LECTURES ON RECONSTRUCTION ALGEBRAS II

MICHAEL WEMYSS

1. Introduction

Last lecture I introduced quivers with relations. Then after choosing a dimension vector α and character θ such that $\sum_{i\in Q_0}\alpha_i\theta_i=0$ we constructed what we called a moduli space. I again emphasize that, given kQ/R, we need to make *two* choices to define the moduli space.

This seminar is aiming to resolve singularities (in particular rational surfaces) so today I'm going to start to go in that direction. First though I have one thing to finish from last time, namely to prove that the spaces we introduced are actually moduli spaces in the strict sense of the word. I'll do this in the first section. I will then spend the rest of the lecture giving the geometric motivation of a reconstruction algebra, and I will highlight many of the subtleties and technicalities we will need to overcome in future.

2. Why its a Moduli Space

In the last lecture, given a dimension vector α and stability θ such that $\sum_{i\in Q_0}\alpha_i\theta_i=0$ we constructed a space $\mathfrak{M}^{ss}_\theta(A,\alpha)=\mathfrak{M}^{ss}_\theta$ and called it a moduli space. In this section we justify the name: we are going to rigorously define what a 'moduli space' is and then apply it to quivers. Proving that the 'moduli spaces' from the last lecture are actually moduli spaces is important since (in some circumstances) it gives us the existence of a *universal bundle* on the space.

First some motivation: in what follows, 'moduli set' means a set of things we would *like* to parameterize by a geometric object. The natural question to ask is

Q1: Does there exist a scheme X whose closed points are the objects in the 'moduli set'
A: Usually no.

Thus we ask

Q2: Does there exist a scheme X whose closed points are 'some' of the objects in the 'moduli set'
A: More often

We clearly have to make this more precise. To do this, for any category $\mathcal{C}$ define $[\mathcal{C},\texttt{Set}]$ to be the category of contravariant functors from $\mathcal{C}$ to $\texttt{Set}$. For any object X in $\mathcal{C}$ define

$$\begin{aligned}\mathrm{Hom}_{\mathcal{C}}(-,X):\ &\mathcal{C}\to\texttt{Set}\\ &C\mapsto\mathrm{Hom}_{\mathcal{C}}(C,X)\end{aligned}$$

in the obvious way, so $\mathrm{Hom}_{\mathcal{C}}(-,X)\in[\mathcal{C},\texttt{Set}]$. We call this *the functor of points of* X since in many examples (but not all!) there exists an object Z of $\mathcal{C}$ with the property that $\mathrm{Hom}_{\mathcal{C}}(Z,X)$ is the set of points of X. For example in the category of groups $\texttt{Gp}$, $\mathbb{Z}$ is an object for which $\mathrm{Hom}_{\texttt{Gp}}(\mathbb{Z},G)=|G|$ as sets, for any group G. Another example would be $\mathbb{Z}[X]$ in the category of rings.

Now Yoneda's Lemma tells us that

$$\begin{array}{ccc}\mathcal{C}&\to&[\mathcal{C},\texttt{Set}]\\ C&\mapsto&\mathrm{Hom}_{\mathcal{C}}(-,C)\end{array}$$

is an embedding, so we can view $\mathcal{C}$ inside the category $[\mathcal{C},\texttt{Set}]$. This may look like we've made things more difficult but in fact it may be the case that in the larger category $[\mathcal{C},\texttt{Set}]$ some constructions are much easier. Anyway,

Definition 2.1. *We call* $F \in [\mathcal{C}, \mathtt{Set}]$ *representable if* F *is naturally isomorphic to* $\mathrm{Hom}_{\mathcal{C}}(-, A)$ *for some object* A *of* $\mathcal{C}$.

Denote the category of affine varieties by `AfVar` then by Yoneda affine varieties are precisely those functors $\mathtt{AfVar} \to \mathtt{Set}$ which are representable. This is all very tautological. Note that affine algebraic groups are (by definition) those representable functors $\mathtt{AfVar} \to \mathtt{Set}$ which take values in the category `Gp` (instead of `Set`).

Now a moduli problem for some class of objects in algebraic geometry consists of

- for every scheme X, a notion of a family parameterized by the scheme X.

We call this a family over X. Note at this stage this is inprecise, but the point is that we specialize this general framework to a precise meaning of 'family over X' whenever we want to do anything. Now the moduli problem is considered solved if there exists a single scheme Y such that the family over Y is universal, in the sense that given any other X, every member of the family over X is uniquely induced by a morphism $X \to Y$.

Denoting the category of schemes by `Sch`, more formally the moduli problem is a contravariant functor

$$\begin{array}{rcl} F : \mathtt{Sch} & \to & \mathtt{Set} \\ S & \mapsto & \text{the set } \{\text{members of the family over S}\} \end{array}$$

and the moduli problem is considered solved if F is representable. This is again tautological: if $F \cong \mathrm{Hom}(-, Y)$ then

$$\{\text{members of the family over X}\} = FX \cong \mathrm{Hom}(X, Y).$$

This leads to the following definition

Definition 2.2. *If a contravariant functor* $F : \mathtt{Sch} \to \mathtt{Set}$ *is represented by a scheme* Y*, we call* Y *the fine moduli space of* F.

This is normally too strong since many moduli problems don't have representable functors. So we compromise:

Definition 2.3. *Given* $F \in [\mathtt{Sch}, \mathtt{Set}]$*, a scheme* Y *is said to be a best approximation to* F *(or sometimes* Y *corepresents* F*) is there is a natural transformation*

$$\alpha : F \to \mathrm{Hom}(-, Y)$$

which is universal amongst the natural transformations from F *to schemes, i.e. given any other* $\beta : F \to \mathrm{Hom}(-, Z)$*, there exists a unique natural transformation*

$$\begin{tikzcd} F \arrow[r, "\alpha"] \arrow[dr, "\beta"'] & \mathrm{Hom}(-,Y) \arrow[d, dotted, "\exists!"] \\ & \mathrm{Hom}(-,Z) \end{tikzcd}$$

such that the diagram commutes. If F *is a moduli functor, we cal* (Y, α) *the moduli space of* F*. If further* (Y, α) *satisfies*

$$\alpha_{\mathrm{Spec}\mathbb{C}} : F(\mathrm{Spec}\mathbb{C}) \to \mathrm{Hom}(\mathrm{Spec}\mathbb{C}, Y)$$

is bijective, we call (Y, α) *a coarse moduli space.*

We now apply this to quivers. To begin we define the notion of a family over X:

Definition 2.4. *A family of* kQ*-modules with dimension vector* $\alpha = (\alpha_i)$ *over a scheme* X *is an assignment, for each vertex* i*, of a vector bundle* $\mathscr{V}_i$ *of rank* α_i*, and for every arrow in* Q *a corresponding morphism of vector bundles.*

If you like, you can think of this as specifying a map $kQ/R \to \mathrm{End}(\oplus_{i \in Q_0} \mathscr{V}_i)$. Or you can also view it as a representation in the category of vector bundles $\mathtt{Vb}X$. The above definition really is a family of representations over X in the obvious way: for any point $x \in X$ if we take the stalk of the bundles (=the fibre) at x then each vertex just becomes a finite dimensional vector space and the morphisms become linear maps such that the relations still hold. This isn't saying anything other than a vector bundle is locally trivial. Thus for every point $x \in X$ we get an actual representation of kQ/R.

We now make our moduli problem precise by defining the families we would like to classify:

Definition 2.5. *A family of semistable kQ/R-modules with dimension vector α over a scheme X is just a family of kQ-modules with dimension vector $\alpha = (\alpha_i)$ as above, in which all members in the family are θ-semistable. We have the similar notion for θ-stability.*

This just means that for every point $x \in X$, the associated stalk (i.e. actual representation) is θ-semistable.

Now every θ-semistable M has a Jordan-Hölder filtration

$$0 = M_0 \subset M_1 \subset \ldots \subset M_{n-1} \subset M_n = M$$

in which every subobject M_i is θ-semistable, and every factor M_i/M_{i-1} is simple (in the category of θ-semistable modules). This is just constructed in the standard way; since M is finite dimensional the process must eventually finish.

It is clear that if M is θ-stable then the JH filtration is just $0 \subset M$ (since by definition M has no θ-semistable subobjects). In more fancy language the θ-stable objects are precisely the simple objects in the category of θ-semistable objects.

Definition 2.6. *Two θ-semistable objects are called S-equivalent (with respect to θ) if their Jordan-Hölder filtrations have isomorphic composition factors*

By the above discussion this collapses in the case of stability: two θ-stable modules M and N are S-equivalent if and only if they are isomorphic, since their JH filtrations are just $0 \subset M$ and $0 \subset N$.

The quotient (=moduli space) defined last time $\mathfrak{M}_\theta^{ss} := \mathscr{R}/\!\!/_{\chi} G$ parameterizes the θ-semistable representations *up to S-equivalence.* I'm not going to explain why this is true, since it involves more GIT than I want to get into. The open set of the quotient which corresponds to the stable points thus parameterizes the θ-stable representations *up to isomorphism.* This answers a question Osamu asked last time.

If θ is generic then stability and semistability coincide (by definition), thus in these cases we are always classifying up to isomorphism. In practice we're only going to be dealing with generic stability conditions.

Now we have defined the moduli problem, so we get the moduli functors

$$\begin{array}{rcl}\mathcal{M}_{kQ,\alpha,\theta}^{ss} : \texttt{Sch} & \rightarrow & \texttt{Set} \\ X & \mapsto & \text{the set } \{\text{families of } \theta\text{-semistable } kQ/R \text{ modules with dim } \alpha \text{ over } X\}/S\text{-equiv} \\ \mathcal{M}_{kQ,\alpha,\theta}^{s} : \texttt{Sch} & \rightarrow & \texttt{Set} \\ X & \mapsto & \text{the set } \{\text{families of } \theta\text{-stable } kQ/R \text{ modules with dim } \alpha \text{ over } X\}/\cong\end{array}$$

Theorem 2.7 (King 5.2)**.** *$\mathfrak{M}_\theta^{ss}$ is a coarse moduli space for the functor $\mathcal{M}_{kQ,\alpha,\theta}^{ss}$.*

Denote the stable points in $\mathfrak{M}_\theta^{ss}$ by $\mathfrak{M}_\theta^{s}$, then

Theorem 2.8 (King, 5.3)**.** *If α is indivisible, $\mathfrak{M}_\theta^{s}$ represents the functor $\mathcal{M}_{kQ,\alpha,\theta}^{s}$, i.e. $\mathfrak{M}_\theta^{s}$ is a fine moduli space.*

Thus for generic θ and indivisible α, $\mathfrak{M}_\theta^{ss}$ is a fine moduli space. This is important as it means we have a universal bundle[1]: since for generic θ and indivisible α

$$\mathcal{M}_{kQ,\alpha,\theta}^{s} \cong \operatorname{Hom}(-, \mathfrak{M}_\theta^{ss})$$

as functors from schemes to sets, apply both sides to the scheme $\mathfrak{M}_\theta^{ss}$. Then

$$1 \in \operatorname{Hom}(\mathfrak{M}_\theta^{ss}, \mathfrak{M}_\theta^{ss}) \cong \mathcal{M}_{kQ,\alpha,\theta}^{s}(\mathfrak{M}_\theta^{ss})$$

---

[1] this is backwards: the theorem is proved by exhibiting such a bundle!

so we have a family of θ-stable kQ/R modules with dimension vector α over $\mathfrak{M}^{ss}_\theta$ corresponding to the identity map. This just means that for every point $x \in \mathfrak{M}^{ss}_\theta$, the representation in this family corresponding to x is just x. We call this family the universal family.

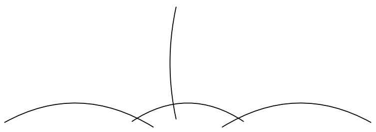
3. Geometric Motivation of Reconstruction Algebras

Before talking about the $\mathrm{SL}(2,\mathbb{C})$ McKay correspondence and its generalization to $\mathrm{GL}(2,\mathbb{C})$ I'll first give some motivation as to what we might regard as being the 'best' possible answer.

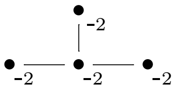
In this section consider a rational normal surface singularity $X = \mathrm{Spec}R$ with minimal resolution $\widetilde{X} \xrightarrow{\pi} X$. From this we have the dual graph, which you should view as a simplified picture of the resolution:

Definition 3.1. *Denote by $\{E_i\}$ the exceptional collection of $\mathbb{P}^1$s. Define the (labelled) dual graph as follows: for every E_i draw a dot, and join two dots if the corresponding $\mathbb{P}^1$'s intersect. Additionally, decorate each vertex with the self-intersection number corresponding to the curve at that vertex.*

In practice what this means is that if we have a collection of $\mathbb{P}^1$'s (which are one-dimensional, so we draw as lines) intersecting as follows:



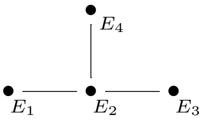
with all curves having self-intersection number (-2), then the dual graph is



The theory of rational normal surfaces is in many ways dictated by the following piece of combinatorial data (the fundamental cycle Z_f) which we can associate to the dual graph:

Definition 3.2 (Artin)**.** *For the dual graph $\{E_i\}$, define the fundamental cycle $Z_f = \sum_i r_i E_i$ (with each $r_i \geq 1$) to be the unique smallest element such that $Z_f \cdot E_i \leq 0$ for all vertices i.*

What this means in practice: for the dual graph



$E_1 \cdot E_1 = -2$
$E_2 \cdot E_2 = -2$
$E_3 \cdot E_3 = -2$
$E_4 \cdot E_4 = -2$

first try the smallest element $Z_r = E_1 + E_2 + E_3 + E_4$:

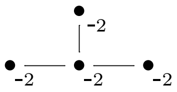
$$\begin{aligned}
Z_r \cdot E_1 &= E_1 \cdot E_1 + E_2 \cdot E_1 + E_3 \cdot E_1 + E_4 \cdot E_1 = (-2) + 1 + 0 + 0 = -1 \leq 0 \\
Z_r \cdot E_2 &= E_1 \cdot E_2 + E_2 \cdot E_2 + E_3 \cdot E_2 + E_4 \cdot E_2 = 1 + (-2) + 1 + 1 = 1 \nleq 0 \\
Z_r \cdot E_3 &= E_1 \cdot E_3 + E_2 \cdot E_3 + E_3 \cdot E_3 + E_4 \cdot E_3 = 0 + 1 + (-2) + 0 = -1 \leq 0 \\
Z_r \cdot E_4 &= E_1 \cdot E_4 + E_2 \cdot E_4 + E_3 \cdot E_4 + E_4 \cdot E_4 = 0 + 1 + 0 + (-2) = -1 \leq 0
\end{aligned}$$

Since it fails against E_2, try $Z_2 = E_1 + 2E_2 + E_3 + E_4$. A similar calculation shows that $Z_2 \cdot E_i \leq 0$ for all curves E_i. Consequently $Z_f = Z_2$, and we write this as $Z_f = \begin{smallmatrix} & 1 & \\ 1 & 2 & 1 \end{smallmatrix}$.

Observe that changing the middle curve in the above example changes the fundamental cycle to be $Z_f = \begin{smallmatrix} & 1 & \\ 1 & 1 & 1 \end{smallmatrix}$, but keeping the middle curve the same and changing any other curve results in the same $Z_f = \begin{smallmatrix} & 1 & \\ 1 & 2 & 1 \end{smallmatrix}$.

I emphasize that Z_f is defined *entirely* in terms of the dual graph. Consequently given a dual graph you can (if you wish) think of Z_f as a purely combinatorial piece of data which we can associate to it, but it is perhaps best to think a little more geometrically.

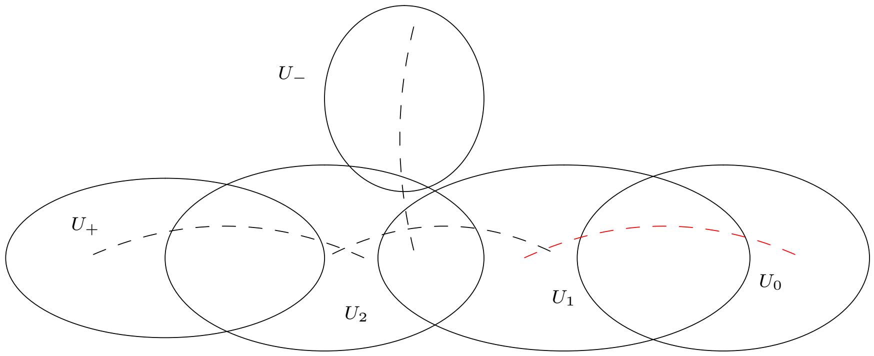
Now in fact



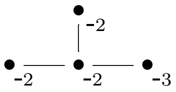
in the above discussion is the dual graph of the minimal resolution of $\mathbb{C}^2/BD_{4\cdot 2}$ where $BD_{4\cdot 2}$ is the binary dihedral group of order 8 inside $\mathrm{SL}(2,\mathbb{C})$:

$$BD_{4.2} := \left\langle \begin{pmatrix}\varepsilon_4 & 0\\ 0 & \varepsilon_4^3\end{pmatrix}, \begin{pmatrix}0 & \varepsilon_4\\ \varepsilon_4 & 0\end{pmatrix} \right\rangle$$

This has been extensively studied by many people. Say we have an open cover of the minimal resolution looking something like:



Now say we want to change the red curve in the minimal resolution into a (-3) curve, i.e. we want the dual graph[2] to become



.

How should we go about doing this? Note first that the fundamental cycle is still $Z_f = \begin{smallmatrix} & 1 & \\ 1 & 2 & 1\end{smallmatrix}$. We want to change the original space as little as possible to achieve our goal, so it would appear sensible to suggest that we only (at worst) change the equation of the open set U_0, and also change how U_0 glues to U_1. The change in glue will give the change in self-intersection number. The rest of the open sets (and their glues) will remain the same, and so we will have the desired configuration of $\mathbb{P}^1$s. I'm actually glossing over the fact that our map down to the singularity also changes, but the quiver takes care of this too so we shouldn't worry.

Here comes the key point:

Remark 3.3. If we change the geometry to accommodate a different self-intesection number, then *provided Z_f does not change* the new geometry will be *very* similar to the old geometry.

This is a subtle change in approach, so I'll emphasize it again. If you are given a group G inside $\mathrm{GL}(2,\mathbb{C})$ then instead of trying to resolve it using the G-Hilbert scheme (which we view as a 'new' space dependent on the group G), we should instead view the resolution as being a very small modification of a space we already understand. It is the (yet to be defined) reconstruction algebra which encodes the difference. Of course at this stage we don't know what space the resolution will be similar to, but the reconstruction algebra will tell us this.

The G-Hilbert scheme turns out to give the minimal resolution, but I do not know of any conceptual reason why this should be true. The groups under consideration can become very large and complicated, but the geometry stays quite simple.

Another point: in the above example if we had changed the middle curve instead of the red curve, you might think it would be more complicated as lots of things would have to

---

[2] In fact this new dual graph corresponds to the non-abelian group $\mathbb{D}_{5,3}$ of order 24

change. However I contest that this is actually the easiest case, since the fundamental cycle has decreased. Since Z_f can only decrease (i.e. improve) or remain the same under changing a self-intersection number, you should view this as saying that the difficulty in the geometry either

(A) remains the same (when Z_f stays the same)
(B) becomes easier (when Z_f changes, i.e. decreases)

As we shall see this is very important, since in many cases for non-abelian subgroups of $\mathrm{GL}(2,\mathbb{C})$ to extract the geometry explicitly from the reconstruction algebra is *precisely* the same level of difficultly as the toric case. The slogan is

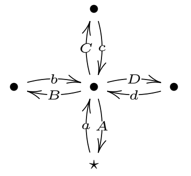
Slogan 3.4. *Take any non-abelian subgroup of* $\mathrm{GL}(2,\mathbb{C})$. *Then if the fundamental cycle* Z_f *is reduced (i.e. consists only of* 1*'s), the geometry is not toric, but it may as well be.*

In practice Z_f is reduced almost all of the time. I'll show how the above slogan works in my next lecture, but for now I'll illustrate case (A) with an example.

4. A COMPUTATION

Earlier I promised to give an example of explicitly resolving a singularity which would be very difficult to do without quivers, and also I promised to give an example of a computation of a non-abelian group action. I'll now do this, and in the process I'll be able to illustrate some of the points I raised in the previous section. At the moment you should view the NC rings that I use in this section as being constructed by magic, but I'll explain in my next lecture where I get them from.

Example 4.1. Consider the group $BD_{4\cdot 2}$ of order 8. This is classical McKay Correspondence territory, so the algebra to consider is the preprojective algebra



$$\begin{gathered} aA = bB = cC = dD = 0 \\ Aa + Bb + Cc + Dd = 0 \end{gathered}$$

This is Morita equivalent to the skew group ring, if you know about these things. We choose dimension vector and stability

$$\alpha = \begin{array}{ccc} & 1 & \\ 1 & 2 & 1 \\ & 1 & \end{array} \qquad \theta = \begin{array}{ccc} & 1 & \\ 1 & 1 & 1 \\ & \text{-5} & \end{array}$$

Notice that $\sum_{i\in Q_0} \alpha_i\theta_i = 0$ so we can form the moduli space. With these choices the computation becomes more complicated than the ones we did before, but not massively so; to now specify an open set we must

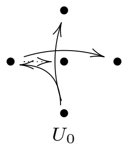
- specify, for each one-dimensional irreducible representation ρ, a non-zero path (which we can change basis to assume to be the identity) from the trivial representation to the vertex ρ.
- specify paths (0 1) and (1 0) from the trivial representation to the 2-dimensional representation.

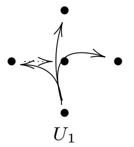
Different choices in the above lead to different open sets. Note that we must be able to make such choices for any θ-stable module M since by definition M is $\star$-generated and so paths leaving the trivial vertex must generate the vector spaces at all other vertices. For a stable M, it must be true that $a \neq 0$ and so after changing basis we can (and will) always assume that $a = (1\ 0)$.

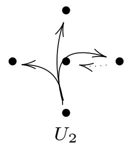
Define the open sets U_0, U_1, U_2, U_+ and U_- by the following conditions:

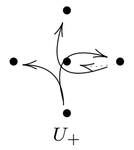
U_0	$aB = 1$	$aC = 1$	$aBbD = 1$	$a = (1\ 0)$	$b = (0\ 1)$
U_1	$aB = 1$	$aC = 1$	$aD = 1$	$a = (1\ 0)$	$b = (0\ 1)$
U_2	$aB = 1$	$aC = 1$	$aD = 1$	$a = (1\ 0)$	$d = (0\ 1)$
U_+	$aB = 1$	$aDdC = 1$	$aD = 1$	$a = (1\ 0)$	$d = (0\ 1)$
U_-	$aDdB = 1$	$aC = 1$	$aD = 1$	$a = (1\ 0)$	$d = (0\ 1)$

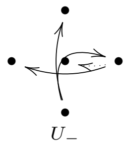
Pictorially we draw this as follows:











where the solid black lines correspond to the identity, and the dotted arrow corresponds to the choice of vector (0 1). These actually cover the moduli, but the proof is a bit messy. Note that there are *lots* of other open covers we could take.

We do the U_0 calculation in full, and just summarize the others. Any stable module in U_0 looks like

$$\begin{tikzcd}
 & \mathbb{C} \arrow[d, shift left, "(c_1\ c_2)"] & \\
\mathbb{C} \arrow[r, shift left, "(0\ 1)"] & \mathbb{C}^2 \arrow[l, shift left, "\begin{pmatrix}1\\B_2\end{pmatrix}"] \arrow[u, shift left, "\begin{pmatrix}1\\C_2\end{pmatrix}"] \arrow[r, shift left, "\begin{pmatrix}D_1\\1\end{pmatrix}"] \arrow[d, shift left, "\begin{pmatrix}A_1\\A_2\end{pmatrix}"] & \mathbb{C} \arrow[l, shift left, "(d_1\ d_2)"] \\
 & \mathbb{C} \arrow[u, shift left, "(1\ 0)"] &
\end{tikzcd}$$

where the variables are scalars, subject only to the quiver relations. Now

- $aA = 0$ implies $A_1 = 0$
- $bB = 0$ implies $B_2 = 0$
- $cC = 0$ implies $c_1 = -c_2C_2$
- $dD = 0$ implies $d_2 = -d_1D_1$

and so plugging this in our module becomes

$$\begin{tikzcd}
 & \mathbb{C} \arrow[d, shift left, "(-c_2C_2\ c_2)"] & \\
\mathbb{C} \arrow[r, shift left, "(0\ 1)"] & \mathbb{C}^2 \arrow[l, shift left, "\begin{pmatrix}1\\0\end{pmatrix}"] \arrow[u, shift left, "\begin{pmatrix}1\\C_2\end{pmatrix}"] \arrow[r, shift left, "\begin{pmatrix}D_1\\1\end{pmatrix}"] \arrow[d, shift left, "\begin{pmatrix}0\\A_2\end{pmatrix}"] & \mathbb{C} \arrow[l, shift left, "(d_1\ -d_1D_1)"] \\
 & \mathbb{C} \arrow[u, shift left, "(1\ 0)"] &
\end{tikzcd}$$

But now there is only one relation left, namely $Aa + Bb + Cc + Dd = 0$. This gives

$$\begin{pmatrix}0 & 0\\ A_2 & 0\end{pmatrix} + \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} + \begin{pmatrix}-c_2C_2 & c_2\\ -c_2C_2^2 & c_2C_2\end{pmatrix} + \begin{pmatrix}d_1D_1 & -d_1D_1^2\\ d_1 & -d_1D_1\end{pmatrix} = \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$$

which yields the four conditions

$$\begin{gathered}
c_2C_2 = d_1D_1\\
c_2 = d_1D_1^2 - 1\\
A_2 = c_2C_2^2 - d_1\\
c_2C_2 = d_1D_1
\end{gathered}$$

The second and third conditions eliminate the variables c_2 and A_2, whereas the first and last conditions are the same. Substituting the second condition into the first we see that this open set is completely parameterized by d_1, D_1 and C_2 subject to the one relation $d_1D_1 = (d_1D_1^2 - 1)C_2$, so U_0 is a smooth hypersurface in $\mathbb{C}^3$.

Similarly we have

$$\begin{tikzcd}
 & & \mathbb{C} \arrow[d, bend left, "(-c_2C_2\ c_2)"] & \\
U_1 & \mathbb{C} \arrow[r, bend left, "(0\ 1)"] & \mathbb{C}^2 \arrow[u, bend left, "\binom{1}{C_2}"] \arrow[l, bend left, "\binom{1}{0}"] \arrow[r, bend left, "\binom{1}{D_2}"] \arrow[d, bend left, "\binom{0}{A_2}"] & \mathbb{C} \arrow[l, bend left, "(-d_2D_2\ d_2)"] \\
 & & \mathbb{C} \arrow[u, bend left, "(1\ 0)"] &
\end{tikzcd}$$

$$\begin{gathered} c_2C_2 = -d_2D_2 \\ 1 + c_2 + d_2 = 0 \\ A_2 = c_2C_2^2 + d_2D_2^2 \\ c_2C_2 = -d_1D_1 \end{gathered}$$

$$\mathbb{C}^3_{d_2,D_2,C_2}/(1+d_2)C_2 = d_2D_2$$

$$\begin{tikzcd}
 & & \mathbb{C} \arrow[d, bend left, "(-c_2C_2\ c_2)"] & \\
U_2 & \mathbb{C} \arrow[r, bend left, "(-b_2B_2\ b_2)"] & \mathbb{C}^2 \arrow[u, bend left, "\binom{1}{C_2}"] \arrow[l, bend left, "\binom{1}{B_2}"] \arrow[r, bend left, "\binom{1}{0}"] \arrow[d, bend left, "\binom{0}{A_2}"] & \mathbb{C} \arrow[l, bend left, "(0\ 1)"] \\
 & & \mathbb{C} \arrow[u, bend left, "(1\ 0)"] &
\end{tikzcd}$$

$$\begin{gathered} b_2B_2 = -c_2C_2 \\ 1 + b_2 + c_2 = 0 \\ A_2 = b_2B_2^2 + c_2C_2^2 \\ b_2B_2 = -c_2C_2 \end{gathered}$$

$$\mathbb{C}^3_{b_2,B_2,C_2}/(1+b_2)C_2 = b_2B_2$$

$$\begin{tikzcd}
 & & \mathbb{C} \arrow[d, bend left, "(c_1\ -c_1C_1)"] & \\
U_+ & \mathbb{C} \arrow[r, bend left, "(-b_2B_2\ b_2)"] & \mathbb{C}^2 \arrow[u, bend left, "\binom{C_1}{1}"] \arrow[l, bend left, "\binom{1}{B_2}"] \arrow[r, bend left, "\binom{1}{0}"] \arrow[d, bend left, "\binom{0}{A_2}"] & \mathbb{C} \arrow[l, bend left, "(0\ 1)"] \\
 & & \mathbb{C} \arrow[u, bend left, "(1\ 0)"] &
\end{tikzcd}$$

$$\begin{gathered} b_2B_2 = c_1C_1 \\ b_2 = c_1C_1^2 - 1 \\ A_2 = b_2B_2^2 - c_1 \\ b_2B_2 = c_1C_1 \end{gathered}$$

$$\mathbb{C}^3_{c_1,B_2,C_1}/(c_1C_1^2 - 1)B_2 = c_1C_1$$

$$\begin{tikzcd}
 & & \mathbb{C} \arrow[d, bend left, "(-c_2C_2\ c_2)"] & \\
U_- & \mathbb{C} \arrow[r, bend left, "(b_1\ -b_1B_1)"] & \mathbb{C}^2 \arrow[u, bend left, "\binom{1}{C_2}"] \arrow[l, bend left, "\binom{B_1}{1}"] \arrow[r, bend left, "\binom{1}{0}"] \arrow[d, bend left, "\binom{0}{A_2}"] & \mathbb{C} \arrow[l, bend left, "(0\ 1)"] \\
 & & \mathbb{C} \arrow[u, bend left, "(1\ 0)"] &
\end{tikzcd}$$

$$\begin{gathered} b_1B_1 = c_2C_2 \\ c_2 = b_1B_1^2 - 1 \\ A_2 = c_2C_2^2 - b_1 \\ b_1B_1 = c_2C_2 \end{gathered}$$

$$\mathbb{C}^3_{b_1,B_1,C_2}/(b_1B_1^2 - 1)C_2 = b_1B_1$$

Note in U_2 above the equation $1+b_2+c_2 = 0$ really means that we have a choice of co-ordinate between b_2 and c_2; thus we could equally well parameterize U_2 as $\mathbb{C}^3_{c_2,B_2,C_2}/c_2C_2 = (1+c_2)B_2$.

Hence we see that the space is covered by 5 open sets, each a smooth hypersurface in $\mathbb{C}^3$. It is also quite easy to write down the glues (I don't have time), and just see the configuration of $\mathbb{P}^1$'s: for example the gluing between U_0 and U_1 is

$$U_0 \ni (d_1, D_1, C_2) \xleftrightarrow{D_1 \neq 0} (-d_1D_1^2, D_1^{-1}, C_2) \in U_1$$

The picture of the glues should (roughly) coincide with the picture I drew earlier.

The next example explains how to change the red $\mathbb{P}^1$ in the previous picture into a (-3)-curve.

Example 4.2. Consider the reconstruction algebra

$$\begin{tikzcd}
 & \bullet \arrow[d, bend left, "c"] & \\
\bullet \arrow[r, bend left, "b"] & \bullet \arrow[u, bend left, "C"] \arrow[l, bend left, "B"] \arrow[r, bend left, "D"] \arrow[d, bend left, "A"] & \bullet \arrow[l, bend left, "d"] \arrow[dl, green, "k_1"] \\
 & \star \arrow[u, bend left, "a"] &
\end{tikzcd}$$

$$\begin{gathered} aA = bB = cC = dD = 0 \\ Aa + Bb + Cc + Dd = 0 \\ k_1aD = dBbD \\ aDk_1 = aCcA \end{gathered}$$

Choose dimension vector and stability as in the previous example. Now notice that *the same* conditions that defined an open cover in the previous example give an open cover here (since the stability cannot 'see' the extra arrows).

Now our old calculation tells us almost everything, except now we have a new variable k_1 inside every open set. The point is that the *only* open set which changes is U_0. The reason for this is quite simple: in the relations $k_1aD = dBbD$ and notice that $aD = 1$ in every open set except U_0. Thus $k_1 = dBbD$ in every open set except U_0 and consequently we can put k_1 in terms of the other variables. Hence k_1 isn't really an extra variable in these open sets, so they do not change.

What happens to U_0? Well by the previous calculation we have

$$\begin{tikzcd}
 & \mathbb{C} \arrow[d, bend left, "(-c_2C_2 \ c_2)"] & \\
\mathbb{C} \arrow[r, bend left, "(0 \ 1)"] & \mathbb{C}^2 \arrow[l, bend left, "\begin{pmatrix} 1 \\ 0 \end{pmatrix}"] \arrow[u, bend left, "\begin{pmatrix} 1 \\ C_2 \end{pmatrix}"] \arrow[r, bend left, "\begin{pmatrix} D_1 \\ 1 \end{pmatrix}"] \arrow[d, bend left, "\begin{pmatrix} 0 \\ A_2 \end{pmatrix}"] & \mathbb{C} \arrow[l, bend left, "(d_1 \ -d_1D_1)"] \arrow[dl, "k_1"] \\
 & \mathbb{C} \arrow[u, bend left, "(1 \ 0)"] &
\end{tikzcd}
\qquad
\begin{array}{c}
c_2C_2 = d_1D_1 \\
c_2 = d_1D_1^2 - 1 \\
A_2 = c_2C_2^2 - d_1 \\
c_2C_2 = d_1D_1 \\
\hline
k_1D_1 = d_1 \\
D_1k_1 = c_2A_2
\end{array}$$

Since $d_1 = k_1D_1$, instead of being given by d_1, D_1, C_2 subject to $d_1D_1 = (d_1D_1^2 - 1)C_2$, the open set is now given by k_1, D_1, C_2 subject to $k_1D_1^2 = (k_1D_1^3 - 1)C_2$. Also, the gluing between U_0 and U_1 has changed to

$$U_0 \ni (k_1, D_1, C_2) \xleftrightarrow{D_1 \neq 0} (\text{-}(k_1D_1)D_1^2, D_1^{-1}, C_2) = (\text{-}k_1D_1^3, D_1^{-1}, C_2) \in U_1$$

Thus we see that the red curve has changed into a (-3)-curve, nothing else in the open cover has changed and so the dual graph is now