\mathcal{S}(X) \times TC^\infty\mathcal{S}(X) \times TC^\infty\mathcal{S}(X) \longrightarrow C^\infty\mathcal{S}(X) \ ,$$

which can be interpreted at a point $S \in \mathcal{S}(X)$ as a linear map

$$\mathcal{R}_S : C^\infty(S) \times C^\infty(S, NS) \times C^\infty(S, NS) \longrightarrow C^\infty(S) \ .$$

This map is given explicitly by the formula

$$\mathcal{R}_S(f, \xi, \eta) = \nabla f \cdot \nabla\xi \cdot \eta - \nabla f \cdot \nabla\eta \cdot \xi \ .$$

1.2.4 Differential forms

Definition 12. A *differential form* of degree r (or simply an *r-form*) on a Frechet manifold $\mathcal{X}$ is a smooth map

$$\omega : \underbrace{T\mathcal{X} \times \cdots \times T\mathcal{X}}_{r} \longrightarrow \mathbb{C}$$

of the rth direct power $T\mathcal{X} \times \cdots \times T\mathcal{X}$ of the tangent bundle $T\mathcal{X}$ such that for any $x \in \mathcal{X}$ its restriction

$$\omega_x : T_x\mathcal{X} \times \cdots \times T_x\mathcal{X} \longrightarrow \mathbb{C}$$

to $T_x\mathcal{X} \times \cdots \times T_x\mathcal{X}$ is an r-multilinear alternating map. In other words, ω_x is an r-multilinear alternating form on $T_x\mathcal{X}$. We denote the space of r-forms on $\mathcal{X}$ by $\Omega^r(\mathcal{X})$. We shall consider smooth functions on $\mathcal{X}$ as forms of degree 0.

In a coordinate neighborhood U of $\mathcal{X}$ we can identify an r-form ω on U with a smooth map from an open subset of a Frechet space F into the vector space $\Omega^r(F)$ of r-multilinear alternating r-forms on F. If $\xi_1, \ldots, \xi_r$ are smooth vector fields on $U \subset F$, we denote by $\omega(\xi_1, \ldots, \xi_r)$ the map from U to $\mathbb{C}$, whose value at $x \in U$ is equal to $\omega_x(\xi_1(x), \ldots, \xi_r(x))$, i.e. the value of the r-form $\omega_x \in \Omega^r(F)$ on vectors $\xi_1(x), \ldots, \xi_r(x)$ in F.

Differential forms on Frechet manifolds share many properties with differential forms on finite-dimensional manifolds. In particular, one can define their exterior derivative and wedge product similar to the finite-dimensional case.

Definition 13. The *exterior derivative* $d\omega$ of an r-form ω on $\mathcal{X}$ is an $(r+1)$-form on $\mathcal{X}$, which can be defined locally as follows. For any smooth vector fields $\xi_0, \xi_1, \dots, \xi_r$ in a coordinate neighborhood $U \subset F$, the value of $d\omega$ on $\xi_0, \xi_1, \dots, \xi_r$ is equal to

$$\begin{aligned} d\omega(\xi_0, \xi_1, \dots, \xi_r) = \sum_{i=0}^{r} (-1)^i \xi_i \left(\omega(\xi_0, \dots, \widehat{\xi_i}, \dots, \xi_r) \right) + \\ + \sum_{\substack{i,j=0 \\ i<j}}^{r} (-1)^{i+j} \omega \left([\xi_i, \xi_j], \xi_0, \dots, \widehat{\xi_i}, \dots, \widehat{\xi_j}, \dots, \xi_r \right) \ . \end{aligned} \tag{1.3}$$

This definition does not depend on the choice of the local data in the sense that there is a unique $(r+1)$-form on $\mathcal{X}$, which respects the given local representations (cf. [47], Ch.V, Prop. 3.2).

Example 22. If f is a 0-form on $\mathcal{X}$, i.e. a smooth map $f : \mathcal{X} \to \mathbb{C}$, then df_x for any $x \in \mathcal{X}$ coincides with the tangent map

$$T_x f : T_x\mathcal{X} \longrightarrow T_{f(x)}\mathbb{C} \ .$$

Moreover, for any vector field ξ on $\mathcal{X}$ we have

$$df(\xi) = \xi f \ .$$

If ω is a 1-form on $\mathcal{X}$, then locally

$$d\omega(\xi, \eta) = \xi\left(\omega(\eta)\right) - \eta\left(\omega(\xi)\right) - \omega\left([\xi, \eta]\right) \ .$$

For a 2-form ω we have locally

$$\begin{aligned} d\omega(\xi, \eta, \zeta) = \xi\left(\omega(\eta, \zeta)\right) + \eta\left(\omega(\zeta, \xi)\right) + \zeta\left(\omega(\xi, \eta)\right) - \\ - \omega\left([\xi, \eta], \zeta\right) - \omega\left([\eta, \zeta], \xi\right) - \omega\left([\zeta, \xi], \eta\right) \ . \end{aligned} \tag{1.4}$$

Definition 14. The *wedge product* of an r-form ω and an s-form ψ on $\mathcal{X}$ is an $(r+s)$-form $\omega \wedge \psi$ on $\mathcal{X}$, which can be defined locally as follows. For any smooth vector fields $\xi_1, \dots \xi_{r+s}$ in a coordinate neighborhood $U \subset F$, the value of $\omega \wedge \psi$ on $\xi_1, \dots \xi_{r+s}$ is equal to

$$(\omega \wedge \psi)(\xi_1, \dots \xi_{r+s}) = \sum_{i=1}^{r+s} (-1)^{\epsilon(\sigma)} \omega(\xi_{\sigma(1)}, \dots, \xi_{\sigma(r)}) \psi(\xi_{\sigma(r+1)}, \dots, \xi_{\sigma(r+s)}) \ ,$$

where the sum is taken over all permutations σ of the numbers $(1, \dots, r+s)$ and $\epsilon(\sigma)$ is the parity of σ.

Again, this definition does not depend on the choice of the local data in the sense that there is a unique $(r+s)$-form $\omega \wedge \psi$ on $\mathcal{X}$, which respects the given local representations.

In particular, the wedge product of a function f and a form ω is equal to $f \wedge \omega = f\omega$. One can easily check that the wedge product of two forms ω and ψ on $\mathcal{X}$ is related to the exterior derivative by the usual formula

$$d(\omega \wedge \psi) = d\omega \wedge \psi + (-1)^{\deg \omega} \wedge d\psi$$

and the square of d is equal to zero: $dd\omega = 0$.

1.2.5 Symplectic and complex structures

Definition 15. A *symplectic structure* on a Frechet manifold $\mathcal{X}$ is a 2-form ω on $\mathcal{X}$, having the following properties:

1. ω is closed, i.e. $d\omega = 0$;

2. ω is non-degenerate at any point $x \in \mathcal{X}$, i.e. for any $\xi \in T_x\mathcal{X}$, $\xi \neq 0$, there exists an $\eta \in T_x\mathcal{X}$ such that $\omega_x(\xi, \eta) \neq 0$.

A Frechet manifold $\mathcal{X}$, provided with a symplectic structure ω, is called *symplectic*.

Remark 1. Note that we have used here the weakest form of the non-degeneracy condition. For Banach manifolds, modelled locally on a Banach space E, a conventional non-degeneracy condition on ω requires that for any $x \in \mathcal{X}$ the linear operator A_x from $T_x\mathcal{X} \sim E$ to the dual space $T_x^*\mathcal{X} \sim E'$, defined by $\omega_x(\cdot, \eta) = A_x(\cdot)(\eta)$, is invertible for any non-zero $\eta \in T_x^*\mathcal{X}$.

Most of Frechet manifolds, considered in this book, are symplectic in the sense of the Def.15. Moreover, they usually have, along with their symplectic structure, a compatible almost complex structure.

Definition 16. An *almost complex structure* on a Frechet manifold $\mathcal{X}$ is a smooth vector bundle automorphism J of the tangent bundle $T\mathcal{X}$, such that for any $x \in \mathcal{X}$ the restriction J_x of J to $T_x\mathcal{X}$ satisfies the condition

$$J_x^2 = -\mathrm{id} \ .$$

A Frechet manifold $\mathcal{X}$, provided with an almost complex structure, is called *almost complex*.

If J is an almost complex structure on a Frechet manifold $\mathcal{X}$, then the isomorphism J can be extended complex linearly to the *complexified* tangent bundle $T^{\mathbb{C}}\mathcal{X} = T\mathcal{X} \otimes \mathbb{C}$, so that $T^{\mathbb{C}}\mathcal{X}$ decomposes into the direct sum of subbundles

$$T^{\mathbb{C}}\mathcal{X} = T^{1,0}\mathcal{X} \oplus T^{0,1}\mathcal{X} \ ,$$

where for any $x \in \mathcal{X}$ the restriction of J_x to $T_x^{1,0}\mathcal{X}$ is given by the multiplication by i, and the restriction of J_x to $T_x^{0,1}\mathcal{X}$ is given by the multiplication by $-i$. Sections of the bundles $T_x^{1,0}\mathcal{X}$ and $T_x^{0,1}\mathcal{X}$ are called otherwise the *vector fields of type* $(1,0)$ and $(0,1)$ respectively.

We call an almost complex structure J on a Frechet manifold $\mathcal{X}$ *integrable* or *formally integrable* complex structure, if the bracket of any two vector fields on $\mathcal{X}$ of type $(1,0)$ is again a vector field of type $(1,0)$.

Remark 2. An almost complex structure J provides a complex structure on every tangent space $T_x\mathcal{X}$, determined by the action of J_x. In particular, any complex Frechet manifold $\mathcal{X}$ has a natural almost complex structure, given by the multiplication by i on $T_x\mathcal{X}$. Such an almost complex structure is automatically integrable. For finite-dimensional manifolds the *Newlander-Nirenberg theorem* asserts that the converse is also true, namely, any almost complex manifold with an integrable almost complex structure is, in fact, *complex*. It means that one can introduce an atlas of local complex charts on this manifold in such a way that the original almost complex structure in these coordinates will be given by the multiplication by i on tangent spaces. For Frechet manifolds this theorem is, in general, not true (cf. [51]), so in order to show that a given Frechet manifold is complex, it's necessary to construct, following Def.5 from Subsec. 1.2.1, an atlas of local complex charts.

The most important class of Frechet manifolds, considered in this book, is that of Kähler Frechet manifolds, i.e. Frechet manifolds, which are both symplectic and complex, and these two structures are compatible in the sense of the following definition.

Definition 17. A complex symplectic Frechet manifold $\mathcal{X}$ is called a *Kähler Frechet manifold*, if its complex structure J and symplectic structure ω are *compatible* in the following sense:

1. $\omega_x(J_x\xi, J_x\eta) = \omega_x(\xi, \eta)$ for any $\xi, \eta \in T_x\mathcal{X}$, $x \in \mathcal{X}$;
2. a symmetric form on $T_x\mathcal{X} \times T_x\mathcal{X}$, defined by

$$g_x(\xi, \eta) := \omega_x(\xi, J_x\eta) \ ,$$

   is positively definite for any $x \in \mathcal{X}$.

Such a form g is called the *Kähler metric* on $\mathcal{X}$.

Bibliographic comments

A key reference to Ch.1 is the Hamilton's paper [32] on the Nash–Moser theorem. Its first part is an excellent introduction to the theory of Frechet manifolds. In our exposition (except for Subsecs.1.2.4,1.2.5) we follow closely that paper. The definition Def.10 of the connection on a Frechet vector bundle is borrowed from [47]. The latter book can be recommended for the readers, interested in the theory of infinite-dimensional manifolds with a special emphasis on the Banach case.

Chapter 2

Frechet Lie groups

Definition 18. A *Frechet Lie group* is a Frechet manifold $\mathcal{G}$, provided with the group structure, such that the multiplication

$$\mathcal{G} \times \mathcal{G} \longrightarrow \mathcal{G} \ , \quad (g, h) \longmapsto g \cdot h \ ,$$

and "taking-the-inverse"

$$\mathcal{G} \longrightarrow \mathcal{G} \ , \quad g \longmapsto g^{-1} \ ,$$

are smooth maps of Frechet manifolds. The *Frechet Lie algebra* of a Frechet Lie group $\mathcal{G}$ is the tangent space $\mathfrak{G} = T_{\mathbf{1}}\mathcal{G}$ at the unit $\mathbf{1}$ of the group $\mathcal{G}$.

For $g \in \mathcal{G}$ denote by

$$\begin{aligned} L_g : \mathcal{G} \to \mathcal{G}, &\quad L_g(h) = g \cdot h \\ R_g : \mathcal{G} \to \mathcal{G}, &\quad R_g(h) = h \cdot g \end{aligned}$$

respectively the *left* and *right translations* on the group $\mathcal{G}$.

Any element ξ of the Lie algebra $\mathfrak{G}$ generates by left translations a vector field X_ξ on $\mathcal{G}$, invariant under these translations. The correspondence $\xi \longleftrightarrow X_\xi$ allows us to consider elements of the Lie algebra $\mathfrak{G}$ as left-invariant vector fields on the Lie group $\mathcal{G}$. The left-invariant vector fields on $\mathcal{G}$ form a Lie algebra with respect to the bracket of vector fields, which induces a Lie algebra bracket on $T_{\mathbf{1}}\mathcal{G} = \mathfrak{G}$ by the identification $\xi \longleftrightarrow X_\xi$ (this justifies the use of the term "Lie algebra" with respect to $T_{\mathbf{1}}\mathcal{G}$). We note that there exists a unique connection $\mathcal{H}$ on $\mathcal{G}$, called the *Cartan–Maurer connection*, such that the left-invariant vector fields are horizontal with respect to $\mathcal{H}$, its curvature being equal to zero. Of course, the choice of the left-invariant vector fields and left translations in this argument was absolutely ambiguous (though traditional), with the same success we could employ here the right-invariant vector fields and right translations.

If in the definition of a Frechet Lie group the group $\mathcal{G}$ is a Banach (resp. Hilbert) manifold, we say that $\mathcal{G}$ is a *Banach* (resp. *Hilbert) Lie group*.

Suppose that for any element ξ of the Lie algebra $\mathfrak{G}$ there exists a unique 1-parameter subgroup $\gamma_\xi : \mathbb{R} \to \mathcal{G}$ of the group $\mathcal{G}$ such that $\gamma'_\xi(0) = \xi$. Then, as in the finite-dimensional case, we can define the *exponential map*

$$\exp : \mathfrak{G} \longrightarrow \mathcal{G}$$

by setting $\exp\xi := \gamma_\xi(\mathbf{1})$. In particular, for Banach Lie groups $\mathcal{G}$ the above condition is always satisfied. Indeed, any element $\xi \in \mathfrak{G}$ is identified with the left-invariant vector field X_ξ, which can be integrated to a 1-parameter group of transformations $\varphi_\xi^t : \mathcal{G} \to \mathcal{G}$. In this case $\gamma_\xi(t) := \varphi_\xi^t(\mathbf{1})$.

We supplement the definition of Frechet fibre bundles, given in Subsec. 1.2.2 (cf. Def.9), with the definition of a principal Frechet bundle. We say that a Frechet Lie group $\mathcal{G}$ *acts on a Frechet manifold* $\mathcal{X}$, if there is a smooth map

$$\mathcal{G} \times \mathcal{X} \longrightarrow \mathcal{X} \ , \quad (g, x) \longmapsto g \cdot x \ ,$$

such that $\mathbf{1} \cdot x = x$ and $(g_1 g_2) \cdot x = g_1 \cdot (g_2 \cdot x)$.

Definition 19. Let $\mathcal{G}$ be a Frechet Lie group, acting on a Frechet manifold $\mathcal{E}$. This manifold is called a *principal Frechet* $\mathcal{G}$*-bundle*, if there is a smooth submersion $\pi : \mathcal{E} \to \mathcal{X}$ onto another Frechet manifold $\mathcal{X}$, such that for any $x \in \mathcal{X}$ there exists an open neighborhood U of x and a diffeomorphism of its preimage $\pi^{-1}(U)$ in $\mathcal{E}$ onto $U \times \mathcal{G}$, satisfying the following conditions:

1. the action of $\mathcal{G}$ on $\mathcal{E}$ corresponds to the natural action of $\mathcal{G}$ on the second factor of $U \times \mathcal{G}$;

2. the following diagram

$$\begin{array}{ccc} \pi^{-1}(U) & \longrightarrow & U \times \mathcal{G} \\ {\scriptstyle \pi}\Big\downarrow & & \Big\downarrow{\scriptstyle \text{projection}} \\ U & \xrightarrow{\text{id}} & U \end{array}$$

   is commutative.

We consider next the two most important examples of Frechet Lie groups, playing a special role in this book.

2.1 Group of currents $C^\infty(X, G)$

2.1.1 Basic properties

Let X be a smooth compact manifold and G is a Lie group. The space $C^\infty(X, G)$ of all smooth maps from X into G is a Frechet manifold, as we have pointed out in Subsec.1.2.1 (cf. Ex.12). Let us recall the definition of the structure of a Frechet manifold on $C^\infty(X, G)$ for this particular case.

The exponential map $\exp : \mathfrak{g} \to G$ determines a local diffeomorphism

$$\exp : \mathfrak{u} \longrightarrow U \ ,$$

mapping an open neighborhood $\mathfrak{u}$ of zero in the Lie algebra $\mathfrak{g}$ onto an open neighborhood U of the unit $e \in G$. Using this diffeomorphism, we can construct a local chart in a neighborhood $\mathcal{U} = C^\infty(X, U)$ of the identity $\mathbf{1} := X \to e \in G$ in $C^\infty(X, U)$. It is given by the homeomorphism

$$\chi : \quad \mathfrak{U} := C^\infty(X, \mathfrak{u}) \longrightarrow C^\infty(X, U) = \mathcal{U} \ ,$$

given by the pointwise application of the exponential map $\exp : \mathfrak{u} \to U$. The inverse map $\varphi_{\mathbf{1}} := \chi^{-1} : \mathcal{U} \to \mathfrak{U}$ yields a homeomorphism of the neighborhood $\mathcal{U}$ of the identity $\mathbf{1} \in C^\infty(X, U)$ onto the open subset $\mathfrak{U}$ in the Frechet space $C^\infty(X, \mathfrak{g})$.

The manifold $C^\infty(X, G)$ is a group with respect to the pointwise multiplication. Using this group structure, we can construct local charts at any point of $C^\infty(X, G)$. To define a local chart at an arbitrary point $\gamma \in C^\infty(X, G)$, denote by $\mathcal{U}_\gamma$ a neighborhood of γ of the form $\mathcal{U}_\gamma := \gamma \cdot \mathcal{U}$ and define a local chart φ_γ in the neighborhood $\mathcal{U}_\gamma$ as the composition map

$$\varphi_\gamma := \varphi_{\mathbf{1}} \circ \gamma^{-1} : \mathcal{U}_\gamma \to \mathfrak{U} ,$$

where the map $\gamma^{-1} : \mathcal{U}_\gamma \to \mathcal{U}$ is given by the multiplication by γ^{-1} from the left. The neighborhoods $\{\mathcal{U}_\gamma\}$ and the maps $\{\varphi_\gamma\}$ with $\gamma \in C^\infty(X, G)$ form an open atlas and a system of local charts on $C^\infty(X, G)$, which defines the structure of a smooth Frechet manifold on $C^\infty(X, G)$, modelled on the Frechet space $C^\infty(X, \mathfrak{g})$.

The pointwise multiplication and taking-the-inverse maps in the group $C^\infty(X, G)$ are smooth with respect to the introduced structure of a Frechet manifold, hence $C^\infty(X, G)$ is a Frechet Lie group, called the *group of currents*.

The Lie algebra of $C^\infty(X, G)$ coincides with the Frechet space $C^\infty(X, \mathfrak{g})$, the Lie bracket in $C^\infty(X, \mathfrak{g})$ being given by the pointwise application of the Lie bracket in $\mathfrak{g}$. The exponential map

$$\exp : C^\infty(X, \mathfrak{g}) \longrightarrow C^\infty(X, G) ,$$

given by the pointwise application of the exponential map $\exp : \mathfrak{g} \to G$, is a local homeomorphism in a neighborhood of zero.

Consider now the most important example of the group $C^\infty(X, G)$, corresponding to the case when $X = S^1$. In this case the group $C^\infty(X, G)$ is called the *loop group* of the Lie group G, and is denoted by

$$LG := C^\infty(S^1, G) .$$

The Lie algebra of LG coincides with the *loop algebra*

$$L\mathfrak{g} := C^\infty(S^1, \mathfrak{g}) .$$

Since all operations in the loop group LG are defined pointwise, one can expect that the properties of LG will be close to the properties of the group G itself. And this is true in most of the cases, but there are still some differences, demonstrated by the examples below.

Consider first the homotopy structure of LG. Let us introduce the *based loop space*

$$\Omega G := LG/G$$

of G, where G in the denominator is identified with the group of constant maps $S^1 \to g_0 \in G$. We can realize ΩG as the closed submanifold of LG, consisting of the maps $\gamma \in LG$, which send the identity $\mathbf{1} \in LG$ to the unit $e \in G$: $\gamma(\mathbf{1}) = e$. Then the loop group LG will be identified with the direct product $\Omega G \times G$. It is well known (cf.,e.g., [36]) that the homotopy groups of ΩG coincide with the homotopy groups of G, shifted by one:

$$\pi_i(\Omega G) \cong \pi_{i+1}(G) \ .$$

It follows that

$$\pi_i(LG) \cong \pi_i(\Omega G) \oplus \pi_i(G) \cong \pi_{i+1}(G) \oplus \pi_i(G) \ .$$

In particular, $\pi_0(LG)$ is equal to $\pi_1(G) \oplus \pi_0(G)$, i.e. the group LG is connected if and only if G is connected and simply connected. The fundamental group of LG coincides with $\pi_2(G) \oplus \pi_1(G) \cong \pi_1(G)$, since $\pi_2(G) = 0$ for any connected compact Lie group G. Hence, *LG is connected and simply connected if the Lie group G itself is connected and simply connected.*

2.1.2 Exponential map of the loop algebra

As we have pointed out, the exponential map

$$\exp : L\mathfrak{g} \longrightarrow LG$$

of the loop algebra $L\mathfrak{g}$ is given by the pointwise application of the exponential map $\exp : \mathfrak{g} \to G$.

If G is a compact Lie group, then it has the following well-known property. Denote by G° the identity connected component of G. Then the *exponential map* $\exp : \mathfrak{g} \to G^\circ$ *is surjective.* This property is a corollary of the fact that every element of G° belongs to some 1-parameter subgroup of G. However, for the loop group LG it is not true, in general.

Consider, for example, the simply connected group $G = \mathrm{SU}(2)$. Then the element

$$LG \ni \gamma : z \longrightarrow \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} \ , \quad z \in S^1 \ ,$$

is not an exponential of any element in the loop algebra $L\mathfrak{g}$.

Indeed, if we suppose that $\gamma = \exp \xi$ for some $\xi \in C^\infty(S^1, \mathfrak{g})$, then the matrix $\gamma(z)$, being a function of $\xi(z)$, should commute with $\xi(z)$ for any $z \in S^1$. It's easy to see that this condition implies that the matrix $\xi(z)$ should be diagonal for any $z \in S^1$, i.e.

$$\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} = \begin{pmatrix} e^{if(z)} & 0 \\ 0 & e^{-if(z)} \end{pmatrix}$$

for some smooth real-valued function f on S^1. In particular, $z = e^{if(z)}$, which is impossible, since the logarithm $\ln(z)$ does not admit a continuous branch on the circle.

However, one can prove the following property of the loop group LG, which may be considered as a substitution of the surjectivity of $\exp : \mathfrak{g} \to G^\circ$.

Proposition 1. *Let G be a connected compact Lie group. Then the exponential map*

$$\exp : L\mathfrak{g} \longrightarrow (LG)^\circ$$

has a dense image in the connected component of the identity $(LG)^\circ$ of the group LG.

Proof. To prove this assertion, we note first that a connected compact Lie group G is the direct product of a torus and a connected semisimple compact Lie group. Our assertion for the torus is easily checked directly, so it is sufficient to consider the case of a semisimple connected compact Lie group G. In this case the group G can be realized as the connected component of the identity of the automorphism group $\mathrm{Aut}\,\mathfrak{g}$ of the Lie algebra $\mathfrak{g}$ (since an arbitrary semisimple connected compact Lie group G is a finite covering over $(\mathrm{Aut}\,\mathfrak{g})^\circ$). If this is the case, then the critical points of the exponential map $\exp : \mathfrak{g} \to G$ lie on a closed hypersurface Γ in $\mathfrak{g}$, dividing $\mathfrak{g}$ into an interior convex domain D, containing 0, and its complement. The image of Γ under the exponential map, denoted by $\exp\Gamma$, is contained in a submanifold of G of codimension ≥ 3.

Consider now an arbitrary loop $\gamma(z) \in (LG)^\circ$, passing through $e \in G$: $\gamma(1) = e$. We assert that it can be approximated by smooth loops in $(LG)^\circ$, which are the *exponentials* in LG (we call a loop $\delta(z)$ in LG an exponential, if it can be represented in the form $\delta = \exp\xi$ for some $\xi \in L\mathfrak{g}$).

By smoothly deforming, if necessary, the loop γ, we can approximate it by a smooth loop $\tilde{\gamma} \in (LG)^\circ$, starting at e, such that $\tilde{\gamma}(e^{it})$ does not intersect $\exp\Gamma$ for $0 < t < 2\pi$. Since the exponential $\exp : \mathfrak{g} \to G$ is locally diffeomorphic along $\tilde{\gamma}(e^{it})$ for $t < 2\pi$, we can, beginning from e, choose a continuous logarithm branch of the loop $\tilde{\gamma}(e^{it})$ for $t < 2\pi$. As a result, we obtain a smooth (but, generally speaking, not closed) path $\xi(e^{it})$, $0 \leq t < 2\pi$, in $\mathfrak{g}$ such that $\exp\xi = \tilde{\gamma}$.

The limit ξ_0 of the path $\xi(e^{it})$ for $t \to 2\pi - 0$ belongs to $\bar{D}$. If $\exp\Gamma$ does not contain e, then ξ_0 cannot belong to $\Gamma = \partial D$, because $\exp\xi_0 = e$. Hence, $\xi_0 \in D$, which forces it to be equal to zero (since, otherwise, exp will be equal to e on the whole orbit of ξ_0 in $D \setminus 0$ under the adjoint action Ad, being a smooth submanifold in $\mathfrak{g}$ of a positive dimension). So $\xi(e^{it})$, $0 \leq t \leq 2\pi$, is a smooth loop in $\mathfrak{g}$ such that $\exp\xi = \tilde{\gamma}$, i.e. we have found a logarithm of $\tilde{\gamma}$ in $\mathfrak{g}$.

If $\exp\Gamma$ contains e, then, in contrast with the considered case, it may happen that the limit $\lim_{t\to 2\pi-0}\xi(e^{it}) = \xi_0$ belongs to Γ. But in such a situation the loop $\tilde{\gamma}$ will not be contractible, i.e. $\tilde{\gamma} \notin (LG)^\circ$, contrary to our assumption. To prove it, note that in this case our path $\xi(e^{it})$ is homotopic to a linear path $\xi_0(e^{it}) := t\frac{\xi_0}{2\pi}$, $0 \leq t \leq 2\pi$, with the same endpoints 0 and ξ_0, as $\xi(e^{it})$. Accordingly, the loop $\tilde{\gamma}$ is homotopic to the loop $\gamma_0(e^{it})$ in G, given by

$$\gamma_0 : S^1 \ni e^{it} \longmapsto \exp\left(t\frac{\xi_0}{2\pi}\right) , \quad 0 \leq t \leq 2\pi .$$

But it is easy to see that γ_0 is not contractible in G. So the loop $\tilde{\gamma}$ is also not contractible in G. □

2.1.3 Complexification

The loop group LG, similar to compact Lie groups, admits the complexification.

Recall that the complexification of a Lie algebra $\mathfrak{g}$ coincides with the complex Lie algebra

$$\mathfrak{g}^{\mathbb{C}} := \mathfrak{g} \otimes \mathbb{C} = \mathfrak{g} + i\mathfrak{g} .$$

Definition 20. We call by the *complexification* of a connected Lie group G a connected complex Lie group $G^{\mathbb{C}}$, having the following properties:

1. the Lie algebra of $G^{\mathbb{C}}$ coincides with the complexification $\mathfrak{g}^{\mathbb{C}}$ of the Lie algebra $\mathfrak{g}$;

2. $G^{\mathbb{C}}$ contains G as a subgroup, i.e. there exists a monomorphism $i : G \to G^{\mathbb{C}}$.

In particular, a group G, which admits the complexification, should have non-trivial homomorphisms into complex Lie groups (the monomorphism i is one of them).

The complexification $G^{\mathbb{C}}$, introduced above, exists and is uniquely defined for any compact connected Lie group G. For example, the complexification of the group $G = S^1$ coincides with the multiplicative group $G^{\mathbb{C}} = \mathbb{C}^* = \mathbb{C}\backslash\{0\}$ of complex numbers, and the complexification of $G = \mathrm{SU}(n)$ coincides with $G^{\mathbb{C}} = \mathrm{SL}(n, \mathbb{C})$. For the non-compact group $\mathrm{SL}(n, \mathbb{R})$ its complexification also coincides with $\mathrm{SL}(n, \mathbb{C})$.

We give an example of a Lie group, which admits no complexification in the above sense. As we have pointed out, the complexification of the group $\mathrm{SL}(2, \mathbb{R})$ coincides with the group $\mathrm{SL}(2, \mathbb{C})$. The group $\mathrm{SL}(2, \mathbb{C})$ is simply connected, while the fundamental group of $\mathrm{SL}(2, \mathbb{R})$ is isomorphic to $\mathbb{Z}$. Let G be a universal covering group of $\mathrm{SL}(2, \mathbb{R})$. Then we have a homomorphism $\pi : G \to \mathrm{SL}(2, \mathbb{R})$, whose kernel is equal to $\mathbb{Z}$. Suppose that G has the complexification $G^{\mathbb{C}}$. Then it should be a covering group of $\mathrm{SL}(2, \mathbb{C})$. Indeed, the composition of π with the natural embedding $i : \mathrm{SL}(2, \mathbb{R}) \hookrightarrow \mathrm{SL}(2, \mathbb{C})$ yields a non-trivial homomorphism of G into the complex group $\mathrm{SL}(2, \mathbb{C})$ with the kernel, equal to $\mathbb{Z}$. This homomorphism extends to a covering homomorphism $G^{\mathbb{C}} \to \mathrm{SL}(2, \mathbb{C})$ with the same kernel. But such a covering cannot exist, since $\mathrm{SL}(2, \mathbb{C})$ is simply connected. The property of the group G, used in this argument, can be reformulated as follows: any homomorphism of G into a connected complex Lie group factors through $\mathrm{SL}(2, \mathbb{R})$ or (still another formulation) the kernel of such a homomorphism should contain $\mathbb{Z}$.

In the case of the loop group $LG = C^\infty(S^1, G)$ of a compact connected Lie group G its complexification coincides with the loop group $LG^{\mathbb{C}} = C^\infty(S^1, G^{\mathbb{C}})$ of the complexified group $G^{\mathbb{C}}$. The group $LG^{\mathbb{C}}$ is a complex Frechet Lie group, modelled on the Frechet Lie algebra $C^\infty(S^1, \mathfrak{g}^{\mathbb{C}})$.

2.2 Group of diffeomorphisms Diff(X)

Let X be a smooth compact manifold and Diff(X) is the group of diffeomorphisms of X. The group Diff(X) is a Frechet manifold, being an open subset in the Frechet manifold $C^\infty(X, X)$. It is a Frechet Lie group with respect to this Frechet manifold structure.

The group Diff(X) is closely related to the group of currents $C^\infty(X, G)$, considered in the previous Sec.2.1. Namely, Diff(X) acts smoothly on the manifold $C^\infty(X, G)$ by the "reparametrization" of maps from $C^\infty(X, G)$.

The Lie algebra of the group Diff(X) coincides with the Frechet Lie algebra

$$C^\infty(X, TX) =: \mathrm{Vect}(X)$$

of smooth tangent vector fields on X.

The exponential map

$$\exp : \mathrm{Vect}(X) \longrightarrow \mathrm{Diff}(X)$$

can be defined, as in the beginning of this Chapter. Namely, any vector field $\xi \in \mathrm{Vect}(X)$ generates a 1-parameter subgroup of diffeomorphisms φ_t^ξ of X, defined as follows. The image $y(t) := \varphi_t^\xi(x)$ of an arbitrary point $x \in X$ under the action of φ_t^ξ coincides with the value at t of the integral path of the ordinary differential equation $y' = \xi(y)$ with the initial condition: $y = x$ for $t = 0$. We set $\exp \xi := \varphi_1^\xi$.

Restrict now to the case of $X = S^1$, which is the most important for us. As we have already remarked in Subsec.1.2.1 (Ex. 14), the group $\mathrm{Diff}(S^1)$ consists of two connected components, and the connected component of the identity $\mathrm{Diff}_+(S^1)$ is formed by the maps from $\mathrm{Diff}(S^1)$, preserving the orientation of S^1.

The Lie algebra of the group $\mathrm{Diff}(S^1)$ coincides with the algebra $\mathrm{Vect}(S^1)$ of smooth tangent vector fields on the circle S^1. Elements $v \in \mathrm{Vect}(S^1)$ can be written in the form $v = v(\theta)\frac{d}{d\theta}$, where $v(\theta)$ is a smooth 2π-periodic function of θ. The bracket of two vector fields $v_1, v_2 \in \mathrm{Vect}(S^1)$ is given by the standard formula

$$\left[v_1(\theta)\frac{d}{d\theta}, v_2(\theta)\frac{d}{d\theta}\right] = \{v_1(\theta)v_2'(\theta) - v_1'(\theta)v_2(\theta)\}\frac{d}{d\theta} .$$

Denote by $\mathrm{Vect}^{\mathbb{C}}(S^1)$ the complexification of the Lie algebra $\mathrm{Vect}(S^1)$, identified with the Frechet vector space $T_{\mathrm{id}}\mathrm{Diff}(S^1)$:

$$\mathrm{Vect}^{\mathbb{C}}(S^1) := \mathrm{Vect}(S^1) \otimes \mathbb{C} .$$

It is convenient to represent the coefficients $v(\theta)$ of vector fields $v = v(\theta)\frac{d}{d\theta}$ from $\mathrm{Vect}^{\mathbb{C}}(S^1)$ by their Fourier series

$$v(\theta) = \sum_{n=-\infty}^{\infty} v_n e^{in\theta} , \quad v_n \in \mathbb{C} .$$

In these terms the real subalgebra $\mathrm{Vect}(S^1)$ of $\mathrm{Vect}^{\mathbb{C}}(S^1)$ is specified by the relations: $v_{-n} = \bar{v}_n$, $n \in \mathbb{Z}$.

The complexified Lie algebra $\mathrm{Vect}^{\mathbb{C}}(S^1)$ has a natural vector space basis, given by the vector fields

$$e_n = ie^{in\theta}\frac{d}{d\theta} , \quad n = 0, \pm 1, \pm 2, \ldots ,$$

satisfying the commutation relations:

$$[e_n, e_m] = (n-m)e_{n+m} , \quad m, n \in \mathbb{Z}.$$

2.2.1 Finite-dimensional subalgebras in $\mathbf{Vect}_\omega(S^1)$

Consider the subalgebra $\mathrm{Vect}_\omega(S^1)$ of $\mathrm{Vect}(S^1)$, consisting of vector fields $v(\theta)\frac{d}{d\theta}$ with real analytic coefficients $v(\theta)$. Such $v(\theta)$ are represented by Fourier series of the form

$$v(\theta) = \sum_{n=-\infty}^{\infty} v_n e^{in\theta} , \quad v_{-n} = \bar{v}_n ,$$

converging in a neighborhood of S^1 in $\mathbb{C}$.

The Lie algebra $\mathrm{Vect}_\omega(S^1)$ has the following interesting property.

Proposition 2. *There are no finite-dimensional Lie subalgebras in* $\mathrm{Vect}_\omega(S^1)$ *of dimension* > 3*. Moreover, for any dimension* $d = 1, 2, 3$ *there exists only one (up to an isomorphism) Lie subalgebra of dimension* d *in* $\mathrm{Vect}_\omega(S^1)$*.*

Proof. To prove this assertion, we note first that the bracket of two (not identically zero) vector fields $v_1, v_2 \in \mathrm{Vect}_\omega(S^1)$ is identically zero if and only if these fields are linearly dependent, i.e. $\lambda_1 v_1 + \lambda_2 v_2 \equiv 0$ for some constants λ_1, λ_2. So any non-trivial commutative subalgebra in $\mathrm{Vect}_\omega(S^1)$ should be one-dimensional. In particular, the rank of any non-trivial subalgebra in $\mathrm{Vect}_\omega(S^1)$ (i.e. the dimension of its Cartan subalgebra) is equal to 1.

We show that any subalgebra $\mathfrak{g}$ of the Lie algebra $\mathrm{Vect}_\omega(S^1)$ of dimension ≥ 3 is *semisimple*, i.e. it contains no non-zero commutative ideals. Suppose, on the contrary, that $\mathfrak{g}$ contains such an ideal, which should be, as we have just noted, one-dimensional. Choose a basis $\{e_1, e_2, e_3, \dots\}$ in $\mathfrak{g}$ so that our ideal is generated by e_1 (by assumption, this basis has, at least, three elements). Then

$$[e_1, e_2] = \lambda e_1 \quad \text{and} \quad [e_1, e_3] = \mu e_1 \ ,$$

where $\lambda, \mu \neq 0$, since e_1, e_2, e_3 are linearly independent. Hence, $[e_1, \mu e_2 - \lambda e_3] = 0$, which implies the linear dependence of e_1, e_2, e_3 in contradiction with our assumption.

Note that the dimension constraint on the Lie algebra $\mathfrak{g}$ in this assertion is essential, since we shall see below that the unique two-dimensional subalgebra, contained in $\mathrm{Vect}_\omega(S^1)$, is not semisimple.

We show next that *any finite-dimensional subalgebra* $\mathfrak{g}$ *in the algebra* $\mathrm{Vect}_\omega(S^1)$ *of dimension* ≥ 3 *is simple*, i.e. it contains no non-trivial ideals. Indeed, any semisimple algebra $\mathfrak{g}$ is decomposed into the direct sum of simple ideals. If $\mathfrak{g}$ is not simple, then it contains an ideal I of dimension less than $\frac{1}{2}\dim\mathfrak{g}$. We choose a basis in $\mathfrak{g}$ of the form $\{e_1, \dots, e_m, f_1, \dots, f_k\}$, so that the vectors $e_1, \dots, e_m$ form a basis of the ideal I. It's clear that $m \geq 2$ (otherwise, the ideal I would be commutative). The brackets

$$[e_1, f_1] \in I \ , \dots, \ [e_1, f_k] \in I \ , [e_1, e_2] \in I$$

are non-zero (otherwise, the corresponding vectors would be linearly dependent) and so form a collection of $k + 1 > m$ non-zero vectors in the m-dimensional subalgebra I. Hence, they are linearly dependent, which implies, as before, that the vectors $e_1, \dots, e_m, f_1, \dots, f_k$ are linearly dependent, contrary to our assumption.

From the list of simple Lie algebras, one can see that only two simple Lie algebras of dimension 3 can have the properties, described above. Namely, it is the non-compact Lie algebra $\mathrm{sl}_2(\mathbb{R})$ and the compact Lie algebra $\mathrm{su}(2)$. By comparing the Lie brackets in the Lie algebras $\mathrm{su}(2)$ and $\mathrm{Vect}_\omega(S^1)$, one shows that the second possibility is not realized. A standard embedding of $\mathrm{sl}_2(\mathbb{R})$ into $\mathrm{Vect}_\omega(S^1)$ realizes $\mathrm{sl}_2(\mathbb{R})$ as the Lie subalgebra in $\mathrm{Vect}_\omega(S^1)$, generated by three vector fields $d/d\theta$, $\cos(\theta)d/d\theta$, $\sin(\theta)d/d\theta$. This subalgebra coincides with the Lie algebra of the Möbius group $\mathrm{PSL}_2(\mathbb{R})$ of all fractional linear automorphisms of the unit disc.

Any two-dimensional subalgebra in $\mathrm{Vect}_\omega(S^1)$ is necessarily non-commutative since, as we have seen before, the vanishing of the bracket of two vector fields in $\mathrm{Vect}_\omega(S^1)$ implies their linear dependence. Since all two-dimensional non-commutative Lie algebras are isomorphic, there exists only one (up to an isomorphism) two-dimensional Lie subalgebra in $\mathrm{Vect}_\omega(S^1)$. One of its realizations inside $\mathrm{Vect}_\omega(S^1)$

is given by the subalgebra, generated by two vector fields $v_1 = \cos(\theta)d/d\theta$, $v_2 = d/d\theta + \sin(\theta)d/d\theta$. □

2.2.2 Exponential map of Vect(S^1)

We analyze now the exponential map

$$\exp : \mathrm{Vect}(S^1) \longrightarrow \mathrm{Diff}_+(S^1)$$

in more detail. Recall that this map associates with a tangent vector field $v = v(\theta)\frac{d}{d\theta}$ on the circle S^1 the diffeomorphism $\exp v := \varphi_1^v$, where φ_t^v is the 1-parameter subgroup of diffeomorphisms in $\mathrm{Diff}_+(S^1)$ with the tangent vector v at the identity $\mathrm{id} \in \mathrm{Diff}_+(S^1)$. In other words, $y_\theta(t) := \varphi_t^v(\theta)$ is a solution of the equation $\frac{dy_\theta}{dt} = v(y_\theta)$ with the initial condition $y_\theta(0) = \theta$.

For finite-dimensional Lie groups one proves easily, using the inverse function theorem, that the map exp (whose derivative at zero is equal to the identity) is locally invertible. However, as we have already pointed out several times before, the inverse function theorem is, in general, not true for Frechet manifolds. By this reason we should not be surprised by the following proposition, proved in [32, 65].

Proposition 3. *The exponential map*

$$\exp :\ \mathit{Vect}(S^1) \to \mathit{Diff}_+(S^1)$$

is neither locally injective, nor locally surjective in any neighborhood of zero.

Proof. We prove first that the exponential is not injective in any neighborhood of zero. Denote by $R_{2\pi/n}$ the rotation of S^1 by the angle $\frac{2\pi}{n}$ and note that this rotation may be chosen arbitrary close to the identity map $\mathrm{id} \in \mathrm{Diff}_+(S^1)$ for sufficiently large n.

Consider 1-parameter subgroups of $\mathrm{Diff}_+(S^1)$ of the form $f \circ S^1 \circ f^{-1}$, where $f \in \mathrm{Diff}_+(S^1)$ and S^1 is identified with the subgroup of rotations in $\mathrm{Diff}_+(S^1)$. Denote by Γ_n the subgroup in $\mathrm{Diff}_+(S^1)$, consisting of diffeomorphisms f, commuting with the rotation $R_{2\pi/n}$:

$$R_{2\pi/n}^{-1} \circ f \circ R_{2\pi/n} = f\ .$$

In other words, it is the subgroup of $(2\pi/n)$-periodic diffeomorphisms in $\mathrm{Diff}_+(S^1)$. An element $f \in \Gamma_n$ can be written in the form

$$f(\theta) = \theta + h(\theta) \bmod 2\pi\ ,$$

where h is a smooth $(2\pi/n)$-periodic function on $\mathbb{R}$ and S^1 is identified with $\mathbb{R}/2\pi\mathbb{Z}$. If $f \in \Gamma_n$, then the 1-parameter subgroup $f \circ S^1 \circ f^{-1}$ contains $R_{2\pi/n}$, since

$$f^{-1} \circ R_{2\pi/n} \circ f = R_{2\pi/n} \in S^1 \Longrightarrow R_{2\pi/n} \in f \circ S^1 \circ f^{-1}\ .$$

Hence, *all 1-parameter subgroups* Γ_n *of the above form intersect in* $R_{2\pi/n}$, so the exponential is not injective near zero.

To prove that the exponential is not a surjection onto a neighborhood of id in $\mathrm{Diff}_+(S^1)$, we use the diffeomorphisms from Γ_n, which are small deformations of the rotation $R_{2\pi/n}$. Such a diffeomorphism $f \in \Gamma_n$ can be given by the formula

$$f(\theta) = \theta + \frac{2\pi}{n} + \epsilon \sin(n\theta) \mod 2\pi \ .$$

For sufficiently large n and sufficiently small $\epsilon > 0$ this diffeomorphism may be made arbitrary close to the identity. The point $\theta = 0$ is a *periodic point of this diffeomorphism of order* n, i.e.

$$f^n(0) = \underbrace{f \circ \cdots \circ f}_{n\,\text{times}}(0) = 0 \mod 2\pi \ ,$$

but $f^n \neq \mathrm{id}$, since the derivative of f^n at zero is equal (by the composition law) to $(1+\epsilon n)^n$. Moreover, for a sufficiently small ϵ the diffeomorphism f is close to the rotation and therefore has no fixed points.

It follows that f *cannot be the exponential of any vector field* $v \in \mathrm{Vect}(S^1)$. Indeed, assuming the opposite, let $f = \exp v$ for some $v \in \mathrm{Vect}(S^1)$. The vector field $v = v(\theta)\frac{d}{d\theta}$ does not vanish, since f has no fixed points. Hence, the vector field $v(\theta)\frac{d}{d\theta}$ may be transformed into a constant field $c\frac{d}{d\theta}$ with the help of a smooth change of variable $\chi = \chi(\theta)$ of the form

$$\chi(\theta) = c\int_0^\theta \frac{dt}{v(t)} \ , \qquad 0 \leq \theta \leq 2\pi \ ,$$

where the normalizing constant $c = 2\pi\left(\int_0^{2\pi}\frac{dt}{v(t)}\right)^{-1}$ is chosen from the condition: $\chi(2\pi) = 2\pi$. This argument shows that the 1-parameter subgroup, generated by the vector v, is conjugate to a rotation R:

$$f = \chi^{-1} \circ R \circ \chi \ .$$

Then $f^n = \chi^{-1} \circ R^n \circ \chi$ and, since $f^n(0) = 0$, the rotation R^n has a fixed point, i.e. $R^n = \mathrm{id}$, which contradicts the relation $f^n \neq \mathrm{id}$. □

Remark 3. The last Proposition asserts that there exist diffeomorphisms in $\mathrm{Diff}_+(S^1)$, which cannot be represented as the exponential of a smooth vector field on the circle. One can ask if there exist diffeomorphisms in $\mathrm{Diff}_+(S^1)$, which cannot be represented as the nth power (with respect to the composition) of a diffeomorphism from $\mathrm{Diff}_+(S^1)$? It's clear that such diffeomorphisms, if they exist, also cannot be represented as the exponentials of smooth vector fields. We try to construct these diffeomorphisms again in the form

$$f(\theta) = \theta + \frac{2\pi}{n} + \epsilon\tilde{h}(\theta) \mod 2\pi \ , \tag{2.1}$$

where $\epsilon > 0$ is sufficiently small (the map f constitutes a diffeomorphism of S^1, when ϵ is less than $1/\max|\tilde{h}'|$). The function $\tilde{h}$, $0 \leq \tilde{h} \leq 1$, is a smooth $2\pi/n$-periodic function on the real line, whose restriction to the interval $[0, 2\pi/n)$ is denoted by h.

Note that the zeros of the function $\tilde{h}$ are n-periodic points of the diffeomorphism f. Then the following assertion is true.

Suppose that h vanishes on the interval $[0, 2\pi/n)$ in a finite number of points, and this number is not divisible by n. Then for a sufficiently small ϵ the diffeomorphism f, given by the formula (2.1) above, can not be represented as the nth power of any diffeomorphism from $\mathrm{Diff}_+(S^1)$.

To prove this assertion, we note that *if g is a diffeomorphism from $\mathrm{Diff}_+(S^1)$, then the number of orbits of n-periodic points of g^n is a multiple of n.* The latter statement is a corollary of the following combinatorial fact: *the number of orbits of k-periodic points of g^n is a multiple of the largest common divisor of n and k, denoted by (n, k)*, which is easy to check by direct calculation.

To deduce our assertion from the statement on the number of n-orbits of g^n, it is sufficient to prove that *our diffeomorphism f has no other n-periodic points apart from those, given by zeros of $\tilde{h}$.* Indeed, suppose for a moment that we have proved already that the set of n-periodic points of f coincides with the set of zeros of $\tilde{h}$. The number of orbits of n-periodic points is equal to the number of zeros of h on the interval $[0, 2\pi/n)$, which is not divisible by n by the assumption. Hence, by the above statement, f cannot be represented in the form g^n for any $g \in \mathrm{Diff}_+(S^1)$.

To prove that the diffeomorphism f has no other n-periodic points apart from the zeros of $\tilde{h}$, suppose, on the contrary, that there exists an n-periodic point θ_0, in which $h(\theta_0) > 0$. Consider the orbit $\{\theta_0, \theta_1, \ldots, \theta_{n-1}, \theta_n = \theta_0\}$ of this point on S^1 under f. Then θ_n may be written in the form

$$\theta_n = f^n(\theta_0) = \theta_0 + \epsilon\left(\tilde{h}(\theta_0) + \tilde{h}(\theta_1) + \cdots + \tilde{h}(\theta_{n-1})\right) \mod 2\pi \ .$$

If $f^n(\theta_0) = \theta_0 \bmod 2\pi$, then $\epsilon\,(h(\theta_0) + \cdots + h(\theta_{n-1})) = 0 \bmod 2\pi$. The coefficient of ϵ in the latter relation is positive and does not exceed n, since $0 \le h \le 1$. Hence, for $\epsilon < \frac{2\pi}{n}$ this relation cannot be true, i.e. $f^n(\theta_0)$ cannot be equal to θ_0 modulo 2π. This contradiction proves that the only n-periodic points of f are those, given by zeros of $\tilde{h}$, which implies that f cannot be represented in the form g^n for any $g \in \mathrm{Diff}_+(S^1)$.

Using the above assertion, one can easily construct concrete examples of diffeomorphisms $f \in \mathrm{Diff}_+(S^1)$, which cannot be represented as the nth power $(n > 1)$ of any diffeomorphism from $\mathrm{Diff}_+(S^1)$. For instance, one can take a diffeomorphism f of the type (2.1) with

$$h(\theta) = \sin^2(n\frac{\theta}{2}) \quad \text{for } 0 \le \theta < 2\pi/n \ .$$

Or, take $h(\theta) = h_0\left(\frac{\pi}{n}(\theta + 1)\right)$, where h_0 is a smooth function on $[-1, 1)$ of the form

$$h_0(t) = (t-1)^2(t+1)^2 \quad \text{or} \quad h_0(t) = e^{1/(t^2-1)} \quad \text{for } -1 \le t < 1 \ .$$

All these diffeomorphisms f cannot be represented as the nth power of any diffeomorphism from $\mathrm{Diff}_+(S^1)$.

2.2.3 Simplicity of $\mathbf{Diff}_+(S^1)$

One of the remarkable properties of the group $\mathrm{Diff}_+(S^1)$ is its simplicity, which means that the only normal subgroups in $\mathrm{Diff}_+(S^1)$ are the identity and the group itself. This fact (which can be anticipated from Prop.2 in Subsec. 2.2.1) was proved by M.R.Herman in [33, 34]. We shall present in this Subsection an idea how to prove the following, somewhat weaker, statement, contained in [33].

Proposition 4. *Any normal subgroup in $\mathit{Diff}_+(S^1)$, containing the rotation subgroup S^1, coincides with the whole group $\mathit{Diff}_+(S^1)$.*

The simplicity property of the group $\mathrm{Diff}_+(S^1)$ is closely related to the following problem, going back to Poincaré and Denjoy: *when a diffeomorphism $f \in \mathit{Diff}_+(S^1)$ is conjugate to a rotation*? We have already touched upon this problem in the proof of Prop.3 in Subsec. 2.2.2. We shall discuss it in more detail after a short digression on the Poincaré rotation number.

Digression 1 (Poincaré rotation number). Let f be an arbitrary diffeomorphism from the group $\mathrm{Diff}_+(S^1)$. Denote by $\tilde{f} : \mathbb{R} \to \mathbb{R}$ its pull-back to $\mathbb{R}$, induced by the universal covering map

$$\mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Z} \approx S^1 \ .$$

Then $\tilde{f}$ is a diffeomorphism of $\mathbb{R}$ of the form $\tilde{f} = \mathrm{id} + h$ with h being a smooth periodic function on the real line with period 1. Denote the set of diffeomorphisms of $\mathbb{R}$ of this form by $\mathrm{Diff}_1(\mathbb{R})$. (Recall that $\tilde{f}$ is determined by f up to an integer additive constant). Note that any shift $\tilde{R}_\lambda : x \mapsto x + \lambda$ of $\mathbb{R}$ by the real number λ projects under the above covering map to the rotation R_α of S^1 by the angle $\alpha \equiv \lambda \bmod 1$.

H.Poincaré has found that any diffeomorphism $\tilde{f} \in \mathrm{Diff}_1(\mathbb{R})$, being iterated sufficiently many times, behaves like a translation $\tilde{R}_\lambda$. More precisely, there exists the uniform limit

$$\frac{\tilde{f}^k - \mathrm{id}}{k} \longrightarrow \lambda \quad \text{for } k \to \infty \ ,$$

where λ is a real number, called the *rotation number* of $\tilde{f}$ and denoted by $\lambda = \tilde{\rho}(\tilde{f})$.

The map $\tilde{\rho} : \mathrm{Diff}_1(\mathbb{R}) \to \mathbb{R}$ is continuous in the C^0-topology. Moreover, for any shift $\tilde{R}_\lambda$ we have the following relations:

$$\tilde{\rho}(\tilde{R}_\lambda) = \lambda \quad \text{and} \quad \tilde{\rho}(\tilde{R}_n \circ \tilde{f}) = n + \tilde{\rho}(\tilde{f}) \quad \text{for any } n \in \mathbb{Z} \ .$$

Therefore, pushing down to S^1, we obtain a correctly defined, continuous map

$$\rho : \mathrm{Diff}_+(S^1) \longrightarrow \mathbb{R}/\mathbb{Z} \approx S^1 \ ,$$

assigning to a diffeomorphism $f \in \mathrm{Diff}_+(S^1)$ its *Poincaré number* $\rho(f) \in S^1$. This number is invariant under conjugations.

If the rotation number of a diffeomorphism $\tilde{f} \in \mathrm{Diff}_1(\mathbb{R})$ is rational, i.e. $\tilde{\rho}(\tilde{f}) = \frac{p}{q}$ for coprime integers p and q, then there is a simple criterion of its conjugacy to a shift, namely: *$\tilde{f}$ is conjugate to the shift $\tilde{R}_{p/q}$ if and only if $\tilde{f}^q = \tilde{R}_p$*.

The situation in the case of an irrational Poincaré number is much more delicate — everything depends on the arithmetic properties of this number. V.I.Arnold (cf.

[4]) gave an example of a diffeomorphism with an irrational Poincaré number, which is not conjugate to a shift, and conjectured that there exists a set $A \subset S^1 \setminus (\mathbb{Q}/\mathbb{Z})$ of a full Haar measure on S^1, such that any diffeomorphism $f \in \mathrm{Diff}_+(S^1)$ with the Poincaré number $\alpha \in A$ is conjugate to the shift R_α. This conjecture was proved by M.R.Herman in [34]. As it was anticipated, the set A in the Herman's theorem has a Diofantine nature and may be described in terms of the decomposition of α into the continuous fraction.

We shall describe here a simpler result by Herman of a similar character, sufficient for the proof of the above Prop. 4.

Recall that, according to the Dirichlet principle, *any irrational number λ may be approximated by rationals so that the following relation holds*

$$\left|\lambda - \frac{p}{q}\right| < \frac{1}{q^2}$$

where $\frac{p}{q} \in \mathbb{Q}$ is an irreducible fraction.

We say that a number λ satisfies the *Diofantine condition* (B_ϵ) with some $\epsilon > 0$, if there exists a constant $C_\epsilon > 0$, such that for all rational numbers p/q the following inequality holds

$$\left|\lambda - \frac{p}{q}\right| \geq \frac{C_\epsilon}{q^{2+\epsilon}} \ .$$

If a number λ satisfies to the Diofantine condition (B_ϵ) for any ϵ (with a constant C_ϵ, depending on ϵ), then λ is called the *Roth number*, and the corresponding $\alpha \in S^1$ form a set of a full Haar measure on the circle. (The numbers, which do not satisfy the condition (B_ϵ) for any $\epsilon > 0$, are called the *Liouville numbers*.)

Lemma 1 (cf. [33]). *Suppose that $\alpha \in S^1 \setminus (\mathbb{Q}/\mathbb{Z})$ satisfies the condition (B_ϵ) for some $\epsilon > 0$. Then there exists a neighborhood U of the rotation R_α in Diff$_+(S^1)$ such that any diffeomorphism $f \in U$ is represented in the form*

$$f = R_\beta \circ \left(g \circ R_\alpha \circ g^{-1}\right)$$

for some $g \in$ Diff$_+(S^1)$ and $\beta \in S^1$.

The proof of this Lemma can be found in [33], we shall only demonstrate how it implies the Prop. 4.

Proof of Proposition 4. Let H be a normal subgroup in $\mathrm{Diff}_+(S^1)$, containing S^1. Take $\alpha \in S^1 \setminus (\mathbb{Q}/\mathbb{Z})$, satisfying the Diofantine condition (B_ϵ) for some $\epsilon > 0$. The rotation $R_\alpha \in H$ (since $H \supset S^1$), and Lemma 1 implies that the whole neighborhood U of R_α belongs to H, due to the normality of H. Hence, the subgroup H is open and so contains a neighborhood of the identity in the group $\mathrm{Diff}_+(S^1)$. It implies that H is also closed, hence it should coincide with the whole group $\mathrm{Diff}_+(S^1)$, due to the connectedness of $\mathrm{Diff}_+(S^1)$. The Proposition is proved. □

Remark 4. We have proved in Prop. 3 from Subsec. 2.2.2 that there are diffeomorphisms from $\mathrm{Diff}_+(S^1)$, which cannot be represented as the exponentials of smooth vector fields on the circle. Using Prop. 4, it's easy to prove that, nevertheless, the exponentials of smooth vector fields generate the whole group $\mathrm{Diff}_+(S^1)$. More precisely, any diffeomorphism $f \in \mathrm{Diff}_+(S^1)$ may be written as the composition

$$f = \exp v_1 \circ \cdots \circ \exp v_k$$

for some vector fields $v_1, \dots, v_k \in \mathrm{Vect}(S^1)$.

Another non-trivial corollary of Prop. 4 is that the group $\mathrm{Diff}_+(S^1)$ *does not admit the complexification.* In other words, *there is no complex Lie group, having the complexified Lie algebra $\mathrm{Vect}^{\mathbb{C}}(S^1)$ as its Lie algebra.*

This statement is the corollary of the following Proposition.

Proposition 5. *There are no non-trivial homomorphisms from the group $\mathrm{Diff}_+(S^1)$ into a connected complex Lie group.*

Proof. Take the Möbius group $\mathrm{PSL}(2,\mathbb{R})$ of fractional linear automorphisms of the unit disc, which can be considered as a subgroup of $\mathrm{Diff}_+(S^1)$. Denote by $G_n := \mathrm{PSL}^{(n)}(2,\mathbb{R})$ the n-fold covering group of $\mathrm{PSL}(2,\mathbb{R})$. More precisely, denote by λ the n-fold covering map of S^1, given by $\lambda : z \mapsto z^n$. Then, by definition, G_n consists of the diffeomorphisms of S^1, which are the n-fold coverings of diffeomorphisms from $\mathrm{PSL}(2,\mathbb{R})$. It means that for any $\varphi \in G_n$ there exists an element $\psi \in \mathrm{PSL}(2,\mathbb{R})$ such that

$$\lambda \circ \varphi = \psi \circ \lambda \ .$$

On the level of Lie algebras, the Lie algebra $\mathrm{sl}(2,\mathbb{R})$ is generated by the vector fields $\frac{d}{d\theta}$, $\sin\theta \frac{d}{d\theta}$, $\cos\theta \frac{d}{d\theta}$, and the Lie algebra of the group G_n (isomorphic to $\mathrm{sl}(2,\mathbb{R})$) is generated by the vector fields $\frac{d}{d\theta}$, $\sin(n\theta)\frac{d}{d\theta}$, $\cos(n\theta)\frac{d}{d\theta}$.

The center of the group G_n consists of rotations $\{R_{2\pi k/n} : k = 0, 1, \dots, n-1\}$. And it can be proved, as in Subsec.2.1.3, that any homomorphism from G_n to a complex connected Lie group should factor through $\mathrm{PSL}(2,\mathbb{R})$. In other words, its kernel contains all rotations from the centre of G_n. It follows that the kernel of any homomorphism from $\mathrm{Diff}_+(S^1)$ into a complex connected Lie group should contain all rotations of the form $\{R_{2\pi k/n} : k = 0, 1, \dots, n-1\}$ for any n, hence, all rotations from S^1. But this kernel is a normal subgroup in $\mathrm{Diff}_+(S^1)$, and any normal subgroup in $\mathrm{Diff}_+(S^1)$, containing S^1, should coincide, according to Prop. 4, with the whole group $\mathrm{Diff}_+(S^1)$. This proves that there are no non-trivial homomorphisms from $\mathrm{Diff}_+(S^1)$ into a connected complex Lie group. □

Bibliographic comments

General properties of Frechet Lie groups and algebras are described in Hamilton's paper [32], already mentioned in the bibliographic comments to Ch.1, and Milnor's paper [55]. (Cf. also [47] for the case of Banach Lie groups.)

Key references to Secs. 2.1 and 2.2 are the Pressley–Segal book [65] and Hamilton's paper [32]. In particular, Prop. 1 is formulated in Sec.3.2 of [65], as well as the example before this Proposition (the proof of Prop.1, proposed by K.A.Trushkin, is borrowed from [80]). The Prop.2 is proved, e.g., in [71] (cf. also [80]). The property of the exponential map of the Lie algebra $\mathrm{Vect}(S^1)$, stated in Prop.3, was pointed out by R.S.Hamilton in [32]. A solution to the problem, considered in Rem.3, was proposed by V.V.Kruglov in [80]. The simplicity of the group of diffeomorphisms $\mathrm{Diff}_+(S^1)$, as well as its weaker version, stated in Prop.4, is due to Herman [33, 34]. Prop.5 on the non-existence of the complexification of $\mathrm{Diff}_+(S^1)$ is proved in [65].

Chapter 3

Flag manifolds and representations

Flag manifolds are finite-dimensional compact Kähler manifolds, homogeneous with respect to a Lie group action. They can be characterized by the existence of two kinds of homogeneous space representations, namely, a "real" one, as a quotient of a compact Lie group G, and a "complex" one, as a quotient of the complexified Lie group $G^{\mathbb{C}}$. The real representation implies that the flag manifold is compact and homogeneous with respect to the G-action by left shifts, and the complex representation implies that it is a complex Kähler manifold.

Flag manifolds are closely related to the representation theory of the group G via the Borel–Weil construction. We present this construction in Subsec. 3.2.2 together with a necessary background from the representation theory of semisimple Lie groups, given in Subsec. 3.2.1. In the last Subsec. 3.2.3 we give an outline of the orbit method, related to the coadjoint representation of G, which stands behind many constructions in this book.

3.1 Flag manifolds

3.1.1 Geometric definition of flag manifolds

To define flag manifolds in $\mathbb{C}$, we fix a decomposition of n into the sum of natural numbers

$$n = k_1 + \cdots + k_r$$

and denote $\mathbf{k} = (k_1, \ldots, k_r)$.

Definition 21. A *flag manifold of type* $\mathbf{k}$ in $\mathbb{C}^n$ is the space

$$\begin{aligned} \mathrm{Fl}_{\mathbf{k}}(\mathbb{C}^n) = \{&\text{collections of flags } \mathbf{E} = (E_1, \ldots, E_r) : \ E_i \text{ are linear subspaces} \\ &\text{in } \mathbb{C}^n\text{:} \ E_1 \subset \ldots \subset E_r \text{ with } \dim E_i = k_1 + \ldots + k_i\} \ . \end{aligned} \tag{3.1}$$

In particular, for $\mathbf{k} = (k, n-k)$ we obtain

$$\mathrm{Fl}_{(k,n-k)}(\mathbb{C}^n) = \{\text{subspaces } E \subset \mathbb{C}^n \text{ of dimension } k\} = \mathrm{Gr}_k(\mathbb{C}^n) \ ,$$

i.e. the flag manifold in this case is the same as the *Grassmann manifold* of k-dimensional subspaces in $\mathbb{C}^n$. For $k = 1$ it coincides with the $(n-1)$-dimensional complex projective space $\mathrm{Fl}_{(1,n-1)}(\mathbb{C}^n) = \mathbb{CP}^{n-1}$.

For $\mathbf{k} = (1, \dots, 1)$ the manifold $\mathrm{Fl}_{\mathbf{k}}(\mathbb{C}^n) =: \mathrm{Fl}(\mathbb{C}^n)$ is called the *full flag manifold*

$$\mathrm{Fl}(\mathbb{C}^n) = \{E_1 \subset E_2 \subset \cdots \subset E_{n-1} \subset E_n = \mathbb{C}^n : \dim E_i = i\} \ .$$

The unitary group $\mathrm{U}(n)$ acts transitively on the flag manifold $\mathrm{Fl}_{\mathbf{k}}(\mathbb{C}^n)$, so that $\mathrm{Fl}_{\mathbf{k}}(\mathbb{C}^n)$ coincides with a homogeneous space of this group. In more detail, fix a basis in $\mathbb{C}^n$ and denote by $\mathbf{E}^0$ the *standard flag* in $\mathrm{Fl}_{\mathbf{k}}(\mathbb{C}^n)$ with $\mathbf{E}^0 = (E_1^0, \dots, E_r^0)$, where E_i^0 is the subspace in $\mathbb{C}^n$, generated by the first $k_1 + \cdots + k_i$ vectors of our basis. The isotropy subgroup of $\mathrm{U}(n)$ at the point $\mathbf{E}^0$ coincides with the direct product

$$\mathrm{U}_{\mathbf{k}}(n) = \mathrm{U}(k_1) \times \cdots \times \mathrm{U}(k_r) \ ,$$

so that the flag manifold $\mathrm{Fl}_{\mathbf{k}}(\mathbb{C}^n)$ is a homogeneous space of $\mathrm{U}(n)$ of the form

$$\mathrm{Fl}_{\mathbf{k}}(\mathbb{C}^n) = \mathrm{U}(n)/\,\mathrm{U}_{\mathbf{k}}(n) = \mathrm{U}(n)/\,\mathrm{U}(k_1) \times \cdots \times \mathrm{U}(k_r) \ . \tag{3.2}$$

On the other hand, the complex general linear group $\mathrm{GL}(n, \mathbb{C})$ is also acting on $\mathrm{Fl}_{\mathbf{k}}(\mathbb{C}^n)$ transitively. The isotropy subgroup at the standard flag $\mathbf{E}^0 \in \mathrm{Fl}_{\mathbf{k}}(\mathbb{C}^n)$ coincides in this case with the subgroup $P_{\mathbf{k}}$ of blockwise upper-triangular matrices of the form

$$\begin{pmatrix} \boxed{*}_{r_1} & * & * \ \cdots & * \\ 0 & \boxed{*}_{r_2} & * \ \cdots & * \\ \vdots & & \ddots & \vdots \\ 0 & 0 & 0 \ \cdots & \boxed{*}_{r_n} \end{pmatrix}$$

So, along with the "real" homogeneous representation (3.2), we obtain for $\mathrm{Fl}_{\mathbf{k}}(\mathbb{C}^n)$ a "complex" representation as a homogeneous space of the group $\mathrm{GL}(n, \mathbb{C})$:

$$\mathrm{Fl}_{\mathbf{k}}(\mathbb{C}^n) = \mathrm{GL}(n, \mathbb{C})/\,P_{\mathbf{k}} \ . \tag{3.3}$$

In the particular cases $\mathbf{k} = (k, n-k)$ and $\mathbf{k} = (1, \dots, 1)$ we get the well known homogeneous representations for the Grassmann manifold

$$\mathrm{Gr}_k(\mathbb{C}^n) = \mathrm{U}(n)/\mathrm{U}(k) \times \mathrm{U}(n-k) = \mathrm{GL}(n, \mathbb{C})/P_{(k,n-k)}$$

and the full flag manifold

$$\mathrm{Fl}(\mathbb{C}^n) = \mathrm{U}(n)/T^n = \mathrm{GL}(n, \mathbb{C})/B_+ \ ,$$

where $T^n = \mathrm{U}(1) \times \cdots \times \mathrm{U}(1)$ is the n-dimensional torus, and $B_+ = P_{(1,\dots,1)}$ is the standard Borel subgroup of upper-triangular matrices.

Note that the flag manifold $\mathrm{Fl}_{\mathbf{k}}(\mathbb{C}^n)$ can be represented also as a homogeneous space of a complex semisimple Lie group by replacing the group $\mathrm{GL}(n, \mathbb{C})$ with $\mathrm{SL}(n, \mathbb{C})$. The corresponding homogeneous representations will take the form

$$\mathrm{Fl}_{\mathbf{k}}(\mathbb{C}^n) = \mathrm{SU}(n)/\mathrm{SU}_{\mathbf{k}}(n) = \mathrm{SL}(n, \mathbb{C})/\mathrm{S}P_{\mathbf{k}} \ ,$$

where

$$\mathrm{SU}_{\mathbf{k}}(n) = \mathrm{S}\left[\mathrm{U}(k_1) \times \ldots \times \mathrm{U}(k_n)\right] = \mathrm{U}(k_1) \times \ldots \times \mathrm{U}(k_n) \cap \mathrm{SL}(n, \mathbb{C}),$$
$$\mathrm{S}P_{\mathbf{k}}(n) = P_{\mathbf{k}} \cap \mathrm{SL}(n, \mathbb{C}) \ .$$

3.1.2 Borel and parabolic subalgebras

To give an invariant definition of flag manifolds, we need some basic notions, related to the Borel and parabolic subalgebras. We recall them here, assuming that a reader is familiar with the basics of the theory of semisimple Lie algebras and groups, presented, e.g., in [76, 75, 28, 67].

Let $G_{\mathbb{C}}$ be a complex semisimple Lie group with the Lie algebra $\mathfrak{g}_{\mathbb{C}}$.

Recall that a *Cartan subalgebra* in $\mathfrak{g}_{\mathbb{C}}$ is a maximal Abelian subalgebra $\mathfrak{h}_{\mathbb{C}}$ in $\mathfrak{g}_{\mathbb{C}}$, for which all the operators $\operatorname{ad} x$, $x \in \mathfrak{h}_{\mathbb{C}}$, are diagonal in $\mathfrak{g}_{\mathbb{C}}$. All Cartan subalgebras in $\mathfrak{g}_{\mathbb{C}}$ are conjugate to each other with respect to the adjoint action of the group $G_{\mathbb{C}}$ on its Lie algebra $\mathfrak{g}_{\mathbb{C}}$. A standard example of the Cartan subalgebra in the case of the general matrix algebra $\mathfrak{g}_{\mathbb{C}} = \mathrm{gl}(n, \mathbb{C})$ is the algebra of all diagonal matrices in $\mathfrak{g}_{\mathbb{C}}$.

We fix now a Cartan subalgebra $\mathfrak{h}_{\mathbb{C}}$ in a complex semisimple Lie algebra $\mathfrak{g}_{\mathbb{C}}$ and consider the adjoint action ad of $\mathfrak{h}_{\mathbb{C}}$ on the Lie algebra $\mathfrak{g}_{\mathbb{C}}$. Note that the operators $\operatorname{ad} h$ for different $h \in \mathfrak{h}_{\mathbb{C}}$ commute with each other. The eigenspaces of the adjoint representation, having the form

$$\mathfrak{g}_{\alpha} = \{\xi \in \mathfrak{g}^{\mathbb{C}} : \operatorname{ad} h(\xi) = \alpha(h)\xi\} \ ,$$

where α is a linear functional on $\mathfrak{h}_{\mathbb{C}}$ (i.e. an element of the dual space $\mathfrak{h}^*_{\mathbb{C}}$), are called the *root subspaces*. The linear functionals α, entering into this definition, are called the *roots* of the algebra $\mathfrak{g}^{\mathbb{C}}$ with respect to the Cartan subalgebra $\mathfrak{h}_{\mathbb{C}}$, and the eigenvectors ξ are called the *root vectors*. In particular, the root subspace $\mathfrak{g}_0$, corresponding to the zero functional $\alpha = 0 \in \mathfrak{h}^*$, coincides with the Cartan subalgebra $\mathfrak{h}_{\mathbb{C}}$ itself.

The Lie algebra $\mathfrak{g}^{\mathbb{C}}$ decomposes into the direct sum of its root subspaces

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_{\alpha} \ , \tag{3.4}$$

where Δ denotes the set of all nonzero roots of the algebra $\mathfrak{g}^{\mathbb{C}}$ with respect to the Cartan subalgebra $\mathfrak{h}_{\mathbb{C}}$. This decomposition, called the *root decomposition*, determines a filtration in $\mathfrak{g}^{\mathbb{C}}$, since

$$[\mathfrak{g}_{\alpha}, \mathfrak{g}_{\beta}] \subset \mathfrak{g}_{\alpha+\beta} \ .$$

A subset $\Pi \subset \Delta$ is called the *set of simple roots*, if any root $\alpha \in \Delta$ can be represented as a linear combination of roots from Π with integer coefficients, such that all of them are either positive, or (all of them are) negative. Such subsets Π, forming bases in $\mathfrak{h}^*_{\mathbb{C}}$, always exist. It can be shown that all of them are conjugate to each other with respect to the coadjoint action of the group $G_{\mathbb{C}}$.

Fix some set $\Pi = \{\alpha_1, \ldots, \alpha_l\}$ of simple roots of the algebra $\mathfrak{g}_{\mathbb{C}}$. The choice of Π defines on $\mathfrak{h}_{\mathbb{C}}^*$ (hence, on Δ) a *partial ordering*, namely, for $\alpha, \beta \in \mathfrak{h}_{\mathbb{C}}^*$ the relation $\alpha \geq \beta$ means that

$$\alpha - \beta = \sum_{i=1}^{l} a_i \alpha_i \quad \text{with} \quad a_i \geq 0 \ .$$

In particular, a root $\alpha \in \Delta$ is called *positive* (notation: $\alpha \in \Delta^+$), if

$$\alpha = \sum_{i=1}^{l} a_i \alpha_i \quad \text{with} \quad a_i > 0 \ .$$

Using the Killing form $(\cdot, \cdot)$ on $\mathfrak{g}_{\mathbb{C}}$, we can identify the dual space $\mathfrak{h}_{\mathbb{C}}^*$ with $\mathfrak{h}_{\mathbb{C}}$, so that any root α can be considered also as an element α^* of $\mathfrak{h}_{\mathbb{C}}$. We associate with a root α of the algebra $\mathfrak{g}_{\mathbb{C}}$ with respect to $\mathfrak{h}_{\mathbb{C}}$ the *dual root* or *co-root* $\alpha^\vee$ by the formula

$$\alpha^\vee = 2\frac{\alpha^*}{(\alpha, \alpha)} \ .$$

It is well known that *a system of simple roots* $\Pi = \{\alpha_1, \ldots, \alpha_l\}$ *and its Cartan matrix*, defined by:

$$c_{ij} := (\alpha_i, \alpha_j^\vee) \ ,$$

uniquely determine the Lie algebra $\mathfrak{g}_{\mathbb{C}}$.

Example 23. Consider as an example the complex semisimple Lie algebra $\mathrm{sl}(n, \mathbb{C})$. Choose in $\mathrm{sl}(n, \mathbb{C})$ the standard Cartan subalgebra $\mathfrak{h}_{\mathbb{C}}$, consisting of diagonal matrices. Denote by E_{ij} the matrix, having 1 at the (i, j)th place, and zeros at all other places. The matrices E_{ij} are the root vectors of the algebra $\mathrm{sl}(n, \mathbb{C})$:

$$\mathrm{ad}(z_1, \ldots, z_n) E_{ij} = (z_i - z_j) E_{ij} \ ,$$

where we denote by $(z_1, \ldots, z_n)$ the diagonal matrix $\mathrm{diag}(z_1, \ldots, z_n)$.

Introduce a functional $\epsilon_i \in \mathfrak{h}_{\mathbb{C}}^*$ by the formula

$$\epsilon_i(z_1, \ldots, z_n) = z_i \ .$$

Then the roots of the algebra $\mathrm{sl}(n, \mathbb{C})$ with respect to $\mathfrak{h}_{\mathbb{C}}$ will have the form

$$\Delta = \{\epsilon_i - \epsilon_j \ : \ i \neq j\} \ .$$

The roots

$$\Pi = \{\epsilon_i - \epsilon_{i+1} \ : \ i = 1, \ldots, n-1\}$$

form a system of simple roots, so that the set of positive roots is given by

$$\Delta^+ = \{\epsilon_i - \epsilon_j \ : \ i < j\} \ .$$

By analogy with the Borel subalgebra of upper-triangular matrices in $\mathrm{gl}(n, \mathbb{C})$, we can define a *standard Borel subalgebra* $\mathfrak{b}_+$ of a complex semisimple Lie algebra $\mathfrak{g}_{\mathbb{C}}$ as

$$\mathfrak{b}_+ = \mathfrak{h}_{\mathbb{C}} \oplus \mathfrak{n}_+ \ ,$$

where $\mathfrak{n}_+$ is a nilpotent subalgebra of the form

$$\mathfrak{n}_+ = \bigoplus_{\alpha\in\Delta^+} \mathfrak{g}_\alpha \ .$$

In the particular case of the algebra $\mathrm{sl}(n,\mathbb{C})$, considered in Ex. 23 above, the subalgebra $\mathfrak{n}_+$ coincides with the subalgebra of above-diagonal matrices, while $\mathfrak{b}_+$ is the subalgebra of upper-triangular matrices.

Definition 22. A *Borel subalgebra* is a subalgebra $\mathfrak{b}$ in $\mathfrak{g}_\mathbb{C}$, conjugate to the standard Borel subalgebra $\mathfrak{b}_+$ with respect to the adjoint action of the group $G_\mathbb{C}$ on $\mathfrak{g}_\mathbb{C}$. (In a more invariant way, a Borel subalgebra is a maximal solvable subalgebra in $\mathfrak{g}_\mathbb{C}$.) Any subalgebra $\mathfrak{p}$ in $\mathfrak{g}_\mathbb{C}$, containing a Borel subalgebra $\mathfrak{b}$, is called *parabolic*.

As in the case of Borel subalgebras, we could define the parabolic subalgebras $\mathfrak{p}$ as subalgebras in $\mathfrak{g}_\mathbb{C}$, which are conjugate to one of standard parabolic subalgebras. These standard subalgebras (their explicit description is given below) are analogous to the parabolic subalgebras $\mathfrak{p}_\mathbf{k}$ of the algebra $\mathrm{gl}(n,\mathbb{C})$, being the Lie algebras of the parabolic subgroups $P_\mathbf{k}$ from Sec. 3.1.1.

Now we define the standard parabolic subalgebras in $\mathfrak{g}_\mathbb{C}$ explicitly. For that fix a set $\Pi = \{\alpha_1,\dots,\alpha_l\}$ of simple roots of the algebra $\mathfrak{g}_\mathbb{C}$ and an arbitrary ordered subset π in the set $\{1,\dots,l\}$. We associate with π a subset of simple roots $\Pi_\pi \subset \Pi$ with indices from π. To define the corresponding standard parabolic subalgebra $\mathfrak{p}_\pi$, we denote by Δ_π the linear span of simple roots from Π_π in Δ and introduce a *reductive Levi subalgebra* of the form

$$\mathfrak{l}_\mathbb{C} = \mathfrak{h}_\mathbb{C} \oplus \bigoplus_{\alpha\in\Delta_\pi} \mathfrak{g}_\alpha \ .$$

We define also a *nilpotent subalgebra* in $\mathfrak{g}_\mathbb{C}$ by setting

$$\mathfrak{u} = \bigoplus_{\alpha\in\Delta^+\setminus\Delta_\pi} \mathfrak{g}_\alpha \ .$$

The *standard parabolic subalgebra* $\mathfrak{p}_\pi$ is by definition

$$\mathfrak{p}_\mathbf{k} = \mathfrak{l}_\mathbb{C} \oplus \mathfrak{u} \ .$$

It contains the standard parabolic subalgebra $\mathfrak{b}_+$ and so is, indeed, parabolic. In the case of the algebra $\mathrm{sl}(n,\mathbb{C})$ the subalgebra $\mathfrak{l}_\mathbb{C}$ coincides with the subalgebra of block-diagonal matrices in $\mathrm{sl}(n,\mathbb{C})$, while $\mathfrak{u}$ is the subalgebra of blockwise above-diagonal matrices.

3.1.3 Algebraic definition of flag manifolds

After this algebraic digression, we can give an invariant definition of flag manifolds of a complex semisimple Lie group $G_\mathbb{C}$.

Definition 23. Let $\mathfrak{p}$ be an arbitrary parabolic subalgebra in $\mathfrak{g}_\mathbb{C}$ and P is the corresponding parabolic subgroup in $G_\mathbb{C}$, having $\mathfrak{p}$ as its Lie algebra. (Otherwise,

P can be defined as the normalizer $N(\mathfrak{p})$ of the subalgebra $\mathfrak{p}$ in $G_{\mathbb{C}}$ with respect to the adjoint representation.) A *flag manifold* of the group $G_{\mathbb{C}}$, associated with the parabolic subalgebra $\mathfrak{p}$, is a homogeneous space of the form

$$F = G_{\mathbb{C}}/P \ . \tag{3.5}$$

Along with the "complex" representation (3.5), taken as the definition of the flag manifold F, there exists also a "real" representation of F as a homogeneous space of a real Lie group. Namely, suppose that the group $G_{\mathbb{C}}$ coincides with the complexification $G^{\mathbb{C}}$ of a compact Lie group G. Then G acts transitively on $G_{\mathbb{C}}/P$ and

$$F = G/G \cap P = G/L \ , \tag{3.6}$$

where the *Levi subgroup* $L = G \cap P$ in the case of the standard parabolic subalgebra $\mathfrak{p}$ has the Lie algebra, given by the real form $\mathfrak{l}$ of the Levi subalgebra $\mathfrak{l}_{\mathbb{C}} = \mathfrak{l}^{\mathbb{C}}$, introduced above in Subsec. 3.1.2. (In a more invariant way, the subgroup L can be defined as the centralizer of a torus in G.)

Hence, we have obtained for the flag manifold F two kinds of representations as a homogeneous space

$$F = G/L = G^{\mathbb{C}}/P \ .$$

The complex representation (3.5) implies that F is a complex manifold, provided with a G-invariant complex structure. The space of tangent vectors of type $(1,0)$ at the origin with respect to this structure can be identified with the subalgebra $\bar{\mathfrak{u}}$ in the decomposition

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{l}^{\mathbb{C}} \oplus \mathfrak{u} \oplus \bar{\mathfrak{u}} \ , \quad \mathfrak{p} = \mathfrak{l}^{\mathbb{C}} \oplus \mathfrak{u} \ ,$$

where the complex conjugation in $\mathfrak{g}^{\mathbb{C}}$ has the property that $\bar{\mathfrak{g}} = \mathfrak{g}$.

The real representation (3.6) implies that F is compact and Kähler. We note also that F is simply connected, if the group G is simply connected. It can be shown that flag manifolds F exhaust all simply connected compact Kähler G-manifolds with the transitive action of a compact semisimple Lie group G (cf. [10, 77]).

Remark 5. The real representation (3.6) implies that that the Lie algebra $\mathfrak{p}$ of the parabolic group P has the form

$$\mathfrak{p} = \mathfrak{l}^{\mathbb{C}} \oplus \mathfrak{u}$$

where $\mathfrak{l}^{\mathbb{C}}$ is the Levi subalgebra and $\mathfrak{u}$ is the nilpotent subalgebra of $\mathfrak{p}$, described in Subsec. 3.1.2 for the case of the standard parabolic subalgebras. The parabolic subalgebras can be defined also in terms of the so called canonical element.

Namely, *for any parabolic subalgebra* $\mathfrak{p}$ *there exists a unique element* ξ *(belonging to the center of the Levi subalgebra* $\mathfrak{l}^{\mathbb{C}}$*), for which the operator* $ad\xi$ *has only imaginary integer eigenvalues, belonging to* $\sqrt{-1}\mathbb{Z}$*. Such an element* ξ *is called the* canonical element *of the parabolic subalgebra* $\mathfrak{p}$. (This fact is proved, e.g., in [15], Theor. 4.4.)

We use this equivalent definition of parabolic subalgebras for the construction of a certain canonical bundle, associated with a flag manifold. The importance of the canonical bundle will become clear in Sec. 7.5, where we show that the loop space ΩG can be considered as a universal flag manifold of the group G.

Denote by $\mathfrak{g}_j$ the eigenspace of the operator $\operatorname{ad}\xi$ with the eigenvalue $\sqrt{-1}j$. In terms of $\mathfrak{g}_j$ the parabolic subalgebra $\mathfrak{p}$ and nilpotent subalgebra $\mathfrak{u}$ can be described as

$$\mathfrak{p} = \bigoplus_{i\geq 0} \mathfrak{g}_i \ , \quad \mathfrak{u} = \bigoplus_{i\geq 1} \mathfrak{g}_i \ .$$

We define now a symmetric space $N = N(F)$, canonically associated with the flag manifold F, by setting

$$N = G/K \ ,$$

where K is a subgroup of G with the Lie algebra

$$\mathfrak{k} = \mathfrak{g} \cap \left[\bigoplus_i \mathfrak{g}_{2i} \right] \ .$$

Since the Lie algebra $\mathfrak{l}$ of the Levi group L is contained in $\mathfrak{g}_0$, there exists a homogeneous bundle

$$F = G/L \longrightarrow G/K = N$$

of the flag manifold F over the associated symmetric space N. So we have constructed for our flag manifold F the associated *symmetric G-space* $N = N(F)$ and *canonical homogeneous bundle* $F \to N$. Note that the symmetric space N is uniquely determined by F, while the canonical bundle $F \to N$ is not uniquely defined, due to the fact that different points of N may have the same stabilizer K. The number of such points is finite, so there exist only a finite number of canonical bundles of the above type.

The importance of flag manifolds is due, in particular, to the fact that all irreducible representations of the group G can be realized in spaces of holomorphic sections of complex line bundles over the flag manifolds of G. This is the Borel–Weil construction, given in Subsec. 3.2.2. To explain this construction, we need some basic facts from the representation theory of complex semisimple Lie groups, collected in the next Subsec. 3.2.1 (cf. for a more detailed exposition [75, 76, 28, 39, 67]).

3.2 Irreducible representations

3.2.1 Irreducible representations of complex semisimple Lie groups

Let $\mathfrak{h}_{\mathbb{C}}$ be a Cartan subalgebra of a complex semisimple Lie algebra $\mathfrak{g}_{\mathbb{C}}$ and $\rho : \mathfrak{g}_{\mathbb{C}} \to \operatorname{End} V$ is a representation of the algebra $\mathfrak{g}_{\mathbb{C}}$ in a complex vector space V.

A *weight of the representation* ρ is a linear functional $\lambda \in \mathfrak{h}_{\mathbb{C}}^*$, for which there exists a vector $v \in V \setminus \{0\}$, called the *weight vector*, such that

$$\rho(h)v = \lambda(h)v \quad \text{for any } h \in \mathfrak{h}_{\mathbb{C}} \ .$$

The linear subspace V_λ, consisting of the vectors $v \in V$, satisfying the relation $\rho(h)v = \lambda(h)v$ for any $h \in \mathfrak{h}_{\mathbb{C}}$, is called the *weight subspace* of weight λ.

Denote by $\Delta_\rho(V) \subset \mathfrak{h}_\mathbb{C}^*$ the set of weights of the representation ρ. There is a *weight decomposition* of ρ, analogous to the root decomposition (3.4) for the adjoint representation $\rho = \mathrm{ad}$ from Subsec. 3.1.2. It has the form

$$V = \bigoplus_{\lambda \in \Delta_\rho(V)} V_\lambda \ ,$$

where V_λ is the weight subspace of weight λ.

Fix a system $\Pi = \{\alpha_1, \dots, \alpha_l\}$ of simple roots of the algebra $\mathfrak{g}_\mathbb{C}$ with respect to $\mathfrak{h}_\mathbb{C}$. Among the weights of a representation the special role is played by the *highest weights*, which are the maximal elements in the set of weights of a representation with respect to the partial ordering on $\mathfrak{h}_\mathbb{C}^*$, introduced in Subsec. 3.1.2. A highest weight Λ of a representation ρ is characterized by the property that its weight vector v is annihilated by the nilpotent subalgebra $\mathfrak{n}_+$, i.e.

$$\rho(\xi)v = 0 \quad \text{for any } \xi \in \mathfrak{n}_+ \ .$$

We associate with a system $\Pi = \{\alpha_1, \dots, \alpha_l\}$ of simple roots of the algebra $\mathfrak{g}_\mathbb{C}$ the dual system of weights $\{\omega_1, \dots, \omega_l\}$, defined by the relation

$$(\omega_i, \alpha_j^\vee) = \delta_{ij} \ ,$$

where $\alpha_j^\vee$ is the co-root, associated with α_j (cf. Subsec. 3.1.2). The elements $\omega_1, \dots, \omega_l \in \mathfrak{h}_\mathbb{C}^*$ are called the *fundamental weights* and form a basis in the space of weights, so that any weight $\lambda \in \mathfrak{h}_\mathbb{C}^*$ can be written in the form

$$\lambda = \sum_j (\lambda, \alpha_j^\vee)\omega_j$$

and is uniquely determined by the coefficients $(\lambda, \alpha_j^\vee)$. A weight λ is called *dominant*, if all the coefficients $(\lambda, \alpha_j^\vee)$ are non-negative integers.

The highest weights characterize uniquely an irreducible representation of a complex semisimple Lie algebra. More precisely, we have the following

Theorem 1. *Let ρ be an irreducible representation of a complex semisimple Lie algebra $\mathfrak{g}_\mathbb{C}$. Then it has a unique highest weight Λ. This weight is dominant and any other weight $\lambda \in \Delta_\rho(V)$ can be written in the form*

$$\lambda = \Lambda - \alpha_{i_1} - \dots - \alpha_{i_k} \ , \quad \text{where } \alpha_{i_j} \in \Pi \ .$$

An irreducible representation is uniquely determined (up to equivalence) by its highest weight.

We add a comment on the last statement of the Theorem. An irreducible representation can be reconstructed from its highest weight Λ in the following way. Let v_Λ be the weight vector, corresponding to the weight Λ. Then by definition

$$\rho(\xi)v_\Lambda = 0 \quad \text{for any } \xi \in \mathfrak{n}_+ \ , \tag{3.7}$$

$$\rho(h)v_\Lambda = \Lambda(h)v_\Lambda \quad \text{for any } h \in \mathfrak{h}_\mathbb{C} \ . \tag{3.8}$$

Consider the vectors, which can be obtained by the action of elements of the nilpotent subalgebra $\mathfrak{n}_- = \bigoplus_{\alpha\in\Delta^+} \mathfrak{g}_{-\alpha}$ on the highest vector v_Λ. More precisely, denote

$$v_{i_1\ldots i_k} = \rho(\xi_{-i_k}) \cdot \cdots \cdot \rho(\xi_{-i_1}) v_\Lambda \ ,$$

where $\xi_{-i} \in \mathfrak{g}_{-\alpha_i}$. Then the vectors $\{v_\Lambda, v_{i_1\ldots i_k}\}$ generate a subspace $\hat{V}$ with a natural action of the representation ρ. The required representation space V is obtained from $\hat{V}$ by taking the quotient with respect to the maximal invariant subspace in $\hat{V}$ (different from $\hat{V}$) and providing it with the induced action of the representation ρ.

In the representation theory of loop groups LG it is customary to use, instead of the highest and dominant weights, the *lowest* and *antidominant weights*, dual to the introduced highest and dominant weights. The main reason for that is that the Borel–Weil construction of irreducible representations of complex semisimple Lie groups, presented in the next Subsec. 3.2.2, is naturally formulated in terms of the lowest and antidominant weights. In order to switch to the lowest and antidominant weights in the above definitions, it's sufficient to replace the nilpotent subalgebra $\mathfrak{n}_+$ with its counter-part $\mathfrak{n}_-$, defined by

$$\mathfrak{n}_- = \bigoplus_{\alpha\in\Delta^+} \mathfrak{g}_{-\alpha} \ .$$

It follows, in particular, that a weight λ is antidominant if and only if the weight $-\lambda$ is dominant. If V is a representation of an algebra $\mathfrak{g}_{\mathbb{C}}$ with a highest weight Λ, then the representation of $\mathfrak{g}_{\mathbb{C}}$ with the lowest weight $-\Lambda$ is realized in the dual vector space V^*. The above Theorem 1 admits an evident reformulation in terms of antidominant lowest weights.

3.2.2 Borel–Weil construction

The Borel–Weil construction, presented in this Subsection, realizes the irreducible representation of a complex semisimple Lie group, associated with a given lowest weight (or a character of the Cartan subgroup), in a space of holomorphic sections of a complex line bundle over the full flag manifold.

Suppose that a Lie group $G^{\mathbb{C}}$ is the complexification of a compact Lie group G and H is its Cartan subgroup. A *character* of H is a homomorphism $\lambda : H \to \mathbb{C}^*$ into the multiplicative group of nonzero complex numbers $\mathbb{C}^*$. The group $X(H)$ of all characters of H is a free Abelian group of rank, equal to $\dim H$. If $\lambda \in X(H)$ is a character of H, then the map λ_*, tangent to λ, is linear, hence, belongs to the dual space $\mathfrak{h}^*$. This defines a monomorphism of the group $X(H)$ into $\mathfrak{h}^*$, which allows to identify a character λ with the corresponding linear functional λ_*.

Suppose now that the subgroup H is a *maximal torus* (i.e. H is a maximal subgroup in $G^{\mathbb{C}}$, isomorphic to the product of several copies of the group $\mathbb{C}^*$). Let $R : G^{\mathbb{C}} \to \mathrm{GL}(V)$ be a linear representation of the group $G^{\mathbb{C}}$. If $\lambda \in X(H)$ is a character of H, then, by analogy with Subsec. 3.2.1, it is called the *weight of the representation* R, if there exists a vector $v \in V \setminus \{0\}$, called the *weight vector*, such that

$$R(h)v = \lambda(h)v \quad \text{for any } h \in H \ . \tag{3.9}$$

The vectors $v \in V$, satisfying the relation (3.9), form the *weight subspace* V_λ, associated with weight λ.

Any representation $R : G^{\mathbb{C}} \to \mathrm{GL}(V)$ of the group $G^{\mathbb{C}}$ admits a *weight decomposition*

$$V = \bigoplus_{\text{weights } \lambda \text{ of } R} V_\lambda \ ,$$

where the summation is taken over the weights $\lambda \in X(H)$ of the representation R. This decomposition is analogous to the weight decomposition from Subsec. 3.2.1 in the case of Lie algebras. Moreover, the weights of the representation R of the group $G^{\mathbb{C}}$ may be identified with the corresponding weights of the associated representation $R_* : \mathfrak{g}^{\mathbb{C}} \to \operatorname{End} V$ of the Lie algebra $\mathfrak{g}^{\mathbb{C}}$, and the associated weight subspaces coincide.

Assume now that the maximal complex torus H is the complexification of some maximal torus T in G. By analogy with Subsec. 3.1.1, we define the full flag manifold F, associated with T, as

$$F = G/T = G^{\mathbb{C}}/B_+ \ , \tag{3.10}$$

where B_+ is the standard Borel subgroup in $G^{\mathbb{C}}$, having the standard Borel subalgebra $\mathfrak{b}_+$ from Subsec. 3.1.2 as its Lie algebra. On the Lie algebra level the homogeneous representations (3.10) correspond to the decompositions

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{t}^{\mathbb{C}} \oplus \mathfrak{n}_+ \oplus \mathfrak{n}_- = \mathfrak{b}_+ \oplus \mathfrak{n}_- \ . \tag{3.11}$$

Let $\lambda \in X(H)$ be a character of H, associated with a lowest weight vector of the algebra $\mathfrak{g}^{\mathbb{C}}$. It can be extended to a holomorphic homomorphism $\lambda : B_+ \to \mathbb{C}^*$ of the Borel subgroup B_+, by setting it equal to 1 on the Lie subgroup N_+, having the nilpotent subalgebra $\mathfrak{n}_+$ as its Lie algebra. We define a *complex homogeneous line bundle L_λ over the flag manifold $F = G^{\mathbb{C}}/B_+$, associated with the character λ*:

$$\begin{array}{c} L_\lambda = G^{\mathbb{C}} \times_{B_+} \mathbb{C} \\ \Big\downarrow \\ F = G^{\mathbb{C}}/B_+ \end{array} \quad ,$$

where $G^{\mathbb{C}} \times_{B_+} \mathbb{C}$ is identified with the quotient $G^{\mathbb{C}} \times \mathbb{C}$ modulo the equivalence relation: $(gb, c) \sim (g, \lambda(b)c)$ for any $g \in G^{\mathbb{C}}$, $b \in B_+$ and $c \in \mathbb{C}$. A section of the line bundle L_λ is identified with a function $f : G^{\mathbb{C}} \to \mathbb{C}$, subject to the relation

$$f(gb) = \lambda(b^{-1}) f(g) \quad \text{for all } g \in G^{\mathbb{C}},\ b \in B_+ \ . \tag{3.12}$$

Denote by Γ_λ the *space of holomorphic sections of the bundle* L_λ or, in other words, the space of holomorphic functions on $G^{\mathbb{C}}$, satisfying the condition (3.12). The group $G^{\mathbb{C}}$ acts from the left on L_λ, hence, on the space Γ_λ.

Theorem 2 (Borel–Weil theorem)**.** *If the weight λ is antidominant, then the representation of the group G in the space of holomorphic sections Γ_λ, constructed above, is the irreducible representation with the lowest weight λ and all irreducible representations of the group G can be realized in this way.*

3.2.3 Orbit method and coadjoint representation

In this Subsection we outline briefly another method of constructing irreducible representations of Lie groups, using the orbits of the coadjoint representation of the group on the dual space of its Lie algebra (the details may be found in Kirillov's book [39]). Though we do not use this method for the construction of representations, we found it useful to explain its idea to motivate the study of coadjoint representations of various infinite-dimensional groups in this book.

We recall first some basic facts on the characters of irreducible representations. Let $T : G \to \mathrm{GL}(V)$ be a finite-dimensional representation of a Lie group G. We define its *character* as a function $\chi_T : G \to \mathbb{C}^*$, given by the formula

$$\chi_T(g) := \operatorname{Tr} T(g) \ , \ g \in G \ .$$

This function is constant on conjugacy classes and depends only on the equivalence class of the representation T. Moreover, it is a homomorphism with respect to the tensor product of representations, i.e. $\chi_{T_1 \otimes T_2} = \chi_{T_1}\chi_{T_2}$. A character of an irreducible representation determines it uniquely up to equivalence.

Let G be a compact Lie group and $L^2(G, dg)$ denotes the space of all square integrable functions on G with respect to the Haar measure dg. Then the characters of all its irreducible unitary representations form an orthonormal basis in a subspace of $L^2(G, dg)$, consisting of functions, constant on conjugacy classes.

The definition of the character χ_T, given above, is valid, evidently, only for finite-dimensional representations T. However, for an infinite-dimensional representation it's often possible to define its character as a distribution on the group G. Namely, denote by $\mathcal{D}(G)$ the space of C^∞-smooth functions on G and suppose that all operators of the form

$$T(f) := \int_G f(g)T(g)\, dg \ , \qquad f \in \mathcal{D}(G) \ ,$$

are of trace class (the definition of the trace class is given in Sec. 5.3 below). Then we can define a character of the representation T as a distribution on the space $\mathcal{D}(G)$ of test functions, or, in other words, as a continuous linear functional on $\mathcal{D}(G)$, acting by the formula

$$\chi_T(f) := \operatorname{Tr} T(f) \ , \qquad f \in \mathcal{D}(G) \ .$$

If, in particular, the group G is semisimple, then the character χ_T can be given by the formula

$$\chi_T(f) = \int_G \chi_T(g) f(g)\, dg \ ,$$

where χ_T is some measurable locally integrable function on G. As in the case of finite-dimensional representations, the character $\chi_T(f)$ is constant on conjugacy classes, i.e.

$$\chi_T(f) = \operatorname{Tr} T(f) = \operatorname{Tr}[T(g)T(f)T(g^{-1})]$$

for any $f \in \mathcal{D}(G)$, $g \in G$. Again, an irreducible representation is uniquely determined (up to equivalence) by its character.

We turn now to the *coadjoint representation* of the group G. Let $\mathfrak{g}$ the Lie algebra of G and $\mathfrak{g}^*$ is its dual space. The adjoint action Ad of the group G on its Lie algebra $\mathfrak{g}$ induces by duality the *coadjoint action* Ad^* of the group G on the space $\mathfrak{g}^*$.

Consider an orbit $F = G \cdot \varphi$ of an arbitrary point $\varphi \in \mathfrak{g}^*$ in $\mathfrak{g}^*$ under the coadjoint action and denote by G_φ the isotropy subgroup at φ. Let $\mathfrak{g}_\varphi$ be the Lie algebra of the group G_φ. Then the tangent space to the orbit F at φ may be identified with the quotient $\mathfrak{g}/\mathfrak{g}_\varphi$.

The orbits F of the coadjoint representation turn out to be symplectic manifolds, provided with a canonical *Kirillov symplectic form* ω_F. This form is generated by a G_φ-invariant 2-form ω_φ on $\mathfrak{g}$, given by the formula

$$\omega_\varphi(\xi, \eta) := \varphi([\xi, \eta]) \ , \quad \xi, \eta \in \mathfrak{g} \ .$$

The kernel of ω_φ on $\mathfrak{g}$ coincides with $\mathfrak{g}_\varphi$, so the form ω_φ can be pushed down to a form on $\mathfrak{g}/\mathfrak{g}_\varphi$ (denoted by the same letter), which is a non-degenerate G_φ-invariant 2-form on $\mathfrak{g}/\mathfrak{g}_\varphi$. So it can be extended to a non-degenerate G-invariant 2-form ω_F on F, which does not depend on the choice of the point φ on the orbit F. Moreover, the form ω_φ satisfies the Jacobi identity, hence, it is a cocycle on $\mathfrak{g}$. This implies that the induced G-invariant 2-form ω_F is closed on F, and so defines a symplectic structure on F.

It may be proved that any G-homogeneous (with respect to the action of a connected Lie group G by symplectic transformations) symplectic manifold M is locally isomorphic to an orbit of the group G or its central extension $\widetilde{G}$ in the coadjoint representation (cf. [46]).

We explain now the idea of the *orbit method*. We want to construct an irreducible unitary representation T from an orbit of the coadjoint representation in $\mathfrak{g}^*$.

Let $F = G \cdot \varphi$ be such an orbit. We construct from it a one-dimensional unitary representation of the group G_φ. In a neighborhood of the identity of G_φ we define it by the formula

$$\chi(\exp \xi) = e^{2\pi i \varphi(\xi)} \ ,$$

where $\exp : \mathfrak{g}_\varphi \to G_\varphi$ is the exponential map. It extends to a representation of the isotropy group G_φ and induces an irreducible unitary representation T_F of the whole group G, if the orbit F is *integral*, i.e. the canonical symplectic form ω_F is an integral form on F (the precise definition of an integral form is given in the beginning of Sec. 8.1).

The character of the irreducible unitary representation T_F is given by the formula

$$\chi_F(\exp \xi) = \frac{1}{p_F(\exp \xi)} \int_F e^{2\pi i \varphi(\xi)} \beta_F(\varphi) \ , \quad \xi \in \mathfrak{g} \ , \tag{3.13}$$

where β_F is the Liouville volume form on F, generated by the symplectic form ω_F, and p_F is some smooth invariant (with respect to conjugations) function on G, equal to 1 at $e \in G$. The formula (3.13) should be understood in the distributional sense, i.e. for any test function $f \in \mathcal{D}(G)$ the integral

$$\chi_F(f) = \operatorname{Tr} T_F(f) = \int_F \left\{ \int_{\mathfrak{g}} \frac{f(\exp \xi)}{p_F(\exp \xi)} e^{2\pi i \varphi(\xi)} \, d\xi \right\} \beta_F(\varphi) \ ,$$

converges (here $d\xi$ is the Lebesgue measure on $\mathfrak{g}$).

In particular, for compact groups G we have $\dim T_F := \chi_F(e) = \operatorname{Vol} F < \infty$, and the integral orbits in this case correspond to flag manifolds. In this case the orbit method is equivalent to the Borel–Weil method from the previous Subsec. 3.2.2.

Bibliographic comments

The content of this Chapter is mostly of the reference character and may be found in a number of books. In particular, general properties of flag manifolds are presented in [6, 15, 61]. The basics of the representation theory of semisimple Lie algebras and groups may be found, e.g., in [76, 75, 28, 67]. The Borel–Weil construction is explained, in particular, in the book [6]. The orbit method is presented in [39, 43].

Chapter 4

Central extensions and cohomologies of Lie algebras and groups

In the first Section of this Chapter (Sec. 4.1) we recall the definition and basic properties of central extensions of Lie algebras and groups. In particular, we point out a relation between central extensions of Lie groups and their projective representations. In Sec. 4.2 we introduce the Lie algebra cohomologies and give several important examples of this notion (including the cohomological interpretation of central extensions). The last Sec. 4.3 is devoted to the Lie group cohomologies and their relation to projective representations.

4.1 Central extensions of Lie groups and projective representations

Definition 24. A *central extension of a Lie algebra* $\mathfrak{G}$ (over the field $\mathbb{R}$) is a Lie algebra $\tilde{\mathfrak{G}}$, which can be included into the exact sequence of Lie algebra homomorphisms

$$0 \longrightarrow \mathbb{R} \longrightarrow \tilde{\mathfrak{G}} \longrightarrow \mathfrak{G} \longrightarrow 0 \ , \tag{4.1}$$

where $\mathbb{R}$ is considered as an Abelian Lie algebra and the image of the monomorphism $\mathbb{R} \to \tilde{\mathfrak{G}}$ is contained in the center of the algebra $\tilde{\mathfrak{G}}$. Two central extensions $\tilde{\mathfrak{G}}_1$ and $\tilde{\mathfrak{G}}_2$ of the same Lie algebra $\mathfrak{G}$ are said to be *equivalent*, if there exist a commutative diagram of Lie algebra homomorphisms

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{R} \arrow[r] \arrow[d, "\mathrm{id}"'] & \tilde{\mathfrak{G}}_1 \arrow[r] \arrow[d] & \mathfrak{G} \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
0 \arrow[r] & \mathbb{R} \arrow[r] & \tilde{\mathfrak{G}}_2 \arrow[r] & \mathfrak{G} \arrow[r] & 0 \ .
\end{tikzcd}$$

The exact sequence (4.1) implies that the Lie algebra $\tilde{\mathfrak{G}}$, as a vector space, is isomorphic to $\tilde{\mathfrak{G}} = \mathfrak{G} \oplus \mathbb{R}$ and the Lie bracket in $\tilde{\mathfrak{G}}$, due to the centrality of the image of $\mathbb{R} \to \tilde{\mathfrak{G}}$, has the form

$$[(\xi, s), (\eta, t)] = [(\xi, 0), (\eta, 0)] = ([\xi, \eta], \omega(\xi, \eta)) \ ,$$

where ω is a skew-symmetric bilinear form on $\mathfrak{G}$, called the *cocycle of the central extension.*

By analogy with Def. 24, we can define central extensions of Lie groups.

Definition 25. A *central extension of a Lie group* $\mathcal{G}$ is a Lie group $\tilde{\mathcal{G}}$, which can be included into the exact sequence of Lie group homomorphisms

$$1 \longrightarrow S^1 \longrightarrow \tilde{\mathcal{G}} \longrightarrow \mathcal{G} \to 1 ,$$

where the image of the circle group under the monomorphism $S^1 \to \tilde{\mathcal{G}}$ is contained in the center of the group $\tilde{\mathcal{G}}$.

Topologically, the map $\tilde{\mathcal{G}} \to \mathcal{G}$ is a principal S^1-bundle. Consider the case, when this S^1-bundle is trivial, i.e. $\tilde{\mathcal{G}} \to \mathcal{G}$ admits a global section $\sigma : \mathcal{G} \to \tilde{\mathcal{G}}$. With the help of this section, we can identify $\tilde{\mathcal{G}}$ with the group $\mathcal{G} \times S^1$, provided with the multiplication

$$(g, \lambda) \cdot (h, \mu) = (gh, \lambda\mu c(g, h)) ,$$

where $c(g, h) = \sigma(g)\sigma(h)\sigma(gh)^{-1}$ is called the *cocycle of the central extension* $\tilde{\mathcal{G}}$.

Central extensions of Lie groups are closely related to their projective representations.

Definition 26. A *projective (unitary) representation of a Lie group* $\mathcal{G}$ is a map

$$\rho : \mathcal{G} \to \mathrm{U}(H)$$

of the group $\mathcal{G}$ into the group of unitary operators, acting in a complex Hilbert space H, satisfying the relation

$$\rho(g_1)\rho(g_2) = c(g_1, g_2)\rho(g_1 g_2) \quad \text{for all } g_1, g_2 \in \mathcal{G} ,$$

where $c(g_1, g_2)$ is a complex number with modulus 1, which is called the *cocycle of the projective representation.*

Another projective representation $\rho' : \mathcal{G} \to \mathrm{U}(H)$ of the same group $\mathcal{G}$ is *equivalent* to ρ, if

$$\rho'(g) = \lambda(g)\rho(g) , \quad g \in \mathcal{G} ,$$

for some $\lambda : \mathcal{G} \to S^1$.

Any projective representation ρ of a Lie group $\mathcal{G}$ determines a true unitary representation $\tilde{\rho}$ of some central extension $\tilde{\mathcal{G}}$ of the group $\mathcal{G}$, which is a topologically trivial S^1-bundle with the cocycle, equal to the cocycle of the projective representation. Namely, we define

$$\tilde{\rho}(g, \lambda) := \lambda\rho(g) \quad \text{for all } g \in \mathcal{G},\ \lambda \in S^1 .$$

Then we'll have

$$\tilde{\rho}\left((g_1, \lambda_1) \cdot (g_2, \lambda_2)\right) = \lambda_1\lambda_2 c(g_1, g_2)\rho(g_1 g_2) = \lambda_1\lambda_2\rho(g_1)\rho(g_2) = \tilde{\rho}(g_1, \lambda_1)\tilde{\rho}(g_2, \lambda_2)$$

for any $g_1, g_2 \in \mathcal{G}$, $\lambda_1, \lambda_2 \in S^1$.

Conversely, any unitary representation $\tilde{\rho}$ of a topologically trivial central extension $\tilde{\mathcal{G}}$, such that $\tilde{\rho}(\lambda) = \lambda \cdot \mathrm{id}$ for any $\lambda \in S^1$, determines a projective representation ρ of the group $\mathcal{G}$, which is defined in the following way. The cocycle c of the central extension $\tilde{\mathcal{G}}$ is given in terms of the trivializing section $\sigma : \mathcal{G} \to \tilde{\mathcal{G}}$ by the formula

$$c(g_1, g_2) = \sigma(g_1)\sigma(g_2)\sigma(g_1 g_2)^{-1} , \quad g_1, g_2 \in \mathcal{G} .$$

Then the map ρ, defined by $\rho(g) := \tilde{\rho}\,(\sigma(g))$, determines a projective representation $\rho : \mathcal{G} \to \mathrm{U}(H)$, since

$$\rho(g_1 g_2) = \tilde{\rho}\,(\sigma(g_1 g_2)) = \tilde{\rho}\left(c(g_1, g_2)^{-1}\sigma(g_1)\sigma(g_2)\right) = c(g_1, g_2)^{-1}\rho(g_1)\rho(g_2)$$

for any $g_1, g_2 \in \mathcal{G}$.

4.2 Cohomologies of Lie algebras

Let $\mathfrak{G}$ be a Lie algebra and $\rho : \mathfrak{G} \to \operatorname{End} V$ is a representation of $\mathfrak{G}$ in a vector space V. In other words, V is a $\mathfrak{G}$-module.

Definition 27. A *q-cochain of the algebra $\mathfrak{G}$ with coefficients in V* is a skew-symmetric continuous q-linear functional on $\mathfrak{G}$ with values in V, i.e. a continuous map

$$\alpha : \underbrace{\mathfrak{G} \times \cdots \times \mathfrak{G}}_{q} \longrightarrow V ,$$

which is skew-symmetric and q-linear. The set of all such cochains is denoted by $C^q(\mathfrak{G}, V)$.

We define the *differential (coboundary map)*

$$\delta_q : C^q(\mathfrak{G}, V) \longrightarrow C^{q+1}(\mathfrak{G}, V)$$

by the formula

$$\begin{aligned} \delta_q\alpha(\xi_1, \dots, \xi_{q+1}) = &\sum_{1 \le i \le q+1} (-1)^i \xi_i \alpha(\xi_1, \dots, \widehat{\xi_i}, \dots, \xi_{q+1}) + \\ &+ \sum_{1 \le i < j \le q+1} (-1)^{i+j-1} \alpha([\xi_i, \xi_j], \xi_1, \dots, \widehat{\xi_i}, \dots, \widehat{\xi_j}, \dots, \xi_{q+1}) \end{aligned} \tag{4.2}$$

for $\alpha \in C^q(\mathfrak{G}, V)$, $\xi_1, \dots, \xi_{q+1} \in \mathfrak{G}$.

It's easy to check that the coboundary maps have the property $\delta_q \circ \delta_{q-1} = 0$, so we obtain a complex

$$\dots \longrightarrow C^{q-1}(\mathfrak{G}, V) \xrightarrow{\delta_{q-1}} C^q(\mathfrak{G}, V) \xrightarrow{\delta_q} C^{q+1}(\mathfrak{G}, V) \longrightarrow \dots .$$

The cohomologies of this complex are called the *cohomologies of the Lie algebra $\mathfrak{G}$ with coefficients in the $\mathfrak{G}$-module V* and denoted by

$$\begin{aligned} H^q(\mathfrak{G}, V) &:= \operatorname{Ker} \delta_q / \operatorname{Im} \delta_{q-1} = \\ &= \frac{\{\xi \in C^q(\mathfrak{G}, V) : \delta_q \xi = 0\}}{\{\xi \in C^q(\mathfrak{G}, V) : \ \xi = \delta_{q-1}\eta \quad \text{for some } \eta \in C^{q-1}(\mathfrak{G}, V)\}} . \end{aligned} \tag{4.3}$$

In the particular case, when V is the basic number field $k = \mathbb{R}, \mathbb{C}$, considered as the trivial $\mathfrak{G}$-module, the cohomologies $H^q(\mathfrak{G}, k)$ are denoted by $H^q(\mathfrak{G})$.

The above expression for the coboundary map looks like exterior derivative of a differential form. This is because differential forms on a smooth manifold X may be considered as cochains of the Lie algebra $\mathrm{Vect}(X)$ with coefficients in the module $C^\infty(X)$ of smooth functions on X, considered as a $\mathrm{Vect}(X)$-module.

Here are several particular examples of Lie algebra cohomologies.

Example 24 (cohomology $H^0(\mathfrak{G}, V)$)**.** Setting $C^{-1}(\mathfrak{G}, V) = 0$, we get

$$H^0(\mathfrak{G}, V) = \mathrm{Ker}\ \left\{\delta_0 : C^0(\mathfrak{G}, V) = V \longrightarrow C^1(\mathfrak{G}, V)\right\} \\ = \{v \in V : \xi v = 0 \text{ for any } \xi \in \mathfrak{G}\} \ . \quad (4.4)$$

In other words, the cohomology $H^0(\mathfrak{G}, V)$ coincides with the set of invariants of $\mathfrak{G}$-module V.

Example 25 (cohomology $H^1(\mathfrak{G})$)**.** In this case the differential $\delta_0 : C^0(\mathfrak{G}) \to C^1(\mathfrak{G})$ is trivial, since the action of $\mathfrak{G}$ on k is trivial. So

$$H^1(\mathfrak{G}) = \mathrm{Ker}\ \left[\delta_1 : C^1(\mathfrak{G}) = \mathfrak{G}^* \longrightarrow C^2(\mathfrak{G})\right] \\ = \{\beta \in \mathfrak{G}^* : \beta([\xi, \eta]) = 0 \text{ for all } \xi, \eta \in \mathfrak{G}\} = (\mathfrak{G}/\left[\mathfrak{G}, \mathfrak{G}\right])^* \ . \quad (4.5)$$

Otherwise speaking, the cohomology $H^1(\mathfrak{G})$ consists of continuous linear functionals on $\mathfrak{G}/\left[\mathfrak{G}, \mathfrak{G}\right]$.

Example 26 (cohomology $H^1(\mathfrak{G}; \mathfrak{G})$)**.** Consider a Lie algebra $\mathfrak{G}$ as a $\mathfrak{G}$-module with respect to the adjoint action ad of $\mathfrak{G}$ on itself. The cohomology $H^1(\mathfrak{G}, \mathfrak{G})$ is interpreted as the set of outer derivations of the algebra $\mathfrak{G}$. Recall that a homomorphism $\phi : \mathfrak{G} \to \mathfrak{G}$ is called the *derivation* of $\mathfrak{G}$, if

$$\phi([\xi, \eta]) = [\phi(\xi), \eta] + [\xi, \phi(\eta)] \ .$$

The *inner derivations*, defined by

$$\xi \longmapsto [\xi, \xi_0] = \mathrm{ad}_{\xi_0}(\xi) \ ,$$

where ξ_0 is a fixed element of $\mathfrak{G}$, may serve as an example.

The set of *outer derivations* coincides, by definition, with the quotient of the set of all derivations of the algebra $\mathfrak{G}$ modulo inner derivations.

Let us show that the cohomology $H^1(\mathfrak{G}, \mathfrak{G})$ coincides with the set of outer derivations of the algebra $\mathfrak{G}$.

Indeed, cochains from $C^1(\mathfrak{G}, \mathfrak{G})$ are given by linear maps $\phi : \mathfrak{G} \to \mathfrak{G}$. The condition $\delta_1\phi = 0$ means that ϕ is a derivation, since

$$\delta_1\phi(\xi, \eta) = \phi([\xi, \eta]) - \xi\phi(\eta) + \eta\phi(\xi) = \phi([\xi, \eta]) - [\xi, \phi(\eta)] - [\phi(\xi), \eta] \ .$$

The cochains from $C^1(\mathfrak{G}, \mathfrak{G})$, belonging to the image of the map $\delta_0 : C^0(\mathfrak{G}, \mathfrak{G}) \to C^1(\mathfrak{G}, \mathfrak{G})$, are inner derivations of the algebra $\mathfrak{G}$, since

$$\xi \in \mathfrak{G} = C^0(\mathfrak{G}, \mathfrak{G}) \quad \Longrightarrow \quad \delta_0\xi(\eta) = -\xi \cdot \eta = [-\xi, \eta] \ .$$

Example 27 (cohomology $H^2(\mathfrak{G})$)**.** The cohomology $H^2(\mathfrak{G})$ may be identified with set of equivalence classes of central extensions of the Lie algebra $\mathfrak{G}$, considered in the previous Sec. 4.1.

Indeed, associate with a cocycle $\omega \in C^2(\mathfrak{G})$ the central extension

$$0 \longrightarrow k \longrightarrow k \oplus \mathfrak{G} \longrightarrow \mathfrak{G} \longrightarrow 0 \ ,$$

where the map $k \to k \oplus \mathfrak{G}$ is an embedding $s \mapsto (s, 0)$, and the map $k \oplus \mathfrak{G} \to \mathfrak{G}$ coincides with the projection $(s, \xi) \mapsto \xi$. The bracket in the algebra $\widetilde{\mathfrak{G}} = k \oplus \mathfrak{G}$ is given by the formula

$$[(s, \xi), (t, \eta)] = (\omega(\xi, \eta), [\xi, \eta]) \ .$$

The Jacoby identity in the algebra $\widetilde{\mathfrak{G}}$ is equivalent to the cocyclicity of ω. Moreover, cohomologous cocycles correspond to equivalent central extensions, and the zero in $H^2(\mathfrak{G})$ corresponds to the trivial central extension $\widetilde{\mathfrak{G}} = k \oplus \mathfrak{G}$.

Example 28 (cohomology $H^3(\mathfrak{G})$)**.** The cohomology $H^3(\mathfrak{G})$ of a semisimple Lie algebra $\mathfrak{G}$ is interpreted as the set of invariant symmetric bilinear forms on $\mathfrak{G}$.

Indeed, with any such form $\langle \cdot, \cdot \rangle$ we can associate an element of $H^3(\mathfrak{G})$, given by the 3-cocycle of the form

$$\mathfrak{G} \times \mathfrak{G} \times \mathfrak{G} \ni (\xi, \eta, \zeta) \longmapsto \langle \xi, [\eta, \zeta] \rangle \ .$$

Apart from the above examples, demonstrating the importance of the cohomologies of Lie algebras, there is one more motivation to introduce such an object. Namely, the cohomologies of a Lie algebra $\mathfrak{G}$ are closely related to the cohomologies of the corresponding Lie group $\mathcal{G}$, considered as a smooth manifold. Let us denote the latter cohomology groups by $H^q_{\mathrm{top}}(\mathcal{G}, k)$. A relation between $H^q_{\mathrm{top}}(\mathcal{G}, k)$ and the cohomologies of the Lie algebra $\mathfrak{G}$ is established in the following way.

Construct first a map of the cochain complex $C^\bullet(\mathfrak{G})$ into the de Rham complex $\Omega^\bullet(\mathcal{G})$ of the group $\mathcal{G}$. Denote by $\Omega^q_{\mathrm{inv}}(\mathcal{G})$ the subspace of differential forms of degree q in $\Omega^q(\mathcal{G})$, invariant under the right translations on $\mathcal{G}$. A form in $\Omega^q_{\mathrm{inv}}(\mathcal{G})$ is uniquely determined by its restriction to the tangent space $T_e\mathcal{G} = \mathfrak{G}$, i.e. there is an isomorphism

$$\Omega^q_{\mathrm{inv}}(\mathcal{G}) \xleftrightarrow{\approx} \Lambda^q(\mathfrak{G}) = C^q(\mathfrak{G}) \ .$$

Moreover, the differential $\delta_q : C^q(\mathfrak{G}) \to C^{q+1}(\mathfrak{G})$ coincides with the restriction of the exterior differential $d_q : \Omega^q(\mathcal{G}) \to \Omega^{q+1}(\mathcal{G})$ to $\Omega^q_{\mathrm{inv}}(\mathcal{G})$. So there is a canonical map

$$H^q(\mathfrak{G}) \longrightarrow H^q_{\mathrm{top}}(\mathcal{G}, k) \ . \tag{4.6}$$

This homomorphism is an isomorphism, when $k = \mathbb{R}$ and $\mathcal{G}$ is a compact Lie group (in this case one can associate with any form on $\mathcal{G}$ a right-invariant form by averaging the original form over $\mathcal{G}$). In the complex case $k = \mathbb{C}$ the above homomorphism is an isomorphism, if $\mathcal{G}$ is a complex semisimple Lie group. The isomorphism (4.6) extends also to some infinite-dimensional Lie groups, in particular, to the loop group $LG = C^\infty(S^1, G)$ of a compact Lie group G ($k = \mathbb{R}$ in this case).

4.3 Cohomologies of Lie groups

Let $\mathcal{G}$ be a Lie group and V is a $\mathcal{G}$-module, i.e. we have a representation $\rho : \mathcal{G} \to \mathrm{GL}(V)$ of the group $\mathcal{G}$ in the vector space V. There are two natural definitions of the cochain complex with values in the $\mathcal{G}$-module V. In the first definition cochains are given by equivariant functions on $\mathcal{G}$ with values in V.

Definition 28. A *q-cochain of the group* $\mathcal{G}$ with values in V is a function

$$\varphi : \underbrace{\mathcal{G} \times \cdots \times \mathcal{G}}_{q+1} \longrightarrow V \ ,$$

which has the following equivariance property

$$\varphi(gg_0, \ldots, gg_q) = g \cdot \varphi(g_0, \ldots, g_q) \ ,$$

where "$\cdot$" in the right hand side denotes the action of the group $\mathcal{G}$ on V, given by the representation ρ. The space of all q-cochains is denoted by $C^q(\mathcal{G}, V)$ and the *differential*

$$\delta_q : C^q(\mathcal{G}, V) \to C^{q+1}(\mathcal{G}, V)$$

is given by the formula

$$\delta_q \varphi(g_0, \ldots, g_{q+1}) = \sum_{i=0}^{q+1} (-1)^i \varphi(g_0, \ldots, \widehat{g_i}, \ldots, g_{q+1}) \ .$$

In the second definition cochains are given by arbitrary functions on $\mathcal{G}$ with values in V.

Definition 29. A *q-cochain on the group* $\mathcal{G}$ with values in V is a function

$$\psi : \underbrace{\mathcal{G} \times \cdots \times \mathcal{G}}_{q+1} \longrightarrow V \ .$$

The space of all q-cochains on $\mathcal{G}$ with values in V is denoted again by $C^q(\mathcal{G}, V)$, but the *differential*

$$\delta_q : C^q(\mathcal{G}, V) \to C^{q+1}(\mathcal{G}, V)$$

is given in this case by the formula

$$\begin{aligned} \delta_q \psi(g_1, \ldots, g_{q+1}) = {} & g_1 \cdot \psi(g_2, \ldots, g_{q+1}) + \\ & + \sum_{i=1}^{q} (-1)^i \psi(g_1, \ldots, g_i g_{i+1}, \ldots, g_{q+1}) + (-1)^{q+1} \psi(g_1, \ldots, g_q) \ . \end{aligned} \tag{4.7}$$

A relation $\varphi \leftrightarrow \psi$ between these two definitions of cochains is established via the formulas

$$\varphi(g_0, \ldots, g_q) = g_0 \cdot \psi(g_0^{-1} g_1, g_1^{-1} g_2, \ldots, g_{q-1}^{-1} g_q) \ , \tag{4.8}$$

$$\psi(g_1, \ldots, g_q) = \varphi(1, g_1, g_1 g_2, \ldots, g_1 g_2 \cdot \ldots \cdot g_q) \ . \tag{4.9}$$

The *cohomologies of the group $\mathcal{G}$ with values in* the $\mathcal{G}$*-module* V in both cases are defined as the cohomologies of the complex $\{C^q(\mathcal{G}, V), \delta_q\}$, i.e.

$$H^q(\mathcal{G}, V) = \frac{\mathrm{Ker}\,[\delta_q : C^q(\mathcal{G}, V) \to C^{q+1}(\mathcal{G}, V)]}{\mathrm{Im}\,[\delta_{q-1} : C^{q-1}(\mathcal{G}, V) \to C^q(\mathcal{G}, V)]} \ .$$

We consider now a relation between 2-dimensional cohomologies of the group $\mathcal{G}$ with its projective representations and central extensions (cf. Sec. 4.1).

Let $\rho : \mathcal{G} \to \mathrm{U}(V)$ be a projective representation of the Lie group $\mathcal{G}$, satisfying the relation

$$\rho(g_1)\rho(g_2) = c(g_1, g_2)\rho(g_1 g_2) \quad \text{for any } g_1, g_2 \in \mathcal{G} \ ,$$

where $c(g_1, g_2)$ is the cocycle of the representation ρ. The associativity of the multiplication in $\mathcal{G}$ and $\mathrm{U}(V)$ implies that c is a 2-cocycle of the group $\mathcal{G}$ with values in the multiplicative group S^1 with the trivial action of the group $\mathcal{G}$, given by $\rho : \mathcal{G} \to 1$. In other words, for any three elements g_1, g_2, g_3 of the group $\mathcal{G}$ we have the relation

$$c(g_2, g_3)c(g_1 g_2, g_3)^{-1}c(g_1, g_2 g_3)c(g_1, g_2)^{-1} = 1 \ ,$$

which means that $\delta_2 c = 1$ (we use here the multiplicative analog of δ_2 from Def. 29).

On the other hand, an equivalent projective representation of the form

$$\rho'(g) = \lambda(g)\rho(g)$$

with $\lambda : \mathcal{G} \to S^1$, corresponds to the cocycle

$$c'(g_1, g_2) = c(g_1, g_2)\lambda(g_1 g_2)\lambda(g_1)^{-1}\lambda(g_2)^{-1} \ ,$$

i.e. to the cocycle $c' \in C^2(\mathcal{G}, S^1)$, cohomologous to the cocycle $c \in C^2(\mathcal{G}, S^1)$. So the class $[c]$ of the cocycle c in the cohomologies $H^2(\mathcal{G}, S^1)$ depends only on the equivalence class of the projective representation ρ. Hence, *the equivalence classes of projective representations of the Lie group $\mathcal{G}$ in a Hilbert space V can be identified with the cohomologies $H^2(\mathcal{G}, S^1)$.*

On the other hand, in Sec. 4.1 we have assigned to any topologically trivial central extension $\tilde{\mathcal{G}} \to \mathcal{G}$ of the group $\mathcal{G}$ its cocycle c, which is the same as a 2-cocycle of the group $\mathcal{G}$ with values in the trivial $\mathcal{G}$-module S^1. Moreover, equivalent central extensions of the group $\mathcal{G}$ correspond to cohomologous cocycles in $H^2(\mathcal{G}, S^1)$. So, the class $[c]$ of the cocycle c in $H^2(\mathcal{G}, S^1)$ depends only on the equivalence class of the central extension $\tilde{\mathcal{G}}$ and we can identify the *set of equivalence classes of (topologically trivial) central extensions of the Lie group $\mathcal{G}$ with the cohomology $H^2(\mathcal{G}, S^1)$.*

Bibliographic comments

The content of this Chapter is also of reference character and may be found in [31, 21, 22]. Central extensions and projective representations, together with cohomologies of Lie algebras and groups, will play an important role in the study of loop groups and diffeomorphism groups in Parts II and III.

Chapter 5

Grassmannians of a Hilbert space

In this Chapter we introduce infinite-dimensional Grassmann manifolds of closed subspaces in a Hilbert space H. We assume that H is polarized, i.e. decomposed into the direct sum of closed (infinite-dimensional) subspaces $H = H_+ \oplus H_-$, and consider Grassmannians, consisting of subspaces, "close" to H_+ in different senses. The most important case is the so called *Hilbert–Schmidt Grassmannian*, introduced in Sec. 5.2. It is a Hilbert Kähler manifold, which has many features of standard finite-dimensional Grassmannians. In particular, it is the homogeneous space of a Hilbert Lie group and can be provided with a natural determinant bundle, constructed in Sec. 5.3.

5.1 Grassmannian $\mathrm{Gr}_b(H)$

Let H be a complex (separable) Hilbert space. Suppose that H is *polarized*, i.e. it is provided with a decomposition into the direct orthogonal sum

$$H = H_+ \oplus H_- \tag{5.1}$$

of closed infinite-dimensional subspaces. Denote by pr_+ (resp. pr_-) the orthogonal projection $\mathrm{pr}_+ : H \to H_+$ (resp. $\mathrm{pr}_- : H \to H_-$).

We usually have in mind a standard example of such a polarized Hilbert space H, given by the Hilbert space $L^2_0(S^1, \mathbb{C})$ of L^2-functions on the unit circle S^1 with zero average value. Functions $f \in L^2_0(S^1, \mathbb{C})$ have Fourier decompositions, converging in L^2-sense, of the form

$$f(z) = \sum_{k=-\infty}^{+\infty} f_k z^k \ , \quad f_0 = 0 \ ,$$

where $z = e^{i\theta}$. For this particular realization of H we take for H_+ (resp. H_-) the subspace, consisting of the functions $f \in L^2_0(S^1, \mathbb{C})$, which have vanishing Fourier coefficients with negative (resp. positive) indices:

$$H_+ = \{f \in H : f(z) = \sum_{k=1}^{\infty} f_k z^k \} \ , \quad H_- = \{f \in H : f(z) = \sum_{k=-\infty}^{-1} f_k z^k \} \ .$$

Definition 30. The Grassmannian $\mathrm{Gr}_b(H)$ consists of all closed subspaces $W \subset H$, such that the orthogonal projection $\mathrm{pr}_+ : W \to H_+$ is a Fredholm operator.

Recall that a linear operator $T : H_1 \to H_2$, mapping a Hilbert space H_1 into a Hilbert space H_2, is called *Fredholm*, if it has finite-dimensional kernel and cokernel. For such an operator one can define its *Fredholm index* by the formula

$$\operatorname{ind} T := \dim(\operatorname{Ker} T) - \dim(\operatorname{Coker} T) .$$

The Fredholm index of T is a topological invariant of T, i.e. it does not change under bounded continuous deformations of T. An equivalent definition: an operator T is Fredholm, if it is invertible modulo compact operators, i.e. if there exists an operator $S : H_2 \to H_1$ such that the operators $\mathrm{id}_{H_1} - ST$ and $\mathrm{id}_{H_2} - TS$ are compact.

We can reformulate Def. 30 in an equivalent way as follows: a subspace $W \in \mathrm{Gr}_b(H)$ iff it coincides with the image of a bounded linear operator

$$w : H_+ \longrightarrow H ,$$

such that the operator $w_+ := \mathrm{pr}_+ \circ w$ is Fredholm.

With respect to the polarization $H = H_+ \oplus H_-$ any linear operator $w \in \operatorname{End} H$ can be written in the block form

$$w = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a : H_+ \to H_+ , & b : H_- \to H_+ \\ c : H_+ \to H_- , & d : H_- \to H_- \end{pmatrix} .$$

In these terms the subspace $W \in \mathrm{Gr}_b(H)$ iff a is Fredholm.

For any $W \in \mathrm{Gr}_b(H)$ denote by

$$U_W = \{W' \in \mathrm{Gr}_b(H) : \text{the orthogonal projection } W' \to W \text{ is an isomorphism}\} .$$

We want to define the structure of a complex Banach manifold on $\mathrm{Gr}_b(H)$, for which the sets U_W will play the role of coordinate neighborhoods. More precisely, we have the following

Proposition 6. *$\mathrm{Gr}_b(H)$ is a complex Banach manifold, having for its local model the Banach space $B(H_+, H_-)$ of bounded linear operators $w : H_+ \to H_-$. The coordinate neighborhoods*

$$U_W = \{W' \in \mathrm{Gr}_b(H) : \textit{the orthogonal projection } W' \to W \textit{ is an isomorphism}\} ,$$

introduced above, form an atlas of $\mathrm{Gr}_b(H)$ and coordinate charts are given by the maps

$$U_W \ni W' \longmapsto w' \in B(W, W^\perp) .$$

Proof. In order to show that the atlas $\{U_W\}$ with given charts does define on $\mathrm{Gr}_b(H)$ the structure of a complex Banach manifold, consider the intersection $U_{W_1} \cap U_{W_2} \neq \emptyset$ of two coordinate neighborhoods. The coordinate change in H, transforming the decomposition $H = W_1 \oplus W_1^\perp$ into the decomposition $H = W_2 \oplus W_2^\perp$, is given by the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : W_1 \oplus W_1^\perp \to W_2 \oplus W_2^\perp ,$$

in which the operators a and d are Fredholm, while b and c are bounded. If a subspace $W \in U_{W_1} \cap U_{W_2}$, then it can be represented, on one hand, as the graph of a bounded operator $w_1 : W_1 \to W_1^\perp$, and, on the other hand, as the graph of a bounded operator $w_2 : W_2 \to W_2^\perp$. The orthogonal projection of W onto the subspaces W_1 and W_2 is an isomorphism, which defines an isomorphism $v : W_1 \to W_2$, so that W is the graph of the operator $w_2 \circ v : W_1 \to W_2^\perp$. It implies that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ w_1 \end{pmatrix} = \begin{pmatrix} v \\ w_2 \circ v \end{pmatrix}$$

as operators from W_1 to $W_2 \oplus W_2^\perp$. In other words, the coordinate change

$$B(W_1, W_1^\perp) \longrightarrow B(W_2, W_2^\perp) \ , \quad w_1 \longmapsto w_2 \ ,$$

which is given by the formula

$$w_2 = (c + dw_1)(a + bw_1)^{-1} \ ,$$

determines a holomorphic map, defined on the open subset $U_{W_1} \cap U_{W_2}$, identified with the subset $\{w_1 \in B(W_1, W_1^\perp) : a + bw_1 \text{ is invertible}\}$. □

Note that the manifold $\mathrm{Gr}_b(H)$ *has a countable number of connected components, numerated by the index of the Fredholm operator* w_+ for a subspace $W \in \mathrm{Gr}_b(H)$, coinciding with the image of a linear operator $w : H_+ \to H$. We say that the subspace W has the *virtual dimension* d, if the index of w_+ is equal to d.

5.2 Hilbert–Schmidt Grassmannian $\mathbf{Gr}_{\mathbf{HS}}(H)$

Recall that a linear operator $T : H_1 \to H_2$, acting from a complex Hilbert space H_1 into another complex Hilbert space H_2, is called a *Hilbert–Schmidt operator*, if for some orthonormal basis $\{e_i\}$ in H_1 the series

$$\sum_i \|Te_i\| < \infty$$

is converging. Note that this condition is satisfied for any orthonormal basis in H_1, if it is satisfied for some orthonormal basis $\{e_i\}$ in H_1. We define the *Hilbert–Schmidt norm* of the operator T by the formula

$$\|T\|_2 = \left(\sum_{i=1}^{\infty} \|Te_i\|^2 \right)^{1/2} .$$

The *Hilbert–Schmidt operators* $T : H_1 \to H_2$ *form a complex Hilbert space* $\mathrm{HS}(H_1, H_2)$ with respect to the introduced norm. Moreover, the space $\mathrm{HS}(H, H)$ of Hilbert–Schmidt operators, acting in a Hilbert space H, is a two-sided ideal in the algebra $B(H)$ of all bounded linear operators in H.

Denote by $\mathrm{GL}(H)$ the group of all linear bounded operators in H, having a bounded inverse.

Definition 31. The *general linear Hilbert–Schmidt group* $\mathrm{GL}_{\mathrm{HS}}(H)$ consists of linear operators $A \in \mathrm{GL}(H)$, such that in their block representation (with respect to polarization $H = H_+ \oplus H_-$)

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

the "off-diagonal" terms b and c are Hilbert–Schmidt operators (for brevity: HS-operators). We denote by $\mathrm{U}_{\mathrm{HS}}(H)$ the intersection of the group $\mathrm{GL}_{\mathrm{HS}}(H)$ with the group $\mathrm{U}(H)$ of all unitary operators in H.

In other words, the group $\mathrm{GL}_{\mathrm{HS}}(H)$ consists of operators $A \in \mathrm{GL}(H)$, for which the "off-diagonal" terms b and c are "small" with respect to the "diagonal" terms a and d.

We introduce now the structure of a Banach Lie group on $\mathrm{GL}_{\mathrm{HS}}(H)$. Namely, consider a subalgebra $B_{\mathrm{HS}}(H)$ of the algebra $B(H)$, consisting of operators of the form

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in B(H) \ ,$$

for which the operators b and c are Hilbert–Schmidt. The algebra $B_{\mathrm{HS}}(H)$ is a Banach algebra with the norm, given by the formula

$$|||A||| := \|A\| + \|b\|_2 + \|c\|_2 \ .$$

The group $\mathrm{GL}_{\mathrm{HS}}(H)$ coincides with the group of invertible elements of the algebra $\mathrm{B}_{\mathrm{HS}}(H)$ and is a complex *Banach Lie group*. Accordingly, $\mathrm{U}_{\mathrm{HS}}(H)$ is a real Banach Lie group, whose complexification coincides with $\mathrm{GL}_{\mathrm{HS}}(H)$.

There is a Grassmann manifold $\mathrm{Gr}_{\mathrm{HS}}(H)$, associated with the group $\mathrm{GL}_{\mathrm{HS}}(H)$.

Definition 32. The *Hilbert–Schmidt Grassmannian* $\mathrm{Gr}_{\mathrm{HS}}(H)$ is the set of all closed subspaces $W \subset H$, such that the orthogonal projection $\mathrm{pr}_+ : W \to H_+$ is a Fredholm operator, and the orthogonal projection $\mathrm{pr}_- : W \to H_-$ is a Hilbert–Schmidt operator.

In other words, $\mathrm{Gr}_{\mathrm{HS}}(H)$ consists of the subspaces $W \subset H$, which differ "little" from the subspace H_+ in the sense that $\mathrm{pr}_+ : W \to H_+$ is an "almost isomorphism" (recall that Fredholm operators are invertible modulo compact operators, cf. Sec. 5.1), and $\mathrm{pr}_- : W \to H_-$ is "small".

Equivalently, a subspace $W \in \mathrm{Gr}_{\mathrm{HS}}(H)$ iff it coincides with the image of a linear operator

$$w : H_+ \longrightarrow H \ ,$$

such that the operator $w_+ := \mathrm{pr}_+ \circ w$ is Fredholm, and $w_- := \mathrm{pr}_- \circ w$ is Hilbert–Schmidt.

It's easy to see that if $W \in \mathrm{Gr}_{\mathrm{HS}}(H)$, then the graph of any HS-operator $w' : W \to W^\perp$ also belongs to $\mathrm{Gr}_{\mathrm{HS}}(H)$. We denote the set of all such subspaces by U_W:

$$U_W = \{W' \in \mathrm{Gr}_{\mathrm{HS}}(H) : \ W' \text{ is the graph of an HS-operator } w' : W \to W^\perp\} \ .$$

As in Sec. 5.1, this definition can be rewritten in the form

$$U_W = \{W' \in \mathrm{Gr}_{\mathrm{HS}}(H) : \text{ the orthogonal projection } W' \to W \text{ is an isomorphism}\} \ .$$

The group $\mathrm{GL_{HS}}(H)$, introduced above, acts in a natural way on $\mathrm{Gr_{HS}}(H)$. Consider, in particular, the action of its unitary subgroup $\mathrm{U_{HS}}(H)$ on $\mathrm{Gr_{HS}}(H)$ and show that it is transitive. It will allow us to obtain a realization of $\mathrm{Gr_{HS}}(H)$ as a homogeneous space of the group $\mathrm{U_{HS}}(H)$, analogous to the realization of the finite-dimensional Grassmannian as a homogeneous space of the unitary group.

To prove that the action of $\mathrm{U_{HS}}(H)$ on $\mathrm{Gr_{HS}}(H)$ is transitive, we should construct for a given subspace $W \in \mathrm{Gr_{HS}}(H)$ an operator $A \in \mathrm{U_{HS}}(H)$ such that $A(H_+) = W$. Consider an isometric operator $w : H_+ \to H$, which has the image, equal to W, and denote by $w^\perp : H_- \to H$ an isometric operator with the image $W^\perp$. Then the operator

$$A = w \oplus w^\perp : H = H_+ \oplus H_- \to H = W \oplus W^\perp$$

defines an isometry of H onto itself and so is unitary. Moreover, it maps H_+ onto W and has the block representation of the form

$$A = \begin{pmatrix} w_+ & w_+^\perp \\ w_- & w_-^\perp \end{pmatrix} .$$

Here, the operator w_+ is Fredholm, and w_- is Hilbert–Schmidt, because $W \in \mathrm{Gr_{HS}}(H)$. Since A is also unitary, it follows that $A \in \mathrm{U_{HS}}(H)$.

The isotropy subgroup of $\mathrm{U_{HS}}(H)$ at $H_+ \in \mathrm{Gr_{HS}}(H)$ coincides with $\mathrm{U}(H_+) \times \mathrm{U}(H_-)$, hence we have the following

Proposition 7. *The Grassmannian $Gr_{HS}(H)$ is a homogeneous space of the group $U_{HS}(H)$ of the form*

$$Gr_{HS}(H) = U_{HS}(H)/\, U(H_+) \times U(H_-) .$$

The Hilbert–Schmidt Grassmannian $\mathrm{Gr_{HS}}(H)$ has the structure of a complex Hilbert manifold, defined in the following way.

Proposition 8. *The Grassmannian $Gr_{HS}(H)$ is a complex Hilbert manifold, having for its local model the Hilbert space of Hilbert Schmidt operators $HS(H_+, H_-)$. The coordinate neighborhoods*

$$U_W = \{W' \in Gr_{HS}(H) : W' \text{ is the graph of an HS-operator } w' : W \to W^\perp\}$$

form an atlas for $Gr_{HS}(H)$, and the coordinate charts are given by the maps

$$U_W \ni W' \longmapsto w' \in HS(W, W^\perp) .$$

This Proposition is proved in the same way, as Prop. 6 from Sec. 5.1.

There is another atlas on $\mathrm{Gr_{HS}}(H)$, which is more natural in some sense. To construct it, we identify H with the Hilbert space $L^2(S^1, \mathbb{C})$. This space has a canonical basis, given by $\{z^k\}$, $k \in \mathbb{Z}$. The subspace H_+ is generated by the elements $\{z^k\}$, $k \in \mathbb{Z}_+$, and H_- by the elements $\{z^k\}$, $k \in \mathbb{Z}_-$, where we denote by $\mathbb{Z}_+$ the subset of nonnegative integers in $\mathbb{Z}$, and by $\mathbb{Z}_-$ its complement in $\mathbb{Z}$.

We take for "coordinate" subspaces in H the closed linear subspaces $H_S \subset H$, generated by vectors $\{z^s\}$, $s \in S$, which are numerated by the subsets $S \subset \mathbb{Z}$, comparable with $\mathbb{Z}_+$. We say that a subset $S \subset \mathbb{Z}$ is *comparable with* $\mathbb{Z}_+$, if the sets $S - \mathbb{Z}_+$ and $\mathbb{Z}_+ - S$ consist of finite number of points. The ensemble of all such subsets $S \subset \mathbb{Z}$ is denoted by $\mathcal{S}$, and the number $|S - \mathbb{Z}_+| - |\mathbb{Z}_+ - S|$ is called the *virtual cardinality* of S. Note that the virtual dimension of the subspace H_S is equal precisely to the virtual cardinality of S.

Lemma 2. *For any $W \in Gr_{HS}(H)$ there exists a subset $S \in \mathcal{S}$, such that the orthogonal projection*

$$pr_S : W \longrightarrow H_S$$

is an isomorphism.

Proof. Indeed, if $W \in \mathrm{Gr}_{\mathrm{HS}}(H)$, then the orthogonal projection $\mathrm{pr}_+ : W \to H_+$ has finite-dimensional kernel and cokernel, so there exists a subset $S_0 \in \mathcal{S}$, containing $\mathbb{Z}_+$, for which the orthogonal projection

$$\mathrm{pr} : W \longrightarrow H_{S_0}$$

is injective. If it's not surjective, then one can find an $s \in S_0$, such that z^s does not belong to $\mathrm{pr}(W)$. In this case we replace S_0 with $S_1 := S_0 \setminus \{s\}$. The projection $\mathrm{pr} : W \to H_{S_1}$ still remains injective. If it's not surjective, we repeat the described procedure. Since the complement of $\mathrm{pr}_+(W)$ in H_+ is finite-dimensional, after a finite number of steps we shall arrive to a subset S, for which the projection $\mathrm{pr}_S : W \to H_S$ is an isomorphism. □

Based on the above Lemma, we can define an atlas on $\mathrm{Gr}_{\mathrm{HS}}(H)$, formed by the open sets $\{U_S\}_{S\in\mathcal{S}}$, where the coordinate neighborhood $U_S = U_{H_S}$ consists of the subspaces, which are the graphs of Hilbert–Schmidt operators $H_S \to H_S^\perp = H_{S^\perp}$ with $S^\perp = \mathbb{Z} - S$.

Since $\mathrm{U}_{\mathrm{HS}}(H)$ acts transitively on the Grassmannian $\mathrm{Gr}_{\mathrm{HS}}(H)$, one can construct an $\mathrm{U}_{\mathrm{HS}}(H)$-invariant Kähler metric on $\mathrm{Gr}_{\mathrm{HS}}(H)$ from an inner product on the tangent space $T_{H_+}\mathrm{Gr}_{\mathrm{HS}}(H)$ at the origin $H_+ \in \mathrm{Gr}_{\mathrm{HS}}(H)$, invariant under the action of the isotropy subgroup $\mathrm{U}(H_+) \times \mathrm{U}(H_-)$.

The tangent space $T_{H_+}\mathrm{Gr}_{\mathrm{HS}}(H)$ coincides with the Hilbert space of Hilbert–Schmidt operators $\mathrm{HS}(H_+, H_-)$, and an invariant inner product on it can be given by the formula

$$(A, B) \longmapsto \mathrm{Re}\,\{\mathrm{tr}(AB^*)\}\,, \quad A, B \in \mathrm{HS}(H_+, H_-)\ .$$

The imaginary part of the complex inner product $\mathrm{tr}(AB^*)$:

$$\omega(A, B) := \mathrm{Im}\,\{\mathrm{tr}(AB^*)\}$$

defines a non-degenerate invariant 2-form on $T_{H_+}\mathrm{Gr}_{\mathrm{HS}}(H)$, which extends to an $\mathrm{U}_{\mathrm{HS}}(H)$-invariant symplectic form on $\mathrm{Gr}_{\mathrm{HS}}(H)$.

This defines on $\mathrm{Gr}_{\mathrm{HS}}(H)$ a *Kähler structure*, making $\mathrm{Gr}_{\mathrm{HS}}(H)$ into a *Kähler Hilbert manifold*.

We shall use in Ch. 9 the "smooth" part $\mathrm{Gr}^\infty(H)$ of the Grassmannian $\mathrm{Gr}_{\mathrm{HS}}(H)$, which can be defined in terms of the open covering $\{U_S\}_{S\in\mathcal{S}}$ in the following way.

Definition 33. The Grassmannian $\mathrm{Gr}^\infty(H)$ consists of the graphs of all bounded linear operators $w : H_S \to H_S^\perp$, $S \in \mathcal{S}$, whose matrix components w_{pq} with $p \in \mathbb{Z} - S$, $q \in S$ are rapidly decreasing, i.e. the quantities $|p - q|^r w_{pq}$ are bounded for each r.

5.3 Plücker embedding and determinant bundle

As in the finite-dimensional case, the Hilbert–Schmidt Grassmannian $\mathrm{Gr}_{\mathrm{HS}}(H)$ may be realized, with the help of the Plücker embedding, as a submanifold in a projective Hilbert space.

In order to define this Plücker embedding, we introduce a notion of an admissible basis for a subspace $W \in \mathrm{Gr}_{\mathrm{HS}}(H)$. Suppose that $W \in \mathrm{Gr}_{\mathrm{HS}}(H)$ has the virtual dimension d. A model example for such a subspace in the case of $H = L_0^2(S^1, \mathbb{C})$ is the subspace $W = z^{-d}H_+$.

Definition 34. A basis in W, consisting of elements $\{w_k\}_{k\geq -d}$, is called *admissible* if:

1. the linear map
$$w : z^{-d}H_+ \longrightarrow W \ ,$$
defined on the basis elements $\{z^k\}_{k\geq -d}$ by the formula $z^k \mapsto w_k$, is a continuous isomorphism;

2. the composition of w with the orthogonal projection onto the subspace $z^{-d}H_+$:
$$\mathrm{pr} \circ w : z^{-d}H_+ \longrightarrow z^{-d}H_+$$
is an operator with determinant.

We recall the definitions of the class Tr of operators with trace and related class $\mathrm{Det} = 1 + \mathrm{Tr}$ of operators with determinant. A linear operator $T : H_1 \to H_2$, acting from a Hilbert space H_1 into a Hilbert space H_2, is called an *operator with trace* or an *operator of trace class*, if for some orthonormal bases $\{e_i\}$ in the space H_1 and $\{f_i\}$ in the space H_2 the series
$$\sum_i (Te_i, f_i)$$
converges. If this condition is satisfied for some orthonormal bases in H_1 and H_2, then it is fulfilled also for any orthonormal bases $\{e_i\}$ in H_1 and $\{f_i\}$ in H_2 and the sum
$$\sum_i (Te_i, f_i)$$
does not depend on the choice of bases. It is called the *trace* of the operator T and denoted by $\mathrm{Tr}\, T$. Operators $T : H \to H$ of trace class, acting in a Hilbert space H, form a two-sided ideal $\mathrm{Tr}(H, H)$ in the algebra $B(H)$ of all bounded linear operators in H, which contains the ideal $\mathrm{HS}(H, H)$ of Hilbert–Schmidt operators. Moreover, it's easy to see that the product of two operators from $\mathrm{Tr}(H, H)$ is a Hilbert–Schmidt operator, i.e. belongs to $\mathrm{HS}(H, H)$. The trace of an operator $T \in \mathrm{Tr}(H, H)$ coincides with the sum of its eigenvalues
$$\mathrm{tr}\, T = \sum_i \lambda_i(T) \ .$$
and behaves like the matrix trace.

If $T : H \to H$ is an operator of the trace class, then one can define for the operator $I - T$, where I is the identity operator, its *determinant* by

$$\det(I - T) := \prod_i (1 - \lambda_i(T)) \ .$$

The product in the right hand side is converging, since for an operator $T : H \to H$ of the trace class the sum $\sum_i |\lambda_i(T)|$ is always finite. Operators of the form $A = I - T$, where $T \in \mathrm{Tr}(H, H)$, are called the *operators with determinant* or *operators of determinant class*, and the set of such operators is denoted by $\mathrm{Det}(H, H)$. It's clear that the class $\mathrm{Det}(H, H)$ is closed under the product of operators.

Coming back to the Def. 34, note that the second condition in this definition means that the isomorphism w is "sufficiently close" to the identity. Moreover, it implies that the orthogonal projection $\mathrm{pr}_S \circ w : z^{-d}H_+ \to H_S$ onto any subspace H_S of virtual dimension d has a determinant, and any two admissible bases in a subspace $W \in \mathrm{Gr}_{\mathrm{HS}}(H)$ are related by the change of variables, which has a determinant.

Using the notion of the admissible basis, we can define the Plücker coordinate of a subspace $W \in \mathrm{Gr}_{\mathrm{HS}}(H)$.

Definition 35. Let W be a subspace of virtual dimension d, having an admissible basis w. The *Plücker coordinate* of W is a function of $S \in \mathcal{S}$ of the following form

$$\pi_S(w) = \begin{cases} \det(\mathrm{pr}_S \circ w) & \text{for } S \in \mathcal{S} \text{ of virtual cardinality } d \ , \\ 0 & \text{for } S \in \mathcal{S} \text{ of any virtual cardinality, other than } d \ . \end{cases}$$

If w' is another admissible basis in W, then

$$\pi_S(w') = \Delta_{ww'} \pi_S(w) \ ,$$

where $\Delta_{ww'}$ is the determinant of the change of variables, relating w with w'. Hence, the *projective class* $[\pi_S(w)]$ *does not depend on the choice of an admissible basis* w *in the subspace* W *and is uniquely determined by the subspace itself.*

In terms of the Plücker coordinate the neighborhoods U_S may be redefined as follows:

$$W \in U_S \Longleftrightarrow \pi_S(w) \neq 0 \text{ for any admissible basis } w \text{ in } W \ .$$

Proposition 9. *The Plücker map*

$$\pi : Gr_{HS}(H) \longrightarrow P(\mathcal{H}) \ , \quad W \longmapsto [\pi_S(w)]_{S \in \mathcal{S}} \ ,$$

determines a holomorphic embedding of the Grassmannian $Gr_{HS}(H)$ *into the projectivization of the Hilbert space* $\mathcal{H} = l^2(\mathcal{S})$.

We omit the proof of this assertion (it may be found in [65], Prop. 7.5.2), and only note that it is based on the relation

$$\sum_{S \in \mathcal{S}} |\pi_S(w)|^2 = \det(w^* w) < \infty \ , \tag{5.2}$$

satisfied for any admissible basis w in $W \in \mathrm{Gr}_{\mathrm{HS}}(H)$.

We shall construct now a holomorphic line bundle over $\mathrm{Gr}_{\mathrm{HS}}(H)$, being an analogue of the determinant bundle over the finite-dimensional Grassmannian.

Let a subspace $W \in \mathrm{Gr}_{\mathrm{HS}}(H)$ has the virtual dimension d. Consider the linear space, consisting of formal semi-infinite forms of the type

$$[\lambda, w] := \lambda w_{-d} \wedge w_{-d+1} \wedge \dots \ ,$$

where $\lambda \in \mathbb{C}$, $w = \{w_k\}_{k \geq -d}$ is an admissible basis in W. If w' is another admissible basis in W, then we shall identify the pair $[\lambda', w']$ with the pair $[\lambda, w]$, if $\lambda' = \lambda \Delta_{ww'}$, where $\Delta_{ww'}$ is the determinant of the change of variables, relating w with w'.

The linear space $\mathrm{Det}\, W$, obtained by taking the quotient of the space of semi-infinite forms of the type $[\lambda, w]$ with respect to the above equivalence relation, is a complex line.

We denote by Det the union of spaces $\mathrm{Det}\, W$ over all $W \in \mathrm{Gr}_{\mathrm{HS}}(H)$.

Proposition 10. *The natural projection*

$$\mathit{Det} \longrightarrow \mathit{Gr}_{\mathit{HS}}(H)$$

is a holomorphic line bundle.

This Proposition follows from the fact that the restriction of Det to any coordinate neighborhood U_S is trivial and the transition function for $U_{S_1} \cap U_{S_2} \neq \emptyset$ is given (in the notation of Sec. 5.1) by the formula

$$[\lambda_1, w_1] \longmapsto [\lambda_2, w_2] \ ,$$

where

$$w_2 = (c + dw_1)(a + bw_1)^{-1}, \quad \lambda_2 = \lambda \det(a + bw_1) \ .$$

This defines the structure of a holomorphic line bundle on Det.

We add several comments on the Plücker embedding and determinant bundle.

Remark 6. The bundle Det can be provided with a natural *Hermitian metric*, given by

$$\|[\lambda, w]\|^2 := |\lambda|^2 \det(w^* w)^2 \ .$$

Remark 7. The Plücker embedding $\pi : \mathrm{Gr}_{\mathrm{HS}}(H) \to P(\mathcal{H})$ may be pulled up to a holomorphic map

$$\tilde{\pi} : \mathrm{Det} \to \mathcal{H} \ ,$$

which is linear on the fibres, so that the *bundle Det will coincide with the inverse image of the tautological line bundle over $P(\mathcal{H})$ with respect to the embedding π.* Moreover, the pulled back map $\tilde{\pi} : \mathrm{Det} \to \mathcal{H}$ will preserve the norms (it follows from the relation (5.2) above).

Remark 8. The holomorphic line bundle Det has no non-trivial (global) sections, on the contrary, the dual bundle Det^* has many such sections. For example, all Plücker coordinates π_S determine holomorphic sections of Det^*. Indeed, the formula $[\lambda, w] \mapsto \lambda \pi_S(w)$ defines a holomorphic function $\mathrm{Det} \to \mathbb{C}$, which is linear on fibres, and induces a global holomorphic section of Det^*.

Note also that the symplectic form of the manifold $\mathrm{Gr}_{\mathrm{HS}}(H)$, constructed in Sec. 5.2, represents the Chern class of the complex line bundle $\mathrm{Det} \to \mathrm{Gr}_{\mathrm{HS}}(H)$. Otherwise speaking, it is *induced by the Fubini-Study form on $P(\mathcal{H})$ under the Plücker embedding $\pi : \mathrm{Gr}_{\mathrm{HS}}(H) \to P(\mathcal{H})$.*

Bibliographic comments

A key reference to this Chapter is the Pressley–Segal book [65]. Most of the assertions are taken from Ch.7 of [65].

Chapter 6

Quasiconformal maps

In this Chapter we introduce quasiconformal maps and prove main existence and uniqueness theorems for such maps. The quasiconformal maps will play a crucial role in Ch. 11, where we study the universal Teichmüller space. For a detailed exposition of the theory of quasiconformal maps cf. [1, 49].

6.1 Definition and basic properties

Let $w : D \to w(D)$ be a homeomorphism, mapping a domain D in the Riemann sphere $\overline{\mathbb{C}}$ onto another domain $w(D)$ in $\overline{\mathbb{C}}$.

Definition 36. Suppose that $w : D \to w(D)$ is a homeomorphism and w has locally L^1-integrable derivatives (in the generalized sense) in D. Then w is called *quasiconformal*, if there exists a measurable complex-valued function $\mu \in L^\infty(D)$ with

$$\|\mu\|_\infty := \text{ess sup}_{z\in D}|\mu(z)| =: k < 1 \ , \tag{6.1}$$

such that the following *Beltrami equation*

$$w_{\bar{z}} = \mu w_z \tag{6.2}$$

holds for almost all $z \in D$.

The function $\mu = \mu_w$ is called the *Beltrami differential* or the *complex dilatation of* w, and the constant k is often indicated in the name of the k*-quasiconformal maps.*

In particular, for $k = 0$ the homeomorphism w determines a conformal map from D onto $w(D)$. For diffeomorphisms w the quasiconformality of w means that infinitesimally it transforms small circles into ellipses, whose eccentricities (the ratio of the large axis to the small one) are bounded by a common constant $K < \infty$, related to the above constant $k = \|\mu\|_\infty$ by the formula

$$K = \frac{1+k}{1-k} \ .$$

The least possible constant K is called the *maximal dilatation* of w and is often included in the name of the K*-quasiconformal maps.*

The term "Beltrami differential" for the complex dilatation μ is motivated by the behavior of μ under conformal changes of variables. Namely, it follows from (6.2) that for a conformal change of variables f we should have

$$\mu(f(z)) = \mu(z)\frac{\overline{f_z(z)}}{f_z(z)}$$

for almost all $z \in D$. In general, we call a functional φ_w, defined on complex-valued functions w, a *differential of type* (m, n) with $m, n \in \mathbb{Z}$, if the quantity $\varphi_w(z)dz^m d\bar{z}^n$ remains invariant under conformal changes of variables. In the sense of this definition the complex dilatation μ_w is a differential of type $(-1, 1)$.

The inverse of a K-quasiconformal map f is again K-quasiconformal. The composition of a K_1-quasiconformal map f with a K_2-quasiconformal map g is a (K_1K_2)-quasiconformal map. This composition property may be deduced from the chain rule for Beltrami differentials. Namely, if f and g are two quasiconformal maps of a domain D with Beltrami differentials μ_f and μ_g respectively, then the following chain rule holds

$$\mu_{f\circ g^{-1}}(g(z)) = \frac{\mu_f(z) - \mu_g(z)}{1 - \mu_f(z)\overline{\mu_g(z)}} \cdot \frac{\overline{g_z(z)}}{g_z(z)} \ , \tag{6.3}$$

for almost all $z \in D$. In particular,

$$\mu_{g^{-1}}(g(z)) = -\mu_g(z) \cdot \frac{\overline{g_z(z)}}{g_z(z)} \ ,$$

so $|\mu_{g^{-1}}(g(z))| = |\mu_g(z)|$ for almost all $z \in D$.

From the chain rule (6.3) we can deduce the following transformation property of Beltrami differentials μ_w with respect to compositions of w with conformal maps f. If f is a conformal map (i.e. $\mu_f \equiv 0$), then

$$\mu_{f\circ w}(z) \equiv \mu_w(z) \ , \quad \mu_{w\circ f} = (\mu_w \circ f)\,\frac{\overline{f_z}}{f_z} \ .$$

These transformation rules for Beltrami differentials imply the following uniqueness property of solutions of the equation (6.2).

Proposition 11. *Suppose that two quasiconformal homeomorphisms w_1 and w_2 in a domain D satisfy the same Beltrami equation*

$$w_{\bar{z}} = \mu w_z$$

for almost all $z \in D$, where μ is a Beltrami differential in D, satisfying the condition (6.1). Then $w_1 \circ w_2^{-1}$ and $w_2 \circ w_1^{-1}$ are conformal. Conversely, the composition $f \circ w_1$ with any conformal map f, defined on $w_1(D)$, satisfies the same Beltrami equation, as w_1.

Quasiconformal homeomorphisms have a good behavior at the boundary, according to the following

Theorem 3 (Mori (cf. [1])). *Let $w : \Delta \to \Delta$ be a K-quasiconformal homeomorphism of the unit disc onto itself, normalized by the condition: $w(0) = 0$. Then the following sharp estimate*

$$|w(z_1) - w(z_2)| < 16|z_1 - z_2|^{1/K}$$

holds for any $z_1 \neq z_2 \in \Delta$. In other words, the homeomorphism w satisfies the Hölder condition of order $1/K$ in the disc Δ.

Mori's theorem implies, in particular, that w extends to a homeomorphism of the closed unit disc $\overline{\Delta}$. Another corollary of Mori's theorem is that K-quasiconformal homeomorphisms w of the unit disc Δ onto itself, normalized by the condition $w(0) = 0$, form a compact family with respect to the topology of normal convergence (i.e. uniform convergence on compact subsets). This result easily extends to general domains $D \subset \overline{\mathbb{C}}$.

Proposition 12. *Consider the family of all K-quasiconformal maps in D, normalized by the condition that any map in the family sends two fixed distinct points $z_1, z_2 \in D$ to another two fixed distinct points ζ_1, ζ_2. Then this family is compact with respect to the topology of normal convergence and any map w in this family satisfies the Hölder condition*

$$|w(z_1) - w(z_2)| < A|z_1 - z_2|^{1/K}$$

on any compact subset in D, where the constant A depends only on K and the compact subset.

In particular, any quasiconformal homeomorphism $w : D_1 \to D_2$ extends to a homeomorphism $w : \bar{D}_1 \to \bar{D}_2$ of the closures and so defines a homeomorphism of the boundaries.

We can ask the converse question: when a given homeomorphism $w : \partial D_1 \to \partial D_2$ extends to a quasiconformal homeomorphism $D_1 \to D_2$. It's convenient to study this problem first in the partial case, when both domains coincide with the upper half-plane: $D_1 = D_2 = H$.

Suppose that $f : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ is a monotone-increasing homeomorphism of the extended real line $\overline{\mathbb{R}} \subset \overline{\mathbb{C}}$, satisfying the normalization condition: $f(\infty) = \infty$. We call it *quasisymmetric*, if there exists a constant $A > 0$, such that the following finite-difference condition

$$\frac{1}{A} \leq \frac{f(x+t) - f(x)}{f(x) - f(x-t)} \leq A \tag{6.4}$$

is satisfied for all $x \in \mathbb{R}$ and all $t > 0$.

This condition can be considered as a variant of the cross ratio condition for quadruples of points. Recall that the *cross ratio* of four different points z_1, z_2, z_3, z_4 on the complex plane is given by the quantity

$$\rho = \rho(z_1, z_2, z_3, z_4) := \frac{z_4 - z_1}{z_4 - z_2} : \frac{z_3 - z_1}{z_3 - z_2}\ .$$

The equality of two cross ratios $\rho(z_1, z_2, z_3, z_4) = \rho(\zeta_1, \zeta_2, \zeta_3, \zeta_4)$ is a necessary and sufficient condition for the existence of a fractional-linear map of the complex plane, transforming the quadruple z_1, z_2, z_3, z_4 into the quadruple $\zeta_1, \zeta_2, \zeta_3, \zeta_4$. In the case of quasiconformal maps the cross ratios of quadruples may change but in a controlled way. The quasisymmetricity condition (6.4) expresses this control in a convenient form. Namely, we choose for a given homeomorphism $f : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ a quadruple of points on $\overline{\mathbb{R}}$ in the form $\vec{x} := (x - t, x, x + t, \infty)$ with the cross ratio $\rho(\vec{x}) =: \rho$ and associate with it the quantity

$$M(\rho(\vec{x})) = M(\rho) := \frac{\rho}{1-\rho} \ .$$

If, in particular, $\rho = 1/2$, then $M(\rho) = 1$. In this case the condition (6.4) means that the corresponding cross ratio of the quadruple $f(\vec{x}) := (f(x-t), f(x), f(x+t), \infty)$ satisfies the inequality

$$\frac{1}{A} \leq M(\rho(f(\vec{x})) \leq A \ .$$

The same condition in terms of $\rho(f(\vec{x}))$ can be rewritten as

$$\frac{1}{A+1} \leq \rho(f(\vec{x})) \leq \frac{A}{A+1}$$

or as

$$\frac{1}{2} - \epsilon \leq \rho(f(\vec{x})) \leq \frac{1}{2} + \epsilon \ ,$$

where $\epsilon = \epsilon(A) := \frac{1}{2} - \frac{1}{A+1}$.

Theorem 4 (Beurling–Ahlfors (cf. [1, 49])). *Suppose that $f : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ is a monotone-increasing homeomorphism of the extended real line $\overline{\mathbb{R}}$ onto itself, satisfying the normalization condition: $f(\infty) = \infty$. Then it can be extended to a quasiconformal homeomorphism $w : H \to H$ if and only if f is quasisymmetric, i.e. if there exists a constant $A > 0$, such that*

$$\frac{1}{A} \leq \frac{f(x+t) - f(x)}{f(x) - f(x-t)} \leq A$$

for all $x \in \mathbb{R}$, $t > 0$.

We have already explained above, where the necessity of the condition (6.4) comes from. The sufficiency of this condition is based on the following remarkable *Beurling–Ahlfors formula*, which gives a quasiconformal extension w to H of the quasisymmetric homeomorphism f:

$$w(x+iy) = \frac{1}{2}\int_0^1 (f(x+ty) + f(x-ty))\, dt + i\int_0^1 (f(x+ty) - f(x-ty))\, dt$$

for $x + it \in H$.

We formulate also an analogue of the above Theorem for the case of the circle S^1. We say that an orientation-preserving homeomorphism $f : S^1 \to S^1$ is *quasisymmetric*, if it satisfies for some $0 < \epsilon < 1$ the inequality

$$\frac{1}{2}(1-\epsilon) \leq \rho(f(z_1), f(z_2), f(z_3), f(z_4)) \leq \frac{1}{2}(1+\epsilon) \tag{6.5}$$

for any quadruple $z_1, z_2, z_3, z_4 \in S^1$ with cross ratio $\rho(z_1, z_2, z_3, z_4) = \frac{1}{2}$.

An analogue of the Beurling–Ahlfors theorem for S^1 asserts that *an orientation-preserving homeomorphism $f : S^1 \to S^1$ can be extended to a quasiconformal homeomorphism $w : \Delta \to \Delta$ if and only if it is quasisymmetric.* Douady and Earle (cf. [19]) have found an explicit extension operator E, which assigns to a quasisymmetric homeomorphism f its extension to a quasiconformal homeomorphism w of Δ and is conformally invariant in the sense that $E(w \circ f) = w \circ E(f)$ for any fractional-linear automorphism of Δ.

The image C of the circle S^1 under a quasiconformal homeomorphism of $\overline{\mathbb{C}}$ is called a *quasicircle* and the domains D_1, D_2, complementary to C in $\overline{\mathbb{C}}$, are called *quasidiscs*. All quasicircles have zero area and their Hausdorff dimension is always less than 2. However, it can be equal to any λ with $1 \leq \lambda < 2$ (cf. [24]).

Remark 9. There is a natural description of quasicircles in terms of quasiconformal reflections. Recall that a *reflection* across a Jordan curve C on $\overline{\mathbb{C}}$, dividing $\overline{\mathbb{C}} \setminus C$ into two domains D_1, D_2, is an orientation-preserving involutive homeomorphism φ of $\overline{\mathbb{C}}$, which maps D_1 onto D_2 (and vice versa) and fixes every point of C. The quasicircles are characterized by the following

Proposition 13. *A Jordan curve C on $\overline{\mathbb{C}}$ is a quasicircle if and only if it admits a quasiconformal reflection across it.*

We omit the proof of this Proposition, referring to the book [49], Theor. 6.1.

There is a simple geometric criterion for the quasicircles, passing through $\infty \in \overline{\mathbb{C}}$. Namely, a Jordan curve C, passing through ∞, is a quasicircle if and only if there exists a constant $c > 0$, for which the following condition is satisfied: for any three finite points z_1, z_2, z_3 on C, such that z_2 lies between z_1 and z_3, we have an inequality

$$|z_1 - z_2| + |z_2 - z_3| < c|z_1 - z_3|$$

(cf. [1, 49]).

6.2 Existence of quasiconformal maps

A key role in the theory of quasiconformal maps is played by the following existence theorem for solutions of the Beltrami equation (6.2).

Theorem 5 (Existence theorem). *For any measurable function μ in a domain $D \subset \overline{\mathbb{C}}$, such that $\|\mu\|_\infty = k < 1$, there exists a quasiconformal map on D, whose complex dilatation agrees with μ almost everywhere on D. In other words, there exists a solution w of the Beltrami equation*

$$w_{\bar{z}} = \mu w_z \ ,$$

satisfied for almost all $z \in D$.

As we have already pointed out earlier (cf. Prop. 11 in Sec. 6.1), any other solution $\tilde{w}$ of the above Beltrami equation has the form

$$\tilde{w} = w \circ f \ ,$$

where f is a conformal map.

The existence theorem implies the following generalization of the Riemann mapping theorem: *Let D_1 and D_2 be two domains in $\overline{\mathbb{C}}$, whose boundaries consist of more than one point. If μ is a measurable function on D_1 with $\|\mu\|_\infty < 1$, then there exists a quasiconformal map of D_1 onto D_2, whose complex dilatation agrees with μ almost everywhere.*

Proof. A detailed proof of Theorem 5 is given in [1], here we only point out its main points. First of all, it's sufficient to prove the existence theorem for the whole plane, since any $\mu \in L^\infty(D)$ with $\|\mu\|_\infty < 1$ can be extended (by setting it equal to zero outside D) to the whole plane, preserving the estimate $\|\mu\|_\infty < 1$.

Starting the proof of the existence theorem for the complex plane, we restrict first to the case, when the complex dilatation μ has a compact support.

We show under this hypothesis that *there exists a unique solution of the Beltrami equation* (6.2)*:*

$$w_{\bar{z}} = \mu w_z \ ,$$

satisfying the conditions:

$$w(0) = 0 \quad \textit{and} \quad w_z - 1 \in L^p \ ,$$

where $p > 2$ is a number, sufficiently close to 2, which will be chosen later.

Introduce the *Cauchy–Green operator*

$$Ph(\zeta) := -\frac{1}{\pi} \int h(z) \left(\frac{1}{z - \zeta} - \frac{1}{z} \right) dx\, dy \ ,$$

where the integral is taken over the complex plane. This operator is correctly defined for functions $h \in L^p$ with $p > 2$ and determines a continuous function (the function $Ph(\zeta)$ is even Hölder-continuous in ζ with Hölder exponent $1 - \frac{2}{p}$).

The partial derivatives of Ph (in the generalized sense) satisfy the equations

$$(Ph)_{\bar{z}} = h \ , \quad (Ph)_z = Th \ ,$$

where T is the *Calderon–Zygmund integral operator*, defined by

$$Th(\zeta) := -\frac{1}{\pi} P.V. \int h(z) \frac{1}{(z - \zeta)^2} dx\, dy \ .$$

Here the integral is taken in the *principal value sense*, i.e.

$$Th(\zeta) := -\frac{1}{\pi} \lim_{\epsilon \to 0} \int_{|z - \zeta| > \epsilon} h(z) \frac{1}{(z - \zeta)^2} dx\, dy \ .$$

The operator Th is correctly defined on functions h of class C_0^2 (i.e. C^2-smooth with compact supports). For such h, the function $Th(\zeta)$ is C^1-smooth. The operator T is also isometric in L^2-sense, i.e.

$$\|Th\|_2 = \|h\|_2 \ .$$

It follows that it can be extended to a bounded linear operator on L^2. Moreover, it can be proved, using the Calderon–Zygmund inequality, that T is bounded on functions $h \in L^p$ with $p > 1$:

$$\|Th\|_p \le C_p \|h\|_p \ ,$$

and $C_p \to 1$ for $p \to 2$. We choose now $p > 2$ in such a way that the inequality $\|\mu\|_\infty C_p < 1$ is satisfied.

We return to the construction of a solution w of the Beltrami equation (6.2), satisfying the conditions: $w(0) = 0$ and $w_z - 1 \in L^p$.

We show first that there could be only one such solution. Suppose that w is such a solution and consider the function

$$W := w - P(w_{\bar z}) \ .$$

Then its partial derivative with respect to $\bar z$ is equal to zero, hence W is an entire function. On the other hand, the condition $w_z - 1 \in L^p$ implies that the derivative of W, equal to $W' = w_z - T(w_{\bar z})$, satisfies the condition $W' - 1 \in L^p$, since $w_{\bar z} = \mu w_z$ belongs to L^p. This is possible only if $W' \equiv 1$, i.e. $W(z) \equiv z + \text{const}$. The constant is equal to zero because of the normalization, so $W(z) \equiv z$ and

$$w = P(w_{\bar z}) + z \ .$$

By differentiating this equality in z, we get for w_z an integral equation

$$w_z = T(w_{\bar z}) + 1 = T(\mu w_z) + 1 \ ,$$

in which the operator $h \mapsto T(\mu h)$ is contractible, since

$$\|T \circ \mu\|_p \le \|\mu\|_\infty C_p < 1 \ .$$

Suppose now that $\tilde w$ is another solution of (6.2), satisfying the conditions $w(0) = 0$ and $w_z - 1 \in L^p$. Then $\tilde w - w$ satisfies the equation

$$\tilde w_z - w_z = T(\mu(\tilde w_z - w_z))$$

which implies, because of the uniqueness of its solution, that $\tilde w_z = w_z$ almost everywhere. It follows from the Beltrami equation that also $\tilde w_{\bar z} = w_{\bar z}$ almost everywhere. Hence, $\tilde w - w$ is a constant, which is equal to zero, due to the normalization.

To prove the existence of a solution w of (6.2), satisfying the conditions $w(0) = 0$ and $w_z - 1 \in L^p$, we use the integral equation

$$h = T(\mu h) + T\mu \ .$$

Its unique L^p-solution yields a desired solution of the Beltrami equation (6.2), given by the formula

$$w = P(\mu(h+1)) + z \ . \tag{6.6}$$

Indeed, since $\mu(h+1) \in L^p$ (recall that μ has a compact support), the function $P(\mu(h+1))$ is correctly defined and continuous. The derivatives of w (in the generalized sense) are equal to

$$w_{\bar z} = \mu(h+1) \ , \quad w_z = T(\mu(h+1)) + 1 = h + 1$$

and $w_z - 1 = h \in L^p$. Hence, w, given by (6.6), satisfies the equation (6.2) and additional conditions $w(0) = 0$ and $w_z - 1 \in L^p$. According to the uniqueness assertion in Prop. 11, the constructed solution w of (6.2) will be uniquely defined, if we suppose additionally that it fixes not only the origin 0, but also two other points, say, $z = 1$ and $z = \infty$. We denote such a normalized solution by $w[\mu]$.

To end the proof, we should get rid of the compactness of the support of the complex dilatation μ. This can be done, using the following trick from [1], Sec. VB.

Note that the case, when $\mu \equiv 0$ in a neighborhood of 0, which is opposite to the case, when μ has a compact support, can be settled down by the reflection with respect to the unit circle S^1. More precisely, given a μ, vanishing in a neighborhood of 0, we set

$$\tilde{\mu}(z) := \mu(\frac{1}{z}) \cdot \frac{z^2}{\bar{z}^2} .$$

Then $\tilde{\mu}$ has a compact support, so we can find a normalized solution $\tilde{w} = w[\tilde{\mu}]$ of the Beltrami equation with the complex dilatation $\tilde{\mu}$, satisfying the additional conditions, indicated in the proof above. Then the "reflected" function

$$w(z) := \frac{1}{\tilde{w}(\frac{1}{z})}$$

will coincide with the normalized solution $w[\mu]$ of the Beltrami equation (6.2).

In the general case we decompose a given complex dilatation μ into the sum $\mu = \mu_\infty + \mu_0$ of complex dilatations μ_∞, having a compact support, and μ_0, equal to zero in a neighborhood of 0. We would like to write $w[\mu]$ as the composition $w[\mu_\infty] \circ w[\mu_0]$ of the corresponding normalized solutions $w[\mu_\infty]$ and $w[\mu_0]$. But this is not possible, unfortunately, due to the composition formula (6.3) for complex dilatations. However, taking into account the formula (6.3), we can write $w[\mu]$ as the composition

$$w[\mu] = w[\lambda] \circ w[\mu_0] ,$$

where the complex dilatation

$$\lambda := \left[\left(\frac{\mu - \mu_0}{1 - \mu\bar{\mu}_0} \right) \left(\frac{w[\mu_0]_z}{\bar{w}[\mu_0]_{\bar{z}}} \right) \right] \circ (w[\mu_0])^{-1}$$

still has a compact support. This concludes the proof of the existence theorem. □

Due to the uniqueness theorem (Prop. 11 in Sec. 6.1), we have the following

Corollary 1. *For any measurable function μ in a domain D with $\|\mu\|_\infty < 1$, there exists a unique normalized quasiconformal map on D, fixing the points $0, 1, \infty$, whose complex dilatation agrees with μ almost everywhere on D.*

Using the existence Theor. 5, it's easy to construct a solution of the Beltrami equation (6.2) in the upper half-plane $H = H_+$. For that it's sufficient to extend the dilatation μ to the lower half-plane $H^* = H_-$ by symmetry, setting

$$\hat{\mu}(z) := \overline{\mu(\bar{z})} \quad \text{for } z \in H_- . \tag{6.7}$$

Then, applying the existence theorem to the Beltrami equation with the dilatation $\hat{\mu}$, we obtain a unique solution w_μ of this equation, fixing the points $0, 1, \infty$. It follows from the uniqueness of the solution that w_μ satisfies the relation

$$w_\mu(\bar{z}) = \overline{w_\mu(z)} \ .$$

So w_μ maps the real axis onto itself and, consequently, preserves the upper half-plane H_+.

Another natural method to solve the Beltrami equation in the upper half-plane is to extend the given potential μ to the whole plane $\mathbb{C}$ by setting

$$\check{\mu}(z) = 0 \quad \text{for } z \in H_- \ .$$

Applying the existence theorem to the Beltrami equation with the dilatation $\check{\mu}$, we obtain a solution w^μ, which is conformal in the lower half-plane H_- and fixes the points $0, 1, \infty$.

The first method of constructing the solution w_μ of the Beltrami equation in H_+ is called *real-analytic*, since in this case w_μ depends real-analytically on μ. Respectively, the second method is called *complex-analytic*, since w^μ depends on μ holomorphically (cf. [56], Ch. 1.2, for a rigorous proof of these assertions).

Both methods are naturally transferred to the Beltrami equation in the unit disc Δ. For that in the first method one should substitute the symmetry transformation (6.7) by the reflection with respect to the unit circle $S^1 := \partial\Delta$. In other words, the dilatation μ, defined in the unit disc $\Delta = \Delta_+$, is extended to its exterior Δ_- by the formula

$$\hat{\mu}\Big(\frac{1}{\bar{z}}\Big) := \overline{\mu(z)} \cdot \frac{z^2}{\bar{z}^2} \quad \text{for } z \in \Delta \ .$$

The existence theorem for the extended dilatation $\hat{\mu}$ yields a quasiconformal homeomorphism $w_\mu : \mathbb{C} \to \mathbb{C}$, which preserves Δ_+ and Δ_- and fixes the points $\pm 1, -i$. The second method provides a quasiconformal homeomorphism $w^\mu : \mathbb{C} \to \mathbb{C}$, which is conformal on Δ_- and fixes the points $\pm 1, -i$.

Remark 10. There is an interesting assertion, due to Mañé, Sad and Sullivan, characterizing quasiconformal homeomorphisms as holomorphic perturbations of the identity. More precisely, we say that a homeomorphism $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is a holomorphic perturbation of the identity, if it can be included into a family of homeomorphisms $f_\lambda : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$, depending on a parameter $\lambda \in \Delta$, such that for every fixed $z_0 \in \mathbb{C}$ the function $f_\lambda(z_0)$ is holomorphic in $\lambda \in \Delta$, and

$$f_0 = \mathrm{id} \ , \quad f_{\lambda_0} = f \quad \text{for some } \lambda_0 \in \Delta \ .$$

It is proved in [52] that any member f_λ of such a family necessarily extends to a quasiconformal homeomorphism $\tilde{f}_\lambda$ of the extended complex plane $\overline{\mathbb{C}}$ with the complex dilatation, not exceeding $(1 + |\lambda|)/(1 - |\lambda|)$.

Conversely, any quasiconformal homeomorphism f of the extended complex plane $\overline{\mathbb{C}}$ is a holomorphic perturbation of the identity. Indeed, if $f = w^\mu$ for some Beltrami differential μ with $\|\mu\|_\infty = k < 1$, then we can include f into a holomorphic family of quasiconformal homeomorphisms, defined by

$$f_\lambda := w^{\lambda\mu/k} \ .$$

Bibliographic comments

Key references to this Chapter are the Ahlfors lectures on quasiconformal mappings [1] and Lehto's book [49]. Most of the assertions can be found in these sources. In particular, we follow Ahlfors' lectures in proving the main existence theorem for quasiconformal maps.

Part II

LOOP SPACES OF COMPACT LIE GROUPS

Chapter 7

Loop space ΩG

Let G be a compact Lie group. Its *loop space* or *based loop space* is a homogeneous space of (right conjugacy classes) of the loop group $LG = C^\infty(S^1, G)$ of the form

$$\Omega G = LG/G \ , \tag{7.1}$$

where $LG = C^\infty(S^1, G)$ is the group of smooth maps of the circle $S^1 = \{|z| = 1\} \subset \mathbb{C}$ into the group G, and G in the denominator is identified with the group of constant maps $S^1 \to g_0 \in G$.

The loop space ΩG is a homogeneous space of the Frechet Lie group LG with a natural action of LG on it by left translations. The origin (neutral element) in ΩG is given by the class $o := [\mathbf{1}] = [G]$ of constant maps.

The space ΩG may be identified (as a homogeneous space) with the subgroup L_1G of maps $\gamma \in LG$ such that

$$\gamma : 1 \in S^1 \longrightarrow \gamma(1) = e \in G \ ,$$

by associating with a class $[\gamma]$ of a loop $\gamma \in LG$ the map $\gamma(1)^{-1}\gamma \in L_1G$. Under this identification ΩG is realized as a closed submanifold (of codimension 1) of the Frechet manifold LG and so is itself a Frechet manifold. We note that this identification of ΩG with L_1G is not canonical, since G is not a normal subgroup in LG.

7.1 Complex homogeneous representation

One of the main features of the loop space ΩG, which plays a key role in the study of its Kähler geometry, is the existence of two kinds of its homogeneous representations. Namely, together with the "real" representation (7.1) of ΩG as a homogeneous space of the real Frechet Lie group LG, there exists also a "complex" representation of ΩG as a homogeneous space of the complex Frechet Lie group $LG^{\mathbb{C}} = C^\infty(S^1, G^{\mathbb{C}})$, where $G^{\mathbb{C}}$ is the complexification of the Lie group G. More precisely, we have the following representation

$$\Omega G = LG^{\mathbb{C}} / L_+G^{\mathbb{C}} \ , \tag{7.2}$$

where $L^+G^{\mathbb{C}} = \mathrm{Hol}(\Delta, G^{\mathbb{C}})$ is the subgroup of maps from $LG^{\mathbb{C}}$, which extend smoothly to holomorphic (and smooth up to the boundary) maps of the unit disc $\Delta \subset \mathbb{C}$ into the group $G^{\mathbb{C}}$.

Let us explain the meaning of the representation (7.2) in the case of the unitary group $G = \mathrm{U}(n)$. In this case $G^{\mathbb{C}} = \mathrm{GL}(n, \mathbb{C})$, and the equality (7.2) means that any complex non-degenerate (i.e. taking values in $\mathrm{GL}(n, \mathbb{C})$) matrix function $T(z)$ on the circle S^1 can be represented in the form

$$T(z) = U(z) \cdot H_+(z) \ , \quad z = e^{i\theta} \ , \tag{7.3}$$

where $U(z)$ is a smooth unitary (i.e. with values in $\mathrm{U}(n)$) matrix function, and $H_+(z)$ smoothly extends to a holomorphic non-degenerate matrix function in the disc Δ. It is a parametric analog of the standard representation of a matrix $T \in \mathrm{GL}(n, \mathbb{C})$ as the product of a unitary and upper-triangular matrices. The representation (7.3) would be unique, if one requires that $U \in L_1\mathrm{U}(n)$. Moreover, the product map $(U, H_+) \mapsto U \cdot H_+$ defines a diffeomorphism $\Omega\mathrm{U}(n) \times L^+\mathrm{GL}(n, \mathbb{C}) \to L\mathrm{GL}(n, \mathbb{C})$.

In the same sense we shall understand the equality (7.2) in the case of an arbitrary compact Lie group G. Namely, we have the following

Theorem 6 (Pressley–Segal)**.** *The product map*

$$\Omega G \times L^+G^{\mathbb{C}} \longrightarrow LG^{\mathbb{C}}$$

is a diffeomorphism of Frechet manifolds.

The proof of this Theorem, given in [65], uses the Grassmann realization of the loop space ΩG and will be given later in Ch. 9, after we introduce the Grassmann model of ΩG.

Remark 11. There is another approach to the proof of this Theorem, based on the Beurling–Helson theorem, describing the shift-invariant subspaces in L^2-spaces on the circle (this approach to the proof of Theorem 6 was proposed to us by A.Fedotov). We explain how to apply this theorem to the proof of Theor. 6 in the scalar case, i.e. for $G = S^1$.

Denote by H^2 the Hardy subspace in $L^2 = L^2(S^1)$, consisting of functions f, which extend holomorphically into the unit disc and have boundary values in the sense of L^2 on the circle S^1. In terms of Fourier decompositions $f \in H^2$ if and only if

$$f(z) = \sum_{n=0}^{\infty} c_n z^n \quad , \quad \sum_0^{\infty} |c_n|^2 := \|f\|^2_{H^2} < \infty \ , \ z \in \Delta \ .$$

Consider the shift operator S in L^2, which is defined by the formula

$$S : f(z) \longmapsto zf(z)$$

and maps H^2 into itself.

Theorem 7 (Beurling–Helson (cf., e.g. [60]))*. Any subspace E in L^2, invariant under the shift operator S, has the following form:*

1. *If $SE = E$, then there exists a measurable subset d in S^1 such that*

$$E = \chi_d L^2 \ ,$$

where χ_d is the characteristic function of the set d.

2. *If $SE \subset E$, but $SE \neq E$, then there exists a function $\theta \in L^2$ such that $|\theta| = 1$ almost everywhere on S^1 and*

$$E = \theta H^2 \ .$$

We return to the relation (7.2). Consider for a function $f \in L\mathbb{C}^*$ (here $\mathbb{C}^* = (S^1)^{\mathbb{C}}$ denotes the multiplicative group of non-zero complex numbers) the subspace E in L^2 of the form

$$E = fH^2 \ .$$

It is invariant under the shift operator S and $SE \neq E$, if $f \notin L^+\mathbb{C}^*$. So by Beurling–Helson theorem

$$fH^2 = \theta H^2$$

for some function $\theta \in L^2$, such that $|\theta| = 1$ almost everywhere on S^1. It implies that

$$f = f \cdot 1 = \theta \cdot h$$

for some $h \in H^2$, which is already the relation, we are looking for. It only remains to show that the functions θ and h may be chosen smooth (and smoothly depending on f), so that $\theta \in \Omega S^1$ and $h \in L^+\mathbb{C}^*$. It may be done as in [65], Ch. 8 (we also discuss this point in Ch. 9).

7.2 Symplectic structure

Since ΩG is a homogeneous space of the loop group LG, it's natural to use geometric structures, invariant under the action of LG, for the study of its Kähler geometry. Such structures are uniquely determined by their values at the origin $o \in \Omega G$. By this reason we start from the description of the tangent space $T_o(\Omega G)$.

The tangent space $T_o(\Omega G)$ coincides with the quotient of the tangent space $T_{\mathbf{1}}(LG) = L\mathfrak{g} = C^\infty(S^1, \mathfrak{g})$ modulo constant maps, i.e.

$$T_o(\Omega G) = L\mathfrak{g}/\mathfrak{g} =: \Omega\mathfrak{g} \ .$$

It is convenient to represent vectors from the complexified tangent space

$$T_o^{\mathbb{C}}(\Omega G) = L\mathfrak{g}^{\mathbb{C}}/\mathfrak{g}^{\mathbb{C}} =: \Omega\mathfrak{g}^{\mathbb{C}}$$

by their Fourier decompositions

$$\Omega\mathfrak{g}^{\mathbb{C}} \ni \xi(z) = \sum_{k \neq 0} \xi_k z^k \ , \ z = e^{i\theta} \ ,$$

where $\xi_k \in \mathfrak{g}^{\mathbb{C}}$ (the term, corresponding to $k = 0$, is eliminated by the factorization modulo $\mathfrak{g}^{\mathbb{C}}$ in $\Omega\mathfrak{g}^{\mathbb{C}}$). A vector $\xi \in T_o^{\mathbb{C}}(\Omega G)$ belongs to the real tangent space $T_o(\Omega G)$ if and only if

$$\xi_{-k} = \bar{\xi}_k \ ,$$

where the "bar" means the complex conjugation in $\mathfrak{g}^{\mathbb{C}}$, for which $\overline{\mathfrak{g}} = \mathfrak{g}$.

We construct now an invariant (with respect to LG-action) symplectic structure on ΩG. Define first its value at the origin or, in other words, the restriction of the symplectic form to the tangent space $T_o(\Omega G) = \Omega\mathfrak{g}$, and then transport it to other points of ΩG with the help of left translations by LG.

To define a symplectic form at the origin, we should fix an invariant inner product $< \cdot, \cdot >$ on the Lie algebra $\mathfrak{g}$ of the group G. Let us recall basic definitions, related to this notion.

Digression 2 (Invariant inner product). The *inner product* on the Lie algebra $\mathfrak{g}$ of the Lie group G is a positively definite symmetric bilinear form on $\mathfrak{g}$. We say that it is *invariant*, if it is invariant under the adjoint action Ad of the group G on its Lie algebra $\mathfrak{g}$, defined in the following way. The group G acts on itself by inner automorphisms of the form

$$G \ni g : \ G \ni h \longrightarrow ghg^{-1} \in G \ .$$

This action fixes the identity $e \in G$ and generates an action of the group G on $T_eG = \mathfrak{g}$, called the *adjoint action* and denoted by $\mathrm{Ad}\, g : \mathfrak{g} \longrightarrow \mathfrak{g}$. Its differential is called the *adjoint representation* of the Lie algebra $\mathfrak{g}$ and has the form

$$\mathrm{ad}\,\xi : \mathfrak{g} \longrightarrow \mathfrak{g}, \qquad \mathrm{ad}\,\xi : \eta \longmapsto [\xi, \eta] \ .$$

An inner product $< \cdot, \cdot >$ on the Lie algebra $\mathfrak{g}$ is invariant iff

$$< (\mathrm{Ad}\, g)\eta, (\mathrm{Ad}\, g)\zeta > = < \eta, \zeta > \quad \text{for any } \eta,\ \zeta \in \mathfrak{g} \ . \tag{7.4}$$

If the group G is connected, this condition is equivalent to a relation on the Lie algebra level, obtained from (7.4) by differentiation:

$$< (\mathrm{ad}\,\xi)\eta, \zeta > + < \eta, (\mathrm{ad}\,\xi)\zeta > = 0$$

or, equivalently,

$$< [\xi, \eta], \zeta > + < \eta, [\xi, \zeta] > = 0 \ .$$

On any Lie algebra $\mathfrak{g}$ there exists an invariant symmetric bilinear form, called the *Killing form*, defined by

$$< \xi, \eta > := \mathrm{tr}(\mathrm{ad}\,\xi\; \mathrm{ad}\,\eta) \ , \quad \xi, \eta \in \mathfrak{g} \ .$$

In particular, for $G = \mathrm{GL}(n, \mathbb{C})$ we have $\mathfrak{g} = \mathrm{gl}(n, \mathbb{C})$ and $< \xi, \eta > := \mathrm{tr}(\xi\, \eta)$. The Killing form is non-degenerate, if the group G is semisimple (e.g. for $G = \mathrm{SL}(n, \mathbb{C})$). If, moreover, G is compact, then the Killing form is negatively definite. Hence, the negation of this form defines an invariant inner product on the Lie algebra $\mathfrak{g}$ of a compact semisimple Lie group G.

We return to the construction of an LG-invariant symplectic form ω on ΩG. Let us fix an invariant inner product $< \cdot, \cdot >$ on the Lie algebra $\mathfrak{g}$ of the group G and define the restriction of the form ω to $T_o(\Omega G) = \Omega\mathfrak{g} = L\mathfrak{g}/\mathfrak{g}$.

Using the inner product $< \cdot, \cdot >$, we introduce, first of all, a 2-form ω_0 on the loop algebra $L\mathfrak{g}$, by setting it equal to

$$\omega_0(\xi, \eta) = \frac{1}{2\pi} \int_0^{2\pi} < \xi(e^{i\theta}), \frac{d\eta(e^{i\theta})}{d\theta} > d\theta \tag{7.5}$$

on vectors $\xi = \xi(e^{i\theta})$, $\eta = \eta(e^{i\theta})$ from the loop algebra $L\mathfrak{g} = C^\infty(S^1, \mathfrak{g})$.

This is a skew-symmetric bilinear form on $L\mathfrak{g}$, which is, due to the invariance of $< \cdot, \cdot >$, invariant under the adjoint action of the group G of constant loops on $L\mathfrak{g}$. It's evident that $\omega_0(\xi, \eta)$ is equal to zero, if at least one of the maps ξ, η is constant. So the form ω_0 can be pushed down to $\Omega\mathfrak{g}$, and the pushed-down form is already non-degenerate (to show that it is non-degenerate, consider its value on $\xi(e^{i\theta}) = \eta'(e^{i\theta}) := \frac{d\eta(e^{i\theta})}{d\theta}$). Hence, we have constructed a skew-symmetric bilinear form ω_0 on $\Omega\mathfrak{g}$, which is invariant under the adjoint action of the group G on $\Omega\mathfrak{g}$. This form can be extended (with the help of left translations) to an LG-invariant non-degenerate 2-form ω on ΩG.

It remains to check that the obtained form ω is closed on ΩG. The closedness condition (cf. Subsec. 1.2.4), due to the invariance of ω, takes on the form

$$\omega([\xi, \eta], \zeta) + \omega([\eta, \zeta], \xi) + \omega([\zeta, \xi], \eta) = 0 \ . \tag{7.6}$$

It is sufficient to check it on vectors $\xi, \eta, \zeta \in L\mathfrak{g}$. In this case the equality (7.6) means that

$$\int_0^{2\pi} \{< [\xi, \eta], \zeta' > + < [\eta, \zeta], \xi' > + < [\zeta, \xi], \eta' >\} \, d\theta = 0 \ . \tag{7.7}$$

Integrating the first integral by parts, we obtain

$$\int_0^{2\pi} < [\xi, \eta], \zeta' > \, d\theta = -\int_0^{2\pi} < [\xi', \eta], \zeta > \, d\theta - \int_0^{2\pi} < [\xi, \eta'], \zeta > \, d\theta \ . \tag{7.8}$$

Due to the invariance of the inner product $< \cdot, \cdot >$ on $\mathfrak{g}$

$$< [\xi', \eta], \zeta > = < \xi', [\eta, \zeta] > = < [\eta, \zeta], \xi' > \ ,$$

and the first term in the right hand side of (7.8) sum to zero together with the second integral in the formula (7.7). By the same reason

$$< [\xi, \eta'], \zeta > = < \eta', [\zeta, \xi] > = < [\zeta, \xi], \eta' >$$

and the second term in the right hand side of (7.8) sum to zero together with the third integral in the formula (7.7). It proves the validity of the equality (7.6), which implies that $d\omega(\xi, \eta, \zeta) = 0$ for all $\xi, \eta, \zeta \in L\mathfrak{g}$.

The choice of the formula (7.5) for the form ω_0 on $L\mathfrak{g}$ looks somewhat ambiguous, but it may be shown that this form is uniquely determined by the invariant inner product $< \cdot, \cdot >$ on $\mathfrak{g}$ in the case of a semisimple Lie group G. More precisely, we have the following

Proposition 14 (Pressley–Segal ([65])). *If the Lie group G is semisimple, then any 2-form ω_0 on $L\mathfrak{g}$, which satisfies the relation* (7.6) *and is invariant under the adjoint action of the group G on $L\mathfrak{g}$, is given by the formula* (7.5) *for some symmetric invariant bilinear form $< \cdot, \cdot >$ on $\mathfrak{g}$.*

Proof. We note, first of all, that bilinear invariant forms on complex semisimple Lie algebras are necessarily symmetric. More precisely, the following assertion is true.

Lemma 3. *If G is a semisimple Lie group with the Lie algebra $\mathfrak{g}$, then any complex-bilinear G-invariant form on the complexified Lie algebra $\mathfrak{g}^{\mathbb{C}}$ is necessarily symmetric.*

In the case of a simple Lie group G the assertion of Lemma follows from the fact that there exists a unique (up to the proportionality) complex-bilinear G-invariant form on G (the Schur's lemma), namely, the Killing form. The case of a semisimple Lie group G is reduced to the considered case (cf. for details [65]).

We turn now to the proof of the Proposition. The form ω_0 on $L\mathfrak{g}$ may be extended to a complex-bilinear form $\omega_0 : L\mathfrak{g}^{\mathbb{C}} \times L\mathfrak{g}^{\mathbb{C}} \to \mathbb{C}$. Since any element $\xi \in L\mathfrak{g}^{\mathbb{C}}$ is represented by the Fourier series

$$\xi = \sum \xi_p z^p \ ,$$

the form ω_0 is uniquely determined by its values on monomials of the type $\xi_p z^p$, i.e. by the forms

$$\omega_{p,q}(\xi, \eta) := \omega_0(\xi z^p, \eta z^q) \ ,$$

defined for $p, q \in \mathbb{Z}$ and $(\xi, \eta) \in \mathfrak{g}^{\mathbb{C}} \times \mathfrak{g}^{\mathbb{C}}$. The forms $\omega_{p,q}$ are G-invariant and so, by Lemma, they are symmetric. Moreover, the skew-symmetricity of ω_0 implies that $\omega_{p,q} = -\omega_{q,p}$. The condition of closedness of ω_0 on $L\mathfrak{g}$, when applied to the monomials $\tilde{\xi} = \xi z^p$, $\tilde{\eta} = \eta z^q$, $\tilde{\zeta} = \zeta z^r$, has the form

$$\omega_0([\tilde{\xi}, \tilde{\eta}], \tilde{\zeta}) + \omega_0([\tilde{\eta}, \tilde{\zeta}], \tilde{\xi}) + \omega_0([\tilde{\zeta}, \tilde{\xi}], \tilde{\eta}) = 0 \ .$$

This equality transforms into the following relation for the forms $\omega_{p,q}$:

$$\omega_{p+q,r}([\xi, \eta], \zeta) + \omega_{q+r,p}([\eta, \zeta], \xi) + \omega_{r+p,q}([\zeta, \xi], \eta) = 0 \ . \tag{7.9}$$

From the symmetricity and G-invariance of the forms $\omega_{p,q}$ we obtain

$$\omega_{q+r,p}([\eta, \zeta], \xi) = \omega_{q+r,p}(\xi, [\eta, \zeta]) = \omega_{q+r,p}([\xi, \eta], \zeta)$$

and, analogously,

$$\omega_{r+p,q}([\zeta, \xi], \eta) = \omega_{r+p,q}(\eta, [\zeta, \xi]) = \omega_{r+p,q}([\xi, \eta], \zeta) \ .$$

Hence, the equality (7.9) may be rewritten in the form

$$\omega_{p+q,r}([\xi, \eta], \zeta) + \omega_{q+r,p}([\xi, \eta], \zeta) + \omega_{r+p,q}([\xi, \eta], \zeta) = 0,$$

equivalent in the case of a semisimple Lie algebra to the relation

$$\omega_{p+q,r} + \omega_{q+r,p} + \omega_{r+p,q} = 0 \ . \tag{7.10}$$

This relation for $q = r = 0$ implies that $\omega_{p,0} = 0$ for all p. Setting $r = -p - q$ in (7.10), we get

$$\omega_{p+q,-p-q} = \omega_{p,-p} + \omega_{q,-q} \ ,$$

whence $\omega_{p,-p} = p\omega_{1,-1}$. Setting $r = n - p - q$ in (7.10), we obtain

$$\omega_{n-p-q,p+q} = \omega_{n-p,p} + \omega_{n-q,q} \ ,$$

implying $\omega_{n-p,p} = p\omega_{n-1,1}$. Hence,

$$\omega_{n-1,1} = \frac{\omega_{0,n}}{n} = 0$$

and so $\omega_{p,q} = 0$, if $p + q \neq 0$. Thus, the form ω_0 on vectors $\xi = \sum \xi_p z^p$, $\eta = \sum \eta_q z^q$ takes the value

$$\omega_0(\xi, \eta) = \sum \omega_{p,q}(\xi_p, \eta_q) = \sum_p \omega_{p,-p}(\xi_p, \eta_{-p}) = \sum_p p\omega_{1,-1}(\xi_p, \eta_{-p}) \ .$$

On the other hand

$$\frac{i}{2\pi}\int_0^{2\pi} \omega_{1,-1}(\xi(\theta), \eta'(\theta))\, d\theta = -\sum_{p,q} \frac{1}{2\pi}\int_0^{2\pi} \omega_{1,-1}(\xi_p e^{ip\theta}, q\eta_q e^{iq\theta})\, d\theta =$$

$$= \sum_p \frac{1}{2\pi}\int_0^{2\pi} p\omega_{1,-1}(\xi_p, \eta_{-p})\, d\theta = \sum_p p\omega_{1,-1}(\xi_p, \eta_{-p}) \ . \quad (7.11)$$

So

$$\omega_0(\xi, \eta) = \frac{1}{2\pi}\int_0^{2\pi} \langle \xi(\theta), \eta'(\theta) \rangle \ d\theta$$

with the invariant inner product on the Lie algebra $\mathfrak{g}$, given by the formula

$$< \xi, \eta > := \omega_{1,-1}(\xi, \eta) \ , \quad (7.12)$$

which concludes the proof of the Proposition. □

Remark 12. There is also a physical motivation behind the formula (7.5) for the symplectic form ω. It comes from the relation with the bosonic open string theory in the flat background space-time (cf. [14]). Mathematically, we consider the space $\Omega\mathbb{R}^d$ of based loops $S^1 \to \mathbb{R}^d$, taking values in the (non-compact) group $\mathbb{R}^d$ of translations of the d-dimensional Euclidean vector space. The loop space $\Omega\mathbb{R}^d$ may be interpreted as the phase space of the bosonic open string theory. More precisely, the configuration space of this theory consists of the smooth maps $q : [0, \pi] \to \mathbb{R}^d$ with all derivatives, vanishing at boundary points. The corresponding phase space consists then of pairs of maps (p, q) of the same type. The symplectic form on this phase space is given by the string analogue of the standard formula

$$\omega(\delta p, \delta q) = \frac{2}{\pi}\int_0^{\pi} \delta p(\sigma) \wedge \delta q(\sigma) d\sigma \ , \quad (7.13)$$

where δp, δq are smooth maps $[0, \pi] \to \mathbb{R}^d$ of the same type, as before, interpreted as tangent vectors to the phase space. A natural map, associating with a pair (p, q) the map $x : [-\pi, \pi] \to \mathbb{R}^d$, given by the formula

$$x(\sigma) = \begin{cases} p(\sigma) + q'(\sigma)\,, & \text{for } 0 \le \sigma \le \pi\,; \\ p(-\sigma) + q'(-\sigma)\,, & \text{for } -\pi \le \sigma \le 0\,, \end{cases}$$

identifies the introduced phase space with the space $\Omega\mathbb{R}^d$. It also converts the standard symplectic form (7.13) on the phase space of string theory into the symplectic form on $\Omega\mathbb{R}^d$, given by the formula, analogous to (7.5) (cf. [14]).

We have assigned to any invariant inner product on the Lie algebra $\mathfrak{g}$ an invariant symplectic structure ω on the loop space ΩG, determined by the formula (7.5). On the other hand, any invariant symplectic structure on the loop space ΩG uniquely determines an invariant inner product on $\mathfrak{g}$, given by the formula (7.12). As we have pointed out in Sec. 4.2, invariant bilinear forms on $\mathfrak{g}$ are parameterized by elements of the cohomology $H^3(\mathfrak{g})$.

Remark 13. We note in passing that the condition of invariance of the form ω with respect to the adjoint action of the group G on the loop algebra $L\mathfrak{g}$ is not essential and plays the role of normalization. Indeed, if ω_0 is an arbitrary 2-form on $L\mathfrak{g}$, satisfying the condition (7.6), then the form

$$g \cdot \omega_0(\xi, \eta) := \omega_0((\operatorname{Ad} g)\xi, (\operatorname{Ad} g)\eta) \quad \text{for } g \in G$$

belongs to the same cohomology class, as ω_0 (it follows from the cocycle identity (7.6)). So the form

$$\int_G g \cdot \omega_0 \, dg \ ,$$

obtained from ω_0 by averaging over the group G, belongs to the same cohomology class, as ω_0, but is already invariant under the adjoint action of constant loops.

7.3 Complex structure

A complex structure on the loop space ΩG is induced from the complex representation

$$\Omega G = LG^{\mathbb{C}}/L_+G^{\mathbb{C}} \ , \tag{7.14}$$

in which $LG^{\mathbb{C}}$ is a complex Lie Frechet group, and $L_+G^{\mathbb{C}}$ is its closed complex subgroup.

This complex structure, denoted by J^0 in the sequel, is LG-invariant, and its restriction to the tangent space $T_o^{\mathbb{C}}(\Omega G) = \Omega\mathfrak{g}^{\mathbb{C}}$ at the origin may be given by an explicit formula. Namely, if $\xi = \sum\limits_{k\neq 0} \xi_k z^k \ \in \Omega\mathfrak{g}^{\mathbb{C}}$, then

$$J^0\xi = -i\sum_{k>0} \xi_k z^k + i\sum_{k<0} \xi_k z^k \ . \tag{7.15}$$

The corresponding tangent space $T_o^{1,0}(\Omega G)$ of $(1,0)$-vectors consists of vectors of the form $\sum_{k<0} \xi_k z^k$, while the space $T_o^{0,1}(\Omega G)$ of $(0,1)$-vectors contains vectors of the form $\sum_{k>0} \xi_k z^k$.

It's clear from the description of $(1,0)$-vectors on ΩG that the complex structure J^0 is formally integrable in the sense of Subsec. 1.2.4, i.e. the bracket of any two $(1,0)$-vector fields on ΩG is again a $(1,0)$-vector field. But we have already pointed out in Subsec. 1.2.4 that the formal integrability of a complex structure in the infinite-dimensional case does not imply the existence of an atlas of coordinate neighborhoods and local complex coordinates on a given manifold. In order to construct local complex coordinates on ΩG, one should use the complex representation (7.14) and the Birkhoff factorization theorem. We formulate next a particular case of this theorem, sufficient for our applications.

Denote by $L^-G^{\mathbb{C}}$ a closed subgroup of $LG^{\mathbb{C}}$, consisting of maps $\gamma \in LG^{\mathbb{C}}$, which extend to holomorphic and smooth up to the circle S^1 maps of the disc Δ_- (equal to the complement of the closed unit disc $\overline{\Delta}$ on the Riemann sphere $\overline{\mathbb{C}}$). We also consider a closed subgroup $L_1^-G^{\mathbb{C}}$ of $L^-G^{\mathbb{C}}$, consisting of maps $\gamma \in L^-G^{\mathbb{C}}$, taking the value $e \in G^{\mathbb{C}}$ at infinity $\infty \in \Delta_-$.

Theorem 8 (Birkhoff theorem ([8, 9], cf. also [65], Ch.8)). *The product map*

$$L^+G^{\mathbb{C}} \times L_1^-G^{\mathbb{C}} \longrightarrow LG^{\mathbb{C}} \tag{7.16}$$

is a diffeomorphism onto a dense open subset in the identity component of $LG^{\mathbb{C}}$.

The Birkhoff theorem implies that for all $\gamma \in LG^{\mathbb{C}}$ in a neighborhood of the identity $\mathbf{1} \in LG^{\mathbb{C}}$ we have a representation

$$\gamma = \gamma_+ \cdot \gamma_- \ ,$$

where $\gamma_+ \in L^+G^{\mathbb{C}}$, $\gamma_- \in L_1^-G^{\mathbb{C}}$. The factors $\gamma_\pm$ are uniquely defined by γ and their product yields a local diffeomorphism (7.16). In particular, it implies that the *loop space ΩG is locally diffeomorphic to the complex Lie Frechet group $L_1^-G^{\mathbb{C}}$.*

7.4 Kähler structure

We show now that the loop space ΩG is a Kähler Frechet manifold. For that, according to Def. 17 from Subsec. 1.2.5, we need to demonstrate that the introduced complex and symplectic structures on ΩG are compatible.

Since both structures are LG-invariant, it's sufficient to check their compatibility only at the origin $o \in \Omega G$. Consider vectors $\xi, \eta \in T_o(\Omega G)$ with Fourier decompositions

$$\xi = \sum_{k\neq 0} \xi_k z^k, \qquad \eta = \sum_{l\neq 0} \eta_l z^l \ .$$

Then

$$\omega_o(\xi,\eta) = \frac{1}{2\pi}\int_0^{2\pi} \left\langle \xi(e^{i\theta}), \eta'(e^{i\theta})\right\rangle d\theta = -\frac{i}{2\pi}\int_0^{2\pi} \sum_{k,l\neq 0} \left\langle \xi_k e^{ik\theta}, l\eta_l e^{il\theta}\right\rangle d\theta =$$
$$= -\frac{i}{2\pi}\sum_{k\neq 0}\int_0^{2\pi} \langle \xi_k, k\eta_{-k}\rangle \, d\theta = -i\sum_{k\neq 0} k\,\langle \xi_k, \eta_{-k}\rangle \ , \tag{7.17}$$

where the inner product $\langle\cdot,\cdot\rangle$ is extended to a complex-bilinear positive definite form on $\mathfrak{g}^{\mathbb{C}}$. (Recall that the form $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}^{\mathbb{C}}$ is positive definite, if $\left\langle \xi, \bar{\xi}\right\rangle > 0$ for any $\xi \in \mathfrak{g}^{\mathbb{C}}\setminus\{0\}$). The above relation implies the first property in Def. 17. To prove the second property in this definition, consider the form

$$g_o^0(\xi,\eta) := \omega_0(\xi, J^0\eta)$$

on $\Omega\mathfrak{g}$ and show that it is positively definite. Note that for $\eta = \xi$ this form can be given by the formula

$$g_o^0(\xi,\xi) = -i\sum_{k>0} k\,\langle\xi_k, i\xi_{-k}\rangle - i\sum_{k<0} k\,\langle\xi_k, -i\xi_{-k}\rangle = 2\sum_{k>0} k\,\langle\xi_k, \xi_{-k}\rangle \ .$$

Since $\xi_{-k} = \overline{\xi}_k$ and the inner product $\langle\cdot,\cdot\rangle$ is positively definite on $\mathfrak{g}^{\mathbb{C}}$, the form $g_o^0(\xi,\xi)$ is also positively definite. Extending it to a LG-invariant positively definite form on ΩG, we get an invariant Kähler metric g^0 on ΩG. So, we have proved that the loop space ΩG is a *Kähler Frechet manifold* with the symplectic structure ω and complex structure J^0.

7.5 Loop space ΩG as a universal flag manifold of a group G

We have pointed out in the beginning of Sec. 7.1 that one of the characteristic properties of the Kähler Frechet manifold ΩG is the existence of two different representations of ΩG:

$$\Omega G = LG/G = LG^{\mathbb{C}}/L^+G^{\mathbb{C}}$$

as a homogeneous space of the real Lie Frechet group LG and its complexification $LG^{\mathbb{C}}$.

We have seen in Ch. 3 that finite-dimensional Kähler manifolds, having a similar property, i.e. being homogeneous spaces of real compact and complex Lie groups simultaneously, are called the flag manifolds. So ΩG may be considered as an infinite-dimensional analogue of flag manifolds. Moreover, we show in this Section that in some sense it may be considered as a universal flag manifold of the group G, since all flag manifolds of G are canonically embedded into ΩG as complex submanifolds.

The real homogeneous representation of a flag manifold

$$F = G/L$$

of the group G may be interpreted otherwise as a representation of F as an orbit of the adjoint action Ad of G on its Lie algebra $\mathfrak{g}$ (or as an orbit of the coadjoint action Ad^* of G on the dual space $\mathfrak{g}^*$). Namely, the orbit of an element $\xi \in \mathfrak{g}$ with respect to the adjoint action has the form

$$G/G(\xi)\ ,$$

where the isotropy subgroup $G(\xi)$ at ξ coincides with the centralizer of ξ, i.e. with

$$G(\xi) = \{g \in G :\ \mathrm{Ad}\,g(\xi) = \xi\}\ .$$

All such orbits are flag manifolds and, conversely, any flag manifold of a compact semisimple Lie group may be represented in this form.

Consider now a natural action of S^1 on the loop space ΩG, identified with the subgroup L_1G in LG, given by the rotation of loops

$$\lambda\cdot\gamma(z) = \gamma(\lambda)^{-1}\gamma(\lambda z)\ , \quad \lambda \in S^1\ ,$$

where $\gamma \in \Omega G$. A loop γ is a fixed point of this S^1-action if and only if

$$\gamma(\lambda z) = \gamma(\lambda)\gamma(z) \quad \text{for all}\, \lambda, z \in S^1 \ .$$

In other words, γ should be a group homomorphism $S^1 \to G$. But if $\gamma : S^1 \to G$ is a homomorphism, so are all the loops, conjugate to γ, i.e. the loops of the form $\gamma_g = g\gamma g^{-1}$ for $g \in G$. The set of all such loops (the conjugacy class of the loop γ) is parameterized by points of the homogeneous space

$$F_\gamma = G/G(\gamma) \ ,$$

where $G(\gamma)$ is the centralizer of the one-parameter subgroup $\gamma(S^1)$ in G. The homogeneous space F_γ can be identified, as we have pointed out above, with a flag manifold of the group G.

So, the *set of fixed points of the S^1-action on ΩG is the disjoint union*

$$\mathrm{Fix}(S^1) = \bigcup_\gamma F_\gamma$$

of flag manifolds F_γ, where γ runs over the set of conjugacy classes of homomorphisms $S^1 \to G$. The flag manifolds F_γ are immersed into ΩG as finite-dimensional Kähler submanifolds.

Remark 14. We can say much more about the constructed embedding of flag manifolds of the group G into the loop space ΩG. Namely, denote by

$$\pi : \Omega G \longrightarrow G \ , \quad \gamma \longmapsto \gamma(-1) \ ,$$

the map, associating with a loop γ its value at the point $-1 \in S^1$. This map is an analogue of the canonical bundle $\pi : F \to N$, considered in Sec. 3.1, Rem. 5.

According to Uhlenbeck [73], the embedding of flag manifolds F of the group G into ΩG respects canonical bundles. In other words, not only the loop space ΩG may be considered as a universal flag manifold of the group G, but also the above canonical bundle $\pi : \Omega G \to G$ may be considered as a universal canonical flag bundle.

More precisely, there exists the following commutative diagram

$$\begin{tikzcd} F \arrow[r, "\Gamma"] \arrow[d, "\pi"'] & \Omega G \arrow[d, "\pi"] \\ N(F) \arrow[r, "\gamma"] & G \end{tikzcd}$$

where $\pi \colon F \to N(F)$ is the canonical bundle over the symmetric space $N(F)$, constructed in Sec. 3.1, and the map Γ is the embedding of a flag manifold into ΩG, constructed above.

The horizontal maps in the this diagram admit a simple description in terms of the canonical element, introduced in Sec. 3.1, Rem. 5. Namely, suppose that the group G has a trivial center, and consider the flag manifold $F = G/L = G^{\mathbb{C}}/P$ with the canonical element ξ. The triviality of the center of G implies that $\exp(2\pi\xi) = e \in G$. So we can define a map $\Gamma \colon F \to \Omega G$ by setting it equal to $\Gamma(o) :=$ a map $\{e^{it} \mapsto$

$\exp(t\xi)\}$ at the neutral element $o \in F$, and transporting it to other points of G/L with the help of left translations by G. On the other hand, there is a natural map $\gamma\colon N(F) \to G$, assigning to a point x of the inner symmetric space $N(F)$, associated with F, the element $\gamma(x)$ of the group G, generating the involution at the point x. Both maps Γ and γ are totally geodesic immersions.

Remark 15. The fixed points of the S^1-action on ΩG can be also interpreted as critical points of some Morse function on ΩG (cf. [65]). Namely, define the *energy* $E : \Omega G \to \mathbb{R}_+$ *of a loop* γ by the formula

$$E(\gamma) = \frac{1}{4\pi}\int_0^{2\pi} < \gamma(e^{i\theta})^{-1}\gamma'(e^{i\theta}), \gamma(e^{i\theta})^{-1}\gamma'(e^{i\theta}) > d\theta\ .$$

It may be shown that the Hamiltonian vector field on ΩG, corresponding to the function E, generates the above S^1-action on ΩG by rotation of loops. So the *critical points of* E *correspond to the fixed points of the* S^1*-action on* ΩG*, i.e. to the homomorphisms* $\gamma : S^1 \to G$.

7.6 Loop space $\Omega_T G$

According to the Borel–Weil theorem (cf. Sec. 3.3), the full flag manifold $F = G/T$ of the group G, where T is a maximal torus in G, plays a special role in the theory of irreducible representations of G. A natural analogue of the full flag manifold in the case of the loop group LG is given by the homogeneous space

$$\Omega_T G = LG/T\ .$$

We list some of the properties of this Kähler Frechet manifold.

In order to define a *symplectic structure* on $\Omega_T G$, we note, first of all, that the loop group LG is diffeomorphic (as a Frechet manifold) to the direct product $G \times \Omega G$. If we identify ΩG with the subgroup $L_1 G$ in LG, then this diffeomorphism will assign to a loop $\gamma \in LG$ the element $(\gamma(1), \gamma(1)^{-1}\gamma) \in G \times \Omega G$. From the group-theoretical point of view, the loop group LG is the semidirect product of G and $L_1 G$. It follows that, as a Frechet manifold, $\Omega_T G$ is diffeomorphic to

$$\Omega_T G = LG/T = G/T \times \Omega G\ .$$

A symplectic structure on $\Omega_T G$ is generated by the symplectic structure on ΩG, introduced in Sec. 7.2, and a canonical symplectic structure on the full flag manifold G/T. Recall that, as we have remarked in the previous Sec. 7.5, the flag manifolds of the group G may be considered as orbits of the coadjoint representation of G on the dual space $\mathfrak{g}^*$ to the Lie algebra $\mathfrak{g}$. Such orbits have a canonical symplectic structure, given by the Kirillov form (cf. Subsec. 3.2.3).

A *complex structure* on $\Omega_T G$ is induced, as in the case of the loop space ΩG, from the "complex" representation of $\Omega_T G$ as a homogeneous space of the complexified loop group $LG^{\mathbb{C}}$, which has the form

$$\Omega_T G = LG^{\mathbb{C}} \big/ B_+ G^{\mathbb{C}}\ , \tag{7.18}$$

where $B_+G^{\mathbb{C}}$ is a subgroup in $L_+G^{\mathbb{C}} = \mathrm{Hol}(\Delta, G^{\mathbb{C}})$, consisting of the maps $\gamma \in \mathrm{Map}(S^1, G^{\mathbb{C}})$, which extend to holomorphic and smooth up to the circle S^1 maps $\gamma : \Delta \to G^{\mathbb{C}}$ of the unit disc, and satisfy the additional condition: $\gamma(0) \in B_+$, where B_+ is the standard Borel subgroup in G. The proof of this assertion is similar to the proof of the complex representation for the loop space ΩG (cf. [65], Ch.8).

The introduced complex structure on $\Omega_T G$ is compatible with the symplectic structure and so defines on $\Omega_T G$ the *structure of a Kähler Frechet manifold*.

Bibliographic comments

A key reference for this Chapter is the Pressley–Segal book [65]. In particular, the proof of the factorization theorem 6 is given in Ch.8 of [65]. Another method of proving this theorem, based on the Beurling–Helson characterization of shift-invariant subspaces in L^2, is due to A.Fedotov (unpublished). We present its idea in the scalar case, though the proof is valid for general matrix functions on the circle. The results in Secs. 7.2,7.3,7.4,7.6 may be found in [65]. An interpretation of the loop space ΩG as a universal flag manifold may be found in [5].

Chapter 8

Central extensions of loop algebras and loop groups

We start this Chapter by recalling a general method of constructing central extensions of Lie groups, acting on a smooth manifold. We then apply this method for the construction of central extensions of loop groups. In the last Section of this Chapter we describe the coadjoint action of the loop groups.

8.1 Central extensions and S^1-bundles

Suppose that a Lie group $\mathcal{G}$ acts by smooth transformations on a smooth simply connected manifold X. We assume that there exists a closed 2-form ω on X, which is invariant under the action of $\mathcal{G}$, such that $\omega/2\pi$ is an *integral form*. In other words, the cohomology class of $\omega/2\pi$ in $H^2(X,\mathbb{R})$ is integral, i.e. contained in $H^2(X,\mathbb{Z})$ (in other words, the integral of $\omega/2\pi$ over any 2-dimensional homology cycle is an integer). We shall construct a natural S^1-bundle over X, associated with these data.

Proposition 15. *Suppose that a Lie group $\mathcal{G}$ acts by smooth transformations on a smooth simply connected manifold X. Assume that ω is a closed $\mathcal{G}$-invariant 2-form on X, such that $\omega/2\pi$ is an integral form. Then there exists a principal S^1-bundle $L \to X$ with a connection ∇, having the curvature, equal to ω.*

The S^1-bundle, which existence is asserted in the Proposition, is used extensively in algebraic geometry and geometric quantization. In geometric quantization the line bundle, associated with the S^1-bundle $L \to X$, is called the *prequantization bundle*.

Proof. In terms of Čech cohomology, any cohomology class in $H^2(X,\mathbb{Z})$ is given by an integer-valued 2-cocycle $\{\nu_{abc}\}$ with respect to an acyclic open covering $\{U_a\}$ of X:

$$U_{abc} = U_a \cap U_b \cap U_c \longmapsto \nu_{abc} \in \mathbb{Z} \ .$$

(We shall assume from now on that all open sets U_a in this covering are contractible and their intersections are connected to guarantee the acyclicity of the covering $\{U_a\}$. This can be always achieved by the refinement of the covering.)

In terms of de Rham cohomology, the integrality condition of the form $\omega/2\pi$ means that there exists an integral closed 2-form ν on X such that

$$\omega = 2\pi\nu + d\beta \ ,$$

where β is an arbitrary 1-form on X. The integral form ν in terms of Čech cohomology is given by an integer-valued cocycle $\{\nu_{abc}\}$. Given such a cocycle, one can recover the form ν by choosing a smooth partition of unity $\{\lambda_a\}$, subordinate to the covering $\{U_a\}$, and setting

$$\nu := \sum_{a,b,c} \nu_{abc}\lambda_a \, d\lambda_b \wedge d\lambda_c \ .$$

We define the required S^1-bundle $L \to X$ by explicit transition functions

$$\varphi_{ab} = \exp\{2\pi i \sum_c \nu_{abc}\lambda_c\}$$

with respect to the covering $\{U_a\}$. It's easy to check that $\{\varphi_{ab}\}$ is a cocycle, i.e. the following relation is satisfied on every triple intersection U_{abc}: $\varphi_{ab}\varphi_{bc}\varphi_{ca} = 1$.

Consider a connection on L, given by the collection of local 1-forms

$$\alpha_a := 2\pi \sum_{b,c} \nu_{abc}\lambda_b \, d\lambda_c \ ,$$

satisfying on double intersections U_{ab} the relation

$$\alpha_b = \alpha_a + i\varphi_{ab}^{-1} d\varphi_{ab} \ .$$

The curvature of this connection is equal to

$$\sum_a \lambda_a \, d\alpha_a = 2\pi\nu \ .$$

So, by adding β to all forms α_a, we obtain a connection ∇ on L, given by the local 1-forms $\alpha_a + d\beta$ and having the curvature, equal to $2\pi\nu + d\beta = \omega$. □

Remark 16. In terms of the sheaf cohomology, the above proof can be rephrased as follows. Denote by $\mathcal{E}$ the sheaf of C^∞-smooth functions on X, and by $\mathcal{E}^*$ the (multiplicative) sheaf of non-vanishing functions in $\mathcal{E}$. We have the following exact sequence of sheafs over X

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{E} \xrightarrow{\exp} \mathcal{E}^* \longrightarrow 0 \ ,$$

where exp is the map $f \longmapsto e^{2\pi i f}$. The corresponding long exact sequence of sheaf cohomology have the form

$$\ldots \longrightarrow H^1(X,\mathcal{E}) \longrightarrow H^1(X,\mathcal{E}^*) \xrightarrow{c_1} H^2(X,\mathbb{Z}) \longrightarrow H^2(X,\mathcal{E}) \longrightarrow \ldots \ .$$

The cohomology $H^1(X,\mathcal{E}^*)$ can be identified with the set of isomorphism classes of complex line bundles on X, and the map $c_1 : H^1(X,\mathcal{E}^*) \longrightarrow H^2(X,\mathbb{Z})$ assigns to a complex line bundle E its 1st Chern class $c_1(E)$. Since the sheaf $\mathcal{E}$ is fine, the cohomologies on the extreme left and extreme right in the above long exact sequence vanish, i.e.

$$H^1(X,\mathcal{E}) = H^2(X,\mathcal{E}) = 0 \ ,$$

and it follows that $c_1 : H^1(X,\mathcal{E}^*) \longrightarrow H^2(X,\mathbb{Z})$ is an isomorphism. Recall that the cohomology class $[\omega/2\pi]$ is integral, i.e. $[\omega/2\pi] \in H^2(X,\mathbb{Z})$. Hence, there exists a complex line bundle $L \to X$ with $c_1(L) = [\omega/2\pi]$.

We prove next that the S^1-bundle $L \to X$, constructed in the above Proposition, is almost uniquely defined.

Proposition 16. *If L and L' are two S^1-bundles over X with connections ∇ and ∇', having the same curvature ω, then there exists a fibrewise isomorphism $\psi : L \to L'$ such that*

$$\psi^* \nabla' = \nabla \ .$$

Such an isomorphism ψ is determined uniquely up to multiplication by an element of S^1.

Proof. Suppose that the bundle L is given by the transition functions $\{\varphi_{ab}\}$ with respect to the covering $\{U_a\}$ of the manifold X, and the bundle L' is given by the transition functions $\{\varphi'_{ab}\}$ with respect to the same covering. If $\psi : L \to L'$ is the required isomorphism, then it should be given locally by functions $\psi_a : U_a \to S^1$, such that

$$\psi_b \varphi_{ab} = \varphi'_{ab} \psi_a \tag{8.1}$$

on double intersections $U_{ab} = U_a \cap U_b$. The condition $\psi^* \nabla' = \nabla$ in terms of local representatives ∇_a, ∇'_a of connections ∇, ∇' means that

$$\nabla'_a = \nabla_a + i\psi_a^{-1} d\psi_a \ . \tag{8.2}$$

We shall construct now the isomorphism ψ, having the required properties. Since $d(\nabla'_a - \nabla_a) \equiv 0$ on U_a, there exist functions $\phi_a : U_a \to \mathbb{R}$ such that

$$d\phi_a = \nabla'_a - \nabla_a \ .$$

The local representatives of connections ∇, ∇' satisfy on double intersections U_{ab} the relations

$$\nabla_b - \nabla_a = i\varphi_{ab}^{-1} d\varphi_{ab} \ , \quad \nabla'_b - \nabla'_a = i{\varphi'}_{ab}^{-1} d\varphi'_{ab} \ , \tag{8.3}$$

which imply that

$$d\varphi_b - d\varphi_a = i d \ln \varphi'_{ab} - i d \ln \varphi_{ab} \iff -id(\varphi_b - \varphi_a) = d \ln \frac{\varphi'_{ab}}{\varphi'_{ab}} \ .$$

Hence

$$d(e^{-i\varphi_b} \varphi_{ab}) = d(\varphi'_{ab} e^{-i\varphi_a}) \ ,$$

whence

$$e^{-i\varphi_b} \varphi_{ab} = \varphi'_{ab} e^{-i\varphi_a} e^{i\mu_{ab}} \quad \text{on} \quad U_{ab} \ ,$$

where μ_{ab} is a real number.

The numbers $\{\mu_{ab}\}$ define a Čech 1-cocycle on X, hence, due to the simply connectedness of X, we can find real numbers $\{m_a\}$, such that $\mu_{ab} = m_b - m_a$. Then the functions

$$\psi_a = e^{-i(\varphi_a + m_a)}$$

satisfy the properties (8.1), (8.2), and so determine the required isomorphism $\psi : L \to L'$.

We analyze next the uniqueness of the constructed isomorphism. Suppose that there exists another isomorphism ψ' of the same type, given by local representatives $\{\psi'_a\}$ with respect to the covering $\{U_a\}$. The relations (8.1) imply that

$$\psi_b\varphi_{ab} = \varphi'_{ab}\psi_a \ , \quad \psi'_b\varphi_{ab} = \varphi'_{ab}\psi'_a \ ,$$

whence

$$\psi_b(\psi'_b)^{-1} = \psi_a(\psi'_a)^{-1} =: h$$

for all a, b, i.e. the local representatives $\{\psi_a\}$ and $\{\psi'_a\}$ differ by a global function $h : X \to S^1$. Then the relations (8.2) imply that

$$d\varphi_a = \nabla'_a - \nabla_a = i\psi_a^{-1}d\psi_a \ , \tag{8.4}$$

$$d\varphi_a = i\psi_a^{-1}h^{-1} \cdot h\, d\psi_a + i\psi_a^{-1}h^{-1}\, dh\psi_a \ , \tag{8.5}$$

whence $dh = 0$, i.e. $h = \text{const}$. □

Using these Propositions, we can construct for a group $\mathcal{G}$, acting by smooth transformations on X, its central extension $\tilde{\mathcal{G}}$, acting on the bundle $L \to X$. We assume again that we are given with a closed $\mathcal{G}$-invariant 2-form ω on X, such that $\omega/2\pi$ is an integral form. Then, by Prop. 15, we can construct the S^1-bundle $L \to X$ with the connection ∇, having the curvature, equal to ω.

Consider for a given $g \in \mathcal{G}$ the pull-back of L under the action of g and provide it with the connection $\nabla_g = g^*\nabla$, having the curvature $g^*\omega = \omega$ (recall that ω is invariant under $\mathcal{G}$). According to Prop. 16, there exists an isomorphism $\psi : L \to g^*L$ such that

$$\psi^*\nabla_g = \psi^*g^*\nabla = \nabla \ .$$

We define $\tilde{\mathcal{G}}$ as a group, consisting of all pairs (g, ψ), where $g \in \mathcal{G}$ and ψ is an isomorphism $L \to g^*L$, for which $\psi^*\nabla_g = \psi^*g^*\nabla = \nabla$. Or, equivalently, we can define $\tilde{\mathcal{G}}$ as a group, consisting of pairs (g, φ), where $g \in \mathcal{G}$ and $\varphi : L \to L$ is a fibrewise isomorphism, covering the action of g on X, and having the property that $\varphi^*\nabla = \nabla$. Note that the fibrewise map $\varphi : L \to L$ of the above type, covering the action of g on X, is uniquely determined by the element g and the image $\varphi(\lambda_0)$ of an arbitrary fixed point $\lambda_0 \in L_{x_0}$, $x_0 \in X$.

8.2 Central extensions of loop algebras and groups

Consider first *central extensions of the loop algebra* $L\mathfrak{g}$. As we have pointed out in Sec. 4.1, any such extension is determined by a cocycle $\omega \in H^2(L\mathfrak{g}, \mathbb{R})$, or, in other words, by a closed bilinear skew-symmetric form $\omega : L\mathfrak{g} \times L\mathfrak{g} \to \mathbb{R}$. We can assume, according to Rem. 11 at the end of Sec. 7.2, that the form ω is invariant under the adjoint action of the group G. Any such form on $L\mathfrak{g}$, according to Prop. 14 from Sec. 7.2, in the case of a semisimple Lie group G is given by the formula

$$\omega(\xi, \eta) = \omega_0(\xi, \eta) = \frac{1}{2\pi}\int_0^{2\pi} < \xi(e^{i\theta}), \eta'(e^{i\theta}) > d\theta \ , \quad \xi, \eta \in L\mathfrak{g} \ ,$$

where $< \cdot, \cdot >$ is an invariant inner product on the Lie algebra $\mathfrak{g}$. This yields a description of all central extensions $\widetilde{L\mathfrak{g}}$ (in the case of a semisimple group G) in terms of the cohomology $H^3(\mathfrak{g}, \mathbb{R})$ (cf. Ex. 28 in Sec. 4.2).

However, not every central extension of the loop algebra $L\mathfrak{g}$ generates a central extension of the loop group LG, even in the case of a simply connected group G. For that the form ω should be integral in the sense of the definition, given in the beginning of Sec. 8.1. More precisely, the following theorem is true.

Theorem 9 (Pressley–Segal [65], Theor. 4.4.1). *If the Lie group G is simply connected, then a central extension $\widetilde{L\mathfrak{g}}$ of the loop algebra $L\mathfrak{g}$ is associated with some central extension $\widetilde{LG}$ of the loop group LG if and only if the corresponding form $\omega/2\pi$ on LG (where ω is the cocycle of the central extension $\widetilde{L\mathfrak{g}}$) is an integral form. In this case the central extension $\widetilde{LG}$ is uniquely determined by the cocycle ω.*

Proof. The sufficiency of the integrality condition of the form $\omega/2\pi$ follows from the argument in the previous Section (cf. Prop. 15). Namely, we apply the construction of Prop. 15 to the case, when the group $\mathcal{G}$ is the loop group LG and the manifold X coincides also with LG. According to Sec. 8.1, we can construct for an integral form $\omega/2\pi$ a complex line bundle L over LG with a connection ∇, having the curvature, equal to ω. Then we define the central extension $\widetilde{LG}$ as the group of bundle automorphisms of L, covering left translations of LG by elements of LG.

We prove the necessity of the integrality condition in the general setting of Sec. 8.1. If a central extension

$$1 \to S^1 \to \tilde{\mathcal{G}} \to \mathcal{G} \to 1$$

of a Lie group $\mathcal{G}$ is generated by a cocycle ω on the Lie algebra $\mathfrak{G}$, then the form $\omega/2\pi$ represents the 1st Chern class of a complex line bundle over X, associated with S^1-bundle $\tilde{\mathcal{G}} \to \mathcal{G}$. Hence, it must be integral.

It remains to prove the uniqueness of the central extension $\tilde{\mathcal{G}}$ of $\mathcal{G}$, corresponding to the cocycle ω. We note first that a central extension $\tilde{\mathcal{G}} \to \mathcal{G}$ is trivial, if the corresponding cocycle ω is trivial. Indeed, in this case the principal S^1-bundle $\tilde{\mathcal{G}} \to \mathcal{G}$ has a flat connection. So we can define a splitting homomorphism $\sigma : \mathcal{G} \to \tilde{\mathcal{G}}$ by associating with an element $g \in \mathcal{G}$ the end-point of a horizontal lift of any path in $\mathcal{G}$, connecting $e \in \mathcal{G}$ with g (recall that $\mathcal{G}$ is simply connected). To prove the uniqueness in the general case, suppose that there are two central extensions $\tilde{\mathcal{G}}$ and $\tilde{\mathcal{G}}'$ of $\mathcal{G}$, corresponding to the same cocycle ω. Then from the two principal S^1-bundles $p : \tilde{\mathcal{G}} \to \mathcal{G}$ and $p' : \tilde{\mathcal{G}}' \to \mathcal{G}$ we can form a "difference" bundle $p'' : \tilde{\mathcal{G}}'' \to \mathcal{G}$, which is a central extension of $\mathcal{G}$, corresponding to the trivial cocycle. To define $\tilde{\mathcal{G}}''$, we first pull back $\tilde{\mathcal{G}}'$ to $\tilde{\mathcal{G}}$ by p to get a subbundle $p^*(\tilde{\mathcal{G}}')$ of the fibre product $\tilde{\mathcal{G}} \times_{\mathcal{G}} \tilde{\mathcal{G}}'$. The circle S^1 is mapped into $\tilde{\mathcal{G}} \times_{\mathcal{G}} \tilde{\mathcal{G}}'$ by the homomorphism $u \mapsto (u, u^{-1})$. We define $\tilde{\mathcal{G}}''$ as the quotient of $p^*(\tilde{\mathcal{G}}')$ by the image of this homomorphism. Now, as we have proved, the difference extension $\tilde{\mathcal{G}}''$ should be trivial, which implies that both central extensions $\tilde{\mathcal{G}}$ and $\tilde{\mathcal{G}}'$ are equivalent. □

Remark 17. Let us discuss in more detail the integrality condition of the form $\omega/2\pi$, required in the above Theorem. We have pointed out earlier in Sec. 7.2 that the form ω is uniquely determined by the choice of an invariant inner product on the semisimple Lie algebra $\mathfrak{g}$. If this algebra $\mathfrak{g}$ is simple, then all invariant inner products on it are proportional to each other and among those, satisfying the integrality condition, there exists a minimal one. It is called the *basic inner product* and the corresponding central extension is called the *basic central extension* of the loop group

LG. The Killing form on $\mathfrak{g}$ satisfies the integrality condition and so is an integer multiple of the basic inner product. (The corresponding integer proportionality coefficient in the case of a simply laced group G coincides with the Coxeter number of G.)

The integrality condition can be also formulated in terms of co-roots $\alpha^\vee$ of the algebra $\mathfrak{g}$. Namely, the *form* $\omega/2\pi$ *is integral if and only if the inner product* $(\alpha^\vee, \alpha^\vee)$ *is an even number for all co-roots* $\alpha^\vee$ *of the algebra* $\mathfrak{g}$ (cf. [65], Sec. 4.4).

Remark 18 ([65], Sec. 4.11). At the end of Sec. 4.2 we have remarked that in the case of the loop algebra $L\mathfrak{g}$ there is an isomorphism

$$H^q(L\mathfrak{g}) = H^q(L\mathfrak{g}, \mathbb{R}) \longrightarrow H^q_{\text{top}}(LG, \mathbb{R}) .$$

This isomorphism can be used for the computation of the *cohomologies of the loop algebra* $L\mathfrak{g}$. Namely, since LG is diffeomorphic to $\Omega G \times G$, the cohomologies $H^*_{\text{top}}(LG, \mathbb{R})$ coincide with the tensor product of cohomologies $H^*_{\text{top}}(\Omega G, \mathbb{R}) \otimes H^*_{\text{top}}(G, \mathbb{R})$.

But in the case of a compact Lie group G, as we have pointed out in Sec. 4.2, we have

$$H^*_{\text{top}}(G, \mathbb{R}) \cong H^*(\mathfrak{g}) .$$

The cohomologies $H^*(\mathfrak{g})$ form an exterior algebra with r generators of odd-dimensional degrees, where r is the rank of G, and the generators correspond to generators of the algebra of invariant polynomials on $\mathfrak{g}$. By this correspondence we associate with an invariant polynomial of degree k a symmetric k-linear function $P : \mathfrak{g}\times\cdots\times\mathfrak{g} \to \mathbb{R}$, and use this function to define a skew-symmetric form S of degree $2k-1$, having the form

$$S(\xi_1, \ldots, \xi_{2k-1}) = \\ = \sum_\sigma (-1)^{\text{sgn}(\sigma)} P([\xi_{\sigma(1)}, \xi_{\sigma(2)}], [\xi_{\sigma(3)}, \xi_{\sigma(4)}], \ldots, [\xi_{\sigma(2k-3)}, \xi_{\sigma(2k-2)}], \xi_{\sigma(2k-1)}) , \quad (8.6)$$

where the summation in the formula (8.6) is taken over all possible permutations σ of the set $\{1, 2, \ldots, 2k-1\}$. If, in particular, $G = \mathrm{U}(n)$, then one can choose for generators of the algebra of invariant polynomials the functions $P_1, \ldots, P_n$ with $P_j(A) = \mathrm{tr}(A^j)$.

The de Rham cohomologies $H^*_{\text{top}}(\Omega G, \mathbb{R})$ (in the case of a simply connected group G) may be computed in terms of the cohomologies $H^*_{\text{top}}(G, \mathbb{R})$. Namely, the cohomologies $H^*_{\text{top}}(\Omega G, \mathbb{R})$ form an algebra of polynomials of even-dimensional classes, obtained from generators of the algebra $H^*_{\text{top}}(G, \mathbb{R})$ with the help of the transgression map. More precisely, consider the evaluation map

$$S^1 \times \Omega G \longrightarrow G , \quad (\theta, \gamma) \longmapsto \gamma(\theta) \in G .$$

The differential forms on G, which are the generators of the algebra $H^*_{\text{top}}(G, \mathbb{R})$, may be first pulled up to $S^1 \times \Omega G$ by the evaluation map, and then integrated over S^1. The obtained even-dimensional classes generate the algebra $H^*_{\text{top}}(\Omega G, \mathbb{R})$. More precisely, the image of the $(2k-1)$-from S from the formula (8.6) under the described transgression map coincides with a $(2k-2)$-form Σ on ΩG, which value at a point $\gamma \in \Omega G$ on vectors $\xi_1, \ldots, \xi_{2k-2} \in \Omega\mathfrak{g}$ is equal to

$$\Sigma_\gamma(\xi_1, \ldots, \xi_{2k-2}) = \frac{1}{2\pi} \int_0^{2\pi} S(\xi_1(\theta), \ldots, \xi_{2k-2}(\theta), \gamma(\theta)^{-1}\gamma'(\theta))\, d\theta .$$

8.3 Coadjoint representation of loop groups

To describe the *coadjoint representation of the loop group* LG of a compact Lie group G, we fix an invariant inner product $< \cdot, \cdot >$ on the Lie algebra $\mathfrak{g}$. It generates an inner product on the loop algebra $L\mathfrak{g}$ by the formula

$$< \xi, \eta >:= \frac{1}{2\pi} \int_0^{2\pi} < \xi(\theta), \eta(\theta) > \, d\theta \ , \quad \xi, \eta \in L\mathfrak{g} \ .$$

The adjoint action of the loop algebra $L\mathfrak{g}$ on the central extension $\widetilde{L\mathfrak{g}}$ of $L\mathfrak{g}$, determined by a cocycle $\omega(\xi, \eta)$, is given by the formula

$$\eta \cdot (\xi, s) := ([\eta, \xi], \omega(\eta, \xi)) \ ,$$

where $\eta \in L\mathfrak{g}$, $(\xi, s) \in \widetilde{L\mathfrak{g}} = L\mathfrak{g} \oplus \mathbb{R}$. It is generated by the adjoint action of the group LG on $\widetilde{L\mathfrak{g}}$, defined by the formula

$$\gamma \cdot (\xi, s) = (\gamma \cdot \xi, s - < \gamma^{-1}\gamma', \xi >) \ ,$$

where $\gamma \in LG$, $(\xi, s) \in \widetilde{L\mathfrak{g}}$ and $\gamma \cdot \xi$ denotes the (pointwise) adjoint action of the loop group LG on its Lie algebra $L\mathfrak{g}$.

Consider the coadjoint action of the loop group LG on the dual space $(\widetilde{L\mathfrak{g}})^*$. We note, first of all, that the dual space of the Frechet space $L\mathfrak{g} = C^\infty(S^1, \mathfrak{g})$ coincides with the space

$$(L\mathfrak{g})^* = \mathcal{D}'(S^1, \mathfrak{g}^*) = \mathcal{D}'(S^1) \otimes \mathfrak{g}^* \ ,$$

i.e. with the space of distributions on S^1 with values in $\mathfrak{g}^*$. Using the invariant inner product on the Lie algebra $\mathfrak{g}$, we can identify this space with the space of distributions on S^1 with values in the Lie algebra $\mathfrak{g}$. Under this identification, the "smooth" part of $(L\mathfrak{g})^*$, consisting of regular distributions in $(L\mathfrak{g})^*$, corresponds to the space $L\mathfrak{g}^* = C^\infty(S^1, \mathfrak{g}^*)$ or the space $L\mathfrak{g} = C^\infty(S^1, \mathfrak{g})$.

We describe first the coadjoint action of the loop group LG on the smooth part of $(\widetilde{L\mathfrak{g}})^* = (L\mathfrak{g})^* \oplus \mathbb{R}$, which coincides with $L\mathfrak{g}^* \oplus \mathbb{R} \cong L\mathfrak{g} \oplus \mathbb{R}$. It is given by the formula

$$\gamma \cdot (\varphi, s) = (\gamma \cdot \varphi + s\gamma'\gamma^{-1}, s) \ ,$$

where $\gamma \in LG$, $(\varphi, s) \in L\mathfrak{g} \oplus \mathbb{R}$, and $\gamma \cdot \varphi$ denotes, as above, the adjoint action of the loop group LG on its Lie algebra $L\mathfrak{g}$. It's easy to see that the map $S(\gamma) := \gamma'\gamma^{-1} \in L\mathfrak{g}$ defines a 1-cocycle in the space $C^1(LG, L\mathfrak{g})$ of 1-cochains on LG with values in $L\mathfrak{g}$, i.e. it satisfies the relation

$$S(\gamma_1\gamma_2) = \gamma_1 \cdot S(\gamma_2) + S(\gamma_1) \ .$$

We describe now the orbits of regular elements (φ, s) from $(L\mathfrak{g})^* \oplus \mathbb{R}$ under the action of the loop group LG. For that note that any element $(\varphi, s) \in L\mathfrak{g} \times \{s\}$, $s \neq 0$, is uniquely determined by a path $\psi : \mathbb{R} \to G$, satisfying the ordinary differential equation

$$\frac{d\psi}{dt}\psi^{-1} = -\frac{\varphi}{s} \iff \frac{d \ln \psi}{dt} = -\frac{\varphi}{s}$$

with the initial condition $\psi(0) = e$. It follows from the periodicity of φ with respect to θ that the shifted $\psi(\theta+2\pi)$ is also a solution of this equation together with $\psi(\theta)$. From the uniqueness theorem we obtain that

$$\psi(\theta + 2\pi) = \psi(\theta) M_\varphi \ ,$$

where the monodromy M_φ is defined by $M_\varphi := \psi(2\pi)$.

The coadjoint action of $\gamma \in LG$ on a regular element $(\varphi, s) \in L\mathfrak{g} \times \{s\}$ in terms of ψ corresponds to

$$\begin{array}{ccc}
\varphi & \xrightarrow{\gamma} & \widetilde{\varphi} = \gamma \cdot \varphi + s\gamma'\gamma^{-1} \\
\downarrow & & \downarrow \\
\psi & \xrightarrow{\gamma} & \widetilde{\psi}(\theta) = \gamma(\theta)\psi(\theta)\gamma(0)^{-1} \\
\downarrow & & \downarrow \\
M_\varphi & \longrightarrow & M_{\widetilde{\varphi}} = \gamma(0) M_\varphi \gamma(0)^{-1}
\end{array}$$

i.e. the coadjoint action of γ on $(\widetilde{L\mathfrak{g}})^*$ generates (in terms of the monodromy M_φ) an inner automorphism of the group G. Hence, we obtain a *1–1 correspondence between the orbits of regular elements of* $(L\mathfrak{g})^* \times \{s\}$ *with respect to the coadjoint action of the loop group* LG *and the conjugacy classes of elements* M_φ *in the group* G. Under this correspondence the isotropy subgroup of an element (φ, s) in the loop group LG corresponds to the centralizer of the monodromy M_φ in the group G.

We note that the orbit of an element (φ, s) is integral, if s is an integer and the corresponding conjugacy class of the monodromy $M \in G$ has the following property. The centralizer of M is a maximal torus T in G (with the Lie algebra $\mathfrak{t}$), in which terms M can be written in the form: $M = \exp \frac{\xi}{s}$ for an element $\xi \in \mathfrak{t} \subset \mathfrak{t}^*$, belonging to the lattice of characters $\widehat{T}$ (cf. [65], Sec. 4.3, for details).

Bibliographic comments

A key reference for this Chapter is the Pressley–Segal book [65]. In particular, the Propositions 15 and 16 are proved in Ch.4 (Prop. 4.5.3) of this book. The Theorem 9 on central extensions of loop groups is contained in Theor. 4.4.1 of [65]. The coadjoint representation of the loop group is described in Sec. 4.3 of [65].

Chapter 9

Grassmann realizations

In this Chapter we introduce the "widest" space of loops, to which the most part of the theory applies, namely, the Sobolev space of "half-differentiable" loops on S^1. This space contains the loop space ΩG, studied in previous sections, as a "smooth" part. In Sec. 9.2 we construct the Grassmann realization of this extended loop space and then apply the same idea to define the Grassmann realization of the "smooth" part ΩG. We end up with the postponed proof of the factorization theorem from Sec. 7.1, using the Grassmann realization of ΩG.

9.1 Sobolev space of half-differentiable loops

We consider first the *Sobolev space of* real-valued *half-differentiable functions* on S^1. This is a Hilbert space

$$V := H_0^{1/2}(S^1, \mathbb{R}) ,$$

which consists of functions $f \in L^2(S^1, \mathbb{R})$ with zero mean value over the circle, having the generalized derivative of order 1/2 in $L^2(S^1, \mathbb{R})$.

It may be shown (cf. [81]) that the Fourier series of a function $f \in H_0^{1/2}(S^1, \mathbb{R})$:

$$f(z) \equiv f(\theta) = \sum_{k \neq 0} f_k z^k \ , \quad f_k = \bar{f}_{-k} \ , \quad z = e^{i\theta} \ ,$$

converges outside a set of zero (logarithmic) capacity and has a finite *Sobolev norm of order* 1/2

$$\|f\|_{1/2}^2 = \sum_{k \neq 0} |k||f_k|^2 = 2 \sum_{k>0} k|f_k|^2 \ .$$

Moreover, by associating with a function $f \in V$ the sequence $\{f_k\}$ of its Fourier coefficients, we establish an isometric isomorphism between the Sobolev space V and the Hilbert space $\ell_2^{1/2}$ of sequences $\{f_k\} \in \ell_2$, satisfying the relations: $f_k = \bar{f}_{-k}$, $f_0 = 0$, and having a finite Sobolev norm: $\sum_{k \neq 0} |k||f_k|^2 < \infty$.

We can consider V as a natural Hilbert extension of the space $\Omega_0 := C_0^\infty(S^1, \mathbb{R})$ of smooth real-valued functions f on S^1, having the zero average over the circle. In terms of their Fourier series, the coefficients f_k of $f \in \Omega_0$ decrease faster than any power k^n with $n \in \mathbb{N}$. In fact, V coincides with the completion of Ω_0 with respect to the Sobolev norm.

The smooth part Ω_0 of V is a Kähler Frechet space, for which a complex and symplectic structures are introduced in the same way, as for the loop space ΩG with a compact Lie group G.

Namely, a *symplectic structure* on Ω_0 is given by the 2-form $\omega : \Omega_0 \times \Omega_0 \to \mathbb{R}$ of the type

$$\omega(\xi, \eta) = \frac{1}{2\pi} \int_0^{2\pi} \xi(\theta) d\eta(\theta) \ .$$

In terms of Fourier decompositions of $\xi, \eta \in \Omega_0$:

$$\xi(z) \equiv \xi(\theta) = \sum_{k \neq 0} \xi_k z^k \ , \quad \eta(z) \equiv \eta(\theta) = \sum_{k \neq 0} \eta_k z^k \ , \quad z = e^{i\theta} \ ,$$

this form has the following expression

$$\omega(\xi, \eta) = -i \sum_{k \neq 0} k \xi_k \eta_{-k} = 2\mathrm{Im} \sum_{k > 0} k \xi_k \bar{\eta}_k \ .$$

A *complex structure* operator J^0 on Ω_0 is given by the *Hilbert transform* $J^0 \in \mathrm{End}(\Omega_0, \Omega_0)$, defined by the formula

$$(J^0 \xi)(\theta) = \frac{1}{2\pi} \mathrm{P.V.} \int_0^{2\pi} k(\theta, \varphi) \xi(\varphi) d\varphi \tag{9.1}$$

with the kernel

$$k(\theta, \varphi) = \cot \frac{1}{2}(\theta - \varphi)$$

(the integral is taken in the principal value sense). In terms of Fourier decompositions the operator J^0 is given by the formula

$$\xi(z) = \sum_{k \neq 0} \xi_k z^k \longmapsto (J^0 \xi)(z) = -i \sum_{k > 0} \xi_k z^k + i \sum_{k < 0} \xi_k z^k \ .$$

The introduced complex structure J^0 is compatible with the symplectic structure ω and, in particular, defines a *Kähler metric* on Ω_0 by the formula

$$g^0(\xi, \eta) := \omega(\xi, J^0 \eta)$$

or, in terms of Fourier decompositions,

$$g^0(\xi, \eta) = 2\mathrm{Re} \sum_{k > 0} k \xi_k \bar{\eta}_k = \sum_{k \neq 0} |k| \xi_k \bar{\eta}_k \ .$$

So, the space $\Omega_0 = C_0^\infty(S^1, \mathbb{R})$ is provided with the structure of a Kähler Frechet space.

The above definitions of the complex structure J^0 and symplectic structure ω on the space Ω_0 extend to its completion V. (For the complex structure operator J^0 it's evident and for the symplectic structure ω follows immediately from the Cauchy–Schwarz inequality.) So, V has the structure of a *Kähler Hilbert space*, provided with the Kähler metric

$$g^0(\xi, \eta) = \omega(\xi, J^0 \eta) = 2\mathrm{Re} \sum_{k > 0} k \xi_k \bar{\eta}_k = \sum_{k \neq 0} |k| \xi_k \bar{\eta}_k \ .$$

The complexification

$$V^{\mathbb{C}} = H_0^{1/2}(S^1, \mathbb{C})$$

of V is a complex Hilbert space and the Kähler metric g^0 on V extends to a *Hermitian inner product* on $V^{\mathbb{C}}$, given by the formula

$$< \xi, \eta >= \sum_{k\neq 0} |k| \xi_k \bar{\eta}_k \ .$$

We extend the symplectic form ω and the complex structure operator J^0 complex linearly to $V^{\mathbb{C}}$.

The space $V^{\mathbb{C}}$ can be decomposed into the direct sum of subspaces

$$V^{\mathbb{C}} = W_+ \oplus W_- \ ,$$

where $W_\pm$ is the $(\mp i)$-eigenspace of the operator $J^0 \in \operatorname{End} V^{\mathbb{C}}$. In other words,

$$W_+ = \{f \in V^{\mathbb{C}} : \ f(z) = \sum_{k>0} f_k z^k\} \ , \quad W_- = \overline{W}_+ = \{f \in V^{\mathbb{C}} : \ f(z) = \sum_{k<0} f_k z^k\} \ .$$

The subspaces $W_\pm$ are isotropic with respect to the symplectic form ω (i.e. $\omega(\xi, \eta) = 0$, if $\xi, \eta \in W_+$ or $\xi, \eta \in W_-$), and the splitting $V^{\mathbb{C}} = W_+ \oplus W_-$ is an orthogonal direct sum with respect to the Hermitian inner product $< \cdot, \cdot >$. The inner product $< \cdot, \cdot >$ has a simple expression in terms of the decomposition $V^{\mathbb{C}} = W_+ \oplus W_-$:

$$< \xi, \eta >= i\omega(\xi_+, \bar{\eta}_+) - i\omega(\xi_-, \bar{\eta}_-) \ ,$$

where $\xi_\pm$ denotes the projection of $\xi \in V^{\mathbb{C}}$ onto the subspace $W_\pm$.

The operator J^0 in terms of the decomposition $V^{\mathbb{C}} = W_+ \oplus W_-$ has the following matrix representation

$$J^0 \longleftrightarrow \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \ .$$

There is another useful realization of the space V in terms of harmonic functions (cf. [58]). Namely, the space V can be identified with the space $\mathcal{D}$ of (real-valued) harmonic functions F in the unit disc Δ, such that $F(0) = 0$, and the Dirichlet integral

$$E(F) := \frac{1}{2\pi} \int_\Delta \left(\left| \frac{\partial F}{\partial x} \right|^2 + \left| \frac{\partial F}{\partial y} \right|^2 \right) dxdy$$

is finite. In other words, $\mathcal{D}$ is the Hilbert space of harmonic functions on Δ, having their first derivatives in $L^2(\Delta)$ and satisfying the normalization condition $F(0) = 0$. The norm of $F \in \mathcal{D}$ is equal, by definition, to the square root of $E(F)$. A map $V \to \mathcal{D}$, given by the Poisson integral, establishes an isometric isomorphism of Hilbert spaces V and $\mathcal{D}$. The inverse map $\mathcal{D} \to V$ associates with a harmonic function $F \in \mathcal{D}$ its boundary values on $\partial\Delta = S^1$ in the Sobolev sense.

We define next the *Sobolev space $H^{1/2}(S^1, GL(n, \mathbb{C}))$ of half-differentiable matrix functions on S^1*. It consists of measurable matrix-valued functions $\gamma : S^1 \to \mathrm{GL}(n, \mathbb{C})$ of the form

$$\gamma = \sum_{k=-\infty}^{\infty} \gamma_k z^k \ , \quad z = e^{i\theta} \ ,$$

with a finite Sobolev norm of order 1/2:

$$\|\gamma\|_{1/2}^2 = \sum_{k=-\infty}^{\infty} |k| \|\gamma_k\|^2 < \infty \ .$$

Accordingly, the space $H\mathrm{GL}(n,\mathbb{C}) := H_0^{1/2}(S^1, \mathrm{GL}(n,\mathbb{C}))$ denotes the subspace of $H^{1/2}(S^1, \mathrm{GL}(n,\mathbb{C}))$, consisting of functions γ with Fourier decompositions of the form

$$\gamma = \sum_{k \neq 0} \gamma_k z^k \ .$$

We define also the group $L_{1/2}(\mathrm{GL}(n,\mathbb{C}))$ of half-differentiable matrix functions. For that we consider the Banach algebra of essentially bounded functions $\gamma \in H^{1/2}(S^1, \mathrm{GL}(n,\mathbb{C}))$, provided with the norm $\|\gamma\|_\infty + \|\gamma\|_{1/2}$. The group of invertible elements in this algebra is called the *group $L_{1/2}(GL(n,\mathbb{C}))$ of half-differentiable matrix functions on S^1*. It is a Banach Lie group.

In the same way one can define the *Sobolev space HG of half-differentiable loops* in a compact Lie group G, when G is realized as a matrix group, i.e. a subgroup of $\mathrm{U}(n)$.

9.2 Grassmann realization

Consider first the Grassmann realization of the group $L_{1/2}(\mathrm{GL}(n,\mathbb{C}))$ of half-differentiable matrix functions on S^1.

Take for a complex Hilbert space H the space $H^{(n)} := L^2(S^1, \mathbb{C}^n)$ with a natural polarization, determined by the subspaces

$$H_+^{(n)} = \{f \in H : f(z) = \sum_{k \geq 0} f_k z^k \text{ with } f_k \in \mathbb{C}^n, z = e^{i\theta}\}$$

and

$$H_-^{(n)} = \{f \in H : f(z) = \sum_{k < 0} f_k z^k \text{ with } f_k \in \mathbb{C}^n\} \ .$$

Associate with a loop $\gamma \in L_{1/2}\mathrm{GL}(n,\mathbb{C})$ the multiplication operator

$$M_\gamma \in \mathrm{End}\, H^{(n)} \ ,$$

which acts on a vector $f \in L^2(S^1, \mathbb{C}^n)$ by the pointwise application of the matrix $\gamma(z) \in \mathrm{GL}(n,\mathbb{C})$ to the vector $f(z) \in \mathbb{C}^n$:

$$(M_\gamma f)(z) := \gamma(z) f(z) \ .$$

Proposition 17. *For any loop $\gamma \in L_{1/2}GL(n,\mathbb{C})$ the multiplication operator M_γ belongs to $GL_{HS}(H^{(n)})$ (cf. Sec. 5.2 for the definition of the Hilbert–Schmidt group GL_{HS}).*

Proof. Let

$$\gamma(z) = \sum_{k \in \mathbb{Z}} \gamma_k z^k \ ,$$

where $\gamma_k \in \mathrm{L}(n, \mathbb{C})$. We choose in $H^{(n)}$ the basis, given by the functions of the form $\epsilon_i z^p$, where $\{\epsilon_i\}$ is a fixed orthonormal basis in $\mathbb{C}^n$, $p \in \mathbb{Z}$. The operator M_γ in this basis has a matrix representation of the form

$$M_\gamma \longleftrightarrow \left(M_{p,q}\right)_{p,q\in\mathbb{Z}}\ , \text{ where } \quad M_{p,q} = \gamma_{q-p} \in \mathrm{L}(n, \mathbb{C})\ .$$

For $M_\gamma \in \mathrm{GL}_{\mathrm{HS}}(H^{(n)})$, it's necessary and sufficient that its components, given by the maps

$$M_\gamma^{+-} : H_+^{(n)} \to H_-^{(n)} \quad \text{and} \quad M_\gamma^{-+} : H_-^{(n)} \to H_+^{(n)}\ ,$$

are Hilbert–Schmidt operators. In terms of the matrix representation $(M_{p,q})_{p,q\in\mathbb{Z}}$ it means that the following inequalities should be satisfied

$$\sum_{p\geq 0, q<0} \|M_{p,q}\|^2 < \infty \quad \text{and} \quad \sum_{p<0, q\geq 0} \|M_{p,q}\|^2 < \infty\ .$$

These relations are equivalent to the inequality

$$\sum_{k\in\mathbb{Z}} k\|\gamma_k\|^2 < \infty\ ,$$

which is satisfied if $\gamma \in L_{1/2}\mathrm{GL}(n, \mathbb{C})$. □

The Grassmann realization of the group $L_{1/2}G$ can be constructed in the same way, when G is realized as a matrix group, i.e. a subgroup of $\mathrm{U}(n)$. For example, if G is a compact semisimple Lie group with the trivial centre, it can be identified with the identity component of the automorphism group of its Lie algebra $\mathfrak{g}$. In this case we can choose for H the Hilbert space $L^2(S^1, \mathfrak{g}^{\mathbb{C}})$, on which the loop group $L_{1/2}G$ acts by the adjoint representation. By identifying $\mathfrak{g}^{\mathbb{C}}$ with $\mathbb{C}^n$ (where n is the dimension of the Lie algebra $\mathfrak{g}$) and fixing an invariant inner product on $\mathfrak{g}$, we realize $L_{1/2}G$ as a subgroup of the loop group $L_{1/2}\mathrm{U}(n)$. Then the above embedding of $L_{1/2}\mathrm{GL}(n, \mathbb{C})$ into $\mathrm{GL}_{\mathrm{HS}}(H^{(n)})$ will map $L_{1/2}\mathrm{U}(n)$ into $\mathrm{U}_{\mathrm{HS}}(H^{(n)})$.

We shall describe now the image of the embedding of $L_{1/2}\mathrm{U}(n)$ into $\mathrm{U}_{\mathrm{HS}}(H^{(n)})$, following [65], Sec. 8.3. This embedding defines an action of $L_{1/2}\mathrm{U}(n)$ on $H^{(n)}$ and, hence, on $\mathrm{Gr}_{\mathrm{HS}}(H^{(n)})$. In particular, the image of this action contains the subspaces $W \in \mathrm{Gr}_{\mathrm{HS}}(H^{(n)})$ of the form $M_\gamma(H_+^{(n)}) := \gamma H_+^{(n)}$, where $\gamma \in L_{1/2}\mathrm{U}(n)$. They have the property that $M_z(W) := zW \subset W$, since the action of γ commutes with the multiplication by z. It turns out that the set of such subsets $W \in \mathrm{Gr}_{\mathrm{HS}}(H^{(n)})$ coincides with the image of the action of $L_{1/2}\mathrm{U}(n)$ on $\mathrm{Gr}_{\mathrm{HS}}(H^{(n)})$.

Before we prove this fact, let's introduce some necessary notations. Denote

$$\mathrm{Gr}_+(H^{(n)}) = \{W \in \mathrm{Gr}_{\mathrm{HS}}(H^{(n)}) : zW \subset W\}\ .$$

We also denote, as in Secs. 7.1,7.3, by $L_{1/2}^{\pm}\mathrm{GL}(n, \mathbb{C})$ the subgroups of $L_{1/2}\mathrm{GL}(n, \mathbb{C})$, consisting of loops γ, which are the Sobolev boundary values of holomorphic maps $\gamma : \Delta_\pm \to \mathrm{GL}(n, \mathbb{C})$.

Proposition 18 ([65]). *The group $L_{1/2}U(n)$ acts transitively on $Gr_+(H^{(n)})$ and the isotropy subgroup of $H_+^{(n)}$ coincides with the group $U(n)$ of constant loops.*

Proof. The assertion about the isotropy subgroup follows from a well known fact: $\gamma H_+^{(n)} = H_+^{(n)}$ if and only if $\gamma \in L_{1/2}^+\mathrm{GL}(n,\mathbb{C})$. The "if" part is evident. To prove the "only if" part, we decompose γ into the sum $\gamma = \gamma_+ + \gamma_-$ with $\gamma_\pm \in L_{1/2}^\pm \mathrm{gl}(n,\mathbb{C})$ (cf., e.g., [58], Theor. 2.1). Then the equality $\gamma H_+^{(n)} = H_+^{(n)}$ will imply that $\gamma_- H_+^{(n)} \subset H_+^{(n)}$, whence $\gamma_- \in H_+^{(n)}$, i.e. $\gamma_- = 0$. If we know that $\gamma \in L_{1/2}\mathrm{U}(n)$ belongs to $L_{1/2}^+\mathrm{GL}(n,\mathbb{C})$, then, by the symmetry principle, γ extends holomorphically to the whole Riemann sphere, which implies that $\gamma = \mathrm{const}$.

To prove the transitivity of the action of $L_{1/2}\mathrm{U}(n)$ on $\mathrm{Gr}_+(H^{(n)})$, we note first that $W \in \mathrm{Gr}_+(H^{(n)})$ *implies that zW has codimension n in W*. Indeed, consider the commutative diagram

$$\begin{array}{ccc} zW & \longrightarrow & W \\ \Big\downarrow & & \Big\downarrow \\ zH_+^{(n)} & \longrightarrow & H_+^{(n)} \end{array} ,$$

where the horizontal arrows are inclusions and the vertical arrows are orthogonal projections. These projections are Fredholm operators, having their index, equal to the virtual dimension of W. Since the inclusion $zH_+^{(n)} \hookrightarrow H_+^{(n)}$ is evidently Fredholm with the index, equal to $-n$, the same is true for the inclusion $zW \hookrightarrow W$.

We choose now an orthonormal basis $\{w_1, \dots, w_n\}$ in the orthogonal complement of zW in W and form an $(n \times n)$-matrix-valued function γ on S^1 from the vector columns $w_1, \dots, w_n$. We assert that $\gamma(\theta)$ is unitary for almost all $\theta \in S^1$. Indeed, write down $w_k(\theta)$ in the form

$$w_k(\theta) = \sum_p w_{kp} e^{ip\theta} \ , \quad w_{kp} \in \mathbb{C}^n \ .$$

Then

$$< w_k(\theta), w_l(\theta) > = \sum_{p,q} < w_{kp}, w_{lq} > e^{i(q-p)\theta} = \sum_r < w_k, z^r w_l >_H e^{ir\theta} = \delta_{kl} \ ,$$

where we have denoted by $< \cdot, \cdot >_H$ the inner product in $H^{(n)}$ to distinguish it from the inner product $< \cdot, \cdot >$ in $\mathbb{C}^n$. This calculation implies that the *multiplication operator M_γ is unitary in $H^{(n)}$ and*

$$M_\gamma(H_+^{(n)} \ominus z^k H_+^{(n)}) = W \ominus z^k W \quad \text{for any } k \ .$$

It follows also that $M_\gamma(H_+^{(n)}) = W$, since $\bigcap_k z^k W = 0$ (which can be proved by the iteration of the codimension assertion).

It remains to check that $M_\gamma \in \mathrm{U_{HS}}(H^{(n)})$. But the component M_γ^{+-} of this operator (we are using the same notation, as in the proof of Prop. 17) is factorized into the composition $H_+^{(n)} \to W \to H_-^{(n)}$, where the second map, given by the orthogonal projection, is a Hilbert–Schmidt operator. The same is true for the component M_γ^{-+} of M_γ. □

This proposition implies that the loop space $H\mathrm{U}(n) = L_{1/2}\mathrm{U}(n)/\mathrm{U}(n)$ can be identified with the Grassmanian $\mathrm{Gr}_+(H^{(n)})$. The same proof realizes the space $\Omega\mathrm{U}(n)$ of smooth loops in $\mathrm{U}(n)$ as a "smooth" part $\mathrm{Gr}_+^\infty(H^{(n)})$ of $\mathrm{Gr}_+(H^{(n)})$. Here,

$$\mathrm{Gr}_+^\infty(H^{(n)}) = \mathrm{Gr}^\infty(H^{(n)}) \cap \mathrm{Gr}_+(H^{(n)}) \ ,$$

and the "smooth" part $\mathrm{Gr}^\infty(H^{(n)})$ was introduced at the end of Sec. 5.2. It can be also shown that the group $L\mathrm{U}(n)$ of smooth loops acts smoothly and transitively on the Grassmanian $\mathrm{Gr}^\infty(H^{(n)})$ and the same is true for the action of $L\mathrm{GL}(n,\mathbb{C})$ on $\mathrm{Gr}^\infty(H^{(n)})$.

An embedding of the loop group LG, where G is a simply connected compact Lie group, into $\mathrm{Gr}_+^\infty(H^{(n)})$ can be constructed in a similar way, if one takes for H the Hilbert space $L^2(S^1,\mathfrak{g}^\mathbb{C})$, on which the group LG acts by the adjoint representation. Identifying $\mathfrak{g}^\mathbb{C}$ with $\mathbb{C}^n$ (where n is the dimension of the Lie algebra $\mathfrak{g}$) and fixing an invariant inner product on $\mathfrak{g}$, we can realize LG as a subgroup of $L\mathrm{U}(n)$. The action of $L\mathrm{U}(n)$ on $\mathrm{Gr}^\infty(H^{(n)})$, described above, realizes $L\mathrm{U}(n)$ as a subgroup of $\mathrm{U}_{\mathrm{HS}}(H^{(n)})$. This embedding generates an embedding of the loop space ΩG into the Grassmann manifold $\mathrm{Gr}^\infty(H^{(n)})$.

9.3 Proof of the factorization theorem

The Grassmann realization of the loop space $\Omega\mathrm{U}(n)$, constructed in the previous Section, allows us to give the postponed proof of the factorization theorem 6 from Sec. 7.1. We recall its formulation.

Theorem 10 ([65])**.** *The product map*

$$\Omega G \times L^+G^\mathbb{C} \longrightarrow LG^\mathbb{C}$$

is a diffeomorphism of Frechet manifolds .

We have pointed out in the proof of Prop. 18 that the complex group $L_{1/2}\mathrm{GL}(n,\mathbb{C})$ acts on the Grassmanian $\mathrm{Gr}_+(H^{(n)})$ and has the stabilizer at $H_+^{(n)}$, equal to the subgroup $L_{1/2}^+\mathrm{GL}(n,\mathbb{C})$. Since the loop group $L_{1/2}\mathrm{GL}(n,\mathbb{C})$ acts transitively on $\mathrm{Gr}_+(H^{(n)})$, we have proved that *the loop group $L_{1/2}GL(n,\mathbb{C})$ coincides with the product*

$$L_{1/2}\mathrm{GL}(n,\mathbb{C}) = L_{1/2}\mathrm{U}(n)\cdot L_{1/2}^+\mathrm{GL}(n,\mathbb{C}) .$$

The same factorization holds for the group $L\mathrm{GL}(n,\mathbb{C})$ of smooth loops. We have to show now that the multiplication map

$$\Omega\mathrm{U}(n)\times L^+\mathrm{GL}(n,\mathbb{C}) \longrightarrow L\mathrm{GL}(n,\mathbb{C})$$

is a diffeomorphism. It is sufficient to prove that the map

$$u : L\mathrm{GL}(n,\mathbb{C}) \longrightarrow \Omega\mathrm{U}(n) ,$$

assigning to a loop γ its unitary component, is smooth. This map is factorized into the composition of two maps: $\gamma\to\tilde\gamma\to u(\gamma)$. The first of them assigns to γ a loop $\tilde\gamma$, which is obtained from γ by projecting the columns $(\gamma_1,\dots,\gamma_n)$ of $\gamma\in L\mathrm{GL}(n,\mathbb{C})$ onto the orthogonal complement $W\ominus zW$ of the subspace zW in W, where $W := \gamma H_+^{(n)}$. The second map $\tilde\gamma\to u(\gamma)$ consists of the orthonormalization of the basis $\{\tilde\gamma_1,\dots,\tilde\gamma_n\}$ of the subspace $W\ominus zW$. The second map is evidently smooth. The smoothness of the first map follows from the smoothness of the projection map

$$C^\infty(S^1,\mathbb{C}^n)\times\mathrm{Gr}^\infty(H^{(n)}) \longrightarrow C^\infty(S^1,\mathbb{C}^n) ,$$

assigning to a smooth vector function f on S^1 its orthogonal projection $\mathrm{pr}_W f$ to a given subspace $W\in\mathrm{Gr}_\infty(H^{(n)})$.

Bibliographic comments

A key reference for this Chapter is the Pressley–Segal book [65]. In Sec. 9.1 we study the Sobolev space of half-differentiable loops on S^1. This space in the scalar case is well-known and widely used in the function theory (cf., e.g., [81]). On the other hand, it's role in geometric quantization became clear quite recently, especially after the paper of Nag and Sullivan [58]. The Grassmann realization of the group $L_{1/2}\mathrm{GL}(n, \mathbb{C})$ of half-differentiable matrix functions on S^1 (Prop. 17) is taken from [65], Sec. 6.3. The Prop. 18 is proved in [65], Theor. 8.3.2. The proof of the factorization theorem in Sec. 9.3 is taken from [65], Theor. 8.1.1.

Part III

SPACES OF COMPLEX STRUCTURES

Chapter 10

Virasoro group and its coadjoint orbits

In this Chapter we introduce the Virasoro group Vir, which is a central extension of the diffeomorphism group of the circle $\mathrm{Diff}_+(S^1)$, and study its coadjoint representation. We are especially interested in the coadjoint orbits, which have, along with the natural symplectic form, also a compatible complex structure. These Kähler coadjoint orbits of Vir are studied in Sec. 10.3 of this Chapter.

10.1 Virasoro group and Virasoro algebra

The Virasoro group is a central extension of the diffeomorphism group of the circle $\mathrm{Diff}_+(S^1)$. To describe it explicitly, we find first central extensions of the Lie algebra $\mathrm{Vect}(S^1)$ of $\mathrm{Diff}_+(S^1)$, being the algebra of tangent vector fields on S^1.

As we have pointed out in Sec. 4.1, any central extension of $\mathrm{Vect}(S^1)$ is determined by some 2-cocycle w on the algebra $\mathrm{Vect}(S^1)$. We extend this cocycle complex-linearly to the complexification $\mathrm{Vect}^{\mathbb{C}}(S^1)$ of the algebra $\mathrm{Vect}(S^1)$. The extended cocycle, denoted by the same letter w, is uniquely determined by its values $w_{m,n} := w(e_m, e_n)$ on the basis vector fields

$$e_m = ie^{im\theta}\frac{d}{d\theta}\ , \quad m = 0, \pm 1, \pm 2, \dots\ ,$$

of $\mathrm{Vect}^{\mathbb{C}}(S^1)$ (cf. Sec. 2.2). The cocycle condition for w, written for three vector fields (e_0, e_m, e_n):

$$w([e_0, e_m], e_n) + w(e_m, [e_0, e_n]) = w(e_0, [e_m, e_n])\ ,$$

implies that the cohomology class $[w]$ does not change under the action of rotations (generated by the vector field e_0). So the cocycle, obtained from w by averaging over S^1, belongs to the same cohomology class, as w. Therefore we can suppose from the beginning that the cocycle w is invariant under rotations, i.e.

$$w([e_0, e_m], e_n) + w(e_m, [e_0, e_n]) = 0$$

on the basis vector fields e_m, e_n. Due to the commutation relations for basis vector fields

$$[e_m, e_n] = (m - n)e_{m+n}\ ,$$

it means that

$$mw_{m,n} + nw_{m,n} = 0 \ . \tag{10.1}$$

The latter relation implies that $w_{m,n} = 0$ for $m + n \neq 0$. So we set $w_m := w_{m,-m}$ and note that $w_{-m} = -w_m$ due to the skew-symmetricity of w. It remains to find out the values of w_m for natural m.

The cocycle condition for w on three basis vector fields (e_m, e_n, e_{m+n}) means that

$$(m-n)w_{m+n} = (m+2n)w_m - (2m+n)w_n \ , \tag{10.2}$$

so we get a finite-difference equation of the 2nd order for the computation of values w_m. In order to find a general solution of (10.2), it's sufficient to find its two particular solutions. But it's easy to see that $w_m = m$ and $w_m = m^3$ are two independent solutions of (10.2). Hence a general solution of (10.2) has the form

$$w_m = \alpha m^3 + \beta m \tag{10.3}$$

with arbitrary complex coefficients α, β.

Note that the cocycle w with $w_m = m$ is a coboundary, since in this case

$$w(e_m, e_n) = d\theta(e_m, e_n) = \theta([e_n, e_m]) \ ,$$

where θ is a 1-cochain on $\mathrm{Vect}^{\mathbb{C}}(S^1)$, defined by: $\theta(e_0) = -\frac{1}{2}$ and $\theta(e_m) = 0$ for $m \neq 0$. So the value of β in the formula (10.3) is not essential. Hence all cocycles w, defining non-trivial central extensions of the algebra $\mathrm{Vect}(S^1)$, up to coboundaries, are proportional to each other. In other words, we have proved the following

Proposition 19. *The cohomology group $H^2(\mathrm{Vect}(S^1), \mathbb{R})$ has dimension 1. A general central extension of the algebra $\mathrm{Vect}(S^1)$ is determined by a cocycle w of the form*

$$w(e_m, e_n) = \begin{cases} \alpha m(m^2-1) & \text{for } m+n=0, \alpha \in \mathbb{R}, \\ 0 & \text{for } m+n \neq 0 \ . \end{cases}$$

We have chosen the parameter $\beta = -\alpha$ in order to annihilate the restriction of the cocycle w to the subalgebra $\mathrm{sl}(2,\mathbb{R})$ in $\mathrm{Vect}(S^1)$, generated by the vectors e_0, e_1, e_{-1} (this subalgebra coincides with the Lie algebra of the Möbius group $\mathrm{PSL}(2,\mathbb{R})$ of diffeomorphisms of the circle S^1, extending to the fractional-linear automorphisms of the unit disc Δ).

We note that the *Gelfand–Fuks cocycle*

$$w(\xi, \eta) = \frac{1}{2\pi}\int_0^{2\pi} \xi'(\theta) d\eta'(\theta) \ , \quad \xi = \xi(\theta)\frac{d}{d\theta}, \ \eta = \eta(\theta)\frac{d}{d\theta} \in \mathrm{Vect}(S^1) \ ,$$

found in [25], has the basis values, equal to $w_m = im^3$, $m \in \mathbb{Z}$.

One can also change the value of α, multiplying the central element by a number. The usual choice for α (based on physical analogies) is $\alpha = \frac{1}{12}$. The corresponding central extension of the algebra $\mathrm{Vect}(S^1)$ is called the *Virasoro algebra* and denoted by vir. The Virasoro algebra is generated (as a vector space) by the basis vector fields $\{e_m\}$ of the algebra $\mathrm{Vect}(S^1)$ and a central element κ, satisfying the commutation relations of the form

$$[e_m, \kappa] = 0 \ , \quad [e_m, e_n] = (m-n)e_{m+n} + \delta_{m,-n}\frac{m^3-m}{12}\kappa \ .$$

This central extension of the Lie algebra $\mathrm{Vect}(S^1)$ corresponds to a central extension of the Lie group $\mathrm{Diff}_+(S^1)$, which we describe next.

Since the Frechet manifold $\mathrm{Diff}_+(S^1)$ is homotopy equivalent to the circle S^1 (cf. Sec. 1.2.1), all S^1-bundles over $\mathrm{Diff}_+(S^1)$ are topologically trivial and any central extension of the group $\mathrm{Diff}_+(S^1)$ is determined by some 2-cocycle c on $\mathrm{Diff}_+(S^1)$ (cf. Sec. 4.1). In other words, such a central extension consists of elements of the form

$$(f, \lambda) \ , \quad f \in \mathrm{Diff}_+(S^1), \ \lambda \in S^1 \ ,$$

and the product is given by the formula

$$(f, \lambda) \cdot (g, \mu) = (f \circ g, \lambda \mu e^{ib(f,g)}) \ ,$$

where $c(f,g) = e^{ib(f,g)}$ is the 2-cocycle on $\mathrm{Diff}_+(S^1)$, defining the central extension. The cocycle condition in terms of b takes the form

$$b(f,g) + b(f \circ g, h) = b(f, g \circ h) + b(g,h) \ . \tag{10.4}$$

An explicit solution of this functional equation, found by Bott [11], has the form

$$b_0(f,g) = \frac{1}{2\pi} \int_0^{2\pi} \ln(f \circ g)' \, d \ln g' \ .$$

Note that the Bott group cocycle corresponds on the Lie algebra level to the Gelfand–Fuks cocycle of the Lie algebra $\mathrm{Vect}(S^1)$.

A general solution of (10.4) coincides with b_0 up to a coboundary, more precisely, it has the form

$$b(f,g) = \alpha b_0(f,g) + a(f \circ g) - a(f) - a(g) \ ,$$

where $\alpha = \mathrm{const} \in \mathbb{R}$, and a is an arbitrary smooth real functional on $\mathrm{Diff}_+(S^1)$.

The central extension of the group $\mathrm{Diff}_+(S^1)$, determined by the Bott cocycle, is called the *Virasoro group* or *Virasoro–Bott group* and is denoted by Vir.

10.2 Coadjoint action of the Virasoro group

Consider the coadjoint action of the diffeomorphism group of the circle $\mathrm{Diff}_+(S^1)$ and its central extension, the Virasoro group Vir, on the dual spaces of their Lie algebras.

We study first the coadjoint action of the diffeomorphism group $\mathrm{Diff}_+(S^1)$ on the space $\mathrm{Vect}^*(S^1)$, dual to the Lie algebra $\mathrm{Vect}(S^1)$ of $\mathrm{Diff}_+(S^1)$. The space $\mathrm{Vect}^*(S^1)$, dual to the Frechet space $\mathrm{Vect}(S^1)$, can be identified with the tensor product

$$\Omega^1(S^1) \otimes_{\mathcal{D}(S^1)} \mathcal{D}'(S^1)$$

over the ring $\mathcal{D}(S^1)$, consisting of all C^∞-smooth (real-valued) functions on S^1. Here, $\Omega^1(S^1)$ is the Frechet space of C^∞-smooth 1-forms on S^1, and $\mathcal{D}'(S^1)$ is the space of distributions on S^1, i.e. of linear continuous functionals on $\mathcal{D}(S^1)$ (note that $\mathcal{D}'(S^1)$ is not a Frechet space!). The above tensor product should be taken in the category of topological vector spaces, we recall its definition for convenience.

Digression 3 (Tensor product of topological vector spaces). The tensor product $E \otimes F$ of topological vector spaces E and F is provided with the projective topology, generated by the seminorms $p \otimes q$, where $\{p\}$ and $\{q\}$ are families of seminorms on E and F respectively. The seminorm $p \otimes q$ is defined as

$$(p \otimes q)(z) = \inf \left\{ \sum_i p(x_i)q(y_i) \ : \ z = \sum x_i \otimes y_i \right\} ,$$

where the infimum is taken over all possible representations of $z \in E \otimes F$ as finite sums of the form $\sum x_i \otimes y_i$ with $x_i \in E$, $y_i \in F$.

The elements of the completion $\widetilde{E \otimes F}$ of the space $E \otimes F$ with respect to this topology in the case of metrizable spaces E and F can be given by series of the form

$$\widetilde{E \otimes F} \ni z = \sum_{i=1}^{\infty} \lambda_i x_i \otimes y_i ,$$

where $\sum_{i=1}^{\infty} |\lambda_i| < \infty$ and the sequences $\{x_i\}$, $\{y_i\}$ tend to zero in E and F respectively.

For the nuclear spaces E and F the topology, introduced on $\widetilde{E \otimes F}$, coincides with the topology of the uniform equicontinuous convergence (i.e. topology of uniform convergence on the sets of the form $S \otimes T$, where S and T are uniformly equicontinuous subsets in E' and F' respectively).

We return to the dual space $\mathrm{Vect}^*(S^1)$, which is identified with the tensor product $\Omega^1(S^1) \otimes_{\mathcal{D}(S^1)} \mathcal{D}'(S^1)$ by the map, associating with an element $(\alpha, \varphi) \in \Omega^1(S^1) \otimes_{\mathcal{D}(S^1)} \mathcal{D}'(S^1)$ a linear continuous functional on $\mathrm{Vect}(S^1)$ of the form

$$T_{(\alpha,\varphi)}(\xi) = \varphi[\alpha(\xi)] , \quad \xi \in \mathrm{Vect}(S^1) .$$

As in Sec. 8.3, we restrict ourselves to the study of the coadjoint action of the group $\mathrm{Diff}_+(S^1)$ on the "smooth" part of the space $\mathrm{Vect}^*(S^1)$, identified with the tensor product of Frechet spaces

$$\Omega^1(S^1) \otimes_{\mathcal{D}(S^1)} \Omega^1(S^1) .$$

An element (α, β) of this space determines a linear continuous functional on $\mathrm{Vect}(S^1)$ by the formula

$$\mathrm{Vect}(S^1) \ni \xi \longmapsto T_{(\alpha,\beta)}(\xi) = \frac{1}{2\pi} \int_0^{2\pi} \beta(\xi(\theta))\alpha(\theta) .$$

In other words, the smooth part of the space $\mathrm{Vect}^*(S^1)$ may be identified with the *space* $Q(S^1)$ *of quadratic differentials* on S^1 of the form

$$q = q(\theta)(d\theta)^2 ,$$

where q is a smooth 2π-periodic function of θ.

From another point of view, one can consider $Q(S^1)$ as a set of pseudometrics on S^1 (the term "pseudo" indicates that the function $q(\theta)$ may have zeros on S^1).

The coadjoint action of the group $\mathrm{Diff}_+(S^1)$ on $Q(S^1)$ coincides with the natural action of the group $\mathrm{Diff}_+(S^1)$ on quadratic differentials

$$\mathrm{Diff}_+(S^1) \ni f \longmapsto K(f)q = q \circ f^{-1} := q(g(\theta))g'(\theta)^2 d\theta^2 \ ,$$

where $g(\theta) = f^{-1}(\theta)$.

We consider next the coadjoint action of the group $\mathrm{Diff}_+(S^1)$ on the dual space vir^* of the Virasoro algebra vir. Since the Virasoro algebra coincides with $\mathrm{vir} = \mathrm{Vect}(S^1) \oplus \mathbb{R}$ (as a vector space), we have $\mathrm{vir}^* = \mathrm{Vect}^*(S^1) \oplus \mathbb{R}$. So the smooth part of vir^* may be identified with the space

$$Q(S^1) \oplus \mathbb{R} = \{(q, s) : \ q \text{ is a quadratic differential}, \ s \in \mathbb{R}\} \ .$$

The coadjoint action of the group $\mathrm{Diff}_+(S^1)$ on $Q(S^1) \oplus \mathbb{R}$ associates with an element $f \in \mathrm{Diff}_+(S^1)$ a linear transformation $\widetilde{K}(f)$ of the space $Q(S^1) \oplus \mathbb{R}$, acting by the formula

$$\widetilde{K}(f)(q, s) = (K(f)q + sS(f) \circ f^{-1}, s) = ((q + sS(f)) \circ f^{-1}, s) \ , \tag{10.5}$$

where S is a 1-cocycle on the group $\mathrm{Diff}_+(S^1)$, satisfying the relation

$$S(f \circ h) = (S(f) \circ h) + S(h) \ . \tag{10.6}$$

A non-trivial particular solution of this equation is given by the *Schwarzian*

$$S[f] = \left(\frac{f'''}{f'} - \frac{3}{2} \left(\frac{f''}{f'} \right)^2 \right) d\theta^2 = d^2 \ln f' - \frac{1}{2}(d \ln f')^2 \ , \tag{10.7}$$

while a general solution has the form

$$S[f] + q \circ f - q \ ,$$

where $q \in Q(S^1)$ is a quadratic differential.

Digression 4 (Schwarzian). A characteristic property of the Schwarzian is its conformal invariance:

$$S \left[\frac{af + b}{cf + d} \right] = S[f]$$

for any fractional-linear transformation $z \mapsto \frac{az+b}{cz+d}$ from the Möbius group $\mathrm{Möb}(S^1) := \mathrm{PSL}(2, \mathbb{R})$. This property follows immediately from the transformation rule for the Schwarzian

$$S[f \circ h] = (S[f] \circ h) \, (h')^2 + S[h] \ , \tag{10.8}$$

which is just a decoded version of (10.6).

The Schwarzian $S[f]$ of a diffeomorphism $f \in \mathrm{Diff}_+(S^1)$ measures its deviation from conformal automorphisms of the unit disc in the sense that

$$S[f] = 0 \Longleftrightarrow f \text{ is fractional-linear} \ .$$

Moreover, one can define the Schwarz derivative $S[f]$ of any conformal map $f : \Delta \to \mathbb{C}$ by the same formula (10.7). Then $S[f]$ measures again the deviation of a conformal map f in Δ from fractional-linear automorphisms of Δ, and the maximal deviation may be explicitly computed. Introduce a natural norm on Schwarz derivatives $S[f]$, coinciding with the hyperbolic norm on quadratic differentials in Δ:

$$\|S[f]\|_2 := \sup_{z\in\Delta} |S[f](z)|(1-|z|^2) \ .$$

There is a following remarkable theorem, known as *Nehari theorem.*

Theorem 11 ((cf. [49], Theor. II.1.3)). *For any conformal map f of the unit disc Δ the following sharp estimate holds*

$$\|S[f]\|_2 \leq 6 \ .$$

The upper bound is attained on the Koebe function $z \mapsto z/(1+z)$.

The infinitesimal variant of the coadjoint representation (10.5) is given by the representation of the Lie algebra $\mathrm{Vect}(S^1)$ on the space $Q(S^1)\oplus\mathbb{R}$, defined by the formula

$$\widetilde{k}(\xi)(q,s) = (-D_{q,s}\xi, s) \ , \tag{10.9}$$

where $\xi = \xi(\theta)\frac{d}{d\theta} \in \mathrm{Vect}(S^1)$, $q = q(\theta)(d\theta)^2 \in Q(S^1)$, and the operator $D_{q,s}$ has the form

$$D_{q,s} = s\frac{d^3}{d\theta^3} + q\frac{d}{d\theta} + \frac{d}{d\theta}q \ .$$

What can be said about the orbits of the coadjoint representation of $\mathrm{Diff}_+(S^1)$? The orbit of a regular element $(q,s) \in Q(S^1)\oplus\mathbb{R}$ under the action of the group $\mathrm{Diff}_+(S^1)$ is completely determined by the isotropy subgroup $G_{q,s}$ with respect to the coadjoint action. The Lie algebra $\mathfrak{g}_{q,s}$ of this subgroup consists of vector fields $\xi = \xi(\theta)\frac{d}{d\theta} \in \mathrm{Vect}(S^1)$, satisfying the condition: $D_{q,s}\xi = 0$. In other words, to describe the subalgebra $\mathfrak{g}_{q,s}$, one should find periodic solutions $\xi(\theta)$ of the linear differential equation

$$s\xi''' + 2q\xi' + q'u = 0 \ . \tag{10.10}$$

Referring for the general solution of this problem to the papers [40, 30], we consider here only its particular case, when a regular element (q,s) has the form $(q(d\theta)^2, s)$ with $q \equiv \mathrm{const} =: c$, $s \neq 0$. In this case the equation (10.10) takes on the form

$$s\xi''' + 2c\xi' = 0 \ , \tag{10.11}$$

which, after the change of variable $\eta := \xi'$, reduces to the equation

$$s\eta'' + 2c\eta = 0 \ .$$

The latter equation has non-trivial periodic solutions only for $2c = n^2$, where n is a natural number, and all these solutions are linear combinations of the functions $\cos n\theta$ and $\sin n\theta$. In other words, the only periodic solutions of the equation (10.11) for $\frac{2c}{s} \neq n^2$ are given by constants, while for $\frac{2c}{s} = n^2$ they are linear combinations of the functions 1, $\frac{1}{n}\cos n\theta$ and $\frac{1}{n}\sin n\theta$.

The isotropy subalgebra $\mathfrak{g}_{q,s}$ in the first case coincides with $\mathbb{R}$, and in the second case with the algebra $\mathrm{sl}(2,\mathbb{R})$. Respectively, the isotropy subgroup $G_{q,s}$ in the first case coincides with the rotation group $S^1 \subset \mathrm{Diff}_+(S^1)$, and in the second case with the group $\mathrm{PSL}^{(n)}(2,\mathbb{R})$, which is the n-fold covering of the Möbius group $\mathrm{M\ddot{o}b}(S^1) = \mathrm{PSL}(2,\mathbb{R})$. We have already encountered this group in Sec. 2.2. Recall that a diffeomorphism $f \in \mathrm{Diff}_+(S^1)$ belongs to the group $\mathrm{PSL}^{(n)}(2,\mathbb{R})$ if and only if there exists a transformation $\varphi \in \mathrm{PSL}(2,\mathbb{R})$ such that

$$\lambda_n \circ f = \varphi \circ \lambda_n$$

where $\lambda_n : z \mapsto z^n$ is the map, defining the n-fold covering of the circle S^1.

It follows from the description of isotropy subgroups that the coadjoint orbit of a constant element $(q,s) = (cd\theta^2, s)$ coincides with the homogeneous space $\mathrm{Diff}_+(S^1)/(S^1)$, when $2c/s$ is not a square of a natural number, and with the homogeneous space $\mathrm{Diff}_+(S^1)/\mathrm{PSL}^{(n)}(2,\mathbb{R})$, when $2c/s = n^2$.

As we have explained earlier in Subsec. 3.2.3, all coadjoint orbits have a natural symplectic structure, given by the Kirillov form. In the case, we are considering, the value of this form at a point $(q,s) \in Q(S^1) \oplus \mathbb{R}$ of an orbit O of the group $\mathrm{Diff}_+(S^1)$ may be computed in the following way. Let $\delta\xi$ and $\delta\eta$ be tangent vectors from $T_{q,s}O$, which are the images of tangent vectors $\xi, \eta \in \mathrm{Vect}(S^1)$ under the map $\widetilde{k}$ from (10.9):

$$\delta\xi = \widetilde{k}(\xi)(q,s) \ , \quad \delta\eta = \widetilde{k}(\eta)(q,s) \ .$$

Then the value of the form ω_O on these vectors is equal to

$$\omega_O(\delta\xi, \delta\eta) = -\int_{S^1} (D_{q,s}\xi)(\theta)\eta(\theta)d\theta \ .$$

Thus every coadjoint orbit of Vir has a symplectic structure. But not all of them can be provided with a compatible complex structure. In fact, among the coadjoint orbits of the group Vir, described above, only the orbits

$$\mathrm{Diff}_+(S^1)/(S^1) \ , \quad \mathrm{Diff}_+(S^1)/\,\mathrm{M\ddot{o}b}(S^1) = \mathrm{Diff}_+(S^1)/\mathrm{PSL}(2,\mathbb{R})$$

are Kähler (cf. [78]). In other words, only these orbits admit $\mathrm{Diff}_+(S^1)$-invariant complex structures, compatible with the symplectic structure ω_O. We shall concentrate our attention on these Kähler orbits.

Example 29. We give now an interesting interpretation of the coadjoint action of the Virasoro group in terms of Hill operators, due to Lazutkin and Pankratova [48].

Recall that a *Hill operator* is a differential operator of the 2nd order, having the form

$$L = \frac{d^2}{d\theta^2} + u(\theta) \ ,$$

where $u = u(\theta)$ is a potential, given by a C^∞-smooth 2π-periodic function on $\mathbb{R}$. The corresponding ordinary differential equation

$$y'' + uy = 0$$

is called the *Hill equation*. Its solutions form a two-dimensional vector space V, provided with a natural symplectic 2-form, given by the Wronskian of two solutions. The shift of a solution y of the Hill equation $Ly = 0$ to the period 2π transforms it into another solution, obtained from y by the action of an operator $M \in \mathrm{SL}(V)$, called the *monodromy matrix* of the operator L.

If $\{y_1, y_2\}$ is a fundamental system of solutions, i.e. a basis in the space V of solutions of the Hill equation, then one can reconstruct the potential u from this system by the Schwarz formula:

$$u(\theta) = \begin{cases} \frac{1}{2}S[y_1/y_2](\theta) \ , & \text{if } y_2(\theta) \neq 0 \ , \\ \frac{1}{2}S[y_2/y_1](\theta) \ , & \text{if } y_1(\theta) \neq 0 \ , \end{cases}$$

where $S[y]$ is the Schwarzian of y.

The diffeomorphism group $\mathrm{Diff}_+(S^1)$ acts in a natural way on the space of Hill operators. Namely, we can associate with any diffeomorphism $f \in \mathrm{Diff}_+(S^1)$, which lifts to a diffeomorphism $\tilde{f}$ of the real line $\mathbb{R}$, a transformation, which sends a given Hill operator $L = \frac{d^2}{d\theta^2} + u(\theta)$ to another Hill operator $f^*L = \frac{d^2}{d\theta^2} + f^*u(\theta)$ with

$$f^*u(\theta) := u(\tilde{f}(\theta)) \cdot (\tilde{f}'(\theta))^2 + \frac{1}{2}S[\tilde{f}](\theta) \ .$$

Under this transformation a solution y of the Hill equation $Ly = 0$ is transferred to a solution z of the Hill equation $(f^*L)z = 0$ with

$$z(\theta) := y(\tilde{f}(\theta)) \cdot (\tilde{f}'(\theta))^{-\frac{1}{2}} \ .$$

Note that, due to the periodicity of the potential u, the action of f on potentials does not depend on the choice of the lift $\tilde{f}$ of the diffeomorphism $f \in \mathrm{Diff}_+(S^1)$ and so defines an action of the group $\mathrm{Diff}_+(S^1)$ on Hill operators. This action coincides with the coadjoint action of the group $\mathrm{Diff}_+(S^1)$ on elements $(u, \frac{1}{2})$ of the space $Q(S^1) \oplus \mathbb{R}$, given by (10.5).

But the action of f on solutions of the Hill equation depends on the choice of the lift $\tilde{f}$, because of the monodromy. In accordance with the above formula, solutions of the Hill equation transform under the action of diffeomorphisms $\tilde{f}$, as densities of order $-1/2$ on the line $\mathbb{R}$.

The constructed action of the group $\mathrm{Diff}_+(S^1)$ on Hill operators was studied in the Lazutkin–Pankratova's paper [48]. The authors formulate, in particular, a conjecture that any Hill operator with the help of the above action can be brought to the *Matieu normal form* of the type:

$$L = \frac{d^2}{d\theta^2} + a\cos(2\pi n\theta) + b \ .$$

10.3 Kähler structure of the spaces $\mathrm{Diff}_+(S^1)/\,\mathrm{M\ddot{o}b}(S^1)$ and $\mathrm{Diff}_+(S^1)/(S^1)$

As we have pointed out in the previous Section, among the coadjoint orbits of the Virasoro group Vir only two are Kähler, namely:

$$\mathcal{R} := \mathrm{Diff}_+(S^1)/(S^1) \quad \text{and} \quad \mathcal{S} := \mathrm{Diff}_+(S^1)/\,\mathrm{M\ddot{o}b}(S^1) \ .$$

In this Section we study their Kähler structure in detail.

As coadjoint orbits of the group Vir, these spaces have a natural symplectic structure ω, given by the Kirillov form.

We introduce now a complex structure J on the space $\mathcal{S} = \mathrm{Diff}_+(S^1)/\mathrm{M\ddot{o}b}(S^1)$, invariant under the action of the diffeomorphism group $\mathrm{Diff}_+(S^1)$ by left translations. Due to its invariance, it's sufficient to define this complex structure only at the origin $o \in \mathcal{S}$.

The tangent space $T_o\mathcal{S}$ may be identified with the quotient of the Lie algebra $\mathrm{Vect}(S^1)$ of tangent vector field on S^1 modulo its subalgebra $\mathrm{sl}(2,\mathbb{R})$. In terms of Fourier decompositions vector fields $v = v(\theta)\frac{d}{d\theta} \in T_o\mathcal{S}$ are given by series of the form

$$v(\theta) = \sum_{n\neq -1,0,1} v_n e^{in\theta} \ , \quad v_n \in \mathbb{C} \ ,$$

subject to the condition: $v_{-n} = \bar{v}_n$. In these terms the restriction of the $\mathrm{Diff}_+(S^1)$-invariant complex structure J to $T_o\mathcal{S}$ is given by the formula

$$Jv(\theta) = -i\sum_{n>1} v_n e^{in\theta} + i\sum_{n<-1} v_n e^{in\theta}$$

for $v = v(\theta)\frac{d}{d\theta} \in T_o\mathcal{S}$. It's easy to see that the constructed complex structure on $\mathcal{S}$ is formally integrable (i.e. the bracket of two tangent vector fields of type $(1,0)$ with respect to this complex structure is again a vector field of type $(1,0)$). Moreover, this complex structure is compatible with the symplectic structure ω on $\mathcal{S}$, mentioned above.

The symplectic form ω on $\mathcal{S}$ together with the complex structure J define a Kähler metric g on $\mathcal{S}$. In terms of Fourier decompositions this metric can be defined in the following way. Suppose that tangent vectors $u, v \in T_o\mathcal{S}$ are given by the Fourier series

$$u = \sum_{n\neq -1,0,1} u_n e_n \quad \text{and} \quad v = \sum_{n\neq -1,0,1} v_n e_n \ . \tag{10.12}$$

Then the value of the metric g on these vectors is equal to

$$g(u,v) = 2\,\mathrm{Re}\left(\sum_{n=2}^{\infty} u_n \bar{v}_n (n^3 - n)\right) \ . \tag{10.13}$$

The infinite series in the right hand side of (10.13) is absolutely converging, if the Fourier series (10.12) correspond to the vector fields u, v of the class $C^{3/2+\epsilon}$ on S^1.

We turn now to the orbit $\mathcal{R} := \mathrm{Diff}_+(S^1)/(S^1)$. It can be identified (as a homogeneous space) with a subgroup of $\mathrm{Diff}_+(S^1)$, consisting of diffeomorphisms $f \in \mathrm{Diff}_+(S^1)$, fixing the point $1 \in S^1$: $f(1) = 1$.

The embedding of the rotation group of the circle S^1 into the Möbius group $\mathrm{M\ddot{o}b}(S^1)$ generates a homogeneous bundle

$$\mathcal{R} = \mathrm{Diff}_+(S^1)/(S^1) \longrightarrow \mathcal{S} \ ,$$

having the unit disc Δ as a fibre.

We describe explicitly the symplectic structure on $\mathcal{R}$, given by the Kirillov form. This form, being invariant under the left translations of the group $\mathrm{Diff}_+(S^1)$, is completely determined by its restriction to the tangent space at the origin $T_o\mathcal{R}$.

The tangent space $T_o\mathcal{R}$ is identified with the space $\mathrm{Vect}_0(S^1)$, consisting of vector fields $v = v(\theta)\frac{d}{d\theta}$, whose coefficients $v(\theta)$ are 2π-periodic functions with zero average:

$$\frac{1}{2\pi}\int_0^{2\pi} v(\theta)d\theta = 0 \ .$$

In terms of Fourier decompositions tangent vectors $v \in T_o\mathcal{R}$ are given by the series of the form $v = \sum_{n\neq 0} v_n e_n$, subject to the condition: $v_{-n} = \bar{v}_n$.

An invariant symplectic structure on $\mathcal{R}$ is defined by a 2-cocycle w on the Lie algebra $\mathrm{Vect}^{\mathbb{C}}(S^1)$, invariant under rotations. Such a cocycle is determined by its values $w(e_m, e_n)$ on the basis elements $\{e_m\}$. These basis values necessarily have the form (cf. Prop. 19 in Sec. 10.1):

$$w(e_m, e_n) = (\alpha m^3 + \beta m)\delta_{m,-n}$$

for some real α, β. Denote the form, corresponding to the parameters α, β, by $w_{\alpha,\beta}$. It's easy to see that it is non-degenerate on $\mathrm{Vect}_0(S^1)$ if and only if

$$\alpha m^3 + \beta m \neq 0 \quad \text{for all natural } m \ .$$

The latter condition is satisfied, if either $\alpha = 0$, $\beta \neq 0$, or $-\beta/\alpha$ is not a square of a natural number. In the first case the form $w_{\alpha,\beta}$ is exact (cf. Sec. 10.1), so we choose the second possibility.

The form $w_{\alpha,\beta}$ defines a symplectic structure on $\mathrm{Vect}_0(S^1)$, which can be written in a more invariant way as

$$w_{\alpha,\beta}(u, v) = \frac{1}{2\pi}\int_0^{2\pi} u(\theta)\left(\beta v'(\theta) - \alpha v'''(\theta)\right) d\theta \ ,$$

where $u, v \in \mathrm{Vect}_0(S^1)$. In terms of Fourier decompositions

$$u = \sum_{n\neq 0} u_n e^{in\theta} \ , \quad v = \sum_{n\neq 0} v_n e^{in\theta} \ ,$$

we get

$$w_{\alpha,\beta}(u, v) = 2\mathrm{Im}\sum_{n\geq 1}(\alpha n^3 + \beta n)\xi_n\bar{\eta}_n \ .$$

The constructed 2-parameter family of symplectic structures on $\mathcal{R}$ has a natural interpretation in terms of the coadjoint action of the group $\mathrm{Diff}_+(S^1)$. Recall that the orbit of an element $(cd\theta^2, s)$ coincides with $\mathcal{R}$, if $2c/s$ is not a square of a natural number. By identifying the homogeneous space $\mathcal{R}$ with the orbit of an element $(cd\theta^2, s)$ and providing it with the canonical symplectic structure, given by the Kirillov form, we shall obtain, for different choices of (c, s) with $2c/s \neq n^2$, the two-parameter family of symplectic structures on $\mathcal{R}$, constructed above.

Introduce a $\mathrm{Diff}_+(S^1)$-invariant complex structure J on the space $\mathcal{R}$. Its restriction to $T_o\mathcal{R} = \mathrm{Vect}_0(S^1)$ is given by the Hilbert transform, which assigns to a tangent vector $v \in \mathrm{Vect}_0(S^1)$ the vector

$$(Jv)(\theta) = \frac{1}{2\pi}\int_0^{2\pi} \cot\frac{\theta - \psi}{2}\, v(\psi)d\psi \ , \quad 0 \leq \theta \leq 2\pi \ .$$

In terms of the Fourier decomposition $v = \sum_{n\neq 0} v_n e_n \in \mathrm{Vect}_0(S^1)$ we get

$$Jv = -i\sum_{n>0} v_n e_n + i\sum_{n<0} v_n e_n \ .$$

The complex structure J is formally integrable, i.e. the bracket of two tangent vector fields of type $(1,0)$ with respect to this complex structure is again a vector field of type $(1,0)$. Moreover, it can be shown that this complex structure is a unique formally integrable $\mathrm{Diff}_+(S^1)$-invariant complex structure on $\mathcal{R}$.

The constructed complex structure J is compatible with all symplectic structures $w_{\alpha,\beta}$, so it generates a 2-parameter family of Kähler metrics $g_{\alpha,\beta}(u,v) := w_{\alpha,\beta}(u, Jv)$ on $\mathcal{R}$, given at the origin by the formula:

$$g_{\alpha,\beta}(u,v) = 2\,\mathrm{Re}\sum_{n\geq 1}(\alpha n^3 + \beta n)u_n \bar{v}_n \ ,$$

where $u = \sum_{n\neq 0} u_n e_n$, $v = \sum_{n\neq 0} v_n e_n \in T_o\mathcal{R}$. Hence, $\mathcal{R}$ is a Kähler Frechet manifold with a 2-parameter family of Kähler metrics $g_{\alpha,\beta}$.

As we know, the existence of a formally integrable complex structure on an infinite-dimensional manifold does not guarantee the existence of an atlas of local complex coordinates on it. We shall introduce local complex coordinates on $\mathcal{R}$, following an idea, proposed by Kirillov and Yuriev [44]. Namely, we shall realize $\mathcal{R}$ as the space of holomorphic univalent functions in the unit disc Δ.

Denote by $\mathcal{A}$ the complex Frechet space of all C^∞-smooth complex-valued functions in the closure $\overline{\Delta}$ of the unit disc Δ, which are holomorphic inside Δ and vanish at the origin. Let $\mathcal{A}_0$ be a subset of $\mathcal{A}$, consisting of all $f \in \mathcal{A}$, which define a C^∞-smooth embedding of the closed disc $\overline{\Delta}$ into $\mathbb{C}$. It is an open subset in $\mathcal{A}$, which inherits a complex Frechet manifold structure. Denote by $\mathfrak{S}$ the set of functions $f \in \mathcal{A}_0$, such that $f'(0) = 1$, which is a smooth hypersurface in $\mathcal{A}_0$. The functions $f \in \mathfrak{S}$ are holomorphic and univalent in Δ, they define C^∞-smooth embeddings $\overline{\Delta} \to \overline{f(\Delta)}$ and satisfy the normalizing conditions: $f(0) = 0$, $f'(0) = 1$. They can be given by power series of the form

$$f(z) = z + c_2 z^2 + c_3 z^3 + \dots \ ,$$

whose coefficients satisfy, according to de Branges theorem, the relations: $|c_k| < k$. The coefficients $\{c_k\}$ may be chosen for local complex coordinates in a neighborhood of $f(z) \equiv z$ in $\mathfrak{S}$.

We construct now a map from $\mathfrak{S}$ to $\mathcal{R}$. For that we associate with a function $f \in \mathfrak{S}$ the contour $K := f(S^1)$. The function $f := f_K$ maps conformally the unit disc $\Delta := \Delta_+$ onto the domain D_K, bounded by the contour K. Denote by

$$g_K : \Delta_- \longrightarrow \overline{\mathbb{C}} \setminus \overline{D}_K$$

the conformal map of the complement $\Delta_- := \overline{\mathbb{C}} \setminus \overline{\Delta}_+$ of the closed unit disc $\overline{\Delta}_+$ on the Riemann sphere $\overline{\mathbb{C}}$ onto the domain $\overline{\mathbb{C}} \setminus \overline{D}_K$, normalized by the conditions:

$$g_K(\infty) = \infty \ , \quad g_K'(\infty) > 0 \ .$$

The map g_K extends to a diffeomorphism of $\partial\Delta_- = S^1$ onto ∂D_K. We associate with $f \in \mathfrak{S}$ the diffeomorphism

$$\gamma_K := f_K^{-1} \circ g_K|_{S^1} \ .$$

In order to construct an inverse map from $\mathcal{R}$ to $\mathfrak{S}$, note that, using an arbitrary diffeomorphism $\gamma \in \mathcal{R}$, we can construct a new complex structure on the Riemann sphere $\overline{\mathbb{C}}$. Indeed, denote by $\overline{\mathbb{C}}_\gamma$ the smooth manifold, obtained by gluing Δ_+ with Δ_- with the help of γ. In other words, $\overline{\mathbb{C}}_\gamma$ is obtained from the disconnected union $\overline{\Delta}_+ \sqcup \overline{\Delta}_-$ by the identification of points from $S^1 = \partial\Delta_+ = \partial\Delta_-$ via the rule:

$$z \in S^1 = \partial\Delta_+ \longleftrightarrow \gamma^{-1}(z) \in S^1 = \partial\Delta_- \ .$$

The complex manifold $\overline{\mathbb{C}}_\gamma$ is diffeomorphic to the Riemann sphere $\overline{\mathbb{C}}$. But, according to the theorem of Ahlfors, there exists a unique complex structure on the Riemann sphere $\overline{\mathbb{C}}$. So the two manifolds are biholomorphic to each other, i.e. there exists a biholomorphic map

$$F : \overline{\mathbb{C}}_\gamma \longrightarrow \overline{\mathbb{C}} \ ,$$

which is uniquely defined, being normalized by the following conditions:

$$F(0) = 0 \ , \quad F(\infty) = \infty \ , \quad F'(0) = 1 \ .$$

The biholomorphism F is given by a pair of functions (f, g), where the function f is holomorphic in Δ_+ and C^∞-smooth up to $S^1 = \partial\Delta_+$, and the function g is holomorphic in Δ_- and C^∞-smooth up to $S^1 = \partial\Delta_-$, while

$$f = g \circ \gamma^{-1} \quad \text{on} \quad S^1 \ .$$

Setting $K := f(S^1)$, we get that $\gamma = \gamma_K \mod S^1$ (since the normalization of F does not fix $\arg g(\infty)$).

As it is pointed out by Lempert [50], one can construct the inverse map by using, instead of the Ahlfors theorem, the factorization theorem of Pflüger [62], which asserts that any diffeomorphism $\gamma \in \mathcal{R}$ may be represented in the form

$$\gamma = f^{-1} \circ g \ ,$$

where f and g have the same properties, as above.

The constructed one-to-one map from $\mathfrak{S}$ to $\mathcal{R}$ is smooth and defines a diffeomorphism

$$\kappa : \mathcal{R} \longrightarrow \mathfrak{S} \ .$$

It's easy to describe its tangent map

$$d_0\kappa : T_0\mathcal{R} \longrightarrow T_1\mathfrak{S} \ .$$

The tangent space $T_1\mathfrak{S}$ is identified with the space Φ, consisting of functions φ, which are holomorphic in Δ, C^∞-smooth up to $\partial\Delta$ and normalized by the conditions: $\varphi(0) = 0$, $\varphi'(0) = 0$. (Indeed, any such vector φ is tangent to the curve $f_t(z) = z + t\varphi(z)$, which is contained in $\mathfrak{S}$ for $0 \le t \le \epsilon$.) The map $d_0\kappa$ associates with a vector $v \in T_0\mathcal{R}$ a function $\varphi \in T_1\mathfrak{S}$ by the formula

$$2\,\mathrm{Re}\,\varphi(e^{i\theta}) = (Jv)(\theta) \ ,$$

where J is the Hilbert transform on $T_0\mathcal{R}$. The Hilbert transform J on $T_0\mathcal{R}$ corresponds to the multiplication by i in the space $T_1\mathfrak{S}$, hence the map, inverse to $d_0\kappa$, is given by the formula: $v(\theta) = -2\operatorname{Im}\varphi(e^{i\theta})$.

It follows from the definition of complex structures on $\mathcal{R}$ and $\mathcal{S}$ that the homogeneous disc bundle $\mathcal{R} \to \mathcal{S}$ is, in fact, holomorphic.

We note also that on the Virasoro group Vir itself there exists a complex structure, induced by the complex structure on $\mathcal{R}$, such that the natural projection

$$\pi : \mathrm{Vir} \longrightarrow \mathcal{R}$$

is a holomorphic $\mathbb{C}^*$-bundle with respect to this complex structure (cf. [50]).

Bibliographic comments

The Virasoro group and Virasoro algebra are considered in different books, dealing with infinite-dimensional groups and algebras. Apart from the Pressley-Segal book [65], see also [38, 22]. The coadjoint representation of the Virasoro group and its orbits are studied in [40, 30]. The study of the Kähler structure of the space $\mathcal{R}$ was initiated by Bowick–Rajeev [14] and Kirillov [41]. A relation between this space and the space of holomorphic univalent functions in the unit disc was established in the Kirillov–Yuriev paper [44].

Chapter 11

Universal Techmüller space

In this Chapter we study the Kähler geometry of the universal Teichmüller space, which can be defined as the space of normalized homeomorphisms of S^1, extending to quasiconformal maps of the unit disc Δ. It may be also realized as an open subset in the complex Banach space of holomorphic quadratic differentials in a disc. All classical Teichmüller spaces $T(G)$, where G is a Fuchsian group, are contained in $\mathcal{T}$ as complex Kähler submanifolds. The homogeneous space $\mathcal{S} = \mathrm{Diff}_+(S^1)/\mathrm{Möb}(S^1)$, introduced in the previous Chapter 10, may be considered as a "smooth" part of $\mathcal{T}$.

11.1 Definition of the universal Techmüller space

Definition 37. A homeomorphism $f : S^1 \to S^1$ is called *quasisymmetric*, if it can be extended to a quasiconformal homeomorphism of the unit disc Δ.

This definition agrees with the definition of a quasisymmetric homeomorphism of S^1 as an orientation-preserving homeomorphism of S^1, satisfying the Beurling–Ahlfors condition (6.5), given in Sec. 6.1. The equivalence of two definitions is established in the Beurling–Ahlfors theorem in Sec. 6.1.

We denote by $\mathrm{QS}(S^1)$ the set of all orientation-preserving quasisymmetric homeomorphisms of S^1. This is a group with respect to the composition of homeomorphisms.

Any diffeomorphism $f \in \mathrm{Diff}_+(S^1)$ extends to a diffeomorphism of the closed unit disc $\overline{\Delta}$, and so to a quasiconformal homeomorphism $\tilde{f}$ (recall that the Jacobian of a diffeomorphism f is equal to $|f_z|^2 - |f_{\bar{z}}|^2$). Hence, $\mathrm{Diff}_+(S^1) \subset \mathrm{QS}(S^1)$. Since the Möbius group $\mathrm{Möb}(S^1)$ of fractional-linear automorphisms of the disc is contained in $\mathrm{Diff}_+(S^1)$, we obtain the following chain of embeddings

$$\mathrm{Möb}(S^1) \subset \mathrm{Diff}_+(S^1) \subset \mathrm{QS}(S^1) \subset \mathrm{Homeo}(S^1) \ .$$

Definition 38. The quotient space

$$\mathcal{T} := \mathrm{QS}(S^1)/\mathrm{Möb}(S^1)$$

is called the *universal Teichmüller space*. It can be identified with the space of *normalized* quasisymmetric homeomorphisms of S^1, fixing the points ± 1 and $-i$.

The reasons for choosing the name "universal Teichmüller space" for the introduced object will become clear later.

As we have just pointed out, we have an inclusion

$$\mathcal{S} = \mathrm{Diff}_+(S^1)/\mathrm{M\ddot{o}b}(S^1) \hookrightarrow \mathcal{T} = \mathrm{QS}(S^1)/\mathrm{M\ddot{o}b}(S^1) \ .$$

Using the existence theorem for quasiconformal maps (Theor. 5 from Sec. 6.2), we can describe the universal Teichmüller space in terms of Beltrami differentials. Denote by $B(\Delta)$ the set of Beltrami differentials in the unit disc Δ. It can be identified, as we have pointed out in Sec. 6.1, with the unit ball in the complex Banach space $L^\infty(\Delta)$.

Given a Beltrami differential $\mu \in B(\Delta)$, we can extend it by symmetry (cf. Sec. 6.2) to the Beltrami differential $\hat{\mu}$ on the whole plane. Theor. 5 from Sec. 6.2 implies the existence of a unique normalized quasiconformal homeomorphism w_μ on the extended complex plane $\overline{\mathbb{C}}$ with complex dilatation $\hat{\mu}$. Moreover, this homeomorphism preserves the unit disc Δ, so we can associate with μ the quasisymmetric homeomorphism $w_\mu|_{S^1}$ of the unit circle S^1. Introduce an equivalence relation between Beltrami differentials in Δ: $\mu \sim \nu$ *if and only if*

$$w_\mu = w_\nu \quad \text{on } S^1 \ .$$

Then *the universal Teichmüller space $\mathcal{T}$ will be identified with the quotient of the space $B(\Delta)$ of Beltrami differentials modulo this equivalence relation*:

$$\mathcal{T} = B(\Delta)/\sim \ .$$

Or, to put it in another words, *$\mathcal{T}$ coincides with the space of normalized quasiconformal self-homeomorphisms of the unit disc Δ*.

We can give still another definition of the universal Teichmüller space $\mathcal{T}$, using the extension of a given Beltrami differential μ by zero outside the unit disc Δ (cf. Sec. 6.2). In more detail, we denote by $\check{\mu}$ the Beltrami differential on the complex plane, obtained by the extension of μ by zero outside Δ. Then by Theor. 5 from Sec. 6.2 we obtain a normalized quasiconformal homeomorphism w^μ of the extended complex plane $\overline{\mathbb{C}}$, which is conformal on the exterior Δ_- of the closed unit disc $\overline{\Delta} \subset \overline{\mathbb{C}}$ and fixes the points $\pm 1, -i$. Recall that the image $\Delta^\mu := w^\mu(\Delta)$ of the unit disc Δ under the quasiconformal map w^μ is called the quasidisc. We associate with the Beltrami differential $\mu \in B(\Delta)$ the normalized quasidisc Δ^μ.

Introduce now another equivalence relation between Beltrami differentials in Δ by saying that two Beltrami differentials μ and ν are equivalent, if $w^\mu|_{\Delta_-} = w^\nu|_{\Delta_-}$. We claim that this new equivalence relation between Beltrami differentials coincides with the previous one. More precisely, we have the following

Lemma 4. *Two Beltrami differentials $\mu, \nu \in B(\Delta)$ are equivalent if and only if*

$$w_\mu|_{S^1} = w_\nu|_{S^1} \Longleftrightarrow w^\mu|_{\Delta_-} = w^\nu|_{\Delta_-} \ .$$

The proof of Lemma will be given below. Note that it implies that *the universal Teichmüller space $\mathcal{T}$ can be identified with the space of normalized quasidiscs in $\overline{\mathbb{C}}$.*

This last definition of $\mathcal{T}$ allows us to consider the elements of $\mathcal{T}$ as univalent holomorphic functions in Δ_- (which extend to quasiconformal homeomorphisms of the extended complex plane $\overline{\mathbb{C}}$ and fix the points ± 1 and $-i$). For such functions it is standard to use an alternative normalization by fixing their Laurent decompositions at ∞ in the form

$$f(z) = z + \frac{b_1}{z} + \frac{b_2}{z^2} + \dots \ .$$

The complex numbers $b_1, b_2, \dots$ play the role of complex coordinates on $\mathcal{T}$. According to the classical *area theorem*, they satisfy the inequality

$$\sum_{n=1}^{\infty} n|b_n|^2 \leq 1 \ .$$

A relation between two different interpretations of Teichmüller space $\mathcal{T}$, namely, as the space of normalized quasisymmetric homeomorphisms of S^1 and the space of normalized quasidiscs in $\overline{\mathbb{C}}$, can be established in the following way.

If f is a given quasisymmetric homeomorphism of S^1, then it can be extended to a quasiconformal homeomorphism of the unit disc Δ, associated with some Beltrami differential μ. Then the corresponding quasidisc

$$\Delta^\mu = w^\mu(\Delta)$$

will not depend on the choice of the quasiconformal extension of f to Δ.

Conversely, let Δ^μ be the quasidisc, corresponding to a quasiconformal map with the complex dilatation μ. Since both maps $w^\mu : \Delta \to \Delta^\mu$ and $w_\mu : \Delta \to \Delta$ are quasiconformal and have the same Beltrami potential μ in Δ, the map $\rho := w^\mu \circ w_\mu^{-1}$ defines a conformal transform of the unit disc Δ onto the quasidisc Δ^μ. Denote this map by ρ_+, and by $\rho_- : \Delta_- \to \Delta_-^\mu$ a conformal map of Δ_- onto the exterior Δ_-^μ of the closed quasidisc $\overline{\Delta^\mu}$, provided by the Riemann mapping theorem. We associate with the quasidisc Δ^μ the quasisymmetric homeomorphism of S^1, given by the formula

$$f := \rho_+^{-1} \circ \rho_- \,|_{S^1} \ .$$

The constructed correspondences preserve the normalizations and so establish a relation between two different interpretations of the universal Teichmüller space $\mathcal{T}$.

We give now the proof of the Lemma, formulated above.

Proof of Lemma. Suppose first that $w^\mu|_{\Delta_-} = w^\nu|_{\Delta_-}$. Then the maps $w^\mu \circ w_\mu^{-1}$ and $w^\nu \circ w_\nu^{-1}$ are both conformal in Δ_+, which they map onto the same quasidisc. Being normalized, they should agree on S^1. But $w^\mu|_{S^1} = w^\nu|_{S^1}$, so we should also have $w_\mu|_{S^1} = w_\nu|_{S^1}$.

Conversely, suppose that $w_\mu|_{S^1} = w_\nu|_{S^1}$. Consider a map w of the extended complex plane $\overline{\mathbb{C}}$, given by

$$w = \begin{cases} w^\mu \circ (w^\nu)^{-1} & \text{on } w^\nu(\overline{\Delta}_-) \ , \\ [w^\mu \circ (w_\mu)^{-1}] \circ [w_\nu \circ (w^\nu)^{-1}] & \text{on } w^\nu(\Delta_+) \ . \end{cases}$$

It follows from the assumption $w_\mu|_{S^1} = w_\nu|_{S^1}$ that w is a homeomorphism of $\overline{\mathbb{C}}$. Moreover, w is conformal on $w^\nu(\Delta_-)$ by construction and w is conformal on $w^\nu(\Delta_+)$, since both maps $w^\mu \circ (w_\mu)^{-1}$ and $w_\nu \circ (w^\nu)^{-1}$ are conformal there. It follows from the quasiconformal extension property (cf. [49], Lemma I.6.1) that w extends to a conformal map of $\overline{\mathbb{C}}$, i.e. to a fractional-linear automorphism of $\overline{\mathbb{C}}$. Since it is normalized, it should be equal to identity, so $w^\mu|_{\Delta_-} = w^\nu|_{\Delta_-}$. □

The universal Teichmüller space $\mathcal{T}$ can be provided with a natural metric, called the *Teichmüller distance*, which can be defined as follows. Representing the points of $\mathcal{T}$ as normalized quasiconformal self-homeomorphisms of Δ, fixing the points ± 1 and $-i$, we can define the distance between two points $[w_1]$, $[w_2]$ of $\mathcal{T}$ as

$$\tau([w_1],[w_2]) := \frac{1}{2}\inf\{\log K_{w_2\circ w_1^{-1}} : w_1 \in [w_1], w_2 \in [w_2]\} ,$$

where K_w is the maximal dilatation of a quasiconformal map w (cf. Sec. 6.1). This metric converts $\mathcal{T}$ into a complete metric space (cf. [49], Sec. III.3.2). Moreover, it can be shown that $\mathcal{T}$ is contractible (cf. [49], Theor. III.3.2).

11.2 Kähler structure of the universal Techmüller space

We shall study the Kähler geometry of the universal Teichmüller space $\mathcal{T}$, using an embedding of $\mathcal{T}$ into the space of quadratic differentials, proposed by L.Bers. This embedding will allow us to introduce complex coordinates on $\mathcal{T}$. It is convenient to use for its definition the model of $\mathcal{T}$ as the space of normalized quasidiscs $\Delta^\mu = w^\mu(\Delta_+)$ or, which is the same, the space of normalized conformal maps w^μ of Δ_-. By using a suitable Möbius transform, we can substitute here the disc Δ_+ by the upper halfplane H_+ and represent $\mathcal{T}$ as the space of normalized quasidiscs $w^\mu(H_+)$, i.e. the images of the upper halfplane H_+ under quasiconformal homeomorphisms w^μ of the extended complex plane $\overline{\mathbb{C}}$, which are conformal on H_- and fix the points $0, 1, \infty$.

Suppose that $[\mu]$ is an arbitrary point of $\mathcal{T}$, represented by a normalized quasidisc $w^\mu(H_+)$, and define a map

$$\Psi : [\mu] \longmapsto \psi[\mu] := S[w^\mu|_{H_-}] , \tag{11.1}$$

where S denotes the Schwarzian (cf. Sec. 10.2). Due to the invariance of the Schwarzian under the Möbius transformations, the image of this map $\psi[\mu]$ depends only on the class $[\mu]$ of the Beltrami differential μ in $\mathcal{T}$ and is a holomorphic function in H_-. The converse is also true: *if* $\psi[\mu] = \psi[\nu]$, *then* $[\mu] = [\nu]$ *in* $\mathcal{T}$. Indeed, consider the conformal map $h := w^\mu \circ (w^\nu)^{-1}$ from $w^\nu(H_-)$ to $w^\mu(H_-)$. Then, applying the transformation rule (10.8) for the Schwarzian on H_-, we shall have

$$S[w^\mu] = S[h \circ w^\nu] = (S[h] \circ w^\nu)\,(w^\nu)'^2 + S[w^\nu] .$$

Since $S[w^\mu] = S[w^\nu]$ in H_-, it follows that $S[h] = 0$ in H_-. So h is a fractional-linear transformation (cf. Sec. 10.2), which is normalized (i.e. fixes the points $0, 1, \infty$). Hence, h is the identity, which implies that $[\mu] = [\nu]$ in $\mathcal{T}$.

The transformation rule for the Schwarzian (10.8) suggests that the image $\psi[\mu]$ of a Beltrami differential $\mu \in B(H_-)$ is a holomorphic quadratic differential in H_-. So the map (11.1) defines an embedding of the universal Teichmüller space $\mathcal{T}$ into the space of holomorphic quadratic differentials in H_-, called the *Bers embedding*.

We have already considered in Sec. 10.2 a natural hyperbolic norm on the space of quadratic differentials. In the case of H_- it is equal to

$$\|\psi\|_2 := \sup_{z \in H_-} 4y^2|\psi(z)|$$

for a quadratic differential ψ. It follows from Theor. 11 in Sec. 10.2 that

$$\|\psi[\mu]\|_2 \leq 6$$

for any Beltrami differential $\mu \in B(H_-)$. Denote by $B_2(H_-)$ the space of holomorphic quadratic differentials in H_- with a finite norm:

$$B_2(H_-) = \{\text{holomorphic quadratic differentials } \psi \text{ on } H_- : \ \|\psi\|_2 < \infty\} \ .$$

So we have an embedding

$$\Psi : \mathcal{T} \longrightarrow B_2(H_-)$$

of $\mathcal{T}$ into a bounded subset in $B_2(H_-)$. It can be shown that it is a homeomorphism (with respect to the topology on $\mathcal{T}$, determined by the Teichmüller distance) onto the image of Ψ (cf. [49], Theor. III.4.1). The image $\Psi(\mathcal{T})$ is an open subset in $B_2(H_-)$, which contains the ball of radius 1/2 (cf. [1]). Moreover, it is known (cf. [20]) that it is a connected contractible set.

Using Bers embedding, we can introduce a complex structure and complex coordinates on the universal Teichmüller space $\mathcal{T}$ by pulling them back from the complex Banach space $B_2(H_-)$. It provides $\mathcal{T}$ with the structure of a complex Banach manifold. Consider now the natural projection of the space of Beltrami differentials to the universal Teichmüller space, defined in the beginning of Sec. 11.1. In our realization of $\mathcal{T}$ this map is given by the projection

$$\Phi : B(H_+) \longrightarrow \mathcal{T} = B(H_+)/\sim \ .$$

Then it is holomorphic with respect to the introduced complex structure on $\mathcal{T}$ (cf. [56], Ch. 3.4). So the composition map

$$F := \Psi \circ \Phi : B(H_+) \longrightarrow B_2(H_-)$$

is also holomorphic.

We study next the tangent structure of this map, i.e. the differential of F. We describe the tangent bundle $T\mathcal{T}$, using the definition of $\mathcal{T}$ in terms of Beltrami differentials

$$\mathcal{T} = B(H_+)/\sim \ .$$

Due to the homogeneity of $\mathcal{T}$ with respect to the right action of quasisymmetric homeomorphisms of $\mathbb{R}$, it's sufficient to determine the tangent space $T_0\mathcal{T}$ at the origin, corresponding to the identity homeomorphism, associated with $\mu = 0$.

Let $\mu \in L^\infty(H_+)$ represents an arbitrary tangent vector from $T_0B(H_+)$. Then for the corresponding quasiconformal map $w^{t\mu}$ we'll have an expansion

$$w^{t\mu}(z) = z + tw_1(z) + o(t)$$

for $t \to 0$, where $o(t) := t\delta(z,t)$ and $\delta(z,t) \to 0$ uniformly in z, when z belongs to a compact subset in $\mathbb{C}$. The term

$$w_1(z) \equiv \dot{w}[\mu](z)$$

represents the first variation of the quasiconformal map $w^{t\mu}$ with respect to μ. We substitute $w^{t\mu}$ into the Beltrami equation and differentiate it with respect to t at

$t=0$. Since $\partial/\partial t$ commutes with $\partial/\partial z$ and $\partial/\partial \bar{z}$ for almost all z, being applied to $w^{t\mu}(z)$ (cf. [2]), we obtain that

$$\frac{\partial}{\partial \bar{z}}\left(\dot{w}[\mu](z)\right)=\mu(z)$$

for almost all z, i.e. $\dot{w}[\mu](z)$ satisfies the $\bar{\partial}$-equation. Hence its solution can be given by the Cauchy-Green formula: if μ has a compact support in $\mathbb{C}$, then any solution is given by

$$-\frac{1}{\pi}\int_{\mathbb{C}}\frac{\mu(\zeta)}{\zeta-z}\,d\xi d\eta \quad \text{for } \zeta=\xi+i\eta$$

plus an arbitrary entire function, which in our case can be only a linear function of the form (cf. [1])

$$A+Bz=(z-1)\int_{\mathbb{C}}\frac{\mu(\zeta)}{\zeta}d\xi d\eta - z\int_{\mathbb{C}}\frac{\mu(\zeta)}{\zeta-1}\,d\xi d\eta\ .$$

Altogether it gives the following formula for $\dot{w}[\mu](z)$

$$w_1(z)=\dot{w}[\mu](z)=-\frac{z(z-1)}{\pi}\int_{\mathbb{C}}\frac{\mu(\zeta)}{\zeta(\zeta-1)(\zeta-z)}\,d\xi d\eta\ , \tag{11.2}$$

which holds for <u>all</u> $\mu\in L^{\infty}(H_+)$ (the restriction on the support of μ being compact is removed by a standard approximation argument, cf. [1]).

We are now able to prove the following

Proposition 20 ([1, 56])**.** *The differential of the map*

$$F=\Psi\circ\Phi : B(H_+)\longrightarrow B_2(H_-)$$

at zero is given by the formula

$$d_0\left(\Psi\circ\Phi\right)[\mu](z)=-\frac{6}{\pi}\int_{H_+}\frac{\mu(\zeta)}{(\zeta-z)^4}\,d\xi d\eta\ ,\quad z\in H_-\ , \tag{11.3}$$

for $\mu\in B(H_+)$.

Proof. Fix $z_0\in H_-$. We want to find the derivative of the function

$$\varphi(t,z):=S[w^{t\mu}](z)=F[t\mu](z)$$

at $t=0$. By denoting $w:=w^{t\mu}$, the derivative with respect to t by "dot", and derivative with respect to z by "prime", we get

$$\dot{\varphi}=\left(\frac{w'''}{w'}-\frac{3}{2}\left(\frac{w''}{w'}\right)^2\right)^{\cdot}=\frac{(w')^3\dot{w}'''-\dot{w}'(w')^2w'''-3\dot{w}''(w')^2w''+3\dot{w}'w'(w'')^2}{(w')^4}\ .$$

For $t=0$ we have $w(z)\equiv z$, so $w'\equiv 1$, $w''=w'''\equiv 0$. Hence, the above formula reduces to

$$\left.\frac{\partial\varphi}{\partial t}\right|_{t=0}=\frac{(w')^3\dot{w}'''}{(w')^4}=\dot{w}'''\ .$$

But the formula (11.2) implies that

$$\dot{w}(z) = -\frac{z(z-1)}{\pi}\int_{H_+}\frac{\mu(\zeta)}{\zeta(\zeta-1)(\zeta-z)}\,d\xi d\eta$$

(note that $\mu \equiv 0$ on H_-). Differentiating this formula three times over z, we obtain the desired formula (11.3). □

In addition to formula (11.3), it may be proved (cf. [56], Theor. 3.4.5) that the operator d_0F is a bounded linear operator and estimate its norm by an absolute constant.

We describe the kernel of the differential d_0F. We note that there is a natural pairing between the space $A_2(H_+)$ of L^1-integrable holomorphic quadratic differentials in H_+ and the space $B(H_+)$ of Beltrami $(-1,1)$-differentials in H_+, denoted by

$$<\mu,\psi>:=\int_{H_+}\mu\psi\ . \tag{11.4}$$

In terms of this pairing, the kernel of d_0F can be identified as follows.

Theorem 12 (Teichmüller lemma)**.** *The kernel of d_0F coincides with the subspace*

$$N \equiv A_2(H_+)^\perp = \{\mu \in L^\infty(H_+) : <\mu,\psi>=0 \text{ for all } \psi \in A_2(H_+)\}\ .$$

The proof of this Lemma may be found in ([1], Sec.IV(D); [56], Sec.3.7).

It will be useful to summarize the previous results also in the case of the unit disc $\Delta = \Delta_+$. The Bers embedding for this case coincides with the map

$$F : B(\Delta_+) \longrightarrow B_2(\Delta_-)\ ,$$

associating with a Beltrami differential $\mu \in B(\Delta_+)$ in the unit disc Δ_+ the restriction $S[w^\mu]|_{\Delta_-}$ of the Schwarzian $S[w^\mu]$ to the exterior $\Delta_- = \{|z|>1\}\cup\infty$ of the closed unit disc $\overline{\Delta}_+$ on the Riemann sphere $\overline{\mathbb{C}}$. The image of this map is contained in the space of holomorphic quadratic differentials in Δ_- with a finite norm

$$\|\psi\|_2 := \sup_{z\in\Delta_-}(1-|z|^2)^2|\psi(z)| < \infty\ .$$

The formula for the differential d_0F is given by

$$d_0F[\mu](z) = -\frac{6}{\pi}\int_{\Delta_+}\frac{\mu(\zeta)}{(\zeta-z)^4}\,d\xi d\eta\ ,\quad z\in\Delta_-\ , \tag{11.5}$$

for $\mu \in L^\infty(\Delta_+)$. The kernel of d_0F is equal to

$$N \equiv A_2(\Delta_+)^\perp = \{\mu \in L^\infty(\Delta_+) : <\mu,\psi>=0 \text{ for all } \psi \in A_2(\Delta_+)\}\ .$$

This definition is equivalent to the following (cf. [56], Sec. 3.7.2)

$$N = \{\mu \in L^\infty(\Delta) : \int_\Delta \frac{\mu(\zeta)}{(\zeta-z)^4}\,d\xi d\eta = 0 \text{ for all } z \in \Delta_-\}\ .$$

The formulas (11.3),(11.5) suggest how a Kähler metric on $\mathcal{T}$ can be defined. Namely, we employ the *Ahlfors map* (cf. [3]): $L^\infty(\Delta) \longrightarrow B_2(\Delta)$, given by

$$L^\infty(\Delta) \ni \mu \longmapsto \varphi[\mu](z) = \int_\Delta \frac{\overline{\mu(\zeta)}}{(1-z\bar\zeta)^4}\, d\xi d\eta \ .$$

It associates with any $\mu \in L^\infty(\Delta)$ a holomorphic quadratic differential $\varphi[\mu]$ with a finite norm $\|\varphi\|_2 = \sup_{z\in\Delta}(1-|z|^2)^2|\varphi(z)| < \infty$. The kernel of this map coincides with $N = A_2(\Delta_+)^\perp$. Now we can define formally a Hermitian metric on $\mathcal{T}$ by setting for two tangent vectors μ, ν in $T_0\mathcal{T} = L^\infty(\Delta)/N$:

$$(\mu,\nu) :=< \mu, \varphi[\nu] >= \int_\Delta \int_\Delta \frac{\mu(z)\overline{\nu(\zeta)}}{(1-z\bar\zeta)^4}\, d\xi d\eta\, dx dy \ . \qquad (11.6)$$

However, this metric is only densely defined. More precisely (cf. [59]), for a general $\mu \in L^\infty(\Delta)$ its image $\varphi[\mu]$ in $B_2(\Delta)$ may be not integrable, i.e. it does not belong, in general, to $A_2(\Delta)$, in which case the integral in (11.6) will diverge. In fact, the formula (11.6) is correctly defined, if the tangent vectors μ, ν in $T_0\mathcal{T}$ are sufficiently smooth. To formulate this smoothness condition more precisely, we realize $\mathcal{T}$ as the space of normalized quasisymmetric homeomorphisms of S^1. Then a tangent vector $\mu \in L^\infty(\Delta) = T_0B(\Delta)$ will correspond under the differential $d_0\Phi$ to the vector field $v = v(\theta)\partial/\partial\theta$ on S^1 of the form

$$v(\theta)\frac{\partial}{\partial\theta} = \dot w[\mu](z)\frac{\partial}{\partial z}\ , \quad z = e^{i\theta}\ ,$$

where $\dot w[\mu]$ is the derivative with respect to t of the one-parameter flow $w_{t\mu}$ of quasisymmetric homeomorphisms:

$$w_{t\mu}(z) = z + t\dot w[\mu](z) + o(t) \quad \text{for } t \to 0\ .$$

Then it may be proved (cf. [59]) that the integral in (11.6) converges, if the tangent vectors μ, ν in $T_0\mathcal{T}$ correspond to $C^{3/2+\epsilon}$-smooth vector fields on S^1. Whenever the metric (11.6) is well-defined, it determines a Kähler metric, in particular, it defines a Kähler metric on the "smooth" part of $\mathcal{T}$.

11.3 Teichmüller spaces $T(G)$ and $\mathbf{Diff}_+(S^1)/\mathbf{M\ddot{o}b}(S^1)$

The universal Teichmüller space $\mathcal{T}$ contains, as its complex submanifolds, all classical Teichmüller spaces $T(G)$, where G is a *Fuchsian group* (cf. [49, 56]). In particular, it is true for all Teichmüller spaces of compact Riemann surfaces. This property of $\mathcal{T}$ motivates the use of the term "universal" in the name of $\mathcal{T}$.

With an arbitrary Fuchsian group G we associate the Riemann surface $X := \Delta/G$, uniformized by the unit disc Δ. By definition, $T(G)$ consists of quasisymmetric homeomorphisms $f \in \mathrm{QS}(S^1)$, which are G-*invariant* in the following sense:

$$f \circ g \circ f^{-1} \text{ belongs to } \mathrm{M\ddot{o}b}(S^1) \text{ for all } g \in G\ ,$$

modulo fractional-linear automorphisms of the disc Δ. If we denote by $\mathrm{QS}(S^1)^G$ the subset of G-invariant quasisymmetric homeomorphisms in $\mathrm{QS}(S^1)$, then

$$T(G) = \mathrm{QS}(S^1)^G/\mathrm{M\ddot{o}b}(S^1)\ .$$

The universal Teichmüller space $\mathcal{T}$ itself corresponds to the Fuchsian group $G = \{1\}$.

The various interpretations of the universal Teichmüller space $\mathcal{T}$, given in Sec. 11.1, are compatible with the notion of G-invariance. In particular, the Teichmüller spaces $T(G)$ admit a description in terms of G-invariant Beltrami differentials. More precisely, denote by $B(\Delta)^G$ the subspace of $B(\Delta)$, consisting of Beltrami differentials μ, satisfying the relation

$$\mu(gz)\frac{\overline{g'(z)}}{g'(z)} = \mu(z) \quad \text{almost everywhere on } \Delta \text{ for all } g \in G \ .$$

Then we'll have, as in Sec. 11.1:

$$T(G) = B(\Delta)^G / \sim \ ,$$

where $\mu \sim \nu$ iff $w_\mu = w_\nu$ on S^1 or,equivalently, $w^\mu|_{\Delta_-} = w^\nu|_{\Delta_-}$.

We can associate with a G-invariant Beltrami differential μ a Fuchsian group G_μ, conjugate to G:

$$G_\mu := w_\mu G w_\mu^{-1} \ ,$$

where w_μ is the quasiconformal homeomorphism of $\overline{\mathbb{C}}$, leaving $\Delta_\pm$ invariant (cf. Sec. 11.1).

We have a natural quasiconformal map of the Riemann surface $X := \Delta/G$ onto another Riemann surface $X_\mu := \Delta/G_\mu$, which is biholomorphic precisely, when $\mu \in \text{Möb}(S^1)$. Hence, one can say that the space $T(G)$ parametrizes, with the help of the map $\mu \mapsto G_\mu$, different complex structures on the Riemann surface $X := \Delta/G$, which can be obtained from the original one by quasiconformal deformations.

On the other hand, we can associate with a G-invariant Beltrami differential $\mu \in B(\Delta)^G$ another conjugated group

$$G^\mu := w^\mu G (w^\mu)^{-1} \ ,$$

operating properly discontinuously on the quasidisc $\Delta^\mu := w^\mu(\Delta)$. Here, w^μ is the quasiconformal homeomorphism of $\overline{\mathbb{C}}$, which is conformal on Δ_- (cf. Sec. 11.1). The group G^μ is a Kleinian group, called otherwise a *quasi-Fuchsian group* (cf. [49, 56]). The Riemann surface X_μ is biholomorphic to Δ^μ/G^μ (cf. [56], Theor. 1.3.5). We note also that the Riemann surface Δ_-^μ/G^μ is biholomorphic to the Riemann surface Δ_-/G, due to the conformality of w^μ on Δ_-.

The definition and main properties of the Bers embedding, given in Sec. 11.2, extend to the Teichmüller spaces $T(G)$. For the case of the unit disc $\Delta \equiv \Delta_+$ the Bers embedding is the map

$$F : B(\Delta_+)^G \longrightarrow B_2(\Delta_-)^G \ ,$$

associating with a Beltrami differential $\mu \in B(\Delta_+)^G$ the restriction $S[w^\mu]|_{\Delta_-}$ of the Schwarzian $S[w^\mu]$ to Δ_-. The image of this map is contained in the space $B_2(\Delta_-)^G$ of G-invariant holomorphic quadratic differentials in Δ_- with a finite norm

$$\|\psi\|_2 := \sup_{z\in\Delta_-} (1 - |z|^2)^2 |\psi(z)| < \infty \ .$$

The formula for the differential d_0F has the form

$$d_0F[\mu](z) = -\frac{6}{\pi}\int_{\Delta_+} \frac{\mu(\zeta)}{(\zeta - z)^4}\, d\xi d\eta\ , \quad z \in \Delta_-\ ,$$

for $\mu \in L^\infty(\Delta_+)^G$. The kernel of d_0F is given by

$$N^G \equiv (A_2(\Delta_+)^G)^\perp = \{\mu \in L^\infty(\Delta_+)^G : <\mu, \psi>= 0 \text{ for all } \psi \in A_2(\Delta_+)\}\ .$$

This definition is equivalent to

$$N^G = \{\mu \in L^\infty(\Delta)^G : \int_\Delta \frac{\mu(\zeta)}{(\zeta - z)^4}\, d\xi d\eta = 0 \text{ for all } z \in \Delta_-\}\ .$$

So the tangent space of $T(G)$ at the origin coincides with the space $L^\infty(\Delta)^G/N^G$.
As in Sec. 11.2, there is the Ahlfors map $L^\infty(\Delta)^G/N^G \longrightarrow B_2(\Delta)^G$, given by

$$L^\infty(\Delta)^G \ni \mu \longmapsto \varphi[\mu](z) = \int_\Delta \frac{\overline{\mu(\zeta)}}{(1 - z\bar{\zeta})^4}\, d\xi d\eta\ .$$

Using this map, we can define the *Weil–Petersson metric* on $T(G)$, as in Sec. 11.2, by setting for two tangent vectors μ, ν in $T_0T(G) = L^\infty(\Delta)^G/N^G$:

$$g_G(\mu, \nu) := \int_{\Delta/G}\int_\Delta \frac{\mu(z)\overline{\nu(\zeta)}}{(1 - z\bar{\zeta})^4}\, d\xi d\eta\, dxdy\ . \tag{11.7}$$

As was pointed out in Sec. 11.2, the image $\varphi[\mu] \in B_2(\Delta)^G$ of the Ahlfors map for a general Fuchsian group G may not belong to the space $A_2(\Delta)^G$ of integrable holomorphic quadratic differentials, so the formula (11.7) for the metric $g_G(\mu, \nu)$ is ill-defined for general Fuchsian groups. But in the case of finite-dimensional Teichmüller spaces $T(G)$ this difficulty does not show up, since in this situation $B_2(\Delta)^G = A_2(\Delta)^G$ (cf. [56]), and the introduced metric coincides with the standard Weil–Petersson metric on the finite-dimensional Teichmüller spaces $T(G)$. Moreover, S.Nag has proved (cf. [59]) that the metric $g_G(\mu, \nu)$ on $T(G)$ can be obtained from the metric (μ, ν) on $\mathcal{T}$ by a certain reduction procedure. This procedure involves a regularization of the integral

$$(\mu, \nu) = \int_\Delta\int_\Delta \frac{\mu(z)\overline{\nu(\zeta)}}{(1 - z\bar{\zeta})^4}\, d\xi d\eta\, dxdy = \int_\Delta \mu \cdot \varphi[\nu]\ . \tag{11.8}$$

To define the regularization, we rewrite the integral (11.8) in the form

$$(\mu, \nu) = \lim_{r \to 1-0} g_r(\mu, \nu)$$

where

$$g_r(\mu, \nu) = \int_{\Delta_r} \mu \cdot \varphi[\nu]\ , \tag{11.9}$$

and $\Delta_r := \{z \in \Delta : |z| < r\}$, $0 < r < 1$.

In the case when μ, ν are G-invariant, i.e. belong to $L^\infty(\Delta)^G/N^G$, the integral (11.8) coincides with

$$n \int_{\Delta/G} \mu \cdot \varphi[\nu] = n g_G(\mu, \nu) \ ,$$

where n is the number of copies of the fundamental domain Δ/G, contained in Δ. Hence, this integral must diverge, if the group G has infinitely many elements. The integral (11.9) by the same argument is proportional to $n_r g_G(\mu, \nu)$, where n_r is the number of copies of the fundamental domain Δ/G, contained in Δ_r. It follows that the integral (11.9) may be regularized by dividing it by a quantity, proportional to n_r. More precisely, the following assertion is true .

Proposition 21 ([59]). *For any finite-dimensional Teichmüller space $T(G)$ its Weil–Petersson metric $g_G(\mu, \nu)$ may be computed by the formula*

$$\frac{g_G(\mu, \nu)}{g_G(\mu_0, \mu_0)} = \lim_{r \to 1-0} \frac{g_r(\mu, \nu)}{g_r(\mu_0, \mu_0)} \ ,$$

where $\mu, \nu \in L^\infty(\Delta)^G$, and $\mu_0 \in L^\infty(\Delta)^G/N^G$ is an arbitrary nonzero tangent vector from $T_0T(G)$.

As we have remarked at the beginning of Sec. 11.1, the universal Teichmüller space $\mathcal{T}$ contains the homogeneous space $\mathcal{S} = \mathrm{Diff}_+(S^1)/\mathrm{Möb}(S^1)$ as its "smooth" part:

$$\mathcal{S} = \mathrm{Diff}_+(S^1)/\mathrm{Möb}(S^1) \hookrightarrow \mathcal{T} = \mathrm{QS}(S^1)/\mathrm{Möb}(S^1) \ .$$

In Sec. 10.3 we have defined the structure of a Kähler–Frechet manifold on $\mathcal{S}$. We recall the definition of the Kähler metric g on this space in terms of Fourier decompositions. For given tangent vectors $u, v \in T_o\mathcal{S}$ with Fourier decompositions

$$u = \sum_{n \neq -1,0,1} u_n e_n \quad \text{and} \quad v = \sum_{n \neq -1,0,1} v_n e_n \ ,$$

the value of g on these vectors is equal to

$$g(u, v) = 2 \,\mathrm{Re} \left(\sum_{n=2}^{\infty} u_n \bar{v}_n \,(n^3 - n) \right) \ . \tag{11.10}$$

As we have noted before, the series on the right hand side is absolutely converging, if the vector fields u, v are of the class $C^{3/2+\epsilon}$ on S^1.

It was pointed out in [59] that the Kähler metric g on $\mathcal{S}$ coincides (up to a constant factor) with the Weil–Petersson metric (11.6) on $\mathcal{S}$, induced by the embedding $\mathcal{S} \hookrightarrow \mathcal{T}$. (Note that the metric (11.6) on the smooth part $\mathcal{S}$ of $\mathcal{T}$ is correctly defined, as we have remarked in Sec. 11.2.) Using the interpretation of tangent vectors from $T_0\mathcal{T}$, given at the end of Sec. 11.2, we can express the equality of these metrics on $\mathcal{S}$ as follows. Given two tangent vectors $u, v \in T_0\mathcal{S}$, written in the form $u = \dot{w}[\mu]\partial/\partial z$, $v = \dot{w}[\nu]\partial/\partial z$, we have

$$g(\mu, \nu) = \lambda \int_\Delta \int_\Delta \frac{\mu(z)\overline{\nu(\zeta)}}{(1 - z\bar{\zeta})^4} \, d\xi d\eta \, dx dy$$

for a suitable choice of the constant λ. By introducing this constant into the definition of the Kähler metric on $\mathcal{S}$, we can make the embedding $\mathcal{S} \hookrightarrow \mathcal{T}$ an isometry.

It is an interesting question, how the smooth part $\mathcal{S}$ is placed inside the universal Teichmüller space $\mathcal{T}$ with respect to the classical Teichmüller spaces $T(G)$. It can be shown (cf. [12]) that the quasidiscs, corresponding to all points of $T(G)$, except the origin, have fractal boundaries (i.e. boundaries of Hausdorff dimension> 1) in contrast with the qiasidiscs, corresponding to points of $\mathcal{S}$, which have C^∞-smooth boundaries.

11.4 Grassmann realization of the universal Teichmüller space

The Grassmann realization of the universal Teichmüller space $\mathcal{T}$ is based on the fact that the group $\mathrm{QS}(S^1)$ of quasisymmetric homeomorphisms of the circle acts on the Sobolev space V of half-differentiable functions on S^1 (cf. Sec. 9.2).

Suppose that $f : S^1 \to S^1$ is a homeomorphism of S^1, preserving its orientation. We define an operator T_f by the formula

$$T_f(\xi) := \xi \circ f - \frac{1}{2\pi} \int_0^{2\pi} \xi\,(f(\theta))\, d\theta$$

for $\xi \in V$. This operator has the following remarkable property.

Proposition 22 ([58]). *The operator T_f acts on V (i.e. $T_f(\xi)$ belongs to V for any $\xi \in V$) if and only if $f \in QS(S^1)$. Moreover, if f extends to a K-quasiconformal homeomorphism of the disc Δ, then the operator norm of T_f does not exceed $\sqrt{K + K^{-1}}$.*

The proof of this assertion, given in [58], uses the interpretation of the space V in terms of harmonic functions in the disc, given at the end of Sec. 9.1.

Transformations of the form T_f with $f \in \mathrm{QS}(S^1)$ preserve the symplectic form ω, i.e. they are symplectic transformations of V.

Proposition 23 ([58]). *If $f \in QS(S^1)$, then*

$$\omega(f^*(\xi), f^*(\eta)) = \omega(\xi, \eta)$$

for any $\xi, \eta \in V$. Moreover, the complex-linear extension of the $QS(S^1)$-action on V to the complexification $V^{\mathbb{C}}$ preserves the "holomorphic" subspace W_+ (cf. Sec. 9.1) if and only if f is a Möbius transformation, i.e. $f \in Möb(S^1)$. In the latter case, T_f acts as a unitary operator on $W_\pm$.

Proof. For homeomorphisms f of the class C^1 the first assertion is a corollary of the change of variables formula. For a general quasisymmetric homeomorphism $f \in \mathrm{QS}(S^1)$ the assertion follows from the fact (cf. [49]) that f may be uniformly approximated by real analytic quasisymmetric homeomorphisms of S^1, having the same quasiconformal constant K as f.

If the action of f on $V^{\mathbb{C}}$ preserves W_+, then it should extend to a map $\Delta \to \Delta$. This map must be a biholomorphism, since f is a homeomorphism, hence, it is a Möbius transformation. It is clear from the definition of the inner product on $V^{\mathbb{C}}$ (cf. Sec. 9.1) that such a transformation acts unitarily on $W_\pm$. □

The symplectic form ω on V is uniquely determined by the invariance property, stated in the above Proposition. In fact, a much stronger assertion is true.

Proposition 24 ([58]). *Suppose that ω_1 is a real-valued continuous bilinear form on V such that*

$$\omega_1(f^*(\xi), f^*(\eta)) = \omega_1(\xi, \eta)$$

for any $f \in$ Möb(S^1) and arbitrary $\xi, \eta \in V$. Then ω_1 is a real multiple of ω, in particular, any form ω_1, satisfying the hypothesis of the Proposition, coincides necessarily with a symplectic form, invariant under quasisymmetric homeomorphisms of S^1.

Proof. Note that both forms ω and ω_1 define the duality maps

$$\Sigma : V \longrightarrow V^* \quad \text{and} \quad \Sigma_1 : V \longrightarrow V^* \ ,$$

given by

$$\Sigma(\xi) := \omega(\cdot, \xi) \ , \quad \Sigma_1(\xi) := \omega_1(\cdot, \xi)$$

for $\xi \in V$. In the case of ω the duality operator Σ coincides, in fact, with the (minus of) J^0. In particular, Σ is a bounded invertible operator, defining an isomorphism between V and its dual.

We consider an intertwining operator

$$M := \Sigma^{-1} \circ \Sigma_1 : V \longrightarrow V \ .$$

It is a bounded linear operator on V, defined by the equality

$$\omega(\xi, M\eta) = \omega_1(\xi, \eta) \ .$$

Note that M commutes with any invertible bounded linear operator on V, preserving the forms ω and ω_1. Indeed, if T is such an operator, then

$$\omega(T\xi, TM\eta) = \omega(\xi, M\eta) = \omega_1(\xi, \eta) = \omega_1(T\xi, T\eta) = \omega(T\xi, MT\eta) \ .$$

Since T is invertible, it implies that

$$\omega(\xi, TM\eta) = \omega(\xi, MT\eta)$$

for any $\xi, \eta \in V$. Since the duality operator Σ, determined by ω, is an isomorphism, the last equality implies that $TM = MT$, as asserted.

We have to show that the intertwining operator M coincides with the scalar operator const $\cdot I$. We prove it by considering the complex-linear extension of M to the complexification $V^{\mathbb{C}}$.

Consider the complexified action $f \mapsto T_f$ of the Möbius group Möb(S^1) on $V^{\mathbb{C}}$. Then its restriction to W_+ can be identified with the standard unitary representation of the group $\mathrm{SL}(2, \mathbb{R})$ on the space of L^2-holomorphic functions in the disc Δ (cf. [58], lemma 4.6), hence, it is irreducible. The same is true for the restriction of $f \mapsto T_f$ to W_-. Moreover, $W_\pm$ are the only irreducible invariant subspaces of the representation $f \mapsto T_f$ of Möb(S^1) on $V^{\mathbb{C}}$.

As we have just proved, the intertwining operator M commutes with all operators $T_f : V^{\mathbb{C}} \to V^{\mathbb{C}}$ with $f \in$ Möb(S^1). Since $W_\pm$ are the only invariant subspaces for all such T_f, the operator M should map W_+ either to W_+ or W_-. If M maps W_+ into W_+, then by Schur's lemma it should be a scalar, which is real, since the operator M was real. If the other possibility (when M maps W_+ into W_-) would realize, we would substitute M by its complex conjugate, mapping W_+ into W_+, which should be again a real scalar. But in this case M cannot map W_+ into W_-, so the second possibility does not occur. □

The Propositions 22 and 23 imply that the quasisymmetric homeomorphisms from $\mathrm{QS}(S^1)$ act on the Hilbert space V by bounded symplectic operators. Hence, we have a map

$$\mathcal{T} = \mathrm{QS}(S^1)/\mathrm{M\ddot{o}b}(S^1) \longrightarrow \mathrm{Sp}(V)/\mathrm{U}(W_+) \ . \tag{11.11}$$

Here, by $\mathrm{Sp}(V)$ we denote the symplectic group of V, consisting of linear bounded symplectic operators on V, and by $\mathrm{U}(W_+)$ its subgroup, consisting of unitary operators, i.e. operators, whose complex-linear extensions to $V^{\mathbb{C}}$ preserve the subspace W_+. We describe these groups in more detail.

Recall that the complexified Hilbert space $V^{\mathbb{C}}$ is decomposed into the direct sum

$$V^{\mathbb{C}} = W_+ \oplus W_-$$

of subspaces

$$W_+ = \{f \in V^{\mathbb{C}} :\ f(z) = \sum_{k>0} x_k z^k\} \ , \quad W_- = \overline{W}_+ = \{f \in V^{\mathbb{C}} :\ f(z) = \sum_{k<0} x_k z^k\} \ .$$

In terms of this decomposition any linear operator $A : V^{\mathbb{C}} \to V^{\mathbb{C}}$ can be written in the block form

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \ ,$$

where

$$a : W_+ \to W_+ \ , \ b : W_- \to W_+ \ , \ c : W_- \to W_- \ , \ d : W_+ \to W_- \ .$$

In particular, the linear operators on $V^{\mathbb{C}}$, obtained by the complex-linear extensions of operators $A : V \to V$, have the block form

$$A = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \ ,$$

where we identify W_- with the complex conjugate $\overline{W}_+$.

An operator $A : V \to V$ belongs to the symplectic group $\mathrm{Sp}(V)$, if it preserves the symplectic form ω. This condition is equivalent to the following relation:

$$A^t J^0 A = J^0 \ ,$$

where

$$J^0 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \ .$$

In other words, the condition $A \in \mathrm{Sp}(V)$ can be written in the form:

$$A = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \in \mathrm{Sp}(V) \Longleftrightarrow \bar{a}^t a - b^t \bar{b} = 1 \ , \ \bar{a}^t b = b^t \bar{a} \ . \tag{11.12}$$

Here a^t, b^t denote the transposed operators

$$a^t : W'_+ \to W'_+ \iff a^t : W_- \to W_- \ , \quad b^t : W'_+ \to W'_- \iff b^t : W_- \to W_+ \ ,$$

where the space W'_+, dual to W_+, is identified with W_- with the help of the inner product $< \cdot , \cdot >$ (cf. Sec. 9.1).

The unitary group $\mathrm{U}(W_+)$ is embedded into $\mathrm{Sp}(V)$ as a subgroup, consisting of block matrices

$$\mathrm{U}(W_+) \ni A = \begin{pmatrix} a & 0 \\ 0 & \bar{a} \end{pmatrix} \ .$$

We return to the map (11.11). The space

$$\mathrm{Sp}(V)/\mathrm{U}(W_+) \ ,$$

standing on the right hand side of the formula (11.11), can be considered as an infinite-dimensional Siegel disc. To justify this assertion, we should study the action of $\mathrm{QS}(S^1)$ on compatible complex structures on the space V.

As we have proved above, Möbius transformations $f \in \mathrm{Möb}(S^1)$ define, via the representation $f \mapsto T_f$, unitary operators in $\mathrm{U}(W_+)$, in particular such transformations preserve the complex structure J_0 on V. If a quasisymmetric homeomorphism f does not belong to $\mathrm{Möb}(S^1)$, it does not preserve the original complex structure J^0, transforming it into another complex structure J_f, which is also compatible with the symplectic form ω. We explain this assertion in more detail.

Any complex structure J on V, compatible with ω, determines a decomposition

$$V^{\mathbb{C}} = W \oplus \overline{W} \tag{11.13}$$

into the direct sum of subspaces, isotropic with respect to ω. This decomposition is orthogonal with respect to the Kähler metric g_J on $V^{\mathbb{C}}$, determined by J and ω. The subspaces W and $\overline{W}$ are identified with, respectively, the $(-i)$- and $(+i)$-eigenspaces of the operator J on $V^{\mathbb{C}}$. Conversely, any decomposition (11.13) of the space $V^{\mathbb{C}}$ into the direct sum of isotropic subspaces determines a complex structure J on $V^{\mathbb{C}}$, which is equal to $-i \cdot I$ on W and $+i \cdot I$ on $\overline{W}$ and is compatible with ω.

This argument shows that the symplectic group $\mathrm{Sp}(V)$ acts transitively on the space $\mathcal{J}(V)$ of complex structures J, compatible with ω. It follows that the space $\mathrm{Sp}(V)/\mathrm{U}(W_+)$ can be identified with the space $\mathcal{J}(V)$. Otherwise, it may be considered as the space of the so called *positive polarizations* of V, i.e. decompositions (11.13) of $V^{\mathbb{C}}$ into the direct sum $V^{\mathbb{C}} = W \oplus \overline{W}$ of isotropic subspaces of $V^{\mathbb{C}}$, orthogonal with respect to the Kähler metric g_J on $V^{\mathbb{C}}$.

We are ready to give a Siegel disc interpretation of the space $\mathrm{Sp}(V)/\mathrm{U}(W_+)$. By definition, the *Siegel disc* is the set of bounded linear operators Z of the form

$$\mathcal{D} = \{Z : W_+ \to W_- \text{ is a symmetric bounded linear operator with } \bar{Z}Z < I\} \ .$$

The symmetricity of Z means, as above, that $Z^t = Z$ and the condition $\bar{Z}Z < I$ means that the symmetric operator $I - \bar{Z}Z$ is positive definite. In order to identify $\mathcal{J}(V)$ with $\mathcal{D}$, consider the action of the group $\mathrm{Sp}(V)$ on $\mathcal{D}$, given by fractional-linear transformations $A : \mathcal{D} \to \mathcal{D}$ of the form

$$Z \longmapsto (\bar{a}Z + \bar{b})(bZ + a)^{-1} \ ,$$

where $A = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \in \mathrm{Sp}(V)$. The invertibility of the operator $bZ + a$ follows from the invertibility of the operator a (cf. (11.12)) and the inequality (cf. (11.12))

$$bZ\bar{Z}\bar{b}^t < b\bar{b}^t < a\bar{a}^t \ .$$

It's evident that $A : \mathbb{D} \to \mathbb{D}$. The isotropy subgroup of the point $Z = 0$ consists of the operators $A \in \mathrm{Sp}(V)$, for which $\bar{b}a^{-1} = 0$, i.e. $b = 0$. This subgroup coincides with $\mathrm{U}(W_+)$. It remains to check that the action of $\mathrm{Sp}(V)$ on $\mathbb{D}$ is transitive, i.e. to construct for a given $Z \in \mathbb{D}$ an operator A, sending $Z = 0$ to this Z. Such an operator may be given by

$$A = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \tag{11.14}$$

with $b = \bar{a}\bar{Z}$ and

$$\bar{a}^t(1 - \bar{Z}Z)a = 1 \Rightarrow (\bar{a}^t)^{-1}a^{-1} = 1 - \bar{Z}Z \Rightarrow a = (1 - \bar{Z}Z)^{-1/2} \ .$$

This proves that the space

$$\mathcal{J}(V) = \mathrm{Sp}(V)/\mathrm{U}(W_+)$$

may be identified with the Siegel disc $\mathcal{D}$.

In Sec. 5.1 we have introduced the Grassmanian $\mathrm{Gr}_b(V^{\mathbb{C}})$, consisting of the images of bounded linear operators $W_+ \to W$. It is clear from the given description of $\mathcal{D}$ that it is embedded in $\mathrm{Gr}_b(V^{\mathbb{C}})$ as a complex submanifold.

Summarizing the argument above, we have the following

Proposition 25 ([58]). *The map*

$$\mathcal{T} = QS(S^1)/M\ddot{o}b(S^1) \hookrightarrow Sp(V)/U(W_+) = \mathcal{D} \hookrightarrow Gr_b(V^{\mathbb{C}})$$

is an equivariant holomorphic embedding of Banach manifolds.

11.5 Grassmann realization of $\mathbf{Diff}_+(S^1)/\mathbf{M\ddot{o}b}(S^1)$ and $\mathbf{Diff}_+(S^1)/(S^1)$

We have constructed in the previous Sec. 11.4 the natural embedding

$$\mathcal{T} = \mathrm{QS}(S^1)/\mathrm{M\ddot{o}b}(S^1) \hookrightarrow \mathrm{Sp}(V)/\mathrm{U}(W_+) = \mathcal{D} \hookrightarrow \mathrm{Gr}_b(V^{\mathbb{C}}) \ .$$

Recall now that in Sec. 10.3 we have identified the space $\mathcal{S}$ with the "smooth" part of the universal Teichmüller space $\mathcal{T}$. Combining the above embedding

$$\mathcal{T} \hookrightarrow \mathrm{Sp}(V)/\mathrm{U}(W_+)$$

with the embedding

$$\mathcal{S} \hookrightarrow \mathcal{T} \ ,$$

we obtain an embedding

$$\mathcal{S} \hookrightarrow \mathrm{Sp}(V)/\mathrm{U}(W_+) \ ,$$

giving a realization of $\mathcal{S}$ in the Grassmann manifold $\mathrm{Gr}_b(V^{\mathbb{C}})$.

However, this result may be significantly strengthened by replacing the Grassmann manifold $\mathrm{Gr}_b(V^{\mathbb{C}})$ with its "regular" part, namely, the Hilbert–Schmidt Grassmanian $\mathrm{Gr}_{\mathrm{HS}}(V)$, introduced in Sec. 5.2.

We recall that this Grassmanian $\mathrm{Gr}_{\mathrm{HS}}(V)$ consists of closed subspaces $W \subset V$ such that the orthogonal projection $\mathrm{pr}_+ : W \to W_+$ is a Fredholm operator, while the orthogonal projection $\mathrm{pr}_- : W \to W_-$ is a Hilbert–Schmidt operator. It was shown in Sec. 5.2 that $\mathrm{Gr}_{\mathrm{HS}}(V)$ is a Kähler Hilbert manifold, having as its local model the Hilbert space $\mathrm{HS}(W_+, W_-)$ of Hilbert–Schmidt operators. Recall (cf. Sec. 5.2) that $\mathrm{Gr}_{\mathrm{HS}}(V)$ is a homogeneous space of the Hilbert–Schmidt unitary group $\mathrm{U}_{\mathrm{HS}}(V)$, more precisely

$$\mathrm{Gr}_{\mathrm{HS}}(V) = \mathrm{U}_{\mathrm{HS}}(V)/\,\mathrm{U}(W_+) \times \mathrm{U}(W_-)\ .$$

We introduce now, by analogy with the group $\mathrm{U}_{\mathrm{HS}}(V)$, the *Hilbert–Schmidt symplectic group* $\mathrm{Sp}_{\mathrm{HS}}(V)$. Recall that the symplectic group $\mathrm{Sp}(V)$ consists of bounded linear operators $A : V^{\mathbb{C}} \to V^{\mathbb{C}}$, having the block representations of the form

$$A = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}\ ,$$

where

$$\bar{a}^t a - b^t \bar{b} = 1\ ,\ \ \bar{a}^t b = b^t \bar{a}\ .$$

By definition, the group $\mathrm{Sp}_{\mathrm{HS}}(V) \subset \mathrm{Sp}(V)$ consists of transformations $A \in \mathrm{Sp}(V)$, for which the operator b is Hilbert–Schmidt. The unitary group $\mathrm{U}(W_+)$ is contained in $\mathrm{Sp}_{\mathrm{HS}}(V)$ as a subgroup

$$\mathrm{U}(W_+) \ni a \longmapsto A = \begin{pmatrix} a & 0 \\ 0 & \bar{a} \end{pmatrix}\ .$$

The diffeomorphism group $\mathrm{Diff}_+(S^1)$ acts on the space V by symplectic transformations, given by the same formula, as in Sec. 11.4:

$$T_f(\xi) := \xi \circ f - \frac{1}{2\pi} \int_0^{2\pi} \xi\,(f(\theta))\, d\theta\ .$$

As before, the transformation T_f preserves the subspace $W_+ \subset V$ if and only if $f \in \mathrm{M\ddot{o}b}(S^1)$, and in this case $T_f \in \mathrm{U}(W_+)$. The correspondence $f \mapsto T_f$ defines an embedding

$$\mathcal{S} \hookrightarrow \mathrm{Sp}_{\mathrm{HS}}(V)/\mathrm{U}(W_+)\ .$$

Moreover, the following result is true.

Proposition 26 ([57]). *The map*

$$\mathcal{S} \hookrightarrow Sp_{HS}(V)/\,U(W_+) = Gr_{HS}(V)$$

is an equivariant holomorphic embedding.

By analogy with Sec. 11.4, we identify the space $\mathrm{Sp}_{\mathrm{HS}}(V)/\mathrm{U}(W_+)$ with the space $\mathcal{J}_{\mathrm{HS}}(V)$ of admissible complex structures on V, compatible with the symplectic form

ω. As in the previous Section, it has a natural realization as a *Hilbert–Schmidt Siegel disc*

$$\mathcal{D}_{\mathrm{HS}} = \{Z : W_+ \to W_- \text{ is a symmetric Hilbert–Schmidt operator with } \bar{Z}Z < I\} .$$

So, the above Proposition yields a holomorphic embedding

$$\mathcal{S} \hookrightarrow \mathrm{Sp}_{\mathrm{HS}}(V)/\mathrm{U}(W_+) = \mathcal{D}_{\mathrm{HS}} .$$

There is another interpretation of the space $\mathcal{S}$ as the space of complex structures, namely, as the space of admissible complex structures on the loop space ΩG.

There is a natural action of the diffeomorphism group of the circle $\mathrm{Diff}_+(S^1)$ on the loop group LG by the reparametrization of loops. It is given by the formula

$$f_*\gamma(\theta) := \gamma\left(f(\theta)\right) - \frac{1}{2\pi}\int_0^{2\pi} \gamma\left(f(\theta)\right) d\theta$$

for $\gamma \in LG$, $f \in \mathrm{Diff}_+(S^1)$. By identifying ΩG with the subgroup $L_1(G)$, it's evident that this action can be pushed down to the action of $\mathrm{Diff}_+(S^1)$ on the loop space ΩG.

From the definition of the symplectic structure ω on ΩG, generated by the form

$$\omega_0(\xi, \eta) = \frac{1}{2\pi}\int_0^{2\pi} < \xi(e^{i\theta}), \eta'(e^{i\theta}) > d\theta \ ,$$

on $L\mathfrak{g}$, it's clear (by the change of variables in the integral) that diffeomorphisms from $\mathrm{Diff}_+(S^1)$ preserve ω, i.e. generate symplectomorphisms of the manifold ΩG.

The complex structure J^0 on ΩG is given at the origin $o \in \Omega G$ by the formula

$$\xi = \sum_{k\neq 0} \xi_k z^k \in \Omega\mathfrak{g}^{\mathbb{C}} \Longrightarrow J^0_o\xi = -i\sum_{k>0}\xi_k z^k + i\sum_{k<0}\xi_k z^k \ ,$$

so the tangent subspaces, consisting of vectors of the type $(1,0)$ and $(0,1)$, have the form

$$T^{1,0}_o(\Omega G) = \{\xi = \sum_{k<0}\xi_k z^k \in \Omega\mathfrak{g}^{\mathbb{C}}\}, \quad T^{0,1}_o(\Omega G) = \{\xi = \sum_{k>0}\xi_k z^k \in \Omega\mathfrak{g}^{\mathbb{C}}\} .$$

A diffeomorphism $f \in \mathrm{Diff}_+(S^1)$ transforms the complex structure J^0 into the complex structure

$$J_f := f_*^{-1} \circ J^0 \circ f_* \ ,$$

where f_* is the tangent map to f.

Proposition 27. *The complex structure J_f with $f \in \mathit{Diff}_+(S^1)$ coincides with the original complex structure J_0 if and only if $f \in \mathit{Möb}(S^1)$.*

Proof. If the diffeomorphism $f \in \mathrm{Diff}_+(S^1)$ does not change the original complex structure, i.e. defines a biholomorphism of ΩG, provided with the complex structure J_0, it means , in particular, that it preserves the tangent space $T^{0,1}_o(\Omega G)$. Hence, such a diffeomorphism should preserve the subspace $L^+G^{\mathbb{C}}$, implying that it extends to a biholomorphism of the unit disc Δ. So, $f \in \mathrm{Möb}(S^1)$. The converse assertion is obvious. ∎

We shall call the complex structures J_f on ΩG, obtained from J^0 by the action of the diffeomorphism group

$$J = f_*^{-1} \circ J^0 \circ f_* \ ,$$

the *admissible* complex structures on ΩG. The Proposition 27 implies that the space of admissible complex structures on ΩG can be identified with the manifold $\mathcal{S}$.

Recall that the complex structure J^0 on ΩG is invariant under the left LG-translations on the space ΩG and compatible with the symplectic structure ω (in the sense of Def. 17 from Sec. 1.2.5). Due to the invariance of ω with respect to the action of the group $\mathrm{Diff}_+(S^1)$, the complex structures J_f are also invariant under the left LG-translations and compatible with ω. In particular, any such complex structure J defines a *Kähler metric* g_f on ΩG by the formula

$$g_f(\xi,\eta) := \omega(\xi, J_f\eta)$$

for any $\xi,\eta \in T_\gamma(\Omega G)$, $\gamma \in \Omega G$.

Consider now the space $\mathcal{R} = \mathrm{Diff}_+(S^1)/(S^1)$. Combining the above embedding

$$\mathcal{S} \hookrightarrow \mathrm{Sp}_{\mathrm{HS}}(V)/\mathrm{U}(W_+) = \mathcal{D}_{\mathrm{HS}}$$

with the holomorphic map

$$\mathcal{R} = \mathrm{Diff}_+(S^1)/(S^1) \longrightarrow \mathcal{S} \ ,$$

we obtain the Grassmann realization of the space $\mathcal{R} = \mathrm{Diff}_+(S^1)/(S^1)$:

$$\mathcal{R} \longrightarrow \mathrm{Sp}_{\mathrm{HS}}(V)/\mathrm{U}(W_+) = \mathcal{D}_{\mathrm{HS}} \ .$$

As in the case of $\mathcal{S}$, the space $\mathcal{R}$ can be also considered as a space of complex structures on the loop space ΩG. Recall that the loop space ΩG, provided with the complex structure J_0, admits the following complex homogeneous representation

$$\Omega G = LG^{\mathbb{C}} \big/ L_+G^{\mathbb{C}} \ .$$

According to Birkhoff theorem (cf. Sec. 7.3), we can identify a neighborhood of the origin in ΩG with a neighborhood of the identity in the loop subgroup $L_1^-G^{\mathbb{C}}$. If a diffeomorphism $f \in \mathrm{Diff}_+(S^1)$ fixes the origin in ΩG and generates a biholomorphism of

$$(\Omega G, J_0) = LG^{\mathbb{C}} \big/ L_+G^{\mathbb{C}} \ ,$$

it generates also a biholomorphism of $L_1^-G^{\mathbb{C}}$. In this case we shall say that the complex structure J_f, associated with $f \in \mathrm{Diff}_+(S^1)$, is *equivalent* to the original complex structure J_0.

Proposition 28. *The complex structure J_f with $f \in \mathit{Diff}_+(S^1)$ is equivalent to the original complex structure J_0 in the above sense if and only if f is a rotation, i.e. $f \in S^1$.*

Proof. If the diffeomorphism $f \in \mathrm{Diff}_+(S^1)$ generates a biholomorphism of

$$(\Omega G, J_0) = LG^{\mathbb{C}} \big/ L_+G^{\mathbb{C}} \ ,$$

fixing the origin, then it leaves the subspace $L^+G^{\mathbb{C}}$ invariant and generates a biholomorphism of $L_1^-G^{\mathbb{C}}$. The first property implies that f extends to a biholomorphism of the unit disc Δ, while the second one implies that f extends to a biholomorphism of its exterior Δ_-, fixing the infinity. Then, by Liouville theorem, $f \in S^1$. □

Bibliographic comments

A key reference for this Chapter is the Nag's book [56]. Most of the assertions in Sec. 11.1, 11.2, 11.3 may be found there. Prop. 21 is proved in the paper [59]. The Grassmann approach to the study of the universal Teichmüller space was initiated by Nag–Sullivan's paper [58]. All assertions from Sec. 11.4 may be found there. Prop. 26 is proved in [57].

Part IV

QUANTIZATION OF FINITE-DIMENSIONAL KÄHLER MANIFOLDS

Chapter 12

Dirac quantization

This Chapter is devoted to the Dirac definition of the geometric quantization of classical mechanical systems. In Sec. 12.1 we recall the notion of classical systems from Hamiltonian mechanics. The geometric quantization of such systems is defined in Sec. 12.2.

12.1 Classical systems

We start from the definition of a classical (mechanical) system — an object to be quantized. A *classical (mechanical) system* is given by a pair $(M, \mathcal{A})$, consisting of the *phase space* M of the system and the *algebra of observables (Hamiltonians)* $\mathcal{A}$.

12.1.1 Phase spaces

Mathematically, the *phase manifold* M is a smooth symplectic manifold of an even dimension $2n$, provided with a symplectic 2-form ω. Locally, it is diffeomorphic (and, in fact, symplectomorphic) to the *standard model* $M_0 := (\mathbb{R}^{2n}, \omega_0)$, where ω_0 is the standard symplectic form on $\mathbb{R}^{2n}$. In the conventional coordinates (p_i, q_i), $i = 1, \dots, n$, on $\mathbb{R}^{2n}$ this form is given by the expression

$$\omega_0 = \sum_{i=1}^{n} dp_i \wedge dq_i \ .$$

The corresponding local coordinates (p_i, q_i), $i = 1, \dots, n$, on M, in which the symplectic form ω takes on the above standard form, are called *canonical*. The coordinates q_i are interpreted as physical "coordinates", while p_i correspond to physical "momenta".

The standard examples of phase spaces, apart from the standard model $M_0 = (\mathbb{R}^{2n}, \omega_0)$, are given by the cotangent bundles and coadjoint orbits of Lie groups.

Example 30 (cotangent bundles)**.** Denote by M the cotangent bundle T^*N of a smooth n-dimensional manifold N, called the *configuration space*. Local canonical coordinates (p_i, q_i) on M have the following meaning: $q := (q_1, \dots, q_n)$ are local coordinates on N, and $p := (p_1, \dots, p_n)$ are coordinates in the fibre T_qN. A symplectic

2-form ω, given in the introduced local coordinates by the standard formula

$$\omega = \sum_{i=1}^{n} dp_i \wedge dq_i \ ,$$

is a correctly defined (global) 2-form on M, as well as a 1-form θ, given in local coordinates by the expression

$$\theta = \sum_{i=1}^{n} p_i dq_i \ .$$

It follows that $\omega = d\theta$, that is ω in this case is exact. To show that θ is a correctly defined (global) 1-form, we note that it can be also defined in an invariant way. Namely, for any $p \in T_q^*N$ and any tangent vector $\xi \in T_{(p,q)}(T^*N)$ it can be given by

$$\theta(\xi) = p(\pi_*\xi) \ ,$$

where $\pi_* : T(T^*N) \to TN$ is the map, tangent to the projection $\pi : T^*N \to N$.

Example 31 (coadjoint orbits)**.** Consider the coadjoint representation of a Lie group G on the dual space $\mathfrak{g}^*$ to the Lie algebra $\mathfrak{g}$ of G. It is given by the formula

$$K : G \longrightarrow \operatorname{End}\mathfrak{g}^* \ , \quad g \longmapsto \left(\operatorname{Ad} g^{-1}\right)^* \ .$$

The orbits of this action (when they are smooth) are symplectic manifolds with the symplectic structure, given by the *Kirillov form*, defined in the following way. Denote by ξ_* the vector field on $\mathfrak{g}^*$, generated by $\xi \in \mathfrak{g}$ via the coadjoint action K. More precisely,

$$\xi_*(x) = K_*(\xi)x \quad \text{for } x \in \mathfrak{g}^* \ ,$$

where $K_* : \mathfrak{g} \to \operatorname{End}\mathfrak{g}^*$ denotes the differential of $K : G \to \operatorname{End}\mathfrak{g}^*$. Then the Kirillov form is defined by the equality

$$\omega(\xi_*(x), \eta_*(x)) := x\big([\xi, \eta]\big) \quad \text{for } \xi, \eta \in \mathfrak{g},\ x \in \mathfrak{g}^* \ .$$

The restriction of this 2-form to a smooth K-orbit defines a symplectic structure on this orbit.

12.1.2 Algebras of observables

An *algebra of observables* $\mathcal{A}$, mathematically, is an arbitrary Lie subalgebra of the Poisson Lie algebra $C^\infty(M, \mathbb{R})$ of smooth real-valued functions on the phase space M with respect to the Poisson bracket, determined by the symplectic 2-form ω.

Recall the definition of this bracket. Given a smooth function $h \in C^\infty(M, \mathbb{R})$, denote by X_h the Hamiltonian vector field on M, associated with h. It is determined by the following relation

$$dh(\xi) = \omega(X_h, \xi) \ ,$$

fulfilled for any vector field ξ on M. Then the Poisson bracket $\{f, g\}$ of two functions $f, g \in C^\infty(M, \mathbb{R})$ is uniquely defined by the relation

$$X_{\{f,g\}} = [X_f, X_g] \ .$$

Example 32 (Heisenberg algebra)**.** In the case of the standard model $M_0 = (\mathbb{R}^{2n}, \omega_0)$ we can take for the algebra of observables $\mathcal{A}$ the *Heisenberg algebra* $\mathrm{heis}(\mathbb{R}^{2n})$. It is the Lie algebra, generated by the coordinate functions p_i, q_i, $i = 1, \ldots, n$ and 1, satisfying the following commutation relations

$$\{p_i, p_j\} = \{q_i, q_j\} = 0 \ ,$$
$$\{p_i, q_j\} = \delta_{ij} \quad \text{for } i, j = 1, \ldots, n \ .$$

We consider $\mathrm{heis}(\mathbb{R}^{2n})$ as a "minimal" algebra of observables on M_0. The opposite extreme is the Poisson algebra $C^\infty(M_0, \mathbb{R})$. The Hamiltonian vector field X_f, corresponding to an observable $f \in C^\infty(M_0, \mathbb{R})$, is given in standard coordinates (p_i, q_i) on M_0 by the formula

$$X_f = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial p_i} \frac{\partial}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial}{\partial p_i} \right) \ .$$

In particular, $X_{p_i} = \frac{\partial}{\partial q_i}$, $X_{q_i} = -\frac{\partial}{\partial p_i}$. The Poisson bracket on M_0 is given by the expression

$$\{f, g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} \right)$$

for $f, g \in C^\infty(M_0, \mathbb{R})$.

Example 33 (Hamiltonian algebra)**.** Let Γ be a Lie group of symplectomorphisms, acting on a phase space M, so that its Lie algebra $\mathrm{Lie}(\Gamma)$ can be regarded as a subalgebra of the Lie algebra of Hamiltonian vector fields on M. If M is simply connected, then $\mathrm{Lie}(\Gamma)$ may be also considered, in the dual way, as a subalgebra of the Poisson algebra $C^\infty(M, \mathbb{R})$. Namely, it can be identified with the algebra $\mathrm{Ham}(\Gamma)$ of Hamiltonians (smooth real functions) on M, generating symplectomorphisms from Γ.

If a Lie group Γ acts on M transitively, such a manifold M is called a *homogeneous symplectic* Γ*-manifold*. It is proved in [46] that any homogeneous symplectic Γ-manifold M is locally equivariantly symplectomorphic to a coadjoint orbit of Γ or its central extension $\tilde{\Gamma}$.

12.2 Quantization of classical systems

Definition 39. Let $(M, \mathcal{A})$ be a classical system. The *Dirac quantization* of $(M, \mathcal{A})$ is given by an irreducible Lie-algebra representation

$$r : \mathcal{A} \longrightarrow \mathrm{End}^* H$$

of the algebra of observables $\mathcal{A}$ in the algebra $\mathrm{End}^* H$ of linear self-adjoint operators, acting in a complex (separable) Hilbert space H, called the *quantization space*. The algebra $\mathrm{End}^* H$ is provided with the Lie bracket, given by the commutator of linear operators of the form

$$\frac{\hbar}{i}[A, B] = \frac{\hbar}{i}(AB - BA) \ .$$

In other words, it is required that

$$r\left(\{f, g\}\right) = \frac{\hbar}{i}\left(r(f)r(g) - r(g)r(f)\right)$$

for any $f, g \in \mathcal{A}$. We also assume the following normalization condition:

$$r(1) = \mathrm{id}\ .$$

If a representation r satisfies all these conditions, except for the irreducubility, it is called a *prequantization* of the system $(M, \mathcal{A})$.

We set $\hbar = 1$ in the sequel for the convenience.

Remark 19. Sometimes it is useful to deal with the *complexified algebra of observables* $\mathcal{A}^{\mathbb{C}}$ instead of $\mathcal{A}$. Its Dirac quantization is given by an irreducible Lie-algebra representation

$$r : \mathcal{A}^{\mathbb{C}} \longrightarrow \mathrm{End}\, H\ ,$$

satisfying the normalization condition and the *conjugation law*

$$r(\bar{f}) = r(f)^* \quad \text{for any } f \in \mathcal{A}\ .$$

In other words, the complex conjugation in $\mathcal{A}^{\mathbb{C}}$ should correspond to the Hermitian conjugation in $\mathrm{End}\, H$.

Remark 20. The quantization operators $r(f)$ in the Dirac definition are usually unbounded. In that case we require that all operators $r(f)$ for $f \in \mathcal{A}$ (or $f \in \mathcal{A}^{\mathbb{C}}$ in the complexified version) are densely defined and, moreover, have a common dense domain of definition in H.

Bibliographic comments

The Dirac definition of geometric quantization of classical systems is presented (with minor modifications) in all books on geometric quantization. A reader may look for a more detailed exposition [29, 37, 42, 70, 79].

Chapter 13

Kostant–Souriau prequantization

It is difficult (and, often, not possible) to construct the Dirac quantization, defined in the previous Chapter, for realistic classical systems. However, there exists a quite general prequantization construction, due to Kostant and Souriau, which is valid for a large class of phase spaces and the "maximal" algebra of observables $\mathcal{A} = C^\infty(M, \mathbb{R})$. We describe it in this Chapter, starting from the simple case of the cotangent bundle.

13.1 Prequantization of the cotangent bundle

Let N be a smooth n-dimensional manifold and $M = T^*N$ denotes its cotangent bundle. Recall (cf. Ex. 30) that the symplectic form ω on T^*N is given by the formula $\omega = d\theta$, where θ is a canonically defined 1-form on M with the local expression $\theta = \sum_{i=1}^n p_i dq_i$. We take for an algebra of observables $\mathcal{A}$ of our system the Poisson algebra $C^\infty(M, \mathbb{R})$ and for the Hilbert prequantization space H the space

$$H = L^2(M, \omega^n)$$

of square integrable functions on M with respect to the Liouville measure, given by ω^n. A representation of $\mathcal{A} = C^\infty(M, \mathbb{R})$ in H is given by the following formula

$$r : f \longmapsto r(f) = f - iX_f - \theta(X_f) \ , \tag{13.1}$$

where $f - \theta(X_f)$ is considered as the multiplication operator on H. Note that this operator, as well as the Hamiltonian vector field X_f, are correctly defined on the subspace $C_0^\infty(M, \mathbb{R})$ of $C^\infty(M, \mathbb{R})$, consisting of smooth functions with compact supports on M.

In particular, for the standard model $N = \mathbb{R}^n$, $M = T^*\mathbb{R}^n \cong \mathbb{R}^{2n}$ the representation (13.1) acts on the coordinate functions in the following way

$$r(p_j) = p_j - iX_{p_j} - \theta(X_{p_j}) = p_j - i\frac{\partial}{\partial q_j} - p_j = -i\frac{\partial}{\partial q_j} \ , \tag{13.2}$$

$$r(q_j) = q_j - iX_{q_j} - \theta(X_{q_j}) = q_j - i\left(-\frac{\partial}{\partial p_j}\right) = q_j + i\frac{\partial}{\partial p_j} \ , \tag{13.3}$$

since $X_{p_j} = \partial/\partial q_j$, $X_{q_j} = -\partial/\partial p_j$. Note that this representation is reducible, even if we restrict it to the "minimal" Heisenberg algebra heis$(\mathbb{R}^{2n})$. Indeed, the operators $i\frac{\partial}{\partial p_j}$ and $p_j + i\frac{\partial}{\partial q_j}$ commute with all operators $r(p_j)$, $r(q_j)$, being non scalar. However, we can make the representation of heis($\mathbb{R}^{2n}$), defined by the above formulas (13.2),(13.3), irreducible by restricting it to the subspace of H, consisting of functions, depending only on (q_j). Then the representation (13.2),(13.3) will reduce to the well known *Heisenberg representation* of heis($\mathbb{R}^{2n}$) in the space $H_{(q)} := L^2(\mathbb{R}^n_{(q)}, d^n q)$, given by

$$r(p_j) = -i\frac{\partial}{\partial q_j} \ , \quad r(q_j) = q_j \ .$$

We can also construct a *dual Heisenberg representation* of heis($\mathbb{R}^{2n}$) in the space $H_{(p)} := L^2(\mathbb{R}^n_{(p)}, d^n p)$, given by

$$r(p_j) = p_j \ , \quad r(q_j) = i\frac{\partial}{\partial p_j} \ .$$

Remark 21. The "physical" explanation of the reducibility of the representation

$$r : \text{heis}(\mathbb{R}^{2n}) \longrightarrow \text{End}^* H \ ,$$

given by (13.1), is that, according to the Heisenberg uncertainty principle, the "physical" quantization space cannot contain the functions, depending on some pair of variables (p_j, q_j) simultaneously, as it occurs in the space $H = L^2(M, \omega^n)$.

13.2 Kostant–Souriau (KS) prequantization

13.2.1 Prequantization map

Suppose now that M is a general smooth symplectic manifold of dimension $2n$ with symplectic form ω. Take the Poisson algebra $C^\infty(M, \mathbb{R})$ as the algebra of observables. We are going to quantize the classical system, represented by the pair $(M, C^\infty(M, \mathbb{R}))$.

Let us begin with some heuristic considerations. Note that the symplectic 2-form ω, being closed, is locally exact, so we can find an open covering $\{U_\alpha\}$ of M, such that

$$\omega = d\theta_\alpha \quad \text{on } U_\alpha$$

for some smooth 1-forms θ_α, defined on U_α. Using these local forms θ_α, we can apply the idea, described in the previous Section 13.1, to construct local representation operators r_α in the spaces $L^2(U_\alpha, \omega^n)$ by the formula (13.1) with $\theta = \theta_\alpha$. It turns out that (under some topological restrictions) we can combine these local representation operators r_α into a unique operator r, which acts, however, not on functions, but on sections of a certain complex line bundle L over M. The structure of this line bundle $L \to M$ is, in fact, prescribed by the local formulas (13.1) with $\theta = \theta_\alpha$. Namely, the local expressions $X^\alpha - i\theta_\alpha(X^\alpha)$ (with X^α being a vector field on U_α) in the right hand sides of the local formulas (13.1) look like local expressions for the covariant derivative of a connection in a line bundle over M. If these expressions do arise from some connection ∇ on a line bundle $L \to M$ (i.e. if they match together on intersections $U_\alpha \cap U_\beta$ up to gauge transformations, given by the transition functions of L), then the local representation operators $r_\alpha(f)$ in the spaces $L^2(U_\alpha, \omega^n)$ will match into a global representation operator

$$r : f \longmapsto f - i\nabla_{X_f} \ , \quad f \in C^\infty(M, \mathbb{R}) \ ,$$

acting on sections of $L \to M$. In this case the curvature of such a connection would be equal to ω. In particular, the 2-form $\frac{1}{2\pi}\omega$, representing the first Chern class $c_1(L)$, should be integral, i.e.

$$\left[\frac{1}{2\pi}\omega\right] \in H^2(M, \mathbb{Z}) \subset H^2(M, \mathbb{R}) \ .$$

From Sec. 8.1 we know that the integrality of $[\frac{1}{2\pi}\omega]$ is not only necessary, but also sufficient for the existence of a line bundle $L \to M$ with a connection ∇. Namely, rephrasing Prop. 15, we have the following

Proposition 29. *Suppose that the manifold M satisfies the following quantization condition: the cohomology class*

$$\left[\frac{1}{2\pi}\omega\right] \quad \textit{is integral in } H^2(M, \mathbb{R}) \ . \tag{13.4}$$

Then there exists a Hermitian line bundle $L \to M$, called the prequantization bundle*, having a Hermitian connection ∇, whose curvature is equal to ω.*

Proof. The only new assertion in this Proposition, compared to Prop. 15, is the Hermiticity of the connection ∇. Recall (cf. Rem. 16) that under the integrality condition (13.4) there exists a complex line bundle $L \to M$, such that $c_1(L) = [\omega/2\pi]$. We take now an arbitrary Hermitian metric and a Hermitian connection ∇' on L. Note that the curvature ω' of ∇' also represents the class $c_1(L)$. Hence,

$$\omega = 2\pi\omega' + d\beta$$

for some 1-form $\beta \in \Omega^1(M, \mathbb{R})$. If the connection ∇' is represented by a 1-form α', we introduce a connection ∇ on L, represented by the 1-form

$$\alpha = 2\pi\alpha' - i\beta \ .$$

This connection is Hermitian and its curvature is equal to ω. □

The Prop. 29 allows us to realize the scheme, described in the beginning of this Section. Namely, suppose that our phase space M satisfies the quantization condition, so that the assertion of Prop. 29 holds. Then there exists a Hermitian line bundle $L \to M$ together with a Hermitian connection ∇, having the curvature, equal to ω. We take for the algebra of observables the Poisson algebra $\mathcal{A} = C^\infty(M, \mathbb{R})$ and define the prequantization space as

$$H = L^2(M, L; \omega^n) \ ,$$

i.e. the Hilbert space of square integrable sections of $L \to M$ with respect to the inner product, given by

$$(s_1, s_2)_H := \int_M < s_1(x), s_2(x) > \omega^n \ ,$$

where $< s_1(x), s_2(x) >$ is the Hermitian product of sections s_1, s_2 of L at $x \in M$. Then the *Kostant-Souriau (KS) prequantization* of the algebra $\mathcal{A}$ in H will be given by the formula

$$r_{\mathrm{KS}} : \mathcal{A} \ni f \longmapsto r(f) = f - i\nabla_{X_f} \ . \tag{13.5}$$

It's easy to check directly (cf. also [29, 37, 42, 70, 72, 79]) that the formula (13.5) defines a representation of the algebra $\mathcal{A} = C^\infty(M, \mathbb{R})$ in the prequantization space H.

Remark 22. There is another interpretation of the Kostant–Souriau operator r_{KS} in terms of the automorphism group $\tilde{\mathcal{G}}$ of the prequantization bundle (L, ∇). An *automorphism* of (L, ∇) is a pair (φ, g), where $\varphi : L \to L$ is a fibrewise isomorphism, preserving the Hermitian metric on L and the connection ∇ (i.e. $\varphi^*\nabla = \nabla$). The projection of φ to M is a symplectomorphism $g : M \to M$, belonging to the group $\mathcal{G}$ of all symplectomorphisms of M. In other words, we have a commutative diagram

$$\begin{array}{ccc} L & \xrightarrow{\varphi} & L \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ M & \xrightarrow[g]{} & M \end{array} \ .$$

According to Prop. 16, the automorphism group $\tilde{\mathcal{G}}$ of the prequantization bundle (L, ∇) can be identified with a central extension of the symplectomorphism group $\mathcal{G}$ by S^1, i.e. there is an exact sequence

$$1 \longrightarrow S^1 \longrightarrow \tilde{\mathcal{G}} \longrightarrow \mathcal{G} \longrightarrow 1 \ .$$

Note that (assuming that M is simply connected) the Lie algebra $\mathrm{Lie}\,\mathcal{G}$ of the group $\mathcal{G}$ can be identified with the Lie algebra of Hamiltonian vector fields on M, generated by Hamiltonians $f \in C^\infty(M, \mathbb{R})$, so that that the Lie algebra $\mathrm{Lie}\,\tilde{\mathcal{G}}$ of the group $\tilde{\mathcal{G}}$ is a central extension of $\mathrm{Lie}\,\mathcal{G}$ by $\mathbb{R}$.

The action of the symplectomorphism group $\mathcal{G}$ on M generates an action of its central extension $\tilde{\mathcal{G}}$ on L. Namely, if an action g on M is generated by a Hamiltonian vector field X_f with $f \in C^\infty(M, \mathbb{R}) = \mathrm{Lie}\,\mathcal{G}$, then the corresponding action $\varphi : C^\infty(M, L) \to C^\infty(M, L)$ on the space of sections of L is generated by

$$\tilde{X}_f(s) := fs - i\nabla_{X_f} s \ . \tag{13.6}$$

Remark 23. In conclusion of this Subsection, we give a description of the $\mathbb{C}^*$-bundle $\dot{L} \to M$, associated with the prequantization bundle $L \to M$. It is sometimes more convenient to use for computations this bundle, rather than $L \to M$. Denote by $\pi : \dot{L} \to M$ the bundle, obtained from the prequantization bundle $\pi : L \to M$ by deleting its zero section. It is a principal $\mathbb{C}^*$-bundle, associated with the line bundle

$\pi : L \to M$. The space $\Gamma(L) := C^\infty(M, L)$ of sections s of $L \to M$ can be identified with the space $\dot\Gamma(L)$ of complex-valued functions $\dot s$ on $\dot L$, subject to the condition

$$\dot s(zp) = \frac{1}{z}\dot s(p)$$

for any $p \in \dot L$ and any $z \in \mathbb{C}^*$. The correspondence between sections s of $L \to M$ and functions $\dot s$ on $\dot L \to M$ is established via the relation

$$s(\pi(p)) = \dot s(p)p \quad \text{for any } p \in \dot L \ .$$

Note that if a section s of $L \to M$ is non-vanishing at some point $x \in M$: $s(x) \neq 0$, then $s(x) \in \dot L$ and, applying the above relation for $p = s(x)$, we obtain that $s(x) = \dot s(s(x))s(x)$, i.e. $\dot s \circ s = 1$ at any point $x \in M$, where $s(x) \neq 0$.

We can introduce a connection $\dot\nabla$ on $\dot L \to M$, associated with the connection ∇ on $L \to M$. In terms of the local representatives θ_α of the connection ∇, the local representatives $\dot\theta_\alpha$ of $\dot\nabla$ are given by

$$\dot\theta_\alpha = \theta_\alpha + i\frac{dz}{z}$$

on $U_\alpha \times \mathbb{C}^*$. It's easy to check that these local forms define a global 1-form, which is the connection form of $\dot\nabla$. This connection generates the horizontal lifting of vector fields on M. Let ξ be such a vector field, then its horizontal lift is a vector field $\dot\xi$ on $\dot L$, such that $\pi_*(\dot\xi) = \xi$ and $\dot\nabla(\dot\xi) = 0$. A correspondence $\xi \leftrightarrow \dot\xi$ between vector fields ξ on M and their horizontal lifts $\dot\xi$ on $\dot L$ has the following properties

$$(\nabla_\xi s)^\cdot = \dot\xi \ , \quad (fs)^\cdot = f\dot s$$

for any vector field ξ on M, section s of L and function $f \in C^\infty(M, \mathbb{R})$.

We can also give an interpretation of the generator (13.6) in terms of the bundle $\dot L$ (cf. [72]). Given a Hamiltonian $f \in C^\infty(M, \mathbb{R})$, we define a vector field η_f on $\dot L$ by local representatives

$$\eta_{f,\alpha} := X_f + (\theta_\alpha(X_f) - f)\,\frac{\partial}{\partial\vartheta}$$

on $U_\alpha \times \mathbb{C}^*$. Here the vector field $\frac{\partial}{\partial\vartheta}$ is the differentiation with respect to the angle coordinate ϑ in the polar representation of the coordinate $z = re^{i\vartheta}$ on $\mathbb{C}^*$. It follows from this definition that the generator (13.6) can be written in terms of $\dot L$ as

$$\tilde X_f(s) = -i\eta_f\dot s \ . \tag{13.7}$$

Remark 24. Using the vector field η_f, introduced in Rem. 23, one can prove that the KS-operator $r_{\mathrm{KS}}(f)$, given by the formula (13.5), is self-adjoint under the assumption that the Hamiltonian vector field X_f is complete. (In this case the vector field η_f is complete too.) Denote by $\dot\varphi^t_f$ the 1-parameter group of transformations of $\dot L$, generated by the vector field η_f. Consider the 1-parameter unitary group of transformations of $\dot\Gamma(L)$ (with respect to the inner product, induced from $\Gamma(L)$), generated by $\dot\varphi^t_f$. It acts by the formula: $\dot s \mapsto \dot s \circ \dot\varphi^t_f$ for $\dot s \in \dot\Gamma(L)$. The operator $r_{\mathrm{KS}}(f)$, given by (13.5), coincides with the generator of this unitary group, according to (13.7). Hence, it is self-adjoint by Stone's theorem. (This argument is due to [72].)

13.2.2 Polarizations

The representation of the algebra $\mathcal{A} = C^\infty(M, \mathbb{R})$ in the prequantization space H, defined by (13.5), is reducible by the same reasons, as in Sec. 13.1. According to the Heisenberg uncertainty principle, we can make this representation irreducible by restricting it to a "half" of the prequantization space H, i.e. to a subspace of H, containing the functions from H, which depend, in terms of the local canonical variables $(p_i, q_i)_{i=1}^n$, only on one variable from each pair (p_j, q_j). This naive idea may be formalized, using the notion of the polarization.

Let (M, ω) be a symplectic manifold of dimension $2n$. We extend its symplectic form ω complex linearly to the complexified tangent bundle $T^{\mathbb{C}}M$.

Definition 40. A *polarization* on M is an integrable involutive Lagrangian distribution P on M. In other words, P is a complex distribution $P : x \mapsto P_x \subset T^{\mathbb{C}}M$ of rank n, satisfying the following conditions: (a) P is involutive, i.e. $[P, P] \subset P$; (b) the restriction of ω to P is identically zero.

For a polarized phase space (M, P), satisfying the quantization condition (13.4), it's natural to choose for the quantization space H the space of *polarized* sections. It is defined as

$$H = L^2_P(M, L; \omega^n) := \{s \in L^2(M, L; \omega^n) : \ \nabla_\xi s = 0 \ \text{ for any } \xi \in P\} \ .$$

There are two distinguished classes of polarizations.

Example 34. A polarization P on a phase space M is called *real*, if $P = \bar{P}$, where "bar" denotes the complex conjugation in $T^{\mathbb{C}}M$. A standard example of such a polarization is the cotangent bundle $M = T^*N$ of a configuration manifold N with local canonical coordinates (p_i, q_i) and polarization P, given by the subbundle of TM, generated by the vector fields $\{\partial/\partial p_i\}$, $i = 1, \dots, n$. (One can take for P the subbundle of TM, generated by the vector fields $\{\partial/\partial q_i\}$, $i = 1, \dots, n$, as well.) The space $L^2_P(M, L; \omega^n)$ of polarized sections in this case consists of sections from $L^2(M, L; \omega^n)$, which do not depend on momenta $\{p_i\}$.

A polarization P is called *Kähler*, if $P \cap \bar{P} = 0$. To give an example of such a polarization, suppose that our phase space (M, ω) is Kähler, i.e. it is provided with a complex structure J, compatible with ω. Then we take for P the subbundle $T^{0,1}M$ of $(0,1)$-vector fields in $T^{\mathbb{C}}M$. In this case the prequantization bundle L can be made holomorphic with the holomorphic structure, determined by the $\bar\partial$-operator, given by the $(0,1)$-part $\nabla^{0,1}$ of the connection ∇. The space $L^2_P(M, L; \omega^n)$ of polarized sections for $P = T^{0,1}M$ coincides with the space $L^2_{\mathcal{O}}(M, L; \omega^n)$ of holomorphic sections of $L \to M$.

Given a polarized phase space (M, P), satisfying the quantization condition (13.4), we can hope to obtain an irreducible representation of the algebra of observables $\mathcal{A}$ by restricting the Kostant–Souriau prequantization map to the space $L^2_P(M, L; \omega^n)$ of polarized sections. Unfortunately, this straightforward idea works only for very special phase spaces and algebras of observables, since in most of the cases the space $L^2_P(M, L; \omega^n)$ of polarized sections is not invariant under the action of the Kostant–Souriau representation. In the next Section we shall demonstrate how the idea of restriction to the space of polarized sections can be realized for the flat space $\mathbb{R}^{2n} \cong \mathbb{C}^n$ and the Heisenberg algebra of observables $\mathrm{heis}(\mathbb{R}^{2n}) =: \mathrm{heis}(\mathbb{C}^n)$. In this case the restriction of Kostant–Souriau representation to the space $L^2_{\mathcal{O}}(\mathbb{C}^n; \omega^n)$ of holomorphic sections yields an irreducible Bargmann–Fock representation of the Heisenberg algebra in $L^2_{\mathcal{O}}(\mathbb{C}^n; \omega^n)$.

Bibliographic comments

The prequantization of the cotangent bundle was known long ago to physisists (cf., e.g., [35]). Its generalization to general manifolds, satisfying the quantization condition, due to B.Kostant and J.-M.Souriau, is presented in all books on geometric quantization (cf. [29, 37, 42, 70, 69, 79]). In these books a more detailed discussion of polarizations may be also found.

Chapter 14

Blattner–Kostant–Sternberg quantization

In this Chapter we present the Blattner–Kostant–Sternberg (BKS) quantization scheme for Kähler manifolds, provided with Kähler polarizations. We start from the simplest example of such a quantization, namely, the Bargmann–Fock quantization of the standard model $(\mathbb{R}^{2n}, \omega_0)$, provided with the Heisenberg algebra of observables. In Secs. 14.2-14.5 we explain how to construct the BKS-quantization of a quantizable Kähler manifold. In Sec. 14.2 we introduce the Fock spaces of half-forms and in Sec. 14.4 define the BKS-pairing between them, using the metaplectic structure, introduced in Sec. 14.3. In Sec. 14.5 we explain how to quantize Kähler phase manifolds, using the BKS-pairing.

14.1 Bargmann–Fock quantization

Let $M_0 = (\mathbb{R}^{2n}, \omega_0)$ be the standard model with standard coordinates (p_j, q_j), $j = 1, \ldots, n$. In these coordinates

$$\omega_0 = \sum_{j=1}^{n} dp_j \wedge dq_j \ ,$$

so that $\omega_0 = d\theta_0$ with $\theta_0 = \sum_{j=1}^{n} p_j \, dq_j$. We identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ by introducing complex coordinates

$$z_j = \frac{p_j + iq_j}{\sqrt{2}} \ , \quad \bar{z}_j = \frac{p_j - iq_j}{\sqrt{2}} \ , \quad j = 1, \ldots, n,$$

(following [70], we have replaced the usual factor $1/2$ in these formulas by $1/\sqrt{2}$ to make the expression for KS-representation more symmetric). In these coordinates

$$\omega_0 = -i \sum_{j=1}^{n} d\bar{z}_j \wedge dz_j \ .$$

The Hamiltonian vector fields, corresponding to coordinates $z_j, \bar{z}_j$, have the form

$$X_{z_j} = -i\frac{\partial}{\partial \bar{z}_j} = \frac{1}{\sqrt{2}i}\left(\frac{\partial}{\partial p_j} + i\frac{\partial}{\partial q_j}\right) \ , \quad X_{\bar{z}_j} = i\frac{\partial}{\partial z_j} = \frac{i}{\sqrt{2}}\left(\frac{\partial}{\partial p_j} - i\frac{\partial}{\partial q_j}\right) \ .$$

In particular, $i\omega(X_{z_j}, X_{\bar{z}_k}) = \delta_{jk}$. Evidently, the vector fields $\{X_{z_1}, \dots, X_{z_n}\}$ span the antiholomorphic tangent space $T^{0,1}(\mathbb{C}^n)$ (which is the Kähler polarization space in the sense of Ex. 34).

The prequantization bundle $L \to \mathbb{C}^n$ is the trivial bundle $\mathbb{C}^n \times \mathbb{C} \to \mathbb{C}^n$. We fix a trivializing section $\lambda_0 : \mathbb{C}^n \to L$ with $< \lambda_0, \lambda_0 >= 1$. The connection ∇ on L is determined by the property

$$\nabla \lambda_0 = -i \sum_{j=1}^{n} p_j \, dq_j \otimes \lambda_0 \ .$$

Following [70], we replace the trivializing section λ_0 by another trivializing section λ_1, given by

$$\lambda_1 := \exp\left(-\frac{1}{4} \sum_{j=1}^{n} (q_j^2 + p_j^2 - 2ip_j q_j) \right) \lambda_0 \ .$$

Then

$$\nabla \lambda_1 = \theta_1 \otimes \lambda_1 \quad \text{with} \quad \theta_1 = -i \sum_{j=1}^{n} \bar{z}_j dz_j \ .$$

In particular, the section λ_1 is covariantly constant along the vector fields from $T^{0,1}(\mathbb{C}^n)$. Hence, any section of L, covariantly constant along $T^{0,1}(\mathbb{C}^n)$, have the form

$$\varphi(z)\lambda_1 \ ,$$

where $\varphi(z)$ is a holomorphic function of $z \in \mathbb{C}^n$. We also have

$$< \lambda_1, \lambda_1 >= \exp\left(-\frac{1}{2} \sum_{j=1}^{n} (q_j^2 + p_j^2) \right) = \exp\left(-|z|^2\right)$$

with $|z|^2 = \sum_j \bar{z}_j z_j$. The inner product in the prequantization space $H = L^2(\mathbb{C}^n, L; \omega_0^n)$ takes on the following form

$$(\varphi\lambda_1, \psi\lambda_1) = \int_{\mathbb{C}^n} \varphi(z)\bar{\psi}(z) e^{-|z|^2} \omega_0^n \ .$$

Following the idea, formulated at the end of Sec. 13.2, we define the quantization space to be the space of polarized sections $L^2_{\mathcal{O}}(\mathbb{C}^n, L; \omega_0^n)$. In our case it coincides with the *Bargmann–Fock space*

$$F(\mathbb{C}^n) = L^2_{\mathcal{O}}(\mathbb{C}^n, e^{-|z|^2/2})$$

of holomorphic square integrable functions on $\mathbb{C}^n$ with the Gaussian weight $e^{-|z|^2/2}$.

The Kostant–Souriau (KS)-operators, associated with observables from the Heisenberg algebra $\mathrm{heis}(\mathbb{R}^{2n}) = \mathrm{heis}(\mathbb{C}^n)$ by formula (13.5), leave the Bargmann–Fock space $F(\mathbb{C}^n)$ invariant and so admit a restriction to this space. To see that, we compute the KS-operators, corresponding to the coordinates $z_j, \bar{z}_j$:

$$r_{\mathrm{KS}}(z_j)(\varphi\lambda_1) = z_j \varphi\lambda_1 \ , \quad r_{\mathrm{KS}}(\bar{z}_j)(\varphi\lambda_1) = \frac{\partial \varphi}{\partial z_j} \lambda_1$$

for $j = 1, \dots, n$. Using the expression for the basis Hamiltonian vector fields, corresponding to coordinates and momenta:

$$X_{p_j} = \frac{i}{\sqrt{2}}\left(\frac{\partial}{\partial z_j} - \frac{\partial}{\partial \bar{z}_j}\right) \ , \quad X_{q_j} = -\frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial z_j} + \frac{\partial}{\partial \bar{z}_j}\right) \ ,$$

we get the expression for the KS-operators, corresponding to the generators of the Heisenberg algebra $\mathrm{heis}(\mathbb{R}^{2n})$:

$$r_{\mathrm{KS}}(p_j)(\varphi\lambda_1) = \frac{1}{\sqrt{2}}\left[\left(z_j + \frac{\partial}{\partial z_j}\right)\varphi\right]\lambda_1 \ , \quad r_{\mathrm{KS}}(q_j)(\varphi\lambda_1) = \frac{1}{\sqrt{2}i}\left[\left(z_j - \frac{\partial}{\partial z_j}\right)\varphi\right]\lambda_1 \ .$$

It is clear from this expression that these operators leave the Bargmann–Fock space invariant. So we can restrict our KS-representation to this space, obtaining a representation r_0 of the Heisenberg algebra $\mathrm{heis}(\mathbb{R}^{2n}) = \mathrm{heis}(\mathbb{C}^n)$ in the Bargmann–Fock space $F(\mathbb{C}^n) = L^2_{\mathcal{O}}(\mathbb{C}^n, e^{-|z|^2/2})$.

This representation, which is called the *Bargmann–Fock representation*, is already irreducible. The easiest way to see that is to use the so called *creation* and *annihilation operators* , given in this case by the formulae

$$a_j^* = r_{\mathrm{KS}}(z_j) = \text{ multiplication by } z_j \ , \quad a_j = r_{\mathrm{KS}}(\bar{z}_j) = \partial/\partial z_j \ ,$$

acting in the Bargmann–Fock space $F(\mathbb{C}^n)$. Denote by $\varphi_0 \equiv 1$ the so called *vacuum vector* in $F(\mathbb{C}^n)$. Note that the Bargmann–Fock space $F(\mathbb{C}^n) = L^2_{\mathcal{O}}(\mathbb{C}^n, e^{-|z|^2/2})$ is generated by vectors, obtained from φ_0 by the action of creation operators a_j^*, i.e. by vectors of the form

$$a_{j_1}^* \dots a_{j_k}^* \varphi_0 \ .$$

To show that the Bargmann–Fock representation r_0 is irreducible, suppose that we have an operator A in $F(\mathbb{C}^n)$, commuting with all creation and annihilation operators a_j^*, a_j of our representation. Then $A\varphi_0$ should be equal to $c\varphi_0$ for some constant c, since $A\varphi_0$ is annihilated by all annihilation operators $a_j = \partial/\partial z_j$. On the other hand,

$$A(a_{j_1}^* \dots a_{j_k}^* \varphi_0) = a_{j_1}^* \dots a_{j_k}^* (A\varphi_0) = c(a_{j_1}^* \dots a_{j_k}^* \varphi_0) \ .$$

These two properties imply that $A = c \cdot \mathrm{id}$, so, by Schur's lemma, the Bargmann–Fock representation r_0 is irreducible.

Unfortunately, the described method of quantization of the standard model $M_0 = (\mathbb{R}^{2n}, \omega_0) = (\mathbb{C}^n, \omega_0)$, provided with the Heisenberg algebra $\mathrm{heis}(\mathbb{R}^{2n}) = \mathrm{heis}(\mathbb{C}^n)$, does not apply to other Kähler phase spaces and polarizations, since the KS-prequantization operators do not preserve, in general, the Fock spaces of holomorphic sections. We describe this situation in more detail in the next Sec. 14.2.1.

14.2 Fock spaces of half-forms

14.2.1 KS-action on Fock spaces

Suppose that our phase space (M, ω) is a Kähler manifold, provided with a compatible complex structure J. Assume that (M, ω) satisfies the quantization condition

(13.4) and $L \to M$ is the prequantization bundle, provided with a Hermitian connection ∇. We introduce a holomorphic structure on L, which is determined by the $\bar{\partial}$-operator, given by the $(0,1)$-component $\nabla^{0,1}$ of the connection ∇ with respect to the complex structure J. The *Fock space*

$$F(M,J) := L^2_{\mathcal{O}}(M,L;\omega^n)$$

is the space of square integrable sections of $L \to M$, holomorphic with respect to the introduced holomorphic structure on L. Denote by $\mathcal{A}$ the Lie algebra of Hamiltonians, which can be identified (under the assumption that M is simply connected) with the Lie algebra of Hamiltonian vector fields on M. Any observable $f \in \mathcal{A}$ generates a (local) 1-parameter group Γ of symplectomorphisms of M, given by

$$\varphi^t_f := \exp(2\pi i t X_f) \ ,$$

where X_f is the Hamiltonian vector field, generated by f. As we have pointed out in Sec. 13.2 (cf. Rem. 22), the action of Γ can be lifted to the action of its central extension $\tilde{\Gamma}$ on L, and this lifted action is generated by the KS-operator $r(f) \equiv r_{\mathrm{KS}}(f)$. More precisely, the lifted action is given by

$$\Phi^t_f := \exp\left(2\pi i t r(f)\right) : L^2(M,L;\omega^n) \longrightarrow L^2(M,L;\omega^n) \ .$$

However, these operators do not preserve, in general, the Fock space $F(M,J)$, since Φ^t_f maps the Fock space $F(M,J)$ into the Fock space $F(M,J^t_f)$, associated with the transformed complex structure $J^t_f := \varphi^t_{f,\star} \circ J \circ \varphi^{-t}_{f,\star}$, which, in general, is not equivalent to J. When this happens, the corresponding KS-operator $r_{\mathrm{KS}}(f)$ does not admit a restriction to $F(M,J)$. If we still want in this case to construct a quantization of $(M,\mathcal{A})$, using the KS-operators, we need to find a method of canonical identification of Fock spaces $F(M,J)$ with different J. In other words, we are looking for a canonical unitary pairing between different Fock spaces $F(M,J)$.

A naive idea would be to have some sort of an integral pairing, given by

$$\int_M < s_1\,, s_2 > \omega^n$$

for $s_1 \in F(M,J_1)$, $s_2 \in F(M,J_2)$. But this idea does not work already for the Bargmann–Fock quantization. In this case sections s_1 and s_2 belong to $L^2_{\mathcal{O}}$-spaces with different weights, more precisely, s_1 belongs to $F(\mathbb{C}^n,J_1) = L^2_{\mathcal{O}}(\mathbb{C}^n, e^{-K_1(z)/2})$ and s_2 belongs to $F(\mathbb{C}^n,J_1) = L^2_{\mathcal{O}}(\mathbb{C}^n, e^{-K_2(z)/2})$, where $K_1(z)$ and $K_2(z)$ denote the Kähler potentials of Kähler metrics, determined by J_1 and J_2. It is clear that the product of these two factors may be not integrable. A better idea is to replace square integrable sections s of $L \to M$ by square integrable "half-forms" $s \otimes \sqrt{\omega^n}$. Then the integral of their product will be finite by the Cauchy inequality. In the next Subsection we realize this approach by formalizing the notion of half-forms.

14.2.2 Half-forms

Bundle of J-frames. Let (M,ω,J) be a Kähler manifold of $\dim_{\mathbb{C}} M = n$. Its complexified tangent bundle $T^{\mathbb{C}}M$ splits into the direct sum

$$T^{\mathbb{C}}M = T^{1,0}_J \oplus T^{0,1}_J$$

of the subbundles, formed by the $(\pm i)$-eigenspaces of the operator J. The *bundle of J-frames*

$$\mathrm{Fr}_J \longrightarrow M$$

is the bundle of frames in $T^{0,1}_J$, i.e. its fibre at $x \in M$ consists of all frames in $T^{0,1}_{J,x}$. The change of frames in the fibre generates a right $\mathrm{GL}(n,\mathbb{C})$-action on Fr_J, making Fr_J a principal $\mathrm{GL}(n,\mathbb{C})$-bundle.

We denote by

$$\mathrm{Fr}^n_J = K^{-1}_J \longrightarrow M$$

the *anti-canonical bundle, associated with* Fr_J, which coincides with the maximal exterior power of Fr_J: $\mathrm{Fr}^n_J = \bigwedge^n(\mathrm{Fr}_J)$. This is a complex line bundle on M, associated to Fr_J by the homomorphism $\det : \mathrm{GL}(n,\mathbb{C}) \to \mathbb{C}^*$. Its sections μ can be identified with functions $\dot{\mu}$ on Fr_J, satisfying the relation

$$\dot{\mu}(X \cdot C) = \det(C^{-1})\dot{\mu}(X) \tag{14.1}$$

for $X = (X_1, \dots, X_n) \in C^\infty(M, \mathrm{Fr}_J)$, $C \in \mathrm{GL}(n,\mathbb{C})$.

We can define a partial connection, acting on sections of the bundle Fr^n_J, following [70, 72]. Suppose that μ is a section of Fr^n_J, identified with the function $\dot{\mu}$ on Fr_J, and ξ is a $(0,1)$-vector field on M, i.e. a section of $T^{0,1}_J$. To define the value of $\nabla_\xi \dot{\mu}$ at a point $x^0 \in M$ on a frame $X^0 \in \mathrm{Fr}_{J,x_0}$, we extend X^0 to a local J-frame $X = (X_1, \dots, X_n)$ in a neighborhood U of x^0, represented by Hamiltonian vector fields $X_1, \dots, X_n$. Then we set

$$(\nabla_\xi \dot{\mu})(X^0) := \xi\, \dot{\mu}(X)|_{x^0} \ ,$$

i.e. the value of $\nabla_\xi \dot{\mu}$ on the frame X^0 at x^0 is equal to the value of the vector field ξ on the function $\dot{\mu}(X)$ at x^0. It can be checked that this definition is correct, i.e. $\nabla_\xi \dot{\mu}$ is again a function on Fr_J, satisfying (14.1), and does not depend on the choice of the local extension X of a J-frame X^0. So we can define $\nabla_\xi \mu$ as the section of Fr^n_J, identified with the function $\nabla_\xi \dot{\mu}$ on Fr_J.

The introduced derivative ∇ has the properties of a *partial connection* (cf. [18]). Namely, for any $(0,1)$-vector fields ξ, η, any functions $f, g \in C^\infty(M, \mathbb{R})$ and any sections μ, ν of Fr^n_J we have:

1. $\nabla_{f\xi + g\eta}\mu = f\nabla_\xi \mu + g\nabla_\eta \mu\,$;
2. $\nabla_\xi(\mu + \nu) = \nabla_\xi \mu + \nabla_\xi \nu\,$;
3. $\nabla_\xi(f\mu) = f\nabla_\xi \mu + (\xi f)\mu\,$.

Moreover, this partial connection satisfies the equality

$$\nabla_\xi \nabla_\eta \mu - \nabla_\eta \nabla_\xi \mu = \nabla_{[\xi,\eta]}\mu \ ,$$

which means that it is flat.

Bundle of half-forms. Denote by $\mathrm{ML}(n,\mathbb{C})$ the *metalinear group*, which is a double covering of $\mathrm{GL}(n,\mathbb{C})$:

$$\rho : \mathrm{ML}(n,\mathbb{C}) \xrightarrow{2:1} \mathrm{GL}(n,\mathbb{C}) \ .$$

Its elements can be identified with the square roots of $(n \times n)$-matrices from $\mathrm{GL}(n, \mathbb{C})$ in the sense that there is a commutative diagram

$$\begin{tikzcd} \mathrm{ML}(n,\mathbb{C}) \arrow[rr, "\chi"] \arrow[dr, "\rho"'] & & \mathbb{C}^* \\ & \mathrm{GL}(n,\mathbb{C}) \arrow[ur, "\det"'] & \end{tikzcd} ,$$

where χ is a unique complex square root of det, such that $\chi(I) = 1$.

Suppose that the principal $\mathrm{GL}(n, \mathbb{C})$-bundle $\mathrm{Fr}_J \to M$ of J-frames can be extended to a principal $\mathrm{ML}(n, \mathbb{C})$-bundle over M. Note that such an extension, in general, may not exist, since there is a topological obstruction for its existence (cf. [79, 29, 70]). This obstruction is an element of the cohomology group $H^2(M, \mathbb{Z}_2)$, moreover, the different choices of such metalinear extensions (if there are any) are parameterized by the elements of $H^1(M, \mathbb{Z}_2)$. So we suppose that this topological obstruction vanishes for our J-frame bundle $\mathrm{Fr}_J \to M$ and it can be extended to a principal $\mathrm{ML}(n, \mathbb{C})$-bundle

$$\widetilde{\mathrm{Fr}_J} \longrightarrow M \ .$$

We call $\widetilde{\mathrm{Fr}_J}$ the *bundle of metalinear J-frames*. It is a principal $\mathrm{ML}(n, \mathbb{C})$-bundle over M together with a double covering bundle epimorphism τ, such that

$$\begin{tikzcd} \widetilde{\mathrm{Fr}_J} \arrow[rr, "\tau"] \arrow[dr, "\mathrm{ML}(n,\mathbb{C})"'] & & \mathrm{Fr}_J \arrow[dl, "\mathrm{GL}(n,\mathbb{C})"] \\ & M & \end{tikzcd} .$$

We denote by

$$\widetilde{\mathrm{Fr}^n_J} = K_J^{-1/2} \longrightarrow M$$

a complex line bundle on M, associated to $\widetilde{\mathrm{Fr}_J} \to M$ by the homomorphism $\chi : \mathrm{ML}(n, \mathbb{C}) \to \mathbb{C}^*$. Its sections ν can be identified with functions $\tilde{\nu}$ on $\widetilde{\mathrm{Fr}_J}$, satisfying the relation

$$\tilde{\nu}(\tilde{X} \cdot \tilde{C}) = \chi(\tilde{C}^{-1})\tilde{\nu}(\tilde{X}) \tag{14.2}$$

for $\tilde{X} \in C^\infty(M, \widetilde{\mathrm{Fr}_J})$, $\tilde{C} \in \mathrm{ML}(n, \mathbb{C})$.

We can define a *partial connection*, acting on sections of the bundle $\widetilde{\mathrm{Fr}^n_J}$, similar to the case of the bundle Fr^n_J. Suppose that ν is a section of $\widetilde{\mathrm{Fr}^n_J}$, identified with the function $\tilde{\nu}$ on $\widetilde{\mathrm{Fr}_J}$, and ξ is a $(0, 1)$-vector field on M. To define the value of $\nabla_\xi \tilde{\nu}$ at a point $x^0 \in M$ on a metalinear frame $\tilde{X}^0 \in \widetilde{\mathrm{Fr}_{J,x_0}}$, we extend the corresponding J-frame $X^0 = \tau(\tilde{X}^0)$ to a local J-frame $X = (X_1, \dots, X_n)$ in a neighborhood of x^0, represented by Hamiltonian vector fields $X_1, \dots, X_n$. Since τ is a double covering, there exists a local metalinear J-frame $\tilde{X}$, defined (perhaps, on a smaller) neighborhood U of x^0, extending $\tilde{X}^0$ and covering X, i.e. $\tau(\tilde{X}) = X$. Then we set

$$(\nabla_\xi \tilde{\nu})(\tilde{X}^0) := \xi \, \tilde{\nu}(\tilde{X})|_{x^0} \ ,$$

i.e. the value of $\nabla_\xi \tilde{\nu}$ on the metalinear frame $\tilde{X}^0$ at x^0 is equal to the value of the vector field ξ on the function $\tilde{\nu}(\tilde{X})$ at x^0. This definition is correct, i.e. $\nabla_\xi \tilde{\nu}$ is again a function on $\widetilde{\mathrm{Fr}}^n_J$, satisfying (14.2), and does not depend on the choices of the extension X and its metalinear lift $\tilde{X}$. So we can define $\nabla_\xi\nu$ as the section of $\widetilde{\mathrm{Fr}}^n_J$, identified with the function $\nabla_\xi\tilde{\nu}$ on $\widetilde{\mathrm{Fr}}_J$. The defined partial connection ∇ on $\widetilde{\mathrm{Fr}}^n_J$ is again flat.

Fock space of half-forms. Consider a line bundle $L\otimes K_J^{-1/2}\to M$. It can be provided with a partial connection ∇, induced by the Hermitian connection on the prequantization bundle L and the partial connection on the anti-canonical bundle $K_J^{-1/2}$, defined above. More precisely, given a $(0,1)$-vector field ξ and a section $\sigma=\lambda\otimes\nu$ of $L\otimes K_J^{-1/2}$ we define

$$\nabla_\xi\sigma=(\nabla_\xi\lambda)\otimes\nu+\lambda\otimes(\nabla_\xi\nu)\ .$$

Denote by $\mathcal{O}_{1/2}(M,J)$ the space of holomorphic sections σ of $L\otimes K_J^{-1/2}\to M$. We want to define an inner product of two sections σ_1,σ_2 in $\mathcal{O}_{1/2}(M,J)$. Locally (in a neighborhood U of an arbitrary point $x\in M$) these sections may be written as

$$\sigma_1=\lambda_1\otimes\nu_1\ ,\quad \sigma_2=\lambda_2\otimes\nu_2$$

for $\lambda_1,\lambda_2\in\mathcal{O}(U,L)$, $\nu_1,\nu_2\in\mathcal{O}(U,K_J^{-1/2})$. We choose a local J-frame $X=(X_1,\dots,X_n)$ on U, so that $\{X_1,\dots,X_n,\bar{X}_1,\dots,\bar{X}_n\}$ form a basis of $T^{\mathbb{C}}M|_U$ and

$$i\omega(X_j,\bar{X}_k)=\delta_{jk}\ ,\quad \omega(X_j,X_k)=\omega(\bar{X}_j,\bar{X}_k)=0\ .$$

Denote by $<\sigma_1,\sigma_2>$ a density on U, defined by

$$<\sigma_1,\sigma_2>:=<\lambda_1(x),\lambda_2(x)>\tilde{\nu}_1(\tilde{X})\overline{\tilde{\nu}_2(\tilde{X})}$$

for $x\in U$ and any metalinear lift $\tilde{X}$ of X (such a lift locally always exists). It may be checked (cf. [70, 72]) that this definition does not depend on the choice of the lift and correctly defines a density, linear in σ_1, anti-linear in σ_2 and positive definite in the sense that $\langle\sigma,\sigma\rangle>0$ for non-vanishing σ.

Introduce a pre-Hilbert space

$$PF_{1/2}(M,J):=\{\sigma\in\mathcal{O}_{1/2}(M,J):\ \int_M<\sigma,\sigma><\infty\}$$

and provide it with the inner product, defined by

$$(\sigma_1,\sigma_2):=\int_M<\sigma_1,\sigma_2>\ .$$

The *Fock space of half-forms* $F_{1/2}(M,J)$ is, by definition, the completion of $PF_{1/2}(M,J)$ with respect to this inner product.

Locally (in a neighborhood U of a point $x\in M$) we can write down the integrand $<\sigma_1,\sigma_2>$ explicitly by choosing local trivializing holomorphic sections λ_0 of L and ν_0 of $K_J^{-1/2}$, subject to the conditions

$$<\lambda_0,\lambda_0>\equiv 1\ ,\quad \tilde{\nu}_0(\tilde{X})\equiv 1$$

in U. In terms of these trivializations, holomorphic sections σ_1,σ_2 of $L\otimes K_J^{-1/2}$ over U will be written as

$$\sigma_1=f_1\cdot\lambda_0\otimes\tilde{\nu}_0\ ,\quad \sigma_2=f_2\cdot\lambda_0\otimes\tilde{\nu}_0$$

for some holomorphic functions f_1, f_2 on U. Then in terms of J-holomorphic local coordinates $(z_1, \dots, z_n)$ in U we'll have

$$< \sigma_1, \sigma_2 >= \left(\frac{i}{2}\right)^n f_1(z)\overline{f_2(z)} d^n z \wedge d^n \bar{z} \ .$$

14.3 Metaplectic structure

14.3.1 Bundle of metaplectic frames

Metaplectic group. The *metaplectic group* $\mathrm{Mp}(2n, \mathbb{R})$ is a connected double covering group of the symplectic group $\mathrm{Sp}(2n, \mathbb{R})$, i.e. there is a 2:1 group homomorphism

$$\rho : \mathrm{Mp}(2n, \mathbb{R}) \longrightarrow \mathrm{Sp}(2n, \mathbb{R}) \ .$$

Such a covering exists, because the fundamental group π_1 of $\mathrm{Sp}(2n, \mathbb{R})$ is equal to $\mathbb{Z}$. To see that, note that $\mathrm{Sp}(2n, \mathbb{R})$ is homeomorphic to

$$\mathrm{U}(n) \times \frac{\mathrm{Sp}(2n, \mathbb{R})}{\mathrm{U}(n)} \cong S^1 \times \mathrm{SU}(n) \times \{\text{Siegel disc}\} \ ,$$

and the second and third factors on the right are simply connected.

Metaplectic structure. Let (M, ω) be a symplectic manifold of dimension $2n$. Denote by $\mathrm{Fr}_\omega \to M$ the principal $\mathrm{Sp}(2n, \mathbb{R})$-*bundle of symplectic frames* on M. A *metaplectic structure* on M is an extension of the bundle $\mathrm{Fr}_\omega \to M$ to a principal $\mathrm{Mp}(2n, \mathbb{R})$-bundle $\widetilde{\mathrm{Fr}}_\omega \to M$, called the *bundle of metaplectic frames* on M. In other words, we have a double covering bundle epimorphism $\tau : \widetilde{\mathrm{Fr}}_\omega \to \mathrm{Fr}_\omega$, which may be included into the following commutative diagram

$$\begin{tikzcd} \widetilde{\mathrm{Fr}}_\omega \arrow[rr, "\tau"] \arrow[dr, "\mathrm{Mp}(n,\mathbb{R})"'] & & \mathrm{Fr}_\omega \arrow[dl, "\mathrm{Sp}(n,\mathbb{R})"] \\ & M & \end{tikzcd} \ .$$

There is a topological obstruction for the existence of the metaplectic structure on M, due to Kostant [46]. Namely, denote by J an almost complex structure on M, compatible with ω, so that $c_1(M)$ is the 1st Chern class of TM with respect to J. Then for the existence of a metaplectic structure on M it is necessary and sufficient that $c_1(M) \bmod 2 \equiv 0 \Longleftrightarrow c_1(M)$ is even in $H^2(M, \mathbb{Z})$. If this condition is satisfied, then the *set of all metaplectic structures on* M *(up to a natural equivalence) is parameterized by* $H^1(M, \mathbb{Z}_2)$.

14.3.2 Bundle of Kähler frames

It is also convenient to introduce the *bundle* $\mathrm{Fr}_K \to M$ *of J-frames for all ω-compatible almost complex structures* J on M. It is a fibre bundle over M with the fibre at $x \in M$, parameterizing J_x-frames on T_xM for all ω_x-compatible almost complex structures J_x on T_xM. This fibre can be identified with

$$\frac{\mathrm{Sp}(2n, \mathbb{R})}{\mathrm{U}(n)} \times \mathrm{GL}(n, \mathbb{C}) \cong \{\text{Siegel disc}\} \times \mathrm{GL}(n, \mathbb{C})$$

in the following way. Given a symplectic frame $(\xi, \eta) := (\xi_1, \dots, \xi_n; \eta_1, \dots, \eta_n)$ at $x \in M$, we can write down any J-frame $X = (X_1, \dots, X_n)$ at x uniquely as (cf. [70])

$$X = \xi U + \eta V \ ,$$

where U, V are complex $n \times n$-matrices, such that the rank of $(2n \times n)$-matrix ${}^t(U, V)$ equals n, ${}^tUV = {}^tVU$, and the matrix $i(V^\dagger U - U^\dagger V)$ is positive definite. The set of such matrices ${}^t(U, V)$ can be identified with with the set: $\{\text{Siegel disc}\} \times \mathrm{GL}(n, \mathbb{C})$, by associating with a matrix ${}^t(U, V)$ a pair of matrices

$$W := (U + iV)(U - iV)^{-1} \ , \quad C := U - iV \ . \tag{14.3}$$

Then C belongs to $\mathrm{GL}(n, \mathbb{C})$ and W belongs to the Siegel disc

$$D := \{W \in \mathrm{L}(n, \mathbb{C}) : {}^tW = W, I - W^\dagger W \text{ is positive definite}\} \ .$$

The structure group of the bundle $\mathrm{Fr}_K \to M$, acting on the left, coincides with $\mathrm{Sp}(2n, \mathbb{R})$. There is also a natural $\mathrm{GL}(n, \mathbb{C})$-action on $\mathrm{Fr}_K \to M$ from the right, given by the frame change. The bundle $\mathrm{Fr}_K \to M$ is associated to the bundle $\mathrm{Fr}_\omega \to M$ of symplectic frames by a natural $\mathrm{Sp}(2n, \mathbb{R})$-action on the fibre.

In a similar way, we introduce the bundle $\widetilde{\mathrm{Fr}}_K \to M$ of all metalinear J-frames on M for all ω-compatible J. It is a fibre bundle with the fibre at $x \in M$, given by

$$\frac{\mathrm{Sp}(2n, \mathbb{R})}{\mathrm{U}(n)} \times \mathrm{ML}(n, \mathbb{C}) \ , \tag{14.4}$$

and the structure group $\mathrm{Mp}(2n, \mathbb{R})$, acting by the homomorphism $\rho : \mathrm{Mp}(2n, \mathbb{R}) \to \mathrm{Sp}(2n, \mathbb{R})$ on the first factor. The bundle $\widetilde{\mathrm{Fr}}_K \to M$ is associated to the bundle $\widetilde{\mathrm{Fr}}_\omega \to M$ of metaplectic frames by the $\mathrm{Mp}(2n, \mathbb{R})$-action. There is a commutative diagram

$$\begin{tikzcd} \widetilde{\mathrm{Fr}}_K \arrow[rr, "\tau"] \arrow[dr] & & \mathrm{Fr}_K \arrow[dl] \\ & M & \end{tikzcd} \ ,$$

where τ is a double covering.

Note that for a fixed ω-compatible almost complex structure J on M the bundle $\mathrm{Fr}_J \to M$ is a subbundle of $\mathrm{Fr}_K \to M$, invariant under the right $\mathrm{GL}(n, \mathbb{C})$-action. The bundle $\widetilde{\mathrm{Fr}}_J \to M$ is a $\mathrm{ML}(n, \mathbb{C})$-invariant subbundle of $\widetilde{\mathrm{Fr}}_K \to M$, which coincides with the inverse image of $\mathrm{Fr}_J \to M$ under the double covering map $\tau : \widetilde{\mathrm{Fr}}_K \to \mathrm{Fr}_K$. In other words, we can say that a metaplectic structure on M, given by the metaplectic frame bundle together with the double covering $\tau : \widetilde{\mathrm{Fr}}_\omega \to \mathrm{Fr}_\omega$, induces metalinear structures on all J-frame bundles simultaneously.

14.4 Blattner–Kostant–Sternberg (BKS) pairing

Lemma 5. *Suppose that J_1, J_2 are two ω-compatible almost complex structures on a symplectic manifold (M, ω). Then they are* transversal *in the sense that*

$$T^{1,0}_{J_1} \oplus T^{0,1}_{J_2} = T^{\mathbb{C}} M \ .$$

Proof. Suppose, on the contrary, that there exists a vector $\xi \neq 0$, such that

$$\xi \in T^{1,0}_{J_1,x} \oplus T^{0,1}_{J_2,x} \quad \text{for some } x \in M \ .$$

Then

$$0 < \omega(\xi, J_1\xi) = \omega(\xi, i\xi) = i\omega(\xi, \xi) \ ,$$

where the inequality on the left is implied by the ω-compatibility of J_1 and the first equality is provided by $\xi \in T^{1,0}_{J_1,x}$. Similarly,

$$0 < \omega(\xi, J_2\xi) = \omega(\xi, -i\xi) = -i\omega(\xi, \xi) \ .$$

So we have simultaneously the two following relations

$$i\omega(\xi, \xi) > 0 \quad \text{and} \quad -i\omega(\xi, \xi) > 0 \ ,$$

contradicting each other. Hence, $T^{1,0}_{J_1,x} \cap T^{0,1}_{J_2,x} = \{0\}$ for any $x \in M$. By dimension counting we obtain that

$$T^{1,0}_{J_1,x} \oplus T^{0,1}_{J_2,x} = T^{\mathbb{C}}_x M \quad \text{for any } x \in M \ .$$

$\square$

Due to the above Lemma 5, we can always choose locally, in a neighborhood U of an arbitrary fixed point $x \in M$, a J_1-frame X_1 and J_2-frame X_2, so that

$$i\omega(X_1^j, \overline{X_2^k}) = \delta_{jk} \ . \tag{14.5}$$

Given two sections σ_1 of $L \otimes K_{J_1}^{-1/2}$ and σ_2 of $L \otimes K_{J_2}^{-1/2}$ on U, we can write them down in the form

$$\sigma_1 = \lambda_1 \otimes \nu_1 \ , \quad \sigma_2 = \lambda_2 \otimes \nu_2 \ .$$

We define a density, similar to that in Subsec. 14.2.2:

$$< \sigma_1, \sigma_2 > := < \lambda_1(x), \lambda_2(x) > \tilde{\nu}_1(\tilde{X}_1)\overline{\tilde{\nu}_2(\tilde{X}_2)} \tag{14.6}$$

where $\tilde{X}_1, \tilde{X}_2$ are metalinear lifts of X_1, X_2, satisfying a metalinear analogue of (14.5). We shall describe this analogue (formula (14.9)) in Rem. 25 below. Now we note only that the definition (14.6) does not depend on the choice of the frames X_1, X_2, satisfying the normalization condition (14.5), and their metaplectic lifts $\tilde{X}_1$, $\tilde{X}_2$, satisfying the metaplectic normalization condition (14.9) below (this fact is proved in [70], Sec.5.1; cf. also [29], Ch.V,Sec.5).

We define the *BKS-pairing* between different Fock spaces of half-forms $F_{1/2}(M, J_1)$ and $F_{1/2}(M, J_2)$ by the formula

$$(\sigma_1, \sigma_2)_{12} := \int_M < \sigma_1, \sigma_2 > \ . \tag{14.7}$$

Suppose now that our almost complex structures J_1 and J_2 are integrable. Then locally, in a neighborhood U of a point $x \in M$, we can write down an explicit formula for the integrand in the above formula. For that we fix a J_1-frame X_1 and a J_2-frame X_2 in U, satisfying the normalization condition (14.5), and their metaplectic lifts

$\tilde{X}_1$, $\tilde{X}_2$, satisfying the metaplectic normalization condition (14.9), and choose local trivializing holomorphic sections λ_0 of L, ν_1 of $K_{J_1}^{-1/2}$ and ν_2 of $K_{J_2}^{-1/2}$, subject to the conditions

$$< \lambda_0, \lambda_0 >\equiv 1 \ , \quad \tilde{\nu}_1(\tilde{X}_1) \equiv 1 \ , \quad \tilde{\nu}_2(\tilde{X}_2) \equiv 1$$

in U. Then holomorphic sections σ_1 of $L \otimes K_{J_1}^{-1/2}$ and σ_2 of $L \otimes K_{J_2}^{-1/2}$ over U will be written as

$$\sigma_1 = f_1 \cdot \lambda_0 \otimes \nu_1 \ , \quad \sigma_2 = f_2 \cdot \lambda_0 \otimes \nu_2 \ ,$$

where f_1 is a J_1-holomorphic function on U, and f_2 is a J_2-holomorphic function on U. Since J_1 and J_2 are transversal, we can find local J_1-holomorphic coordinates $(z_1, \dots, z_n)$ and J_2-holomorphic coordinates $(w_1, \dots, w_n)$ in U, such that $(\partial/\partial z_1, \dots, \partial/\partial z_n; \partial/\partial \bar{w}_1, \dots, \partial/\partial \bar{w}_n)$ form a local basis of $T^{\mathbb{C}}M$ over M. Then

$$< \sigma_1, \sigma_2 >= \left(\frac{i}{2}\right)^n f_1(z)\overline{f_2(w)} d^n z \wedge d^n \bar{w} \ .$$

Remark 25 ([70]). To describe the metaplectic analogue of (14.5), suppose that our frames X_1 and X_2 are written in terms of a single symplectic frame $(\xi, \eta) := (\xi_1, \dots, \xi_n; \eta_1, \dots, \eta_n)$, as in Subsec. 14.3.1:

$$X_1 = \xi U_1 + \eta V_1 \ , \quad X_2 = \xi U_2 + \eta V_2 \ .$$

Then Eq. (14.5) can be written in the form

$$V_2^\dagger U_1 - U_2^\dagger V_1 = -iI \ .$$

In terms of the matrices

$$W_j = (U_j + iV_j)(U_j - iV_j)^{-1} \ , \quad C_j := U_j - iV_j \ , \quad j = 1, 2 \ ,$$

this condition means that

$$I - W_2^\dagger W_1 = 2(C_2^\dagger)^{-1} C_1^{-1} \ . \tag{14.8}$$

Note that $Z := W_2^\dagger W_1$ belongs to the matrix disc

$$\tilde{D} := \{Z \in \mathrm{L}(n, \mathbb{C}) : I - Z^\dagger Z \text{ is positive definite}\} \ .$$

Consider the map $\tilde{D} \to \mathrm{GL}(n, \mathbb{C})$, given by $Z \mapsto I - Z$. Since $\tilde{D}$ is contractible (moreover, convex), this map can be uniquely extended to a map $Z \mapsto \widetilde{I - Z}$, sending $\tilde{D}$ to $\mathrm{ML}(n, \mathbb{C})$ and taking the value $\tilde{I}$ at $Z = 0$ (where we denote by $\tilde{I}$ the unit element in $\mathrm{ML}(n, \mathbb{C})$). Suppose that the metalinear lifts $\tilde{X}_1, \tilde{X}_2$ of our frames X_1, X_2 are described, according to (14.4), by pairs $(W_1, \tilde{C}_1)$, $(W_2, \tilde{C}_2)$, where $W_j \in D$, $\tilde{C}_j \in \mathrm{ML}(n, \mathbb{C})$ for $j = 1, 2$. Then the metalinear analogue of (14.5) has the form

$$\widetilde{I - W_2^\dagger W_1} = 2(\tilde{C}_2^\dagger)^{-1} \tilde{C}_1^{-1} \ . \tag{14.9}$$

14.5 Blattner–Kostant–Sternberg (BKS) quantization

14.5.1 Lifting the φ_f^t-action

Let (M, ω, J) be a Kähler manifold, and $f \in C^\infty(M, \mathbb{R})$ is an observable on M, for which the Hamiltonian vector field X_f is complete, i.e. the 1-parameter flow $\varphi_f^t = \exp(2\pi i t X_f)$, generated by X_f, is defined for all $t \in \mathbb{R}$. Hence, $\{\varphi_f^t\}$ is a 1-parameter group of symplectomorphisms of M. The flow φ_f^t generates a natural flow on the space of ω-compatible complex structures on M, given by

$$J \longmapsto J_f^t := \varphi_{f,\star}^t \circ J \circ \varphi_{f,\star}^{-t} \ ,$$

and a natural flow, denoted by the same letter φ_f^t, on the bundle $\mathrm{Fr}_K \to M$ of all J-frames on M.

By the covering homotopy property, this flow can be lifted to a 1-parameter flow $\tilde{\varphi}_f^t$ on the bundle $\widetilde{\mathrm{Fr}}_K \to M$ of all metalinear J-frames on M, yielding a 1-parameter flow of bundle isomorphisms

$$\tilde{\varphi}_f^t : \widetilde{\mathrm{Fr}}_J \longrightarrow \widetilde{\mathrm{Fr}}_{J_f^t} \ .$$

We are going to define an extension of the φ_f^t-flow to the Fock spaces of half-forms, denoted by

$$\mathcal{H}_t \equiv \mathcal{H}_f^t := F_{1/2}(M, J_f^t) \ .$$

φ_f^t-action on $K_J^{-1/2}$. Define first a φ_f^t-action on the bundle $K^{-1/2}$ over the space of ω-compatible complex structures on M. Let ν be a section of $K_J^{-1/2}$, identified with the function $\tilde{\nu}$ on the bundle $\widetilde{\mathrm{Fr}}_J$. Denote by $\varphi_f^t \nu$ a section of $K_t^{-1/2} \equiv K_{J_f^t}^{-1/2}$, identified with the function $\widetilde{\varphi_f^t \nu}$, defined by

$$\widetilde{\varphi_f^t \nu}(\tilde{X}) = \tilde{\nu}(\tilde{\varphi}_f^{-t} \tilde{X})$$

for any metalinear frame $\tilde{X} \in \widetilde{\mathrm{Fr}}_t \equiv \widetilde{\mathrm{Fr}}_{J_f^t}$.

φ_f^t-action on sections of L. By Rem. 22, the φ_f^t-flow on M can be lifted to a φ_f^t-flow on sections of L. More precisely, the generator of the φ_f^t-action on L

$$\mathcal{P}_f(\lambda) := i \frac{d}{dt} \left(\varphi_f^t \lambda\right)\Big|_{t=0}$$

is equal to

$$\mathcal{P}_f(\lambda) = r_{\mathrm{KS}}(f)(\lambda) = f\lambda - i\nabla_{X_f}\lambda \ .$$

φ_f^t-action on the Fock space of half-forms. Recall (cf. Subsec. 14.2.2) that the Fock space of half-forms $\mathcal{H}$ is defined as

$$\mathcal{H} = F_{1/2}(M, J) \ .$$

Suppose that an element σ of $\mathcal{H}$ is written in the form

$$\sigma = \lambda \otimes \nu \ ,$$

where $\lambda \in \mathcal{O}(M, L)$, $\nu \in \mathcal{O}(M, K_J^{-1/2})$. Then by definition

$$\varphi_f^t \sigma := \varphi_f^t \lambda \otimes \varphi_f^t \nu \ .$$

By linearity and continuity this definition extends to arbitrary sections in $\mathcal{H}$, so we obtain a Hilbert space isomorphism

$$\varphi_f^t : \mathcal{H} \longrightarrow \mathcal{H}_t$$

with the inverse map, given by φ_f^{-t}. It may be shown (cf. [70]) that $\varphi_f^t : \mathcal{H} \to \mathcal{H}_t$ is unitary.

14.5.2 Quantization of quantizable observables

Let f be an observable on M with a complete Hamiltonian vector field X_f. Suppose first that f is *quantizable*, i.e. the associated flow $\varphi_{f,\star}^t$ preserves the complex structure $J \iff \varphi_f^t$ is a J-holomorphic map. Otherwise speaking, f is quantizable iff $[X_f, T_J^{0,1}M] \subset T_J^{0,1}M$. The quantizable observables form a subalgebra of the Lie algebra $\mathcal{A}$ of all observables. If f is quantizable, then the φ_f^t-flow preserves $\mathcal{H}$, i.e. we have a 1-parameter group of unitary operators $\varphi_f^t : \mathcal{H} \to \mathcal{H}$, and we can define the *quantized observable* $\mathcal{Q}_f$ by

$$\mathcal{Q}_f(\sigma) := i \frac{d}{dt} \left(\varphi_f^t \sigma \right) \Big|_{t=0} \tag{14.10}$$

for any $\sigma \in \mathcal{H}$.

We can describe the operator $\mathcal{Q}_f$ in a more explicit way as follows. Suppose that ξ is any vector field on M, preserving J, i.e. $[\xi, T_J^{0,1}M] \subset T_J^{0,1}M$. Define a *partial Lie derivative* L_ξ of half-forms with respect to ξ. Namely, for any half-form ν, identified with the function $\tilde{\nu}$ on the bundle $\widetilde{\mathrm{Fr}}_J$, we identify $L_\xi \nu$ with the function $\widetilde{L_\xi \nu}$, given by the formula

$$\widetilde{L_\xi \nu}(\tilde{X})|_x \equiv (L_\xi \tilde{\nu})(\tilde{X})|_x = \frac{d}{dt} \, |_{t=0} \left(\tilde{\nu}(\tilde{\varphi}_f^t \tilde{X})|_{\varphi_f^t x} \right)$$

for any metalinear J-frame $\tilde{X}$. In other words, the L_ξ-derivative of the function $\tilde{\nu}$, evaluated on a metalinear J-frame $\tilde{X}$ at a point x, is equal to the $\frac{d}{dt}$-derivative at $t = 0$ of the function $\tilde{\nu}$, evaluated on the metalinear J-frame $\tilde{\varphi}_f^t \tilde{X}$ at the point $\varphi_f^t x$.

The derivative L_ξ has the properties of the Lie derivative, but it can be taken only along the vector fields ξ, preserving J. The operator $\mathcal{Q}_f$ can be written in terms of partial Lie derivative as

$$\mathcal{Q}_f(\lambda \otimes \nu) = (-i \nabla_{X_f} \lambda + f \lambda) \otimes \nu - i \lambda \otimes L_{X_f} \nu \ .$$

Locally, we can compute the second term on the right as follows. Denote by $X = (X^1, \ldots, X^n)$ a local J-frame on an open set U, consisting of Hamiltonian $(0,1)$-vector fields X^j. Then

$$[X_f, X^j](x) = \sum_{k=1}^{n} a_k^j(x) X^k$$

for some smooth matrix function $A := (a_k^j)$ on U. Denote by $\tilde{X}$ a metalinear lift of X over U and choose a local section $\tilde{\nu}_0$ of $K_J^{-1/2}$, so that $\tilde{\nu}_0(\tilde{X}) \equiv 1$. Any $\sigma \in F_{1/2}(U, J)$ can be written in the form

$$\sigma = \lambda \otimes \tilde{\nu}_0$$

for some holomorphic section $\lambda \in \mathcal{O}(U, L)$. Then (cf. [70], Sec.6.2)

$$L_{X_f}\tilde{\nu}_0 = -\frac{1}{2}\mathrm{tr}\, A \cdot \tilde{\nu}_0 \ ,$$

so that

$$\mathcal{Q}_f(\lambda \otimes \tilde{\nu}_0) = \left(-i\nabla_{X_f}\lambda + f\lambda - i\frac{1}{2}\mathrm{tr}\, A \cdot \lambda\right) \otimes \tilde{\nu}_0 \ .$$

It can be shown (cf. [70, 72]) that the map $f \mapsto \mathcal{Q}_f$ is a Lie-algebra representation

$$\{\text{Lie algebra of quantizable observables}\} \xrightarrow{\mathcal{Q}} \mathrm{End}^*\mathcal{H}$$

in the Fock space of half-forms $\mathcal{H} = F_{1/2}(M, J)$.

14.5.3 Quantization of general observables

Assume that for an observable f the integrals, defining the BKS-pairing $\mathcal{H}\times\mathcal{H}_t \to \mathbb{C}$, are finite, so we have a unitary operator

$$U_t : \mathcal{H}_t \longrightarrow \mathcal{H} \ .$$

In its terms the BKS-pairing, defined by formula (14.7), may be written as

$$(\sigma, \sigma_t)_{0t} = (\sigma, U_t\sigma_t)$$

for $\sigma \in \mathcal{H} \equiv \mathcal{H}_0$, $\sigma_t \in \mathcal{H}_t$.

Consider a unitary operator

$$\Phi_f^t := U_t \circ \varphi_f^t : \mathcal{H} \longrightarrow \mathcal{H}$$

and define a self-adjoint *quantized observable* $\mathcal{Q}_f$ by

$$\mathcal{Q}_f := i\frac{d}{dt}\,\Phi_f^t\Big|_{t=0} : \mathcal{H} \longrightarrow \mathcal{H} \ .$$

Then the map $f \mapsto \mathcal{Q}_f$ defines an irreducible Lie-algebra representation

$$\mathcal{Q} : \mathcal{A} \longrightarrow \mathrm{End}^*\mathcal{H}$$

of the algebra of observables $\mathcal{A}$ in the Fock space of half-forms $\mathcal{H}$ (under the assumption that the BKS-pairing is finite for all observables $f \in \mathcal{A}$).

Bibliographic comments

The BKS-quantization is presented in several books on geometric quantization. We follow mainly the Snyatycki book [70], dealing with different kinds of polarizations. We also recommend the Guillemin–Sternberg book [29], devoted mostly to real polarizations, and Tuynman lecture notes [72]. Our goal here was to present the BKS-quantization scheme without going too much into details (which may be found in [70, 29, 72]).

Part V

QUANTIZATION OF LOOP SPACES

Chapter 15

Quantization of the loop space of a vector space

In this Chapter we solve the geometric quantization problem for the classical system $(\Omega\mathbb{R}^d, \mathcal{A}_d)$, where the phase space $\Omega\mathbb{R}^d$ consists of smooth loops in the d-dimensional vector space $\mathbb{R}^d$, and the algebra of observables $\mathcal{A}_d$ is the Lie algebra of the Frechet Lie group $\mathcal{G}_d$, being the semi-direct product of the loop group $\widetilde{L\mathbb{R}^d}$ and the diffeomorphism group $\mathrm{Diff}_+(S^1)$ of the circle.

We start from the quantization of the "enlarged" system, obtained from $(\Omega\mathbb{R}^d, \mathcal{A}_d)$ by enlarging both the phase space and the algebra of observables. More precisely, we enlarge the phase space $\Omega\mathbb{R}^d$ to the Sobolev space V^d of half-differentiable vector-functions (a vector analogue of the Sobolev space V, introduced in Sec. 9.1), and the algebra of observables $\mathcal{A}_d$ to the Lie algebra $\mathcal{A}$ of the Hilbert Lie group $\mathcal{G}$, being the semi-direct product of the Heisenberg group $\mathrm{Heis}(V^d)$ and the symplectic Hilbert–Schmidt group $\mathrm{Sp}_{\mathrm{HS}}(V^d)$. The group $\mathcal{G}$ may be considered as a Hilbert-space (symplectic) analogue of the standard group of motions of the d-dimensional vector space $\mathbb{R}^d$. The latter group is the semi-direct product of the group of translations of $\mathbb{R}^d$ and the group of rotations of $\mathbb{R}^d$. In the case of the Hilbert space V the role of translation group is played by the Heisenberg group, and the group of rotations is replaced by the symplectic group $\mathrm{Sp}_{\mathrm{HS}}(V)$.

To simplify the formulas, we set $d = 1$ in the most part of this Chapter, replacing it with a general d only in Sec. 15.6, where the quantization of $\Omega\mathbb{R}^d$ is completed. The last Sec. 15.7 is devoted to the quantization of the universal Teichmüller space.

15.1 Heisenberg representation

15.1.1 Fock space

Consider the *Sobolev space*

$$V := H_0^{1/2}(S^1, \mathbb{R})$$

of half-differentiable functions on the circle S^1 (cf. Sec. 9.1) and its complexification

$$V^{\mathbb{C}} = H_0^{1/2}(S^1, \mathbb{C}) .$$

A natural complex structure operator J_0 on $V^{\mathbb{C}}$, introduced in Sec. 9.1, generates a decomposition of $V^{\mathbb{C}}$ into the direct sum of subspaces

$$V^{\mathbb{C}} = W_+ \oplus W_- =: W_0 \oplus \overline{W_0} \ , \tag{15.1}$$

where $W_\pm$ is the $(\mp i)$-eigenspace of the operator $J^0 \in \operatorname{End} V^{\mathbb{C}}$. The subspaces $W_\pm$ are isotropic with respect to the symplectic form ω on $V^{\mathbb{C}}$. Moreover, the splitting (15.1) is an orthogonal direct sum with respect to the Hermitian inner product on $V^{\mathbb{C}}$, defined by

$$< z, w >= i\omega(z_+, \bar{w}_+) - i\omega(z_-, \bar{w}_-) \ ,$$

where $z_\pm$ (resp. $w_\pm$) denotes the projection of $z \in V^{\mathbb{C}}$ (resp. $w \in V^{\mathbb{C}}$) onto the subspace $W_\pm$.

We introduce the *Fock space* $F_0 \equiv F(V^{\mathbb{C}}, J_0)$ as the completion of the algebra of symmetric polynomials on W_0 with respect to a natural norm.

In more detail, denote by $S(W_0)$ the algebra of symmetric polynomials in variables $z \in W_0 \equiv W_+$ and introduce an inner product on $S(W_0)$, induced by the Hermitian product $< \cdot, \cdot >$ on $V^{\mathbb{C}}$. This inner product on monomials is given by the formula

$$< z_1 \cdot \ldots \cdot z_n, z'_1 \cdot \ldots \cdot z'_n >= \sum_{\{i_1,\ldots,i_n\}} < z_1, z'_{i_1} > \cdot \ldots \cdot < z_n, z'_{i_n} > \ ,$$

where the summation is taken over all permutations $\{i_1, \ldots, i_n\}$ of the set $\{1, \ldots, n\}$ (the inner product of monomials of different degrees is set to 0 by definition). This inner product is extended by linearity to the whole algebra $S(W_0)$. The completion $\widehat{S(W_0)}$ of $S(W_0)$ with respect to the introduced norm is called the *Fock space* $F_0 \equiv F(V^{\mathbb{C}}, J_0)$ over $V^{\mathbb{C}}$ with respect to the complex structure J^0

$$F_0 = F(V^{\mathbb{C}}, J^0) := \widehat{S(W)} \ .$$

If $\{w_n\}$, $n = 1, 2, \ldots$, is an orthonormal base of W_0, then one can take for an orthonormal base of F_0 the family of polynomials of the form

$$P_K(z) = \frac{1}{\sqrt{K!}} < z, w_1 >^{k_1} \cdot \ldots \cdot < z, w_n >^{k_n} \ , \quad z \in W_0 \ ,$$

where $K = (k_1, \ldots, k_n)$, $k_i \in \mathbb{N}$, and $K! = k_1! \cdot \ldots \cdot k_n!$.

Recall that, according to Sec. 11.4, any complex structure J on V, compatible with ω, determines a decomposition

$$V^{\mathbb{C}} = W_J \oplus \overline{W}_J =: W \oplus \overline{W} \tag{15.2}$$

into the direct sum of subspaces W and $\overline{W}$, isotropic with respect to ω. The subspaces W and $\overline{W}$ are identified, respectively, with the $(-i)$- and $(+i)$-eigenspaces of the operator J on $V^{\mathbb{C}}$. The complex structure J and the symplectic form ω determine together a Kähler metric g_J and the associated inner product $< \cdot, \cdot >_J$ on $V^{\mathbb{C}}$. The decomposition (15.2) is orthogonal with respect to the Kähler metric g_J on $V^{\mathbb{C}}$, determined by J and ω.

Using the decomposition (15.2), we can define the *Fock space* $F_J \equiv F(V^{\mathbb{C}}, J)$ as the completion of the algebra of symmetric polynomials on W with respect to the norm, generated by $< \cdot, \cdot >_J$:

$$F_J = F(V^{\mathbb{C}}, J) := \text{ completion of } S(W) \text{ with respect to } < \cdot, \cdot >_J \ .$$

15.1.2 Heisenberg algebra and Heisenberg group

The *Heisenberg algebra* heis(V) of the Hilbert space V is a central extension of the Abelian Lie algebra V, generated by the coordinate functions. In other words, it coincides, as a vector space, with

$$\mathrm{heis}(V) = V \oplus \mathbb{R} \ ,$$

and is provided with the Lie bracket

$$[(x,s),(y,t)] := (0, \omega(x,y)) \ , \quad x,y \in V, \ s,t, \in \mathbb{R} \ .$$

The Heisenberg algebra heis(V) is the Lie algebra of the *Heisenberg group* Heis(V), which coincides with a central extension of the Abelian group V. In other words, Heis(V) is the direct product

$$\mathrm{Heis}(V) = V \times S^1 \ ,$$

provided with the group operation, given by

$$(x,\lambda)\cdot(y,\mu) := \left(x+y, \lambda\mu e^{i\omega(x,y)}\right) \ .$$

15.1.3 Heisenberg representation

Representation of the Heisenberg algebra. We are going to construct an irreducible representation of the Heisenberg algebra heis(V) in the Fock space $F_J = F(V^{\mathbb{C}}, J)$, where $V^{\mathbb{C}} = W \oplus \overline{W}$ and F_J is the completion of the symmetric algebra $S(W)$ with respect to the norm, generated by $< \cdot, \cdot >_J$. We can consider elements of $S(W)$ as holomorphic functions on $\overline{W}$ by identifying $z \in W$ with a holomorphic function $\bar{w} \mapsto < w, z >$ on $\overline{W}$. Accordingly, F_J may be considered as a subspace of the space $\mathcal{O}(\overline{W})$ of functions, holomorphic on $\overline{W}$ (provided with the topology of uniform convergence on compact subsets).

With this convention we can define the *Heisenberg representation*

$$r_J : \mathrm{heis}(V) \longrightarrow \mathrm{End}\, F_J$$

of the Heisenberg algebra heis(V) in the Fock space $F_J = F(V^{\mathbb{C}}, J)$ by the formula

$$v \longmapsto r_J(v)f(\bar{w}) := -\partial_v f(\bar{w}) + < w, v >_J f(\bar{w}) \ , \tag{15.3}$$

where ∂_v is the derivation operator in the direction of $v \in V^{\mathbb{C}}$. Extending r_J to the complexified algebra heis$^{\mathbb{C}}(V)$ by the same formula (15.3), we'll have for $v = \bar{z} \in \overline{W}$

$$r_J(\bar{z})f(\bar{w}) := -\partial_{\bar{z}} f(\bar{w}) \ ,$$

and for $z \in W$

$$r_J(z)f(\bar{w}) := < w, z >_J f(\bar{w}) \ .$$

For the central element $c \in \mathrm{heis}(V)$ we set

$$c \longmapsto r_J(c) := \lambda \cdot I \ ,$$

where λ is an arbitrary fixed non-zero constant.

Introduce *creation* and *annihilation operators* on F_J, defined for $v \in V^{\mathbb{C}}$ by

$$a_J^*(v) := \frac{r_J(v) - ir_J(Jv)}{2} \ , \quad a_J(v) := \frac{r_J(v) + ir_J(Jv)}{2} \ . \tag{15.4}$$

In particular, for $z \in W$

$$a_J^*(z)f(\bar{w}) = < w, z >_J f(\bar{w}) \ , \tag{15.5}$$

and for $\bar{z} \in \overline{W}$

$$a_J(\bar{z})f(\bar{w}) = -\partial_{\bar{z}} f(\bar{w}) \ . \tag{15.6}$$

Choosing an orthonormal basis $\{e_n\}$ of W, we can introduce the operators

$$a_n^* := a^*(e_n) \ , \quad a_n := a(\bar{e}_n) \ , \quad n = 1, 2, \ldots,$$

and $a_0 := \lambda \cdot I$.

A vector $f_J \in F_J \setminus \{0\}$ is called the *vacuum*, if $a_n f_J = 0$ for $n = 1, 2, \ldots$. In other words, the vacuum is a non-zero vector, annihilated by all operators a_n. It is uniquely defined by r_J (up to a multiplicative constant) and in the case of the initial Fock space $F_0 = F(V, J_0)$ we take $f_0 \equiv 1$. By acting on the vacuum f_J by creation operators a_n^*, we can define the action of the representation r_J on any polynomial, which implies the irreducibility of r_J.

Moreover, any irreducible representation $r : \mathrm{heis}^{\mathbb{C}}(V) \to \mathrm{End}\, F$ of the algebra $\mathrm{heis}^{\mathbb{C}}(V)$, having a vacuum f, is equivalent to the Heisenberg representation r_0. Indeed, vectors of the form $(a_1^*)^{k_1} \cdot \ldots \cdot (a_n^*)^{k_n} f$, obtained from the vacuum by the action of creation operators, are linearly independent and generate the whole representation space F. Assigning to a polynomial $P(z) = P(z_1, \ldots, z_n)$ in the Fock space F_0 the vector of the form $P(a_1^*, \ldots, a_n^*)f$ in the space F, we obtain an intertwining map from F_0 into F. This map can be made unitary by introducing a Hermitian inner product on F, for which the vectors $(a_1^*)^{k_1} \cdot \ldots \cdot (a_n^*)^{k_n} f$ form an orthogonal base.

Representation of the Heisenberg group. The Heisenberg representation r_J of the algebra $\mathrm{heis}^{\mathbb{C}}(V)$ may be integrated to an *irreducible unitary representation R_J of the Heisenberg group* $\mathrm{Heis}^{\mathbb{C}}(V)$ in the Fock space F_J. The integrated representation is given by the formula

$$R_J(\bar{z})f(\bar{w}) = f(\bar{w} - \bar{z})$$

for $\bar{z} \in \overline{W}$, and by

$$R_J(z)f(\bar{w}) = e^{<w,z>_J} f(\bar{w})$$

for $z \in W$. In particular, the creation operator $a^*(z)$ generates the multiplication operator $f(\bar{w}) \mapsto e^{<w,z>_J} f(\bar{w})$ and the annihilation operator $a(\bar{z})$ generates the translation operator $f(\bar{w}) \mapsto f(\bar{w} - \bar{z})$.

The constructed representation of the group $\mathrm{Heis}^{\mathbb{C}}(V)$ in F_J may be conveniently described in terms of the so called *coherent states*, given by the functions in F_J of the form

$$\epsilon_z(\bar{w}) := e^{<z,w>_J} \ ,$$

parameterized by vectors $z \in W$. The action of the representation of $\mathrm{Heis}^{\mathbb{C}}(V)$ on coherent states is given by the formula

$$v \in V \longmapsto R_J(v)\epsilon_z = e^{-<w,z>_J - \frac{1}{2}<w,w>_J}\epsilon_{z+w}$$

for $v = w + \bar{w}$. We have

$$< \epsilon_z, \epsilon_{z'} >_{F_J} = e^{<z,z'>_J} \tag{15.7}$$

and

$$< R_J(v)\epsilon_z, R_J(v)\epsilon_{z'} >_{F_J} = < \epsilon_z, \epsilon_{z'} >_{F_J} \ .$$

The Fock space F_J may be defined in terms of coherent states as the completion of the complex vector space, generated by vectors $\{\epsilon_z\}$, $z \in W$, with respect to the norm, given by the inner product (15.7).

Using these properties of coherent states, it may be proved (cf. [65], Sec. 9.5) that the defined representation of the Heisenberg group in the Fock space F_J is unitary and irreducible.

15.2 Action of Hilbert–Schmidt symplectic group on Fock spaces

Recall the definition of the symplectic Hilbert–Schmidt group $\mathrm{Sp_{HS}}(V)$ from Sec. 11.5. In terms of the block representation, generated by the decomposition

$$V^{\mathbb{C}} = W_+ \oplus W_- = W_0 \oplus \overline{W_0} \ ,$$

the elements A of $\mathrm{Sp_{HS}}(V)$ are written in the form

$$A = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \ ,$$

where

$$\bar{a}^t a - b^t \bar{b} = 1 \ , \ \bar{a}^t b = b^t \bar{a} \ ,$$

and the operator b is Hilbert–Schmidt. The unitary group $\mathrm{U}(W_+)$ is embedded into $\mathrm{Sp_{HS}}(V)$ as a subgroup of operators of the form

$$A = \begin{pmatrix} a & 0 \\ 0 & \bar{a} \end{pmatrix} \ .$$

In Subsec. 15.1.3 we have constructed the Heisenberg representations r_J of the Heisenberg algebra $\mathrm{heis}^{\mathbb{C}}(V)$ in Fock spaces F_J. A general *theorem of Shale* (cf. [68]) asserts that the *representations r_0 in F_0 and r_J in F_J are unitary equivalent if and only if $J \in Sp_{HS}(V)$*. In other words, for $J \in \mathrm{Sp_{HS}}(V)$ there exists a unitary intertwining operator $U_J : F_0 \to F_J$ such that

$$r_J = U_J \circ r_0 \circ U_J^{-1} \ .$$

The $\mathrm{Sp_{HS}}(V)$-action, defined by

$$\mathrm{Sp_{HS}}(V) \ni A \longmapsto U_J : F_0 \to F_J \quad \text{with } J = A \cdot J^0 \ ,$$

defines a projective (unitary) action of the group $\mathrm{Sp_{HS}}(V)$ on the Fock bundle

$$\mathcal{F} := \bigcup_{J\in\mathcal{D}_{\mathrm{HS}}} F_J \longrightarrow \mathcal{D}_{\mathrm{HS}} = \frac{\mathrm{Sp_{HS}}(V)}{\mathrm{U}(W_0)} ,$$

covering the $\mathrm{Sp_{HS}}(V)$-action on the Siegel disc D_{HS} (cf. Sec. 11.5). An explicit description of this projective action is given in [66].

15.3 Hilbert–Schmidt symplectic algebra representation

The algebra $\mathrm{sp_{HS}}(V)$ is the Lie algebra of symplectic Hilbert–Schmidt group $\mathrm{Sp_{HS}}(V)$. It follows from the definition of this group (cf. Sec. 15.2) that $\mathrm{sp_{HS}}(V)$ consists of linear operators A in $V^{\mathbb{C}}$, which have the following block representation (with respect to the decomposition $V^{\mathbb{C}} = W_0 \oplus \overline{W_0}$)

$$A = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} ,$$

where α is a bounded skew-Hermitian operator and β is a symmetric Hilbert–Schmidt operator. The complexified Lie algebra $\mathrm{sp_{HS}}(V)^{\mathbb{C}}$ consists of operators of the form

$$A = \begin{pmatrix} \alpha & \beta \\ \bar{\gamma} & -\alpha^t \end{pmatrix} ,$$

where α is a bounded operator, while β and $\bar{\gamma}$ are symmetric Hilbert–Schmidt operators.

The infinitesimalization of the projective $\mathrm{Sp_{HS}}(V)$-action on the Fock bundle $\mathcal{F}$, described in the previous Sec. 15.2, yields a projective representation of $\mathrm{sp_{HS}}(V)$ in the Fock space $F_0 \equiv F_{J_0}$. Its complexified version is given by the formula (cf. [66])

$$\mathrm{sp_{HS}}(V^{\mathbb{C}}) \ni A = \begin{pmatrix} \alpha & \beta \\ \bar{\gamma} & -\alpha^t \end{pmatrix} \longmapsto \rho(A) = D_\alpha + \frac{1}{2}M_\beta + \frac{1}{2}M_\gamma^* .$$

Here, D_α for $\alpha : W_0 \to W_0$ is the derivation of F_0 in the α-direction, defined by

$$D_\alpha f(\bar{w}) =< \alpha w, \partial_{\bar{w}} > f(\bar{w}) .$$

The operator M_β for $\beta : \overline{W_0} \to W_0$ is the multiplication operator on F_0, defined by

$$M_\beta f(\bar{w}) =< \bar{\beta} w, \bar{w} > f(\bar{w}) ,$$

and the operator M_γ^* is the adjoint of M_γ:

$$M_\gamma^* f(\bar{w}) =< \gamma\partial_w, \partial_{\bar{w}} > f(\bar{w}) .$$

This is a projective representation with the cocycle

$$[\rho(A_1), \rho(A_2)] - \rho([A_1, A_2]) = \frac{1}{2}\mathrm{tr}(\bar{\gamma}_2\beta_1 - \bar{\gamma}_1\beta_2) . \tag{15.8}$$

Note that the constructed Lie-algebra representation of $\mathrm{sp_{HS}}(V)$ is intertwined with the Heisenberg representation r_0 of $\mathrm{heis}(V)$ on F_0 (cf. [66]).

15.4 Twistor interpretation

15.4.1 Twistor bundle

Let us call a complex structure J on V *admissible*, if it can be obtained from a reference complex structure J_0 by the action of the $\mathrm{Sp}_{\mathrm{HS}}(V)$ group. Such structures are parameterized by points of the Siegel disc

$$\mathcal{D}_{\mathrm{HS}} = \mathrm{Sp}_{\mathrm{HS}}(V)/\mathrm{U}(W_0)\ .$$

The *twistor bundle* $\pi : \mathcal{Z} \to V$ is, by definition, the vector bundle of admissible complex structures on V. Its fibre $Z_x \cong \mathcal{D}_{\mathrm{HS}}$ at $x \in V$ is formed by the restrictions J_x of admissible complex structures J to the tangent space $T_xV \cong V$. The twistor bundle is a trivial bundle on V, and the admissible complex structures on V may be considered as its translation-invariant sections. In particular, we have a natural projection $p : \mathcal{Z} \to \mathcal{D}_{\mathrm{HS}}$, assigning to a point $z = (x, J_x)$ the translation-invariant complex structure $J = J_x$ on V. The fibre $p^{-1}(J)$ of this projection is identified with the space (V, J), i.e. with the space V, provided with the complex structure J. The introduced maps may be united into the following *twistor diagram*

$$\begin{tikzcd} \mathcal{Z} \arrow[r, "p"] \arrow[d, "\pi"'] & \mathcal{D}_{\mathrm{HS}} \\ V & \end{tikzcd} .$$

The *twistor space* $\mathcal{Z}$ has a natural complex structure. To define it, consider a decomposition of the tangent bundle $T\mathcal{Z}$ into the direct sum

$$T\mathcal{Z} = \mathcal{V} \oplus \mathcal{H} \tag{15.9}$$

of the *vertical subbundle* $\mathcal{V}$, identified with the tangent bundle to the fibres of π, and the *horizontal subbundle* $\mathcal{H}$, identified with the tangent bundle to the fibres of p. The complex structure $\mathcal{J}$ on $\mathcal{Z}$ is the direct sum

$$\mathcal{J}_z = \mathcal{J}_z^v \oplus \mathcal{J}_z^h$$

of the natural complex structure $\mathcal{J}_z^v$ on the vertical space $\mathcal{V}_z$, identified (by p_*) with the tangent space $T_{p(z)}\mathcal{D}_{\mathrm{HS}}$ to the Siegel disc $\mathcal{D}_{\mathrm{HS}}$, and the complex structure $\mathcal{J}_z^h = J_{\pi(z)}$ on the horizontal space $\mathcal{H}_z$, identified (by π_*) with the tangent space $T_{\pi(z)}V$. Note that the map p is holomorphic with respect to the introduced complex structure (while π is not!).

We note that with respect to the decomposition (15.9) the Heisenberg group $\mathrm{Heis}(V)$ acts on the twistor space $\mathcal{Z}$ horizontally, preserving the fibres of p, and the symplectic group $\mathrm{Sp}_{\mathrm{HS}}(V)$ acts on $\mathcal{Z}$ vertically (this action is induced by the action of $\mathrm{Sp}_{\mathrm{HS}}(V)$ on the Siegel disc $\mathcal{D}_{\mathrm{HS}}$).

15.4.2 Fock bundle

The Fock space $F_J = F(V, J)$ can be characterized in terms of the twistor diagram as the Fock space $F(p^{-1}(J))$ of holomorphic functions on the fibre $p^{-1}(J)$ (in variables

$\bar{w} \in \overline{W}_J$) with respect to the complex structure on $\mathcal{Z}$, introduced above. The *Fock bundle*

$$\mathcal{F} = \bigcup_{J \in \mathcal{D}_{\mathrm{HS}}} F_J \longrightarrow \mathcal{D}_{\mathrm{HS}}$$

is a Hermitian holomorphic Hilbert-space bundle over $\mathcal{D}_{\mathrm{HS}}$. Since $\mathcal{D}_{\mathrm{HS}}$ is contractible (even convex), it is trivial on $\mathcal{D}_{\mathrm{HS}}$. Moreover, the holomorphic map $U_J : F_0 \to F_J$, defined in Sec. 15.2, establishes an explicit holomorphic trivialization of $\mathcal{F}$. Note that the trivialization map $U_J : F_0 \to F_J$ is equivariant with respect to the action of the $\mathrm{Sp}_{\mathrm{HS}}(V)$ group.

In Sec. 15.3 a projective representation ρ of the Lie algebra $\mathrm{sp}_{\mathrm{HS}}(V)$ in the Fock space F_0 was constructed. Using this representation, we can define a linear connection on the Fock bundle $\mathcal{F}$, whose curvature coincides with the cocycle of the representation ρ.

Using the description of the Lie algebra $\mathrm{sp}_{\mathrm{HS}}(V)$, given in Sec. 15.3, we can decompose it into the direct sum

$$\mathrm{sp}_{\mathrm{HS}}(V) = \mathfrak{u}(W_0) \oplus \mathfrak{m} \ . \tag{15.10}$$

Here, $\mathfrak{u}(W_0)$ is the Lie algebra of the unitary group $\mathrm{U}(W_0)$, identified with the set of matrices

$$\begin{pmatrix} \alpha & 0 \\ 0 & -\alpha^t \end{pmatrix} ,$$

where α is a bounded skew-Hermitian operator. The linear subspace $\mathfrak{m} \cong T_0\mathcal{D}_{\mathrm{HS}}$ is identified with the set of matrices

$$\begin{pmatrix} 0 & \beta \\ \bar{\beta} & 0 \end{pmatrix} ,$$

where β is a symmetric Hilbert–Schmidt operator. Note that the adjoint action of $\mathrm{U}(W_0)$ on $\mathrm{sp}_{\mathrm{HS}}(V)$ preserves the subspace $\mathfrak{m}$.

According to the general theory of invariant connections (cf. [45], Ch. II.11), the decomposition (15.10) together with the projective representation ρ determine an $\mathrm{Sp}_{\mathrm{HS}}(V)$-invariant connection $\mathbf{A}$ on the Fock bundle $\mathcal{F}$ with the curvature, given by the cocycle of ρ.

The original quantization problem from Sec. 12.2 can be reformulated in twistor terms as follows: *construct a quantization Hilbert-space bundle $\mathcal{H} \to \mathcal{D}_{HS}$ together with a flat unitary connection on it.* The connection in this definition may be considered as an infinitesimal analogue of the BKS-operator from Sec. 14.4. In the next Sec. 15.5 we consider in more detail a relation between the twistor and Dirac quantizations of the system $(V, \mathcal{A})$, where $\mathcal{A}$ is the semi-direct product of the Heisenberg algebra $\mathrm{heis}(V)$ and the symplectic Hilbert–Schmidt algebra $\mathrm{sp}_{\mathrm{HS}}(V)$.

15.5 Quantization bundle

In this Section we construct a quantization bundle $\mathcal{H} \to \mathcal{D}_{\mathrm{HS}}$ over $\mathcal{D}_{\mathrm{HS}}$. From finite-dimensional considerations in Ch. 14, it is clear that a good candidate for $\mathcal{H}$ should be the Fock bundle of half-forms, which we are going to define next.

15.5.1 Bundle of half-forms

We define first a *bundle of half-forms*

$$K^{-1/2} \longrightarrow \mathcal{D}_{\mathrm{HS}}$$

on the Siegel disc $\mathcal{D}_{\mathrm{HS}}$.

Namely, consider on the Siegel disc $\mathcal{D}_{\mathrm{HS}}$ the following analogue of the Poincaré metric:

$$g_Z(\xi, \eta) = \mathrm{tr}\left\{(1 - \bar{Z}Z)^{-2}\xi\bar{\eta}\right\}$$

for $Z \in \mathcal{D}_{\mathrm{HS}}$, $\xi, \eta \in T_Z^{1,0}\mathcal{D}_{\mathrm{HS}} \cong E_{\mathrm{HS}}$. It is a correctly defined Kähler metric on $\mathcal{D}_{\mathrm{HS}}$ with Kähler potential $K(Z, \bar{Z}) := -\mathrm{tr}\,\log(1 - \bar{Z}Z)$. Moreover, it is invariant under the action of the group $\mathrm{Sp}_{\mathrm{HS}}(V)$ on the Siegel disc (cf. Sec. 11.4).

The canonical bundle $K \to \mathcal{D}_{\mathrm{HS}}$ is the restriction of the determinant bundle $\mathrm{Det} \to \mathrm{Gr}_{\mathrm{HS}}(V)$, defined in Sec. 5.3, to the Siegel disc $\mathcal{D}_{\mathrm{HS}}$. The metric g on $\mathcal{D}_{\mathrm{HS}}$ induces a Hermitian metric $\tilde{g}$ on K, given by the formula

$$\|(\lambda, Z)\|^2 = |\lambda|^2 \det(1 - \bar{Z}Z)^2 \tag{15.11}$$

for $\lambda \in \mathbb{C}$, $Z \in \mathcal{D}_{\mathrm{HS}}$.

There is a natural action of a central extension $\widetilde{\mathrm{Sp}_{\mathrm{HS}}(V)}$ of symplectic group $\mathrm{Sp}_{\mathrm{HS}}(V)$ on the canonical bundle K, covering the action of $\mathrm{Sp}_{\mathrm{HS}}(V)$ on the Siegel disc $\mathcal{D}_{\mathrm{HS}}$. If $\tilde{A} \in \widetilde{\mathrm{Sp}_{\mathrm{HS}}(V)}$ projects to

$$A = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \in \mathrm{Sp}_{\mathrm{HS}}(V) ,$$

then $\tilde{A}$ acts on K by the formula

$$\tilde{A} \cdot (\lambda, Z) = \left(\lambda \det(1 + \bar{a}^{-1}\bar{b}Z)^2, A \cdot Z\right) ,$$

where $A \cdot Z = (aZ + b)(\bar{b}Z + \bar{a})^{-1}$. The canonical connection on K, determined by the metric (15.11), is invariant under this $\widetilde{\mathrm{Sp}_{\mathrm{HS}}(V)}$-action on K.

The anticanonical bundle $K^{-1} \to \mathcal{D}_{\mathrm{HS}}$ of $\mathcal{D}_{\mathrm{HS}}$ coincides with the restriction of the dual determinant bundle $\mathrm{Det}^* \to \mathrm{Gr}_{\mathrm{HS}}(V)$, defined in Sec. 5.3, to $\mathcal{D}_{\mathrm{HS}}$. Since the Siegel disc $\mathcal{D}_{\mathrm{HS}}$ is contractible, the anticanonical bundle K^{-1} has a square root $K^{-1/2} \to \mathcal{D}_{\mathrm{HS}}$. The metric $\tilde{g}$ on K induces a Hermitian metric on $K^{-1/2}$, given by the formula

$$\|(\lambda, Z)\|^2 = |\lambda|^2 \det(1 - \bar{Z}Z)^{-1} . \tag{15.12}$$

The group $\widetilde{\mathrm{Sp}_{\mathrm{HS}}(V)}$ acts on $K^{-1/2}$ by the formula

$$\tilde{A} \cdot (\lambda, Z) = \left(\lambda \det(1 + \bar{a}^{-1}\bar{b}Z)^{-1}, A \cdot Z\right) .$$

The canonical connection $\mathbf{B}$ on $K^{-1/2} \to \mathcal{D}_{\mathrm{HS}}$, generated by Hermitian metric (15.12) is invariant under the action of $\widetilde{\mathrm{Sp}_{\mathrm{HS}}(V)}$ on $K^{-1/2}$.

15.5.2 Quantization bundle

By definition, the *quantization bundle* $\mathcal{H}$ coincides with the *Fock bundle of half-forms* on $\mathcal{D}_{\text{HS}}$, given by the tensor product of the Fock bundle $\mathcal{F}$ and the bundle of half-forms $K^{-1/2}$:

$$\mathcal{H} := \mathcal{F} \otimes K^{-1/2} \longrightarrow \mathcal{D}_{\text{HS}} \ .$$

We provide it with the tensor product connection

$$\mathbf{C} := \mathbf{A} \otimes 1 + 1 \otimes \mathbf{B} \ .$$

15.6 Twistor quantization of the loop space $\Omega\mathbb{R}^d$

In this Section we apply the construction of quantization bundle, described in Sec. 15.5, to the original system $(\Omega\mathbb{R}^d, \mathcal{A}_d)$. As in Sec. 9.2, we can embed the phase space $\Omega\mathbb{R}^d$ into the Sobolev space V^d of half-differentiable loops in $\mathbb{R}^d$. The space V^d coincides with the Sobolev space of half-differentiable vector-functions $S^1 \to \mathbb{R}^d$, defined in the same way, as its scalar analogue V (cf. also [17], Sec. VI.5.1). The embedding of $\Omega\mathbb{R}^d$ into V^d realizes the loop algebra $\widetilde{L\mathbb{R}^d}$ as a subalgebra of the Heisenberg algebra $\text{heis}(V^d)$ and the Lie algebra $\text{Vect}(S^1)$ as a subalgebra of the symplectic Lie algebra $\text{sp}_{\text{HS}}(V^d)$. Moreover, under the above embedding the diffeomorphism group $\text{Diff}_+(S^1)$ is realized as a subgroup of $\text{Sp}_{\text{HS}}(V^d)$. We have also, according to Sec. 11.5, a holomorphic embedding

$$\mathcal{S} = \text{Diff}_+(S^1)/\text{Möb}(S^1) \hookrightarrow \text{Sp}_{\text{HS}}(V^d)/\text{U}(W_+^d) = \mathcal{D}_{\text{HS}} \ .$$

the space $\mathcal{S}$ into the Siegel disc $\mathcal{D}_{\text{HS}}$.

Denote by

$$\mathcal{F} \longrightarrow \mathcal{S}$$

the Fock bundle over $\mathcal{S}$, obtained from the Fock bundle $\mathcal{F} \to \mathcal{D}_{\text{HS}}$ by restricting it to $\mathcal{S}$. We still have the Heisenberg representations

$$r_J : \widetilde{L\mathbb{R}^d} \longrightarrow \text{End}^* F_J$$

for $J \in \mathcal{S}$, defined by the same formulas, as in Sec. 15.1. The projective $\text{Sp}_{\text{HS}}(V^d)$-action on the Fock bundle yields a projective $\text{Diff}_+(S^1)$-action on $\mathcal{F} \to \mathcal{S}$. This action of $\text{Diff}_+(S^1)$ on $\mathcal{F} \to \mathcal{S}$ was constructed in [27]. Its infinitesimal version is a projective representation

$$\rho : \text{Vect}(S^1) \longrightarrow \text{End}^* F_0 \ .$$

It can be described explicitly in terms of the basis $\{e_n\}$ of the complexified algebra $\text{Vect}^{\mathbb{C}}(S^1)$ (cf. Sec. 2.2).

Denote by

$$L_n := \rho(e_n)$$

the operators in F_0, corresponding to the basis elements of $\text{Vect}^{\mathbb{C}}(S^1)$. They are called otherwise the *Virasoro operators* and can be computed explicitly, using the formulas, given in Sec. 15.3. The cocycle of representation ρ in the basis $\{e_n\}$ is equal to (cf. [14])

$$[\rho(e_m), \rho(e_n)] - \rho([e_m, e_n]) = \frac{d}{12}(m^3 - m)\delta_{m,-n} \ . \tag{15.13}$$

This cocycle coincides with the curvature of the connection $\mathbf{A}$ on the Fock bundle $\mathcal{F} \to \mathcal{S}$, defined in Sec. 15.4.2.

Consider the anticanonical bundle $K^{-1/2} \to \mathcal{S}$, obtained by the restriction of the bundle $K^{-1/2} \to \mathcal{D}_{\text{HS}}$ (cf. Sec. 15.5.1) to $\mathcal{S}$. The curvature of the canonical connection $\mathbf{B}$ on $K^{-1/2} \to \mathcal{S}$ in the basis $\{e_n\}$ was computed in [13]. It is equal to

$$R_{\mathbf{B}}(e_m, e_n) = -\frac{26}{12}(m^3 - m)\delta_{m,-n} \ . \tag{15.14}$$

We define the quantization bundle, as in Sec. 15.5.2, to be the Fock bundle of half-forms

$$\mathcal{H} := \mathcal{F} \otimes K^{-1/2} \longrightarrow \mathcal{S}$$

and provide it with the tensor product connection

$$\mathbf{C} := \mathbf{A} \otimes 1 + 1 \otimes \mathbf{B} \ .$$

The curvature of $\mathbf{C}$ is equal to the sum of the curvatures of connections $\mathbf{A}$ and $\mathbf{B}$, i.e.

$$R_{\mathbf{C}}(e_m, e_n) = \frac{d - 26}{12}(m^3 - m)\delta_{m,-n} \ . \tag{15.15}$$

It vanishes precisely, when $d = 26$. For this dimension our system $(\Omega\mathbb{R}^d, \mathcal{A}_d)$, where the algebra of observables $\mathcal{A}_d$ is the semi-direct product of the loop algebra $\widetilde{L\mathbb{R}^d}$ and $\text{Vect}(S^1)$, admits the twistor quantization.

To derive from an obtained solution of the twistor quantization problem a solution of the original quantization problem , i.e. a representation of the algebra of observables $\mathcal{A}_d$ in the Fock space of half-forms $\mathcal{H}_0 = F_0 \otimes K_0^{-1/2}$, identified with the fibre of the quantization bundle at the origin $o \in \mathcal{S}$, we should proceed along the same lines, as in the BKS-quantization method in Sec. 14.5. Namely, the representation of the Heisenberg algebra in the fibres of the Fock bundle $\mathcal{F}$ extends to a representation in the fibres of the quantization bundle $\mathcal{H}$. The group $\text{Diff}_+(S^1)$ acts projectively on the bundle $\mathcal{H}$ and this action intertwines with representations of the Heisenberg algebra in the fibres. The Kostant–Souriau operators L_n, corresponding to the basis elements of the algebra $\text{Vect}(S^1)$, do not preserve, in general, the spaces F_0 and $\mathcal{H}_0$, since the symplectic diffeomorphisms φ^t, corresponding to L_n, transform the spaces F_0 and $\mathcal{H}_0$ into the spaces F_t and $\mathcal{H}_t$, associated with the complex structure $J^t = \varphi^t_* \circ J^0 \circ (\varphi^t_*)^{-1}$. However, by integrating the flat Hermitian connection on the quantization bundle $\mathcal{H}$, one can construct a unitary operator U_t, identifying $\mathcal{H}_t$ with $\mathcal{H}_0$. The composition $U_t \circ L_n$ acts now in $\mathcal{H}_0$, and, after the differentiation, yields the required representation of the algebra $\text{Vect}(S^1)$ in $\mathcal{H}_0$.

15.7 Quantization of the universal Teichmüller space

In the previous Section we have defined the Fock bundle

$$\mathcal{F} \longrightarrow \mathcal{S}$$

over the smooth part $\mathcal{S} = \text{Diff}_+(S^1)/\text{Möb}(S^1)$ of the universal Teichmüller space $\mathcal{T} = \text{QS}(S^1)/\text{Möb}(S^1)$. This bundle is provided with a projective action of the diffeomorphism group $\text{Diff}_+(S^1)$, covering the natural action of $\text{Diff}_+(S^1)$ on the base $\mathcal{S}$. The infinitesimal version of this action yields a projective representation of the Lie algebra $\text{Vect}(S^1)$ in the Fock space $\mathcal{H}_0$. We can consider this construction as a geometric quantization of the phase space $\mathcal{S}$ with the algebra of observables, given by the Virasoro algebra vir, the quantization being given by the projective representation of $\text{Vect}(S^1)$ in $\mathcal{H}_0$. Note that it can be obtained by restriction to $\mathcal{S}$ of the analogous construction over the Hilbert–Schmidt Siegel disc $\mathcal{D}_{\text{HS}} = \text{Sp}_{\text{HS}}(V)/\text{U}(W_+)$, given in Subsec. 15.4.2. Recall that we have constructed there the Fock bundle

$$\mathcal{F} \longrightarrow \mathcal{D}_{\text{HS}}$$

over $\mathcal{D}_{\text{HS}}$, provided with the projective action of the symplectic group $\text{Sp}_{\text{HS}}(V)$, covering the natural action of $\text{Sp}_{\text{HS}}(V)$ on $\mathcal{D}_{\text{HS}}$. The infinitesimal version of this action yielded the projective representation of the symplectic algebra $\text{sp}_{\text{HS}}(V)$ in the Fock space $\mathcal{H}_0$, described in Sec. 15.3. This construction may be considered as a geometric quantization of the phase space $\mathcal{D}_{\text{HS}} = \text{Sp}_{\text{HS}}(V)/\text{U}(W_+)$ with the algebra of observables, given by a central extension of the Lie algebra $\text{sp}_{\text{HS}}(V)$, the quantization being given by the projective representation of $\text{sp}_{\text{HS}}(V)$ in $\mathcal{H}_0$.

Unfortunately, the described quantization procedure does not apply to the whole universal Teichmüller space $\mathcal{T} = \text{QS}(S^1)/\text{Möb}(S^1)$. According to Prop. 25 from Sec. 11.4, we can still embed this space into the infinite-dimensional Siegel disc $\mathcal{D} = \text{Sp}(V)/\text{U}(W_+)$, but we cannot construct a Fock bundle over $\mathcal{D} = \text{Sp}(V)/\text{U}(W_+)$ with a projective action of the whole symplectic group $\text{Sp}(V)$. The reason is that, according to the theorem of Shale (cf. Sec. 15.2), it is possible only for the Hilbert–Schmidt symplectic subgroup $\text{Sp}_{\text{HS}}(V)$ of $\text{Sp}(V)$. So one should look for another way of quantizing the universal Teichmüller space $\mathcal{T} = \text{QS}(S^1)/\text{Möb}(S^1)$. It seems that a natural way to do that is to use the quantized calculus of A.Connes and D.Sullivan. We now present briefly the idea of this approach in application to our problem, borrowed from Ch.IV of the Connes' book [16].

Recall that in Dirac's approach (cf. Sec. 12.2), we quantize a classical system $(M, \mathcal{A})$, consisting of the phase space M, which is a symplectic manifold, and the algebra of observables $\mathcal{A}$, which is a Poisson Lie algebra, consisting of smooth functions on M. The quantization of this system is given by a representation π of $\mathcal{A}$ in a Hilbert space H, sending the Poisson bracket $\{f, g\}$ of two functions $f, g \in \mathcal{A}$ into the commutator $[\pi(f), \pi(g)]$ (times $1/i$) of the corresponding operators. In Connes' approach the algebra of observables $\mathcal{A}$ is an associative involutive algebra, provided with an exterior differential d. Its quantization is, by definition, a representation of $\mathcal{A}$ in H, sending the differential df of a function $f \in \mathcal{A}$ into the commutator $[S, \pi(f)]$ of the operator $\pi(f)$ with a symmetry operator S, which is self-adjoint and of square 1.

If the algebra of observables $\mathcal{A}$ consists of smooth functions on the phase manifold M, this new formulation is essentially equivalent to that of Dirac. Indeed, the differential df of an observable $f \in \mathcal{A}$ is symplectically dual to the Hamiltonian vector field X_f, so we can reproduce the Poisson Lie algebra from the associative algebra with the exterior differential. On the other hand, a symmetry operator S on the polarized quantization space $H = H_+ \oplus H_-$ is given by the rule: $S = \pm I$ on $H_\pm$. But in the case, when $\mathcal{A}$ contains non-smooth functions, the Dirac definition does not work, while Connes quantization still makes sense, as we shall demonstrate on examples below.

Before that, we formalize the definition of Connes quantization. Suppose that our Hilbert space H is provided with a polarization $H = H_+ \oplus H_-$. We can associate with it a self-adjoint symmetry operator S such that

$$H_\pm = \{x \in H : Sx = \pm x\}$$

and $S^2 = I$. Suppose that the algebra of observables of our physical system $\mathcal{A}$ is an associative involutive algebra over $\mathbb{C}$ (in other words, $\mathcal{A}$ is an algebra with conjugation). A *Fredholm module* over $\mathcal{A}$ is an involutive representation π of $\mathcal{A}$ in the Hilbert space H, such that the commutator $[S, \pi(a)]$ is a compact operator for any $a \in \mathcal{A}$.

We demonstrate now that the notion of a Fredholm module provides a natural concept for the quantization of algebras of observables, containing non-smooth functions. Consider the following example, in which $\mathcal{A}$ coincides with the algebra $L^\infty(S^1)$ of bounded functions on the circle S^1. Any function $f \in \mathcal{A}$ defines a bounded multiplication operator in the Hilbert space $H = L^2(S^1)$:

$$M_f : h \in H \longmapsto fh \in H \ .$$

The operator S in this case is given by the Hilbert transform $S : L^2(S^1) \to L^2(S^1)$. The differential of a general function $f \in \mathcal{A}$ is not defined in the classical sense, but we can still define its quantum analogue by setting

$$d^q f := [S, M_f] \ .$$

The correspondence between functions $f \in \mathcal{A}$ and operators M_f on H has the following remarkable properties (cf. [64]):

1. The differential $d^q f$ is a finite rank operator if and only if f is a rational function.

2. The differential $d^q f$ is a compact operator if and only if the function f has a vanishing mean oscillation.

3. The differential $d^q f$ is a bounded operator if and only if the function f has a bounded mean oscillation.

This list may be supplemented by further function-theoretic properties of functions in $\mathcal{A}$, which have nice operator-theoretic characterizations (cf. [16], Ch.IV).

How this idea can be applied to the quantization of the universal Teichmüller space $\mathcal{T} = \mathrm{QS}(S^1)/\mathrm{M\ddot{o}b}(S^1)$? Let us switch for convenience from S^1 to the real line

$\mathbb{R}$, so that $\mathcal{T}$ will be identified with the space $\mathrm{QS}(\mathbb{R})/\mathrm{Möb}(\mathbb{R})$ of normalized quasisymmetric homeomorphisms of $\mathbb{R}$. Our main Sobolev space $H^{1/2}(\mathbb{R}) := H^{1/2}(\mathbb{R},\mathbb{R})$ of half-differentiable functions on the real line $\mathbb{R}$ has a simple description in terms of the quantum differential. Namely, the symmetry operator S is again given by the Hilbert transform

$$(Sf)(s) = \frac{1}{\pi i}\,\mathrm{P.V.}\int \frac{f(t)}{s-t}\,dt\ , \quad f \in L^2(\mathbb{R})\ , \tag{15.16}$$

where the integral is taken in the principal value sense.

The quantum differential $d^q f = [S, M_f]$ of a function $f \in L^\infty(\mathbb{R})$ is an operator on $L^2(\mathbb{R})$, given by

$$(d^q f)h(s) = \frac{1}{\pi i}\int k(s,t)h(t)\,dt$$

with the kernel, equal to

$$k(s,t) = \frac{f(s)-f(t)}{s-t}\ , \quad s,t \in \mathbb{R}\ .$$

Note that the quasiclassical limit of this operator, defined by taking the value of the kernel on the diagonal, i.e. for $s \to t$, coincides with the multiplication operator $h \mapsto f'h$, and the quantization means in this case the replacement of the derivative by its finite-difference analogue.

Then $f \in H^{1/2}(\mathbb{R})$ *if and only if its quantum differential* $d^q f$ *is a Hilbert–Schmidt operator on* $L^2(\mathbb{R})$; moreover, the Hilbert–Schmidt norm of $d^q f$ coincides with the $H^{1/2}(\mathbb{R})$-norm of f (cf. [58], Prop. 6.1). This result may be considered as a hint that the Dirac's quantization method can still be realized in the case of $\mathcal{T}$, when reformulated in terms of the quantized calculus.

The idea of how to do it, may be grasped from studying the action of the group $\mathrm{QS}(\mathbb{R})$ of quasisymmetric homeomorphisms of $\mathbb{R}$ on $H^{1/2}(\mathbb{R})$. Introduce an operator L, sending 1-forms on $\mathbb{R}$ to functions on $\mathbb{R}$, defined by

$$L\varphi(s) = \int \log|s-t|\varphi(t)\ .$$

The operator L can be considered as a "generalized inverse" to the exterior derivative d, since it is related to d by the following identities

$$d \circ L = S\ , \quad L \circ d = S\ ,$$

where the Hilbert transform S acts on 1-forms by the same formula (15.16) as above, i.e. by the integration with kernel $(s-t)^{-1}$. To describe the action of $\mathrm{QS}(\mathbb{R})$ on $H^{1/2}(\mathbb{R})$ in terms of the quantized calculus means to study its action on operators L and S.

A quasisymmetric homeomorphism $h \in \mathrm{QS}(\mathbb{R})$ transforms the operators L and S into

$$L^h := h \circ L \circ h^{-1}\ , \quad S^h := h \circ S \circ h^{-1}\ .$$

In [58] the perturbations $L^h - L$ and $S^h - S$ are explicitly computed. Namely, denote by $K^h(s,t)$ the kernel, defined by

$$K^h(s,t) = \log\frac{h(s)-h(t)}{s-t}\ .$$

Then $L^h - L$ is an integral operator with the kernel $K^h(s,t)$. Note that the quasiclassical limit of this kernel, i.e. its value on the diagonal $\{s = t\}$, coincides with $\log h'(s)$.

The quantized analogue of $L^h - L$ is given by $d^q(L^h - L)$, which is an integral operator with kernel $d_s K^h(s,t)$, having the quasiclassical limit, equal to $\frac{h''}{h'}ds$. The quantized version of $S^h - S$ is given by $d^q(S^h - S)$, which is an integral operator with kernel $d_t d_s K^h(s,t)$, having the quasiclassical limit, equal to $\frac{1}{6}\text{Schwarzian}(h)ds^2$.

Bibliographic comments

In Sec. 15.1 we have collected well known facts about the Fock spaces and Heisenberg representations. They can be found in a number of books and papers, starting from Berezin's book [7]. In Sec. 15.2 we study the projective action of the Hilbert–Schmidt symplectic group on Fock spaces. This study was initiated by Shale [68] (cf. also [66, 65, 74]). The projective representation of the Hilbert–Schmidt symplectic algebra in the Fock space was computed by Segal [66]. The Section 15.4, devoted to the twistor interpretation of our construction, is based on [63, 17]. The twistor quantization of the loop space $\Omega\mathbb{R}^d$ was initiated by Bowick–Rajeev [14]. In particular, they have found in [14] that the twistor quantization problem for $\Omega\mathbb{R}^d$ can be solved in the critical dimension $d = 26$. The last Section 15.7 is based on Ch.IV of the Connes' book [16] and Nag–Sullivan's paper [58].

Chapter 16

Quantization of the loop space $\Omega_T G$

In this Chapter we solve the geometric quantization problem for the phase space, represented by the Kähler-Frechet manifold $\Omega_T G$. The role of the algebra of observables $\mathcal{A}$ is played by the Lie algebra $\widehat{L\mathfrak{g}} \rtimes \mathrm{vir}$, an extension of the Lie algebra $L\mathfrak{g} \rtimes \mathrm{Vect}(S^1)$. The latter is the Lie algebra of the Frechet Lie group $LG \rtimes \mathrm{Diff}_+(S^1)$, the semi-direct product of the loop group LG and the diffeomorphism group $\mathrm{Diff}_+(S^1)$ of the circle.

In the most part of this Chapter we assume that G is a simply connected and simple Lie group.

16.1 Representations of loop algebras

In the loop space case the role of the Heisenberg algebra and its Heisenberg representation from Ch. 15 is played by central extensions $\widetilde{L\mathfrak{g}}$ of the loop algebras $L\mathfrak{g}$ and its lowest weight representations.

16.1.1 Affine algebras

The S^1-action plays a central role in the representation theory of the loop algebras and groups. To take care of this action, it is convenient to extend the loop algebra $L\mathfrak{g}$ to the *extended loop algebra* $\mathbb{C} \oplus L\mathfrak{g}$, the generator of U(1)-action being denoted by e_0 in accordance with Sec. 10.1. In the same way we extend the loop group LG to the *extended loop group* $\mathrm{U}(1) \ltimes LG$ by taking the semi-direct product of LG with the circle group $S^1 \equiv \mathrm{U}(1)$.

Suppose that $\mathfrak{g}_{\mathbb{C}}$ is a complex simple Lie algebra and fix a Cartan subalgebra $\mathfrak{h}_{\mathbb{C}}$ in $\mathfrak{g}_{\mathbb{C}}$. The corresponding *root decomposition* of the extended Lie algebra $\mathbb{C}e_0 \oplus L\mathfrak{g}_{\mathbb{C}}$ with respect to the Cartan subalgebra $\mathbb{C}e_0 \oplus \mathfrak{h}_{\mathbb{C}}$ has the form

$$\mathbb{C}e_0 \oplus L\mathfrak{g}_{\mathbb{C}} = \mathbb{C}e_0 \oplus \left[\bigoplus_{n\in\mathbb{Z}} \mathfrak{h}_{\mathbb{C}} z^n\right] \oplus \left[\bigoplus_{(n,\alpha)} \mathfrak{g}_{\alpha} z^n\right] , \tag{16.1}$$

where $\mathfrak{g}_\alpha$ are the root subspaces of the Lie algebra $\mathfrak{g}_{\mathbb{C}}$. The pairs $a = (n, \alpha)$, where $n \in \mathbb{Z}$ and α is a root of $\mathfrak{g}_{\mathbb{C}}$ with respect to $\mathfrak{h}_{\mathbb{C}}$, are called the *roots* of the algebra

$L\mathfrak{g}_{\mathbb{C}}$. They can be considered as linear functionals on the Lie algebra $\mathbb{C}e_0 \oplus \mathfrak{h}_{\mathbb{C}}$. If, in particular, we introduce a functional $\delta \in (\mathbb{C}e_0 \oplus \mathfrak{h}_{\mathbb{C}})^*$ by setting:

$$\delta(e_0) = 1 \ , \quad \delta(\mathfrak{h}_{\mathbb{C}}) = 0 \ ,$$

then the whole set of roots of $\mathbb{C}e_0 \oplus L\mathfrak{g}_{\mathbb{C}}$ with respect to $\mathbb{C}e_0 \oplus \mathfrak{h}_{\mathbb{C}}$ will be described as

$$\hat{\Delta} = \{\alpha + n\delta : \alpha \in \Delta,\, n \in \mathbb{Z}\} \cup \{n\delta : n \in \mathbb{Z}\} \ ,$$

where Δ is the set of roots of $\mathfrak{g}_{\mathbb{C}}$ with respect to $\mathfrak{h}_{\mathbb{C}}$. Accordingly, the set of positive roots of $\mathbb{C}e_0 \oplus L\mathfrak{g}_{\mathbb{C}}$ with respect to $\mathbb{C}e_0 \oplus \mathfrak{h}_{\mathbb{C}}$ is identified with

$$\hat{\Delta}^+ = \{\alpha + n\delta : \alpha \in \Delta,\, n > 0\} \cup \{n\delta : n > 0\} \cup \Delta^+ \ ,$$

where Δ^+ is the set of positive roots of $\mathfrak{g}_{\mathbb{C}}$ with respect to $\mathfrak{h}_{\mathbb{C}}$. If $\{\alpha_1, \dots, \alpha_l\}$ is a system of simple roots of $\mathfrak{g}_{\mathbb{C}}$ with respect to $\mathfrak{h}_{\mathbb{C}}$, and A is the highest root in Δ^+, then any root in $\hat{\Delta}^+$ may be written in the form

$$n_0\alpha_0 + n_1\alpha_1 + \cdots + n_l\alpha_l$$

with non-negative integer coefficients $n_0, n_1, \dots, n_l$, where $\alpha_0 := \delta - A$. We call $\{\alpha_0, \alpha_1, \dots, \alpha_l\}$ a system of *affine simple roots* in $\hat{\Delta}$.

We associate with any root $a = (n, \alpha)$ the *root subspace* $\mathfrak{g}_{(n,\alpha)}$ in $L\mathfrak{g}_{\mathbb{C}}$, defined by

$$\begin{aligned} \mathfrak{g}_{(n,\alpha)} &= \mathfrak{g}_\alpha z^n \quad \text{for } \alpha \neq 0 \ , \\ \mathfrak{g}_{(n,0)} &= \mathfrak{h}_{\mathbb{C}} z^n \quad \text{for } \alpha = 0 \ . \end{aligned}$$

The loop analogue of the decomposition of the Lie algebra $\mathfrak{g}_{\mathbb{C}}$

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} \oplus \mathfrak{n}^+ \oplus \mathfrak{n}^- \ ,$$

where $\mathfrak{n}^\pm$ are nilpotent subalgebras of $\mathfrak{g}_{\mathbb{C}}$ of the form

$$\mathfrak{n}^+ = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha \quad , \quad \mathfrak{n}^- = \bigoplus_{\alpha \in \Delta^-} \mathfrak{g}_\alpha \ ,$$

has the form

$$L\mathfrak{g}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} \oplus N^+\mathfrak{g}_{\mathbb{C}} \oplus N^-\mathfrak{g}_{\mathbb{C}} \ ,$$

where

$$N^+\mathfrak{g}_{\mathbb{C}} = \mathfrak{n}^+ \oplus \left[\bigoplus_{n>0} \mathfrak{g}_{\mathbb{C}} \cdot z^n \right] \quad , \quad N^-\mathfrak{g}_{\mathbb{C}} = \mathfrak{n}^- \oplus \left[\bigoplus_{n<0} \mathfrak{g}_{\mathbb{C}} \cdot z^n \right] \ .$$

The loop analogues of the Borel subalgebras have the form

$$B^\pm\mathfrak{g}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} \oplus N^\pm\mathfrak{g}_{\mathbb{C}} \ .$$

We introduce now a central extension $\widetilde{L\mathfrak{g}_{\mathbb{C}}}$ of the loop algebra $L\mathfrak{g}_{\mathbb{C}}$. Recall (cf. Sec. 8.2) that such an extension is determined by a 2-cocycle on $L\mathfrak{g}_{\mathbb{C}}$, given by the formula

$$\omega(\xi, \eta) = \omega_0(\xi, \eta) = \frac{1}{2\pi} \int_0^{2\pi} < \xi(e^{i\theta}), \eta'(e^{i\theta}) > d\theta \ , \quad \xi, \eta \in L\mathfrak{g}_{\mathbb{C}} \ ,$$

where $< \cdot, \cdot >$ is an invariant inner product on the Lie algebra $\mathfrak{g}_{\mathbb{C}}$. As a vector space,

$$\widetilde{L\mathfrak{g}_{\mathbb{C}}} = L\mathfrak{g}_{\mathbb{C}} \oplus \mathbb{C}\, c \ ,$$

with commutation relations

$$[\xi + s\, c, \eta + t\, c] = [\xi, \eta] + \omega(\xi, \eta)\, c$$

for $\xi, \eta \in L\mathfrak{g}_{\mathbb{C}}$, $s, t \in \mathbb{C}$. We denote the corresponding central extension of the loop group $LG_{\mathbb{C}}$ (cf. Sec. 8.2) by $\widetilde{LG_{\mathbb{C}}}$.

The representations of the loop algebra $L\mathfrak{g}$ and the loop group LG, which we consider here, are projective and intertwine with the S^1-action. It means that they arise, in fact, from representations of the *affine algebra*

$$\widehat{L\mathfrak{g}_{\mathbb{C}}} = \mathbb{C}e_0 \oplus \widetilde{L\mathfrak{g}_{\mathbb{C}}} = \mathbb{C}e_0 \oplus L\mathfrak{g}_{\mathbb{C}} \oplus \mathbb{C}\, c$$

and the *affine group*

$$\widehat{LG_{\mathbb{C}}} := \mathbb{C}^* \ltimes \widetilde{LG_{\mathbb{C}}} \ .$$

The root decomposition of the affine algebra $\widehat{L\mathfrak{g}_{\mathbb{C}}}$ has the form

$$\widehat{L\mathfrak{g}_{\mathbb{C}}} = \widehat{\mathfrak{h}_{\mathbb{C}}} \oplus N^+\mathfrak{g}_{\mathbb{C}} \oplus N^-\mathfrak{g}_{\mathbb{C}} \ ,$$

where

$$\widehat{\mathfrak{h}_{\mathbb{C}}} = \mathbb{C}e_0 \oplus \widetilde{\mathfrak{h}_{\mathbb{C}}} = \mathbb{C}e_0 \oplus \mathfrak{h}_{\mathbb{C}} \oplus \mathbb{C}\, c \ .$$

Accordingly,

$$\widehat{B^{\pm}\mathfrak{g}_{\mathbb{C}}} = \widehat{\mathfrak{h}_{\mathbb{C}}} \oplus N^{\pm}\mathfrak{g}_{\mathbb{C}} \ .$$

Having a root $\alpha \in \mathfrak{h}_{\mathbb{C}}^*$, we extend it to $\widehat{\mathfrak{h}_{\mathbb{C}}}^*$ by setting $\alpha(\, c) = \alpha(e_0) = 0$. We also extend the functional $\delta \in (\mathbb{C}e_0 \oplus \mathfrak{h}_{\mathbb{C}})^*$ to $\widehat{\mathfrak{h}_{\mathbb{C}}}^*$ by setting $\delta(\, c) = 0$. It's also useful to introduce a functional $\beta \in \widehat{\mathfrak{h}_{\mathbb{C}}}^*$, defined by

$$\beta(\, c) = 1 \ , \quad \beta(e_0) = 0 \ , \quad \beta(\mathfrak{h}_{\mathbb{C}}) = 0 \ .$$

With any system $\alpha_0, \alpha_1, \dots, \alpha_l$ of affine simple roots we can associate a corresponding system of co-roots $\alpha_0^\vee, \alpha_1^\vee, \dots, \alpha_l^\vee$, where $\alpha_j^\vee$, $j = 1, \dots, l$, are the co-roots, associated with simple roots α_j of the algebra $\mathfrak{g}_{\mathbb{C}}$, and

$$\alpha_0^\vee = -A^\vee + \frac{2\, c}{< A, A >}$$

is the affine co-root, associated with the highest root $A \in \Delta^+$.

Denote by $\{\omega_1, \dots, \omega_l\}$ the system of fundamental weights of the algebra $\mathfrak{g}_{\mathbb{C}}$, dual to the simple root system $\alpha_1, \dots, \alpha_l$. We can introduce the corresponding system $\{\hat{\omega}_0, \hat{\omega}_1, \dots, \hat{\omega}_l\}$ of fundamental weights of $\widehat{L\mathfrak{g}_{\mathbb{C}}}$, dual to the system $\alpha_0, \alpha_1, \dots, \alpha_l$ of affine simple roots, defined by

$$\hat{\omega}_i(\alpha_k^\vee) = \delta_{ik} \quad \text{for} \quad 0 \le i, k \le l \ , \quad \hat{\omega}_i(e_0) = 0 \ .$$

Then

$$\hat{\omega}_0 = \frac{1}{2} < A, A > \beta \ , \quad \hat{\omega}_j = \omega_j + < \omega_j, A > \beta \ , \quad 1 \le j \le l \ .$$

16.1.2 Highest weight representations of affine algebras

Suppose that $\rho : \widehat{L\mathfrak{g}_\mathbb{C}} \to V$ is a representation of the loop algebra $\widehat{L\mathfrak{g}_\mathbb{C}}$, i.e. an $\widehat{L\mathfrak{g}_\mathbb{C}}$-module. Consider for any linear form on $\widehat{\mathfrak{h}_\mathbb{C}}$, i.e. an element $\lambda \in \left(\widehat{\mathfrak{h}_\mathbb{C}}\right)^*$, the subspace

$$V_\lambda = \{v \in V : \rho(h)v = \lambda(h)v \quad \text{for } h \in \widehat{\mathfrak{h}_\mathbb{C}}\} \ .$$

If $V_\lambda \neq 0$, then λ is called the *weight* of ρ, and the subspace V_λ is the *weight subspace* of ρ, corresponding to λ. Any vector $v \in V_\lambda \setminus \{0\}$ is called the *weight vector* of ρ.

A weight $\lambda \in \left(\widehat{\mathfrak{h}_\mathbb{C}}\right)^*$ is *dominant integral*, if $\lambda(\alpha_i^\vee)$ is a non-negative integer for any affine co-root $\alpha_i^\vee$, $0 \leq i \leq l$. Any such weight can be written in the form

$$\lambda = n_0\hat{\omega}_0 + \ldots + n_l\hat{\omega}_l + s\delta \ , \tag{16.2}$$

where $n_i = \lambda(\alpha_i^\vee)$, $0 \leq i \leq l$, and $s = \lambda(e_0) \in \mathbb{C}$. Respectively, an *anti-dominant integral weight* $\lambda \in \left(\widehat{\mathfrak{h}_\mathbb{C}}\right)^*$ takes non-positive integer values on affine co-roots $\alpha_i^\vee$, $0 \leq i \leq l$, and can be written in the same form (16.2) with non-positive integer coefficients n_i, $0 \leq i \leq l$.

Given a weight $\lambda \in \left(\widehat{\mathfrak{h}_\mathbb{C}}\right)^*$, we can extend it to the Borel subalgebra $\widehat{B^+\mathfrak{g}_\mathbb{C}}$ by setting it equal to zero on $N^+\mathfrak{g}_\mathbb{C}$. Consider an $\widehat{L\mathfrak{g}_\mathbb{C}}$-module of the form

$$\hat{V} \equiv \hat{V}_\lambda = \mathcal{U}(\widehat{L\mathfrak{g}_\mathbb{C}}) \otimes_{\mathcal{U}(\widehat{B^+\mathfrak{g}_\mathbb{C}})} \mathbb{C}_\lambda \ ,$$

where the symbol "$\mathcal{U}$" stands for the universal enveloping algebra, and $\mathbb{C}_\lambda$ denotes the 1-dimensional $\widehat{B^+\mathfrak{g}_\mathbb{C}}$-module, i.e. the complex line $\mathbb{C}$, provided with an action of the Borel subalgebra $\widehat{B^+\mathfrak{g}_\mathbb{C}}$, given by: $z \longmapsto \lambda(b)z$ for $b \in \widehat{B^+\mathfrak{g}^\mathbb{C}}$, $z \in \mathbb{C}$. Since

$$\widehat{L\mathfrak{g}_\mathbb{C}} = N^-\mathfrak{g}_\mathbb{C} \oplus \widehat{B^+\mathfrak{g}_\mathbb{C}} \ ,$$

the Poincaré–Birkhoff–Witt theorem implies that

$$\mathcal{U}(\widehat{L\mathfrak{g}_\mathbb{C}}) \cong \mathcal{U}(N^-\mathfrak{g}_\mathbb{C}) \otimes \mathcal{U}(\widehat{B^+\mathfrak{g}_\mathbb{C}}) \ .$$

So we have a natural isomorphism

$$\hat{V}_\lambda \cong \mathcal{U}(N^-\mathfrak{g}_\mathbb{C}) \otimes \mathbb{C}_\lambda \ .$$

Denote by $V \equiv V_\lambda$ the quotient of $\hat{V}$ modulo the maximal submodule in $\hat{V}$, strictly contained in $\hat{V}$ (in other words, the maximal submodule, not containing $1 \otimes 1$). This V, together with the natural action of $\widehat{L\mathfrak{g}_\mathbb{C}}$, is called the *standard representation of the Lie algebra* $\widehat{L\mathfrak{g}_\mathbb{C}}$ *with the highest weight* λ and the weight vector $1 \otimes 1$.

More generally, we shall say that a representation $\rho : \widehat{L\mathfrak{g}_\mathbb{C}} \to \operatorname{End} V_\lambda$ of the affine algebra $\widehat{L\mathfrak{g}_\mathbb{C}}$ is the *highest weight representation* with weight $\lambda \in \widehat{\mathfrak{h}_\mathbb{C}}^*$, if there exists a *highest weight vector* $v_\lambda \in V_\lambda$ such that:

1. $\rho(h)v_\lambda = \lambda(h)v_\lambda \quad$ for any $h \in \widehat{\mathfrak{h}_\mathbb{C}}$;

2. $\rho(n)v_\lambda = 0 \quad$ for any $n \in N^+\mathfrak{g}_\mathbb{C}$;

3. V_λ is spanned by vectors $\rho(b)v_\lambda$ with $b \in \widehat{B^-\mathfrak{g}_\mathbb{C}}$.

The highest weight vector v_λ plays the role, analogous to that of the vacuum in the Heisenberg representation.

In analogous way one can define the *lowest weight representation* of the affine algebra $\widehat{L\mathfrak{g}_\mathbb{C}}$. For that one should replace in the above definition the nilpotent subalgebra $N^+\mathfrak{g}_\mathbb{C}$ by the nilpotent subalgebra $N^-\mathfrak{g}_\mathbb{C}$ and the Borel subalgebra $\widehat{B^-\mathfrak{g}_\mathbb{C}}$ by the Borel subalgebra $\widehat{B^+\mathfrak{g}_\mathbb{C}}$.

The standard $\widehat{L\mathfrak{g}_\mathbb{C}}$-module V_λ, defined above, is an irreducible highest weight representation of $\widehat{L\mathfrak{g}_\mathbb{C}}$, if $\lambda \in \left(\widehat{\mathfrak{h}_\mathbb{C}}\right)^*$ is an integral dominant weight. Moreover, it was proved in [23], that if $\lambda(e_0)$ is real, then V_λ admits a positive-definite (contravariant) Hermitian inner product. We denote by $H \equiv H_\lambda$ the completion of $V \equiv V_\lambda$ with respect to this inner product. The space H_λ will play the role of the Fock space, associated with the weight λ.

16.2 Representations of loop groups

We present here some general properties of irreducible representations of the affine group $\widehat{LG^\mathbb{C}}$ and the Borel–Weil construction for $\widehat{LG^\mathbb{C}}$.

16.2.1 Irreducible representations of affine groups

Consider the affine group

$$\widehat{LG} := \mathrm{U}(1) \ltimes \widetilde{LG}$$

and fix a maximal torus $\widehat{T}$ in $\widehat{LG}$, given by

$$\widehat{T} := \mathrm{U}(1) \times T \times S$$

Here, the first factor $\mathrm{U}(1) = S^1$ is the group of rotations, the second factor T is a maximal torus in G, and the third one $S = S^1$ is a central subgroup in $\widetilde{LG}$.

Any irreducible representation of the affine group $\widehat{LG}$ has a unique highest weight λ, which is a character of the maximal torus $\widehat{T}$. This character has the form

$$\lambda = (n, \lambda_0, h) \ ,$$

where $n \in \mathbb{Z}$ is an eigenvalue of the S^1-rotation operator e_0, called the *energy* of the representation, λ_0 is a character of T, and $h \in \mathbb{Z}$ is an eigenvalue of the central subgroup action, called the *level* of the representation. The highest weights of $\widehat{LG}$ are integral and dominant and the isomorphism classes of irreducible representations of $\widehat{LG}$ are in 1:1 correspondence with the set of integral dominant weights.

There is a similar characterization of irreducible representations of the affine group $\widehat{LG}$ in terms of lowest weights.

16.2.2 Borel–Weil construction

Consider the full flag loop space (cf. Sec. 7.6)

$$\Omega_T G = LG/T = LG^{\mathbb{C}}/B^+G^{\mathbb{C}} \ .$$

In terms of central extensions, $\Omega_T G$ may be written in the form

$$\Omega_T G = \widetilde{LG^{\mathbb{C}}}/\widetilde{B^+G^{\mathbb{C}}} \ .$$

Suppose that λ is a lowest weight of the maximal torus $\widetilde{T} = T \times S$. We extend it to $\widetilde{B^+G^{\mathbb{C}}}$ by setting $\lambda = 1$ on the nilpotent subgroup $N^+G^{\mathbb{C}}$ in

$$\widetilde{B^+G^{\mathbb{C}}} = \widetilde{T^{\mathbb{C}}} \times N^+G^{\mathbb{C}} \ .$$

Define a holomorphic line bundle $L = L_\lambda$ over $\Omega_T G$ by

$$L = \widetilde{LG^{\mathbb{C}}} \times_{\widetilde{B^+G^{\mathbb{C}}}} \mathbb{C} \longrightarrow \Omega_T G = \widetilde{LG^{\mathbb{C}}}/\widetilde{B^+G^{\mathbb{C}}} \ ,$$

where $\widetilde{B^+G^{\mathbb{C}}}$ acts on the complex line $\mathbb{C}$ by the character λ

$$\widetilde{B^+G^{\mathbb{C}}} \ni b :\longmapsto \lambda(b) z \ .$$

Denote by $\Gamma = \Gamma_\lambda$ the vector space of holomorphic sections of $L = L_\lambda$. Sections $s \in \Gamma$ can be identified with holomorphic functions $\dot{s} : \widetilde{LG^{\mathbb{C}}} \to \mathbb{C}$, satisfying the condition

$$\dot{s}(\gamma b^{-1}) = \lambda(b)\dot{s}(\gamma)$$

for any $b \in \widetilde{B^+G^{\mathbb{C}}}$, $\gamma \in \widetilde{LG^{\mathbb{C}}}$. The group $\widetilde{LG^{\mathbb{C}}}$ acts in a natural way on L and on Γ, and this action defines a holomorphic representation of $\widetilde{LG^{\mathbb{C}}}$ on Γ. We note that Γ *is non-trivial* (i.e. contains non-zero holomorphic sections of L) *if and only if the weight λ is anti-dominant* (cf. [65], Prop. 11.3.1). Under this condition it may be proved (cf. [65], Prop. 11.1.1) that the corresponding representation of the loop group $\widetilde{LG}$ is an irreducible lowest weight representation of $\widetilde{LG}$ with the lowest weight λ. Moreover, it can be proved (cf. [65], Prop. 11.2.3) that any irreducible representation of the group $\widetilde{LG}$ is essentially equivalent to some Γ_λ.

Note that Γ contains a 1-dimensional subspace of sections, invariant under the action of the nilpotent subgroup $N^-G^{\mathbb{C}}$. Indeed, it follows from the representation (7.18) in Sec. 7.6 that $\Omega_T G$ contains a dense open orbit, containing the origin $o \in \Omega_T G$, which can be identified with the subgroup $N^-G^{\mathbb{C}}$. Hence, any $N^-G^{\mathbb{C}}$-invariant section in Γ is uniquely determined by its value at o. We take for the vacuum the lowest weight vector $v = v_\lambda$, which is an $N^-G^{\mathbb{C}}$-invariant section in Γ, equal to 1 at the origin o.

There is a Hermitian inner product, defined on a dense subspace of Γ. Namely, consider the anti-dual space $\overline{\Gamma}^*$ and introduce a complex-linear map $\beta : \overline{\Gamma}^* \to \Gamma$, which value on the element $\xi \in \overline{\Gamma}^*$ is a section $\beta(\xi) \in \Gamma$, identified with the function $\dot{\beta}(\xi)$ on $\widetilde{LG^{\mathbb{C}}}$, defined by

$$\dot{\beta}(\xi)(\gamma) := \xi(\gamma \cdot v) \quad \text{for } \gamma \in \widetilde{LG^{\mathbb{C}}} \ .$$

Using this map, we define a Hermitian inner product of two elements $\xi, \eta \in \overline{\Gamma}^*$ by

$$< \xi, \eta > := \eta \left(\overline{\beta(\xi)} \right) .$$

The constructed inner product on $\overline{\Gamma}^*$ is positive definite and we denote by $H = H_\lambda$ the completion of $\overline{\Gamma}^*$ with respect to this inner product, so that $\overline{\Gamma}^* \subset H \subset \Gamma$. The space H plays the role of the Fock space, associated with the lowest weight λ.

The elements ϵ_γ of $\overline{\Gamma}^*$ with $\gamma \in \widetilde{LG}$, defined by

$$\epsilon_\gamma(s) := \overline{\dot{s}(\bar{\gamma}^{-1})} , \quad s \in \Gamma ,$$

play the role of the coherent states. They have the inner product, equal to

$$< \epsilon_{\gamma_1}, \epsilon_{\gamma_2} >= v(\gamma_2 \bar{\gamma}_1^{-1}) ,$$

and generate a dense subset in $\overline{\Gamma}^*$.

16.3 Twistor quantization of $\Omega_T G$

There are two different approaches to the geometric quantization of the loop space $\Omega_T G$. One method is to replace the original classical system $(\Omega_T G, \widehat{L\mathfrak{g}} \rtimes \mathrm{vir})$ by an enlarged system. One can do it by enlarging first the phase space $\Omega_T G$ to the Sobolev space HG of half-differentiable loops in G (cf. Sec. 9.1), and then embedding HG into the space $VG := H^{1/2}(S^1, \mathrm{GL}(V))$, using a faithful representation V of the group G. Accordingly, the algebra of observables $\widehat{L\mathfrak{g}} \rtimes \mathrm{vir}$ should be enlarged to an algebra $\mathcal{A}$, which is an extension of the semi-direct product of the algebra $H\mathfrak{g}$, embedded into $V\mathfrak{g} := H^{1/2}(S^1, \mathrm{End}(V))$, and the Lie algebra of the symplectic Hilbert–Schmidt group $\mathrm{Sp}_{\mathrm{HS}}(V)$, acting on VG and $V\mathfrak{g}$ by change of variables. We obtain the quantization of the original system by first quantizing the enlarged system and then by restricting this quantization to the original system. The described method was used in Ch. 15 for the quantization of $\Omega\mathbb{R}^d$. In this Chapter we follow a more direct approach, based on the Goodman–Wallach construction of a projective action of the diffeomorphism group $\mathrm{Diff}_+(S^1)$ on representations of the affine algebra $\widehat{L\mathfrak{g}^{\mathbb{C}}}$ and affine group $\widehat{LG^{\mathbb{C}}}$.

16.3.1 Projective representation of Vect(S^1)

The projective action of $\mathrm{Diff}_+(S^1)$, mentioned in the introduction to this Section, can be generated by exponentiating a projective representation of the Lie algebra $\mathrm{Vect}(S^1)$, constructed in this Subsection.

Choose an orthonormal base $\{e_\alpha\}$, $\alpha = 1, \ldots, N$, of the Lie algebra $\mathfrak{g}$ with respect to an invariant inner product $< \cdot, \cdot >$ on $\mathfrak{g}$. Then the elements

$$e_\alpha(n) := e_\alpha z^n , \quad z = e^{i\theta},\ \alpha = 1, \ldots, N,\ n \in \mathbb{Z} ,$$

form a basis in the vector space $L\mathfrak{g}^{\mathbb{C}}$.

Introduce for $k \in \mathbb{Z}$ the *Casimir operators*, given by the formal series

$$\Delta_k := \frac{1}{2} \sum_{n \in \mathbb{Z}} \sum_{\alpha=1}^{N} :\, e_\alpha(n)\, e_\alpha(k-n))\, : \quad ,$$

where the *normally ordered product* $: \cdot :$ is defined by the rule

$$: e(m)e(n) := \begin{cases} e(m)e(n) & \text{for } m \leq n \\ e(n)e(m) & \text{for } m > n \ . \end{cases}$$

The Casimir operators Δ_k are correctly defined, when applied to any element $v \in V$, since in this case the power series reduces to a finite sum (cf. [23]). In other words, the Casimir operators determine endomorphisms of V. The operator Δ_k is homogeneous of order k with respect to the action of the operator e_0 in the sense that

$$e_0 \Delta_k v = \Delta_k (e_0 + k) v \quad \text{for any } v \in V \ .$$

Moreover, for any $\xi \in \mathfrak{g}^{\mathbb{C}}$ and any $n \in \mathbb{Z}$ the following relation between operators on V holds

$$[\xi(n), \Delta_k] = n \left(c + \frac{1}{2} \right) \xi(n+m) \ .$$

Given a $\lambda \in (\widehat{L\mathfrak{h}^{\mathbb{C}}})^*$, denote by λ_0 its restriction to the Cartan subalgebra $\mathfrak{h}$, and set $\rho = \sum_{j=1}^{l} \omega_j$. Then we have the following

Proposition 30. *([26]) The operators* $\Delta_0 + \left(c + \frac{1}{2}\right) e_0$ *and*

$$[\Delta_m, \Delta_n] + \left(c + \frac{1}{2} \right) (n-m) \Delta_{m+n}$$

commute with the action of $\widetilde{L\mathfrak{g}^{\mathbb{C}}}$ *on* V*. Moreover,*

$$\Delta_0 = -\mu e_0 + \left(\frac{1}{2} < \lambda_0, \lambda_0 + 2\rho > + \mu \lambda(e_0) \right) I \ ,$$
$$[\Delta_m, \Delta_n] = \mu(m-n)\Delta_{m+n} + \delta_{m,-n} \nu m (m^2 - 1) \ ,$$

where $\mu := \lambda(c) + \frac{1}{2}$, $\nu := \frac{\dim \mathfrak{g}}{12} \lambda(c) \mu$.

Using the introduced Casimir operators, we construct a projective action of $\mathrm{Vect}(S^1)$ on V. More precisely, recall (cf. Sec. 10.1) that the Virasoro algebra vir is a central extension of the Lie algebra $\mathrm{Vect}(S^1)$. As a vector space, vir $= \mathrm{Vect}(S^1) \oplus \mathbb{R}\kappa$, and the Lie bracket is given by

$$[\xi + s\kappa, \eta + t\kappa] = [\xi, \eta] + \omega(\xi, \eta)\kappa \ ,$$

where $\xi, \eta \in \mathrm{Vect}(S^1)$, $s, t \in \mathbb{R}$, and ω is the Gelfand–Fuks cocycle, defined on the basis elements $\{e_n\}$ by

$$\omega(e_m, e_n) = \delta_{m,-n} \frac{n(n^2-1)}{12} \ .$$

Then the following Theorem is true.

Theorem 13. *([26]) Let $(V,\pi) \equiv (V_\lambda, \pi_\lambda)$ be a highest weight representation of $\widehat{L\mathfrak{g}^{\mathbb{C}}}$ with the dominant integral weight λ. Introduce the operators*

$$D_k := -\frac{1}{\mu}\Delta_k \quad \textit{for } k \in \mathbb{Z} \ .$$

Then the representation π of $\widetilde{L\mathfrak{g}^{\mathbb{C}}}$ on V can be extended to a representation $\hat{\pi}$ of the algebra $\widetilde{L\mathfrak{g}^{\mathbb{C}}} \rtimes vir$ on V by setting

$$\hat{\pi}(e_k) = D_k \ , \quad \hat{\pi}(\kappa) = \frac{\dim \mathfrak{g}}{12\mu}\lambda(c)I \ .$$

Moreover, V can be provided with a positive definite Hermitian form, contravariant with respect to $\widetilde{L\mathfrak{g}^{\mathbb{C}}} \rtimes vir$.

The operator $D_0 = \hat{\pi}(e_0)$ from Theor. 13, which is given by the formula

$$D_0 = \pi(e_0) - \lambda(e_0) - \frac{<\lambda_0, \lambda_0 + 2\rho>}{2\lambda(c)+1} \ ,$$

is diagonalizable on V with eigenvalues

$$\mu_i = -i - \frac{<\lambda_0, \lambda_0 + 2\rho>}{2\lambda(c)+1} \ , \quad i = 0, 1, \ldots, \ .$$

The eigenspaces of D_0 are finite-dimensional and mutually orthogonal. Denote by T the closure of $I - D_0$, then T is a self-adjoint operator, bounded from below by I and having a compact inverse T^{-1}. So by spectral theorem, all its powers T^t with $t \in \mathbb{R}$ are correctly defined and we can set

$$\|v\|_t := \|T^t v\| \quad \text{for any } v \in V \ .$$

Denote by $H^t \equiv H^t_\lambda$ the completion of $V \equiv V_\lambda$ with respect to the norm $\|\cdot\|_t$ and set

$$H^\infty \equiv H^\infty_\lambda = \bigcap_{t\in\mathbb{R}} H^t_\lambda \ , \quad H^{-\infty} \equiv H^{-\infty}_\lambda = \bigcup_{t\in\mathbb{R}} H^t_\lambda \ .$$

The inner product on H defines a sesquilinear pairing between H^∞ and $H^{-\infty}$, and the operator T^t yields an isomorphism between H^s and H^{t-s}, defining a pairing between them, given by

$$(u,v) := (T^t u, T^{-t} v) \quad \text{for } u \in H^t, v \in H^{-t} \ ,$$

where the inner product on the right is taken in H.

16.3.2 Goodman–Wallach construction

We extend a natural right action of $\mathrm{Diff}_+(S^1)$ on $L\mathfrak{g}^{\mathbb{C}}$ by change of variables to $\widetilde{L\mathfrak{g}^{\mathbb{C}}}$, demanding that $\mathrm{Diff}_+(S^1)$ acts trivially on the central subalgebra in $\widetilde{L\mathfrak{g}^{\mathbb{C}}}$. For $f \in \mathrm{Diff}_+(S^1)$ we denote the action of f on $\widetilde{L\mathfrak{g}^{\mathbb{C}}}$ by: $\xi \mapsto \xi_f$ for $\xi \in \widetilde{L\mathfrak{g}^{\mathbb{C}}}$.

Given a highest weight representation $(V,\pi) \equiv (V_\lambda, \pi_\lambda)$ of $\widetilde{L\mathfrak{g}^{\mathbb{C}}}$ we define an action of $f \in \mathrm{Diff}_+(S^1)$ on (V,π) by setting

$$f : \pi \longmapsto \pi_f \ , \quad \text{where } \pi_f(\xi)v := \pi(\xi_f)v$$

for $\xi \in \widetilde{L\mathfrak{g}^{\mathbb{C}}}$, $v \in V$. Note that for $v \in H^\infty$ the image $\pi(\xi_f)v$ is again in H^∞. The main result of [26] asserts that representations π and π_f are unitary equivalent. More precisely, we have the following

Theorem 14. *(Goodman–Wallach [26]) There is a unitary projective action σ of $\mathit{Diff}_+(S^1)$ on $H \equiv H_\lambda$ such that the map*

$$\mathit{Diff}_+(S^1) \times H^n \longrightarrow H^n \ , \quad (f,v) \longmapsto \sigma(f)v \ ,$$

is continuous for any $n \geq 0$, and

$$\sigma(f)\pi_f(\xi)v = \pi(\xi)\sigma(f)v$$

for any $v \in H^\infty$, $f \in \mathit{Diff}_+(S^1)$, $\xi \in \widetilde{L\mathfrak{g}^{\mathbb{C}}}$.

Moreover, in [26] it is proved that this $\mathrm{Diff}_+(S^1)$-action on H is uniquely defined up to projective equivalence. More precisely, suppose that τ is another projective action of $\mathrm{Diff}_+(S^1)$ on H, such that $\tau_f H^\infty \subset H^\infty$ for any $f \in \mathrm{Diff}_+(S^1)$, which intertwines π with π_f, i.e.

$$\tau_f \pi_f(\xi) = \pi(\xi)\tau_f$$

for any $f \in \mathrm{Diff}_+(S^1)$, $\xi \in \widetilde{L\mathfrak{g}^{\mathbb{C}}}$. Then there exists a continuous map $\mu : \mathrm{Diff}_+(S^1) \to S^1$, such that $\tau_f = \mu(f)\sigma_f$.

16.3.3 Twistor quantization of $\Omega_T G$

In Subsec. 16.2.2 we have constructed for any lowest weight λ of the loop algebra $\widetilde{L\mathfrak{g}}$ a holomorphic line bundle $L \equiv L_\lambda \to \Omega_T G$ and the space $\Gamma \equiv \Gamma_\lambda$ of its holomorphic sections, on which the representation of $\widetilde{LG}$ with lowest weight λ is realized. We denoted by $H \equiv H_\lambda$ the completion of $\overline{\Gamma}^*$ with respect to the natural norm on $\overline{\Gamma}^*$.

This construction depends on the complex structure on $\Omega_T G$, which is provided by the complex representation

$$\Omega_T G = LG^{\mathbb{C}}/B^+G^{\mathbb{C}} \ .$$

Denote this complex structure by J^0 and the corresponding spaces of sections Γ_λ and H_λ respectively by Γ_0 and H_0, so that we have a representation π_0 of $\widetilde{LG}$ in Γ_0.

If we change this complex structure to J_f by the action of a diffeomorphism $f \in \mathrm{Diff}_+(S^1)$, then we can again, using the Borel–Weil construction, realize the lowest weight representation π_f of the group $\widetilde{LG}$, corresponding to the lowest weight λ, in the space Γ_f of sections of L, holomorphic with respect to the complex structure J_f on $\Omega_T G$. Denote the corresponding completion of $\overline{\Gamma}_f^*$ by H_f.

By the Goodman–Wallach construction, there is a projective unitary action

$$U_f : \Gamma_0 \longrightarrow \Gamma_f$$

of the group $\mathrm{Diff}_+(S^1)$, intertwining the representations π_0 and π_f:

$$\pi_f U_f(v) = U_f \pi_0(v) \quad \text{for } v \in H_0 \ .$$

It is uniquely defined by the normalization condition on the lowest weight vectors: $U_f v_0 = v_f$, and defines a continuous unitary operator

$$U_f : H^0 \longrightarrow H^f \ .$$

So we have again, as in Sec. 15.4, a holomorphic Hilbert space bundle

$$H = \bigcup_{f \in \mathcal{S}} H_f$$

and a projective unitary action of the group $\mathrm{Diff}_+(S^1)$ on H, given by $f \mapsto U_f$, which covers the natural $\mathrm{Diff}_+(S^1)$-action on $\mathcal{S}$. The infinitesimalization of this action yields a projective unitary representation ρ of lowest weight λ of the Lie algebra $\mathrm{Vect}(S^1)$ in the space H_0, constructed in Subsec. 16.3.1.

Having a projective representation ρ of $\mathrm{Vect}(S^1)$, we can construct a $\mathrm{Diff}_+(S^1)$-invariant connection $\mathbf{A}$ on the bundle $H \to \mathcal{S}$, whose curvature at the origin $o \in \mathcal{S}$ coincides with the cocycle of ρ, given in the basis $\{e_k\}$ by (cf. [53, 54])

$$[\rho(e_m), \rho(e_n)] - \rho\left([e_m, e_n]\right) = \frac{c(\mathfrak{g})}{12}(m^3 - m)\delta_{m,-n} \ ,$$

where

$$c(\mathfrak{g}) = \frac{h \dim \mathfrak{g}}{h + \kappa(\mathfrak{g})} \ ,$$

and $\kappa(\mathfrak{g})$ is the *dual Coxeter number* of $\mathfrak{g}$ (cf., e.g., [76]).

The construction of the connection $\mathbf{A}$ is similar to that in Subsec. 15.4.2. Namely, we have again a splitting of the Lie algebra $\mathrm{Vect}(S^1)$ into the direct sum

$$\mathrm{Vect}(S^1) = \mathrm{sl}(2, \mathbb{R}) \oplus \mathfrak{m} \ ,$$

where $\mathrm{sl}(2, \mathbb{R})$ is the Lie algebra of $\mathrm{Möb}(S^1)$ and $\mathfrak{m} \cong T_0\mathcal{S}$. This splitting is, in fact, induced by the splitting (15.10) from Subsec. 15.4.2, under the embedding of $\mathrm{Vect}(S^1)$ into $\mathrm{sp}_{\mathrm{HS}}(H_0)$. The above splitting together with the projective representation $\rho : \mathrm{Vect}(S^1) \to \mathrm{End}\,(H_0)$ determine, as in Subsec. 15.4.2, a $\mathrm{Diff}_+(S^1)$-invariant connection $\mathbf{A}$ on the bundle $H \to \mathcal{S}$, whose curvature at the origin $o \in \mathcal{S}$ coincides with the cocycle of ρ.

Consider now, as in Sec. 15.5.2, the quantization bundle

$$\mathcal{H} := H \otimes K^{-1/2} \to \mathcal{S}$$

and provide it with the tensor-product connection $\mathbf{C}$:

$$\mathbf{C} := \mathbf{A} \otimes 1 + 1 \otimes \mathbf{B} \ ,$$

where $\mathbf{B}$ is the connection on $K^{-1/2}$, defined in Subsec. 15.5.1. The curvature of $\mathbf{C}$ in the basis $\{e_k\}$ is equal to

$$R_{\mathbf{C}}(e_m, e_n) = \frac{c(\mathfrak{g}) - 26}{12}(m^3 - m)\delta_{m,-n} \ ,$$

which vanishes precisely for $c(\mathfrak{g}) = 26$. Under this condition we get a flat unitary connection on $\mathcal{H}$. By integrating it, we obtain a unitary action of $\mathrm{Diff}_+(S^1)$ on $\mathcal{H}$, yielding the geometric quantization of the system $(\Omega_T G, \mathcal{A})$ in H_0.

Bibliographic comments

In Sec. 16.1 we follow mostly the papers [23, 26]. The Borel–Weil construction of the lowest weight representations of the loop group is explained in Chap. 11 of Pressley–Segal's book [65]. The projective action of the diffeomorphism group $\mathrm{Diff}_+(S^1)$ on the lowest weight representations of the loop algebra is studied in detail in Goodman–Wallach's paper [26]. Its infinitesimal version, i.e. the projective representation of the Lie algebra $\mathrm{Vect}(S^1)$, given by the Casimir operators, is well known and may be found, for example, in the books [38],[65]. The geometric quantization of loop spaces of compact Lie groups was first considered by Mickelsson [53, 54].

REFERENCES

Bibliography

[1] L.Ahlfors, *Lectures on Quaiconformal Mappings*, Van Nostrand, Princeton, 1966.

[2] L.Ahlfors, L.Bers, Riemann's mapping theorem for variable metrics, Ann. Math. **72**(1960), 385-404.

[3] L.Ahlfors, Some remarks on Teichmüller's space of Riemann surfaces, Ann. Math. **74**(1961), 171-191.

[4] V.I.Arnold, Small denominators I; on the mapping of a circle into itself, Trans. Amer. Math. Soc. **46**(1961), 213-284.

[5] M.F.Atiyah, Instantons in two and four dimensions, Commun. Math. Phys. **93**(1984), 437-451.

[6] R.J.Baston, M.G.Eastwood, *The Penrose Transform: its Interaction with Representation Theory*, Clarendon Press, Oxford, 1989.

[7] F.A.Berezin, *The Method of Second Quantization*, Academic Press, N.Y.-London, 1966.

[8] G.D.Birkhoff, Singular points of ordinary differential equations, Trans. Amer. Math. Soc. **10**(1909), 436-470.

[9] G.D.Birkhoff, Equivalent singular points of ordinary linear differential equations, Math. Ann. **74**(1913), 134-139.

[10] A.Borel, Kählerian coset spaces of semi-simple Lie groups, Proc. Nat. Acad. Sci. USA **40**(1954), 1147-1151.

[11] R.Bott, On the characteristic classes of groups of diffeomorphisms, Enseign. math. **23**(1977), N 3-4, 209-220.

[12] R.Bowen, Hausdorff dimension of quasicircles, Publ. Math. IHES **50**(1979), 259-273.

[13] M. J. Bowick, A. Lahiri, The Ricci curvature of $\mathrm{Diff}\, S^1/\mathrm{SL}(2,\mathbb{R})$, J. Math. Phys. **29**(1988), 1979-1981.

[14] M.J.Bowick, S.G.Rajeev, The holomorphic geometry of closed bosonic string theory and $\mathrm{Diff}\, S^1/S^1$, Nucl. Phys. **B293** (1987), 348-384.

[15] F.E.Burstall, J.H.Rawnsley, *Twistor Theory for Riemannian Symmetric Spaces. With Applications to Harmonic Maps of Riemann Surfaces*, Lecture Notes in Math. **1424**, 1990.

[16] A.Connes, *Geometrie Non Commutative*, Intereditions, Paris, 1990.

[17] J.Davidov, A.G.Sergeev, Twistor quantization of loop spaces, Steklov Inst. Proc. **217**(1997), 1-90.

[18] S.K.Donaldson, P.B.Kronheimer, *Geometry of Four-Manifolds*, Oxford University Press, Oxford, 1990.

[19] A.Douady, C.J.Earle, Conformally natural extension of homeomorphisms of the circle, Acta Math. **157**(1986), 23-48.

[20] C.J.Earle, J.Eells, On the differential geometry of Teichmüller spaces, J.Analyse Math. **19**(1967), 35-52.

[21] B.L.Feigin, D.B.Fuks, Cohomology of Lie groups and algebras. In: Itogi nauki i techn. Sovr. probl. matem. Fund. Naprav. **21**, VINITI, Moscow, 1988, 121–209 (In Russian)

[22] D.B.Fuks, *Cohomology of Infinite-Dimensional Lie Algebras*, Consultants Bureau, N.Y., 1986.

[23] H.Garland, The arithmetic theory of loop algebras, J. Algebra **53**(1978), 480–581.

[24] F.W.Gehring, J.Väisälä, Hausdorff dimension and quasiconformal mappings, J. London Math. Soc. **6**(1973), 504-512.

[25] I.M.Gelfand, D.B.Fuks, Cohomology of the Lie algebra of vector fields on the circle, Funct. Anal and its Appl. **2**(1968), 342-343.

[26] R.Goodman, N.R.Wallach, Structure and unitary cocycle representations of loop groups and the group of diffeomorphisms of the circle, J. reine angew. Math. **B347**(1984), 67-133.

[27] R.Goodman, N.R.Wallach, Projective unitary positive-energy representations of Diff(S^1), J. Funct. Anal. **63** (1985), 299-321.

[28] M.Goto, F.D.Grosshans, *Semisimple Lie Algebras*, M.Dekker, N.Y.-Basel, 1978.

[29] V.Guillemin, S.Sternberg, *Geometric Asymptotics*, Amer. Math. Soc., Providence, R.I., 1977.

[30] L.Guieu, Nombre de rotation, structures géométriques sur un cercle et groupe de Bott–Virasoro, Ann. Inst. Fourier **46**(1996), 971-1009.

[31] A.Guichardet, *Cohomologie des groupes topologiques et des algebres de Lie*, Cedic/Nathan, Paris, 1980.

[32] R.S.Hamilton, The inverse function theorem of Nash and Moser, Bull. Amer. Math. Soc. **7**(1982), 65-222.

[33] M.R.Herman, Sur la groupe des difféomorphismes du tore, Ann. Inst. Fourier **23**, N 2, 75-86.

[34] M.R.Herman, Sur la conjugaison différentiable des difféomorphismes du cercle a des rotations, Publ. Math. IHES **49**(1979), 5-234.

[35] L.Van Hove, Sur certaines représentations unitaires des groupes infini de transformations, Mem. Acad. Roy. Belgique **26**(1951), 1-102.

[36] S.T.Hu, *Homotopy Theory*, Academic Press, N.Y., 1959.

[37] N.E.Hurt, *Geometric Quantization in Action*, D.Reidel, Dordrecht-Boston-London, 1983.

[38] V.G.Kac, A.K.Raina, *Highest Weight Representations of Infinite Dimensional Lie Algebras*, World Scientific, Singapore, 1987.

[39] A.A.Kirillov, *Elements of the Theory of Representations*, Springer Verlag, Berlin-Heidelberg-N.Y., 1976.

[40] A.A.Kirillov, Infinite dimensional Lie groups: their orbits, invariants and representations. The geometry of moments, Lect. Notes Math. **970**(1982), 101-123.

[41] A.A.Kirillov, Kähler structure on K-orbits of the diffeomorphism group of the circle, Funct. Anal. and its Appl. **21**(1987), 122-125.

[42] A.A.Kirillov, Geometric quantization, In: Dynamical Systems, IV. Encycl. of Math. Sciences **4**, Springer Verlag, Berlin-Heidelberg-N.Y., 1990; 137-172.

[43] A.A.Kirillov, The orbit method, I: geometric quantization; II: infinite-dimensional Lie groups and Lie algebras, Contemp. Math. **145**(1993), 1-63.

[44] A.A.Kirillov, D.V.Yuriev, Kähler geometry of the infinite-dimensional homogeneous space $M = \mathrm{Diff}_+(S^1)/\mathrm{Rot}(S^1)$, Funct. Anal. and its Appl. **21**(1987), 284-294.

[45] S.Kobayashi, K.Nomizu, *Foundations of Differential Geometry*, Interscience Publishers, N.Y.–London, 1963.

[46] B.Kostant, Quantization and unitary representations, Lect. Notes Math. **170**(1970), 87-208.

[47] S.Lang, *Differential Manifolds*, Springer Verlag, Berlin–Heidelberg–N.Y., 1985.

[48] L.P.Lazutkin, N.F.Pankratova, Normal forms and versal deformations for Hill's equations, Funct. Anal. and its Appl. **9**(1975), 306-311.

[49] O.Lehto, *Univalent Functions and Teichmüller Spaces*, Springer Verlag, Berlin, 1987.

[50] L.Lempert, The Virasoro group as a complex manifold, Math. Res. Lett. **2**(1995), 475-495.

[51] L.Lempert, Loop spaces as complex manifolds, J. Differ. Geom. **38**(1993), 519-543.

[52] R.Mañe, P.Sad, D.Sullivan, On the dynamics of rational maps, Ann. Sci. École Norm. Super. **16**(1983), 193-217.

[53] J.Mickelsson, String quantization on group manifolds and the holomorphic geometry of Diff S^1/S^1, Commun. Math. Phys. **112**(1987), 653-661.

[54] J.Mickelsson, *Current algebras and groups*, Plenum Press, N.Y., 1989.

[55] J.W.Milnor, Remarks on infinite dimensional Lie groups. In: Relativity, groups and topology II; Les Houches Session XL, 1983 (eds. B.S.de Witt, R.Stora), North-Holland, Amsterdam, 1984; 1010-1057.

[56] S.Nag, *The Complex Analytic Theory of Teichmüller Spaces*, Wiley Interscience, New York, 1988.

[57] S.Nag, A period mapping in universal Teichmüller space, Bull. Amer. Math. Soc. **26**(1992), 280-287.

[58] S.Nag, D.Sullivan, Teichmüller theory and the universal period mapping via quantum calculus and the $H^{1/2}$ space on the circle, Osaka J. Math. **32**(1995), 1-34.

[59] S.Nag, A.Verjovsky, Diff(S^1) and the Teichmüller spaces, Comm. Math. Phys. **130**(1990), 123-138.

[60] N.K.Nikol'skii, *Treatise on the Shift Operator*, Springer Verlag, Berlin-Heidelberg-N.Y., 1986.

[61] A.L.Onishchik, *Topology of Transitive Transformations Groups*, J.A.Barth, Leipzig-Berlin-Heidelberg, 1994.

[62] A.Pflüger, Über die Konstruktion Riemannscher Flächen durch Verhäftung, J. Indian Math. Soc. **24**(1961), 401-412.

[63] A.D.Popov, A.G.Sergeev, Symplectic twistors and geometric quantization of strings, In: Algebraic geometry and its applications; Aspects of Math., vol. E25 (eds. A.Tikhomirov, A.Tyurin), Vieweg Verlag, Wiesbaden, 1994; 137-157.

[64] S.Power, Hankel operators on Hilbert space, Research Notes in Math. **64**(1982).

[65] A.Pressley, G.Segal, *Loop Groups*, Clarendon Press, Oxford, 1986.

[66] G.Segal, Unitary representations of some infinite dimensional groups, Commun. Math. Phys. **80**(1981), 301-342.

[67] J.-P.Serre, *Lie Algebras and Lie Groups*, Springer Verlag, Berlin-Heidelberg-N.Y., 1992.

[68] D.Shale, Linear symmetries of free boson field, Trans. Amer. Math. Soc. **103**(1962), 149-167.

[69] J.-M.Souriau, *Structure des Systémes Dynamiques*, Dunod, Paris, 1970.

[70] J.Sniatycki, *Geometric Quantization and Quantum Mechanics*, Springer Verlag, Berlin-Heidelberg-N.Y., 1980.

[71] M.S.Strigunova, Finite-dimensional subalgebras of the Lie algebra of vector fields on a circle, Steklov Inst. Proc. **236**(2002), 325-329.

[72] G.M.Tuynman, *Geometric Quantization*, CWI Syllabus, Amsterdam, 1985.

[73] K.Uhlenbeck, Harmonic maps into Lie groups (classical solutions of the chiral model), J. Diff. Geom. **30**(1989), 1-50.

[74] M.Vergne, Second quantification et groupe symplectique, C. R. Acad. Sci. Ser. A **285**(1977), 191-194.

[75] E.B.Vinberg, V.V.Gorbacevich, A.L.Onishchik, *Structure of Lie Groups and Algebras*, In: Lie Groups and Lie Algebras, III. Encycl. of Math. Sciences **41**, Springer Verlag, Berlin-Heidelberg-N.Y., 1994.

[76] E.B.Vinberg, A.L.Onishchik, *Lie Groups and Algebraic Groups*, Springer Verlag, Berlin-Heidelberg-N.Y., 1990.

[77] H.C.Wang, Closed manifolds with complex homogeneous structure, Amer. J. Math. **76**(1954), 1-32.

[78] E.Witten, Coadjoint orbits of the Virasoro group, Commun. Math. Phys. **114**(1988), 1-53.

[79] N.M.J.Woodhouse, *Geometric Quantization, 2nd ed.*, Clarendon Press, Oxford, 1992.

[80] V.A.Zaitseva, V.V.Kruglov, A.G.Sergeev, M.S.Strigunova, K.A.Trushkin, Remarks on the loop groups of compact Lie groups and the diffeomorphism group of the circle, Steklov Inst. Proc. **224**(1999), 124-136.

[81] A.Zygmund, *Trigonometric Series I, II, 3rd ed.*, Cambridge Univ. Press, Cambridge, 1987.

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