Singularity Seminar 2008

Organized by Yukari Ito and Osamu Iyama
October 2008 -- February 2009

Title of the talks in the seminar:

October 1. (Michael Wemyss) Quiver representation and the mouli space I
October 6. (Michael Wemyss) Quiver representation and the mouli space II
October 15. (Michael Wemyss) Quiver representation and the mouli space III
November 12. (Osamu Iyama) Special CM module
November 17. (Takehiko Yasuda)
    Noncommutative resolution via Frobenius morphisms and D-modules
November 19. (Kentaro Nagao)
    Mutations and noncommutative Donaldson-Thomas invariants
November 26. (Michael Wemyss) GL(2) McKay correspondence
November 26. (Alvaro Nolla de Celis) Dihedral groups and G-Hilb
January 28. (Yukari Ito) Special McKay correspondence I
January 28. (Kouta Yamaura)
    Structure of AR-quiver of representation-finite self-injective algebras I
February 4. (Yukari Ito) Special McKay correspondence II
February 4. (Kouta Yamaura)
    Structure of AR-quiver of representation-finite self-injective algebras II
February 23. (Yuhi Sekiya) G-Hilbert scheme and Groebner bases

Contents of this note

LECTURES ON RECONSTRUCTION ALGEBRAS I

MICHAEL WEMYSS

1. Introduction

Noncommutative algebra (=quivers) can be used to solve both explicit and non-explicit problems in algebraic geometry, and these lectures will try to explain some of the features of both approaches. I want to use these notes to give a gentle (!) introduction to the subject, and will try and make them as self-contained as possible. Since I want to eventually end up doing non-toric geometry, throughout I shall never adopt the language of toric geometry, even if the example I am considering is toric. First some motivation:

From a noncommutative perspective we would like to take a singularity $X = \text{Spec } R$ and produce a NC ring A from which we can extract resolution(s) of X. We can then ask whether the NC ring has some geometrical meaning, and if so whether this gives information about A. We can also ask what A says about X and its resolutions.

From a more geometric perspective we may already have some resolution Y of X and would like produce other resolutions, for example by flopping certain curves. We may also want to describe the derived category of Y. This can sometimes be done using noncommutative algebra.

In practice however things are not quite as simple as this, since most of the time a specific problem will be a mixture of the two above problems. Sometimes it is easier to solve the problem using the geometry, sometimes it is easier using quivers. Thus geometry can give us results in noncommutative algebra and noncommutative algebra can give us results in geometry; it is the process of playing the two sides off each other which gives us the strongest results.

Today I'm going to define quivers and tell you how to think of them, then following King [King1] I'll talk about their moduli space(s) of finite dimensional representations. Time permitting I'll then show how to calculate the moduli spaces in some easy examples.

2. Quivers and Representations

Any algebra with a finite number of generators and a finite number of relations (i.e. almost all algebras you can think off) can be written as a quiver with relations[1]. You want to do this since the quiver gives you a way to visualize the algebra, and more importantly it gives you a way to visualize the finite dimensional modules (see later).

Definition 2.1. *A quiver Q is just a finite directed graph.*

At this stage loops, double arrows,... are all allowed, and the directed graph need not be connected. For example

$$Q = \begin{tikzcd} \bullet \arrow[r, shift left] & \bullet \arrow[l, shift left] \arrow[loop above] \arrow[r] & \bullet & \bullet & \bullet \arrow[l, shift left] \arrow[l, shift right] \arrow[loop right] \end{tikzcd}$$

is an example of a quiver. A small technical point: for every vertex i we actually also add in a trivial loop at that vertex and denote it by e_i, but we do not draw these loops. In the above example, the loops drawn are the non-trivial loops.

Denoting the vertices of Q by Q_0 and the arrows by Q_1, you can view the directed graph Q as simply a piece of combinatorial data (Q_0, Q_1, h, t) where h and t are maps $Q_1 \to Q_0$. The map h (the 'head') assigns to an arrow its head, and the map t (the 'tail') assigns to an arrow its tail.

[1] This *cannot* be done in a unique way

Definition 2.2. *A non-trivial path of length n in Q is just a sequence of arrows $a_1 \cdots a_n$ in Q with $h(a_i) = t(a_{i+1})$ for all $1 \leq i \leq n-1$. We call this path a cycle if $h(a_n) = t(a_1)$.*

We want to add more structure to the combinatorial data of a quiver by producing an algebra:

Definition 2.3. *For a given quiver Q, the path algebra kQ is defined to be the k-algebra with basis given by the paths, with multiplication*

$$pq := \begin{cases} pq & h(p) = t(q) \\ 0 & \text{else} \end{cases} \qquad e_i p := \begin{cases} p & t(p) = i \\ 0 & \text{else} \end{cases} \qquad pe_i := \begin{cases} p & h(p) = i \\ 0 & \text{else} \end{cases}$$

for any paths p and q.

This is an algebra, with identity $1_{kQ} = \sum_{i \in Q_0} e_i$. Note we are using the convention that pq means p then q; be aware that some savage barbarians[2] use the opposite convention. Note that by the definition of multiplication the path algebra is often noncommutative: for example if

$$Q = \bullet \xrightarrow{a} \bullet \xrightarrow{b} \bullet$$

then $ab \neq ba$ since $ba = 0$. In fact in this example kQ is easy to describe: the basis of kQ is e_1, e_2, e_3, a, b, ab. Its not hard to convince yourself that

$$kQ \cong \begin{pmatrix} k & k & k \\ 0 & k & k \\ 0 & 0 & k \end{pmatrix}.$$

Exercise 2.4. Let Q be a quiver, then kQ is finite dimensional if and only if Q has no non-trivial cycles.

For quivers Q without cycles, the resulting path algebras kQ have been used in geometry although their use is generally limited to projective varieties; since in these talks we are going to be resolving singularities we need to make one more definition:

Definition 2.5. *For a given quiver Q, a relation is simply a k-linear combination of paths in Q. Given a finite number of relations, we can form their two sided ideal R in the path algebra, and we thus define the algebra kQ/R to be a quiver with relations.*

We can assume (by removing arrows if necessary) that the length of every path in every relation is greater than or equal to two. Note that with relations it is possible that kQ/R can be finite dimensional even when Q has cycles, though in these lectures most of the examples will involve infinite dimensional algebras.

In practice you should think of the relation $p - q$ as saying 'going along path p is the same as going along path q', since $p = q$ in the quotient kQ/R.

Now as is standard in ring theory (and geometry), we tend to study a ring by instead studying its module category (=coherent sheaves), since this is an abelian category and so we have the machinery of homological algebra at our disposal. Representation theorists would tell us that we are we're studying the ring's representations - I'll now make this more precise.

Definition 2.6. *Let kQ/R be a quiver with relations. A finite dimensional representation of kQ/R is the assignment to every vertex i of Q a finite dimensional vector space V_i, and to every arrow a a linear map $f_a : V_{t(a)} \to V_{h(a)}$, such that the relations R between the linear maps hold. Denote $\alpha_i = \dim V_i$ and let $\alpha = (\alpha_i)$ be the collection of all the α_i. We call α the dimension vector of the representation.*

[2] you know who you are

For example let kQ/R be $\begin{tikzcd} \bullet \arrow[r, "a"] & \bullet \arrow[r, "b"] & \bullet \arrow[ll, bend left, "c"] \end{tikzcd}$ subject to $bc = 0$. Denoting

$$M := \begin{tikzcd} \mathbb{C} \arrow[r, "4"] & \mathbb{C} \arrow[r, "(1\ 0)"] & \mathbb{C}^2 \arrow[ll, bend left, "\binom{0}{1}"] \end{tikzcd} \qquad N := \begin{tikzcd} \mathbb{C} \arrow[r, "1"] & \mathbb{C} \arrow[r, "1"] & \mathbb{C} \arrow[ll, bend left, "3"] \end{tikzcd}$$

then M is a representation of dimension vector $(1, 1, 2)$ whereas N is *not* a representation of dimension vector $(1, 1, 1)$.

We also have the obvious notion of a morphism between two representations:

Definition 2.7. *Let $V = (V_i, f_a)$ and $W = (W_i, g_a)$ be finite dimensional representations of kQ/R. A morphism ψ from V to W is given by specifying, for every vertex i, a linear map $\psi_i : V_i \to W_i$ such that for every arrow $a \in Q_1$,*

$$\begin{tikzcd} V_{t(a)} \arrow[r, "\psi_{t(a)}"] \arrow[d, "f_a"] & W_{t(a)} \arrow[d, "g_a"] \\ V_{h(a)} \arrow[r, "\psi_{h(a)}"] & W_{h(a)} \end{tikzcd}$$

commutes.

Note ψ is an isomorphism if and only if each ψ_i is a linear isomorphism. Also note we have the obvious notion of a subrepresentation. It is fairly clear that in this way the finite dimensional representations form a category, which we denote by $\texttt{fRep}(kQ, R)$

The whole point to all this is the following:

Lemma 2.8. *Let $A = kQ/R$ be a quiver with relations. Denote by $\texttt{fdmod}A$ the finite dimensional modules of A. Then there is a categorical equivalence*

$$\texttt{fRep}(kQ, R) \approx \texttt{fdmod}A$$

Proof. This is actually quite tautological. Given a representation (V_i, f_a) then $\oplus_{i \in Q_0} V_i$ is the corresponding module. Conversely given any finite dimensional module W, setting $W_i = e_i W$ (where e_i is the trivial path at vertex i) gives us the corresponding representation. $\square$

Thus we now see the benefit of writing an algebra A as a quiver with relations, as by the above lemma we have a way to visualize the finite dimensional modules of A.

3. Moduli and GIT

In this section we consider a quiver with relations $A = kQ/R$ and define various moduli spaces of finite dimensional representations. In the process we have to take a very fast detour through the world of geometric invariant theory (GIT).

For a fixed dimension vector α we may consider all representations of $A = kQ/R$ with dimension vector α:

$$\mathscr{R} := \mathrm{Rep}(A, \alpha) = \{\text{representations of } A \text{ of dimension } \alpha\}$$

This is an affine variety, so denote the co-ordinate ring by $k[\mathscr{R}]$. The variety (hence the co-ordinate ring) carries a natural action of $G := \prod_{i \in Q_0} \mathrm{GL}(\alpha_i)$ acting on an arrow a as $g \cdot a = g_{t(a)}^{-1} a g_{h(a)}$. Actually its really an action of PGL since the diagonal one-parameter subgroup $\Delta = \{(\lambda 1, \cdots, \lambda 1) : \lambda \in k^*\}$ acts trivially, but this won't concern us much. Anyway, by linear algebra the *isomorphism classes* of representations of $A = kQ/R$ are in natural one-to-one correspondence with the orbits of this action.

To understand this space is normally an impossible problem (e.g. wild quiver type), so we want to throw away some representations and take what is known as a GIT quotient.

To make a GIT quotient we need to add the extra data of a character χ of G. Now the characters χ of $G=\prod_{i\in Q_0}\mathrm{GL}(\alpha_i)$ are given by powers of the determinants

$$\chi(g)=\prod_{i\in Q_0} det(g_i)^{\theta_i}$$

for some collection of integers $\theta_i\in\mathbb{Z}^{Q_0}$. Since such a χ determines and is determined by the θ_i, we usually denote χ by χ_θ. Now consider the map

$$\begin{array}{rcl}\theta:\texttt{fdmod}A & \to & \mathbb{Z}\\ M & \mapsto & \sum_{i\in Q_0}\theta_i\,\dim M_i\end{array}$$

This is additive on short exact sequences, so really its a map $K_0(\texttt{fdmod}A)\to\mathbb{Z}$.

Now assume that our character satisfies $\chi_\theta(\Delta)=\{1\}$ (this is need to use Mumford's numerical criterion [King, 2.5]). It not too hard to see that this condition translates into $\sum_{i\in Q_0}\theta_i\alpha_i=0$. Hence for these χ_θ, $\theta(M)=0$ if M has dimension vector α.

We arrive at the key definition [King,1.1]

Definition 3.1. *Let $\mathscr{A}$ be an abelian category, and $\theta:K_0(\mathscr{A})\to\mathbb{Z}$ an additive function. We call θ a character of $\mathscr{A}$. An object $M\in\mathscr{A}$ is called θ-semistable if $\theta(M)=0$ and every subobject $M'\subseteq M$ satisfies $\theta(M')\geq 0$. Such an object M is called θ-stable if the only subobjects M' with $\theta(M')=0$ are M and 0. We call θ generic if every M which is θ-semistable is actually θ-stable.*

For $A=kQ/R$ as before, we are interested in the above definition for the case $\mathscr{A}=\texttt{fdmod}A$. We shall see how this works in practice in the next section. The reason King gave the above definition is that it is equivalent to the other notion of stability from GIT, which we now describe:

$\mathscr{R}$ is an affine variety with an action of a linearly reductive group $G=\prod_{i\in Q_0}\mathrm{GL}(\alpha_i)$. Since G is reductive, we have a quotient

$$\mathscr{R}\to\mathscr{R}/\!\!/G=\mathrm{Spec}k[\mathscr{R}]^G$$

which is dual to the inclusion $k[\mathscr{R}]^G\to k[\mathscr{R}]$. Its the reductiveness of the group which ensures that $k[\mathscr{R}]^G$ is a finitely generated k-algebra, and so $\mathrm{Spec}k[\mathscr{R}]^G$ is really a variety, not just a scheme. Virtually by definition the above is a categorical quotient (quite a weak condition); further its actually a good quotient (if you don't know what this means, don't worry)

To make a GIT quotient we have to add to this picture the extra data of χ, some character of G.

Definition 3.2. *$f\in k[\mathscr{R}]$ is a semi-invariant of weight χ if $f(g\cdot x)=\chi(g)f(x)$ for all $g\in G$ and all $x\in\mathscr{R}$. We write the set of such f as $\mathscr{R}^{G,\chi}$. We define*

$$\mathscr{R}/\!\!/_{\chi}G:=\mathrm{Proj}\left(\bigoplus_{n\geq 0}k[\mathscr{R}]^{G,\chi^n}\right)$$

Definition 3.3. *$x\in\mathscr{R}$ is called χ-semistable (in the sense of GIT) if there exists some semi-invariant f of weight χ^n with $n>0$ such that $f(x)\neq 0$, otherwise $x\in\mathscr{R}$ is called unstable.*

The set of semistable points $\mathscr{R}^{ss}$ forms an open subset of $\mathscr{R}$; in fact we have a morphism

$$q:\mathscr{R}^{ss}\to\mathscr{R}/\!\!/_{\chi}G$$

which is a good quotient. One more definition:

Definition 3.4. *$x\in\mathscr{R}$ is called χ-stable (in the sense of GIT) if it is χ-semistable, the G orbit containing x is closed in $\mathscr{R}^{ss}$ and further the stabilizer of x is finite.*

In fact q is a geometric quotient on the stable locus $\mathscr{R}^s$, meaning that $\mathscr{R}^s /\!\!/_{\chi} G$ really is an orbit space.

The point in the above discussion is the following result [King1, 3.1], which says the two notions are the same

Proposition 3.5. *Let $M \in \operatorname{Rep}(A, \alpha) = \mathscr{R}$, choose θ as in Definition 3.1. Then M is θ-semistable (in the sense of Definition 3.1) if and only if M is χ_θ-semistable (in the sense of GIT). The same holds replacing semistability with stability.*

Thus we use the machinery from the GIT side to define for quivers the following:

Definition 3.6. *For $A = kQ/R$ choose dimension vector α and character θ satisfying $\sum_{i \in Q_0} \alpha_i \theta_i = 0$. Denote $\operatorname{Rep}(A, \alpha) = \mathscr{R}$ and $G = \operatorname{GL}(\alpha)$. We define*

$$\mathfrak{M}^{ss}_\theta(A, \alpha) := \mathscr{R} /\!\!/_{\chi_\theta} G := \operatorname{Proj}\left(\bigoplus_{n \geq 0} k[\mathscr{R}]^{G, \chi^n} \right)$$

and call it the moduli space of θ-semistable representations of dimension vector α.

This is by definition projective over the ordinary quotient $\mathscr{R} /\!\!/ G = \operatorname{Spec} k[\mathscr{R}]^G$. We make some remarks

- (i) If $k[\mathscr{R}]^G = k$ then $\mathfrak{M}^{ss}_\theta(A, \alpha)$ is a projective variety.
- (ii) In the resolution of singularities we ideally would like the zeroth piece $\operatorname{Spec} k[\mathscr{R}]^G$ to be the singularity since then the moduli space is projective over it! However, even in cases where we use NC rings to resolve singularities, $\operatorname{Spec} k[\mathscr{R}]^G$ might not be the thing we want; see Example 4.6 later.
- (iii) Note that $\mathfrak{M}^{ss}_\theta(A, \alpha)$ may be empty.
- (iv) One way to compute this space is to compute semi-invariants, but this in general is quite hard.

One small point before we continue: we can't just call $\mathfrak{M}^{ss}_\theta(A, \alpha)$ a moduli space, we really have to justify that it *is* a moduli space, i.e. why it parameterizes certain objects. We shall describe this more precisely in a future section. For now though we shall concern ourselves with showing how to calculate the moduli space in some examples:

4. Examples

The last section was quite abstract, here we show how it works in practice. For $A = kQ/R$, we may want to construct a space X from A as a moduli space of θ-stable A-modules. What this means [King2]:

> "To specify such a moduli space we must give a dimension vector α and a weight vector (or 'character') θ satisfying $\sum_{i \in Q_0} \theta_i \alpha_i = 0$. The moduli space of θ-stable A-modules of dimension vector α is then the parameter space for those A-modules which have no proper submodules with any dimension vector β for which $\sum_{i \in Q_0} \theta_i \beta_i \leq 0$."

For computational ease I will only compute moduli with dimension vector $(1, \dots, 1)$ in this section; I will return and do a computation of some other dimension vectors in a future section. There are many different (and better) ways to view the following example, but here I give the easiest:

Example 4.1. Consider the quiver

$$\bullet \rightrightarrows \bullet$$

with no relations. Choose $\alpha = (1, 1)$ and $\theta = (-1, 1)$. With these choices, since $\sum \theta_i \alpha_i = 0$ we can form the moduli space. Now a representation of dimension vector $\alpha = (1, 1)$ is θ-semistable by definition if $\theta(M') \geq 0$ for all subobjects M'. But the only possible subobjects in this example are of dimension vector $(0, 0)$, $(0, 1)$ and $(1, 0)$, and θ is ≥ 0 on all but the last (in fact its easy to see that θ is generic in this example). Thus a representation of dimension vector $(1,1)$ is θ-semistable if and only if it has no submodules of dimension vector $(1,0)$. Now take an arbitrary representation M of dimension vector $(1,1)$

$$M = \begin{tikzcd} \mathbb{C} \arrow[r, shift left, "a"] \arrow[r, shift right, "b"'] & \mathbb{C} \end{tikzcd}.$$

Notice that M has a submodule of dimension vector $(1,0)$ if and only if $a=b=0$, since the diagram

$$\begin{tikzcd} \mathbb{C} \arrow[r, "\cong"] \arrow[d, shift right, "0"'] \arrow[d, shift left, "0"] & \mathbb{C} \arrow[d, shift right, "b"'] \arrow[d, shift left, "a"] \\ 0 \arrow[r, "0"] & \mathbb{C} \end{tikzcd}$$

must commute. Thus by our choice of stability θ,

M is θ-semistable $\iff$ M has no submodule of dim vector $(1,0)$ $\iff$ $a \neq 0$ or $b \neq 0$.

and so we see that the semistable objects parametrize $\mathbb{P}^1$ via the ratio $(a:b)$, so the moduli space is just $\mathbb{P}^1$. Another way to see this: we have two open sets, one corresponding to $a \neq 0$ and the other to $b \neq 0$. After changing basis we can set them to be the identity, and so we have

$$U_0 = \{ \ \mathbb{C} \overset{1}{\underset{b}{\rightrightarrows}} \mathbb{C} \ : b \in \mathbb{C}\} \quad U_1 = \{ \ \mathbb{C} \overset{a}{\underset{1}{\rightrightarrows}} \mathbb{C} \ : a \in \mathbb{C}\}$$

Now the gluing is given by, whenever $U_0 \ni b \neq 0$

$$U_0 \ni b = \ \mathbb{C} \overset{1}{\underset{b}{\rightrightarrows}} \mathbb{C} \ = \ \mathbb{C} \overset{b^{-1}}{\underset{1}{\rightrightarrows}} \mathbb{C} \ = b^{-1} \in U_1$$

which is evidently just $\mathbb{P}^1$.

This lecture series is devoted to resolving singularities, so we warm up by blowing up the origin in $\mathbb{C}^2$:

Example 4.2. Consider the quiver with relations

$$\begin{tikzcd} \bullet \arrow[r, shift left=2, "a"] \arrow[r, "b" description] & \bullet \arrow[l, shift left=2, "t"] \end{tikzcd} \qquad atb = bta$$

and again choose dimension vector $(1,1)$ and stability $\theta_0 = (-1,1)$. Exactly as above if

$$M = \begin{tikzcd} \mathbb{C} \arrow[r, shift left=2, "a"] \arrow[r, "b" description] & \mathbb{C} \arrow[l, shift left=2, "t"] \end{tikzcd}$$

then

M is θ-semistable $\iff$ M has no submodule of dim vector $(1,0)$ $\iff$ $a \neq 0$ or $b \neq 0$.

For the first open set in the moduli U_0 (when $a \neq 0$): after changing basis so that $a = 1$ we see that the open set is parameterized by the two scalars b and t subject to the single relation (substituting $a = 1$ into the quiver relations) $tb = bt$. But this always holds so it isn't really a relation, thus the open set U_0 is just $\mathbb{C}^2$ with co-ordinates b, t. We write this as $\mathbb{C}^2_{b,t}$. Similarly for the other open set:

$$\begin{array}{cc} \begin{tikzcd} \mathbb{C} \arrow[r, shift left=2, "1"] \arrow[r, "b" description] & \mathbb{C} \arrow[l, shift left=2, "t"] \end{tikzcd} & \begin{tikzcd} \mathbb{C} \arrow[r, shift left=2, "a"] \arrow[r, "1" description] & \mathbb{C} \arrow[l, shift left=2, "t"] \end{tikzcd} \\ U_0 = \mathbb{C}^2_{b,t} & U_1 = \mathbb{C}^2_{a,t}. \end{array}$$

Now the gluing is given by, whenever $b \neq 0$

$$U_0 \ni (b,t) = \begin{tikzcd} \mathbb{C} \arrow[r, shift left=2, "1"] \arrow[r, "b" description] & \mathbb{C} \arrow[l, shift left=2, "t"] \end{tikzcd} = \begin{tikzcd} \mathbb{C} \arrow[r, shift left=2, "b^{-1}"] \arrow[r, "1" description] & \mathbb{C} \arrow[l, shift left=2, "bt"] \end{tikzcd} = (b^{-1}, bt) \in U_1$$

and so we see that this is just the blowup of the origin of $\mathbb{C}^2$.

Exercise 4.3. What does the stability $\theta_1 = (1,-1)$ give us in the above example?

Example 4.4. Consider the group $\frac{1}{3}(1,1) := \langle \left(\begin{smallmatrix} \varepsilon_3 & 0 \\ 0 & \varepsilon_3 \end{smallmatrix} \right) \rangle$ where ε_3 is a primitive third root of unity. This acts on $\mathbb{C}^2$ giving us a quotient singularity $\mathbb{C}[x,y]^{\frac{1}{3}(1,1)}$. Consider the quiver with relations (the reconstruction algebra)

$$\begin{tikzcd} \bullet \arrow[r, shift left=4, "c_1"] \arrow[r, shift left=2, "c_2"] & \bullet \arrow[l, "a_1"] \arrow[l, shift left=2, "a_2"] \arrow[l, shift left=4, "k_1"] \end{tikzcd} \qquad \begin{array}{ll} c_1 a_2 = c_2 a_1 & a_2 c_1 = a_1 c_2 \\ c_1 k_1 = c_2 a_2 & k_1 c_1 = a_2 c_2 \end{array}$$

Choose dimension vector (1,1). We are going to calculate the moduli space for stability $\theta_0 = (-1,1)$, then calculate the moduli space for stability $\theta_1 = (1,-1)$.
(i) Take $\theta_0 = (-1,1)$. As in the examples above, for a module

$$M = \mathbb{C} \begin{array}{c} \xrightarrow{c_1} \\ \xrightarrow{c_2} \\ \xleftarrow{a_1} \\ \xleftarrow{a_2} \\ \xleftarrow{k_1} \end{array} \mathbb{C}$$

to be semistable requires $c_1 \neq 0$ or $c_2 \neq 0$ and so we have two open sets $U_0 = (c_1 \neq 0)$ and $U_1 = (c_2 \neq 0)$. Changing basis so that these are 1, by the relations we have

$$\begin{array}{cc} \mathbb{C} \begin{array}{c} \xrightarrow{1} \\ \xrightarrow{c_2} \\ \xleftarrow{a_1} \\ \xleftarrow{c_2 a_1} \\ \xleftarrow{c_2^2 a_1} \end{array} \mathbb{C} & \qquad \mathbb{C} \begin{array}{c} \xrightarrow{c_1} \\ \xrightarrow{1} \\ \xleftarrow{c_1^2 k_1} \\ \xleftarrow{c_1 k_1} \\ \xleftarrow{k_1} \end{array} \mathbb{C} \\ \\ U_0 = \mathbb{C}^2_{c_2, a_1} & \qquad U_1 = \mathbb{C}^2_{c_1, k_1}. \end{array}$$

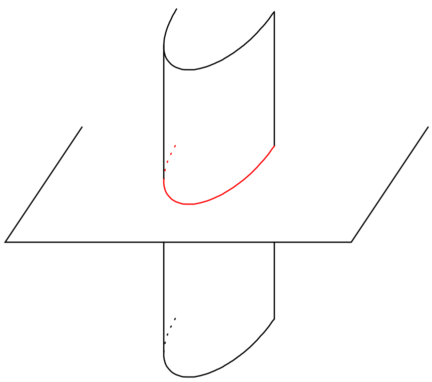
Now the gluing is given by, whenever $c_2 \neq 0$

$$U_0 \ni (c_2, a_1) = \mathbb{C} \begin{array}{c} \xrightarrow{1} \\ \xrightarrow{c_2} \\ \xleftarrow{a_1} \\ \xleftarrow{c_2 a_1} \\ \xleftarrow{c_2^2 a_1} \end{array} \mathbb{C} = \mathbb{C} \begin{array}{c} \xrightarrow{c_2^{-1}} \\ \xrightarrow{1} \\ \xleftarrow{c_2 a_1} \\ \xleftarrow{c_2^2 a_1} \\ \xleftarrow{c_2^3 a_1} \end{array} \mathbb{C} = (c_2^{-1}, c_2^3 a_1) \in U_1$$

since we read off the co-ordinates in U_1 in the c_1 and k_1 positions. Thus by inspection we see that our space is $\mathscr{O}_{\mathbb{P}^1}(-3)$, the minimal resolution.
(ii) Take $\theta_1 = (1,-1)$. Its clear that we now have 3 open sets $U_0 = (a_1 \neq 0)$, $U_1 = (a_2 \neq 0)$, $U_2 = (k_1 \neq 0)$. Consider first U_0: after changing basis so that $a_1 = 1$, we have

$$\mathbb{C} \begin{array}{c} \xrightarrow{c_1} \\ \xrightarrow{c_2} \\ \xleftarrow{1} \\ \xleftarrow{a_2} \\ \xleftarrow{k_1} \end{array} \mathbb{C} \qquad \begin{array}{cc} c_1 a_2 = c_2 & a_2 c_1 = c_2 \\ c_1 k_1 = c_2 a_2 & k_1 c_1 = a_2 c_2 \end{array}$$

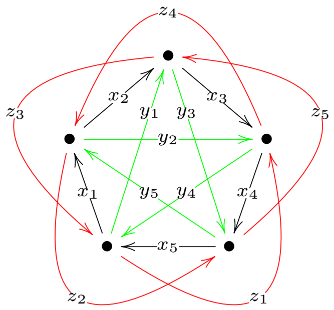
which is parameterized by the three variables c_1, a_2, k_1 subject to the one relation $c_1 k_1 = c_1 a_2^2$ i.e. $c_1(k_1 - a_2^2) = 0$. This is singular in dimension 1! If we draw U_0, it looks something like



It has two components, namely the $c_1 = 0$ component and the $k_1 = a_2^2$ component. The $k_1 = a_2^2$ component is the one that we want, since it ends up giving us (part of) the minimal resolution.

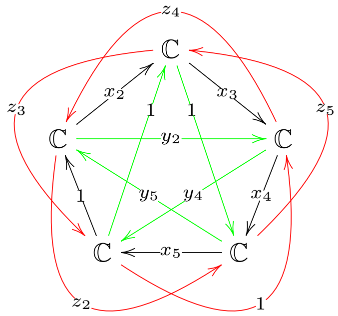
From the above example we see that a moduli space may not be smooth and might have components. Note that in the above example there is one component which is particularly nice, however the next example shows that a moduli space may be both irreducible and singular.

Example 4.5. Consider the group $\frac{1}{5}(1,2,3) := \left\langle \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & \varepsilon^2 & 0 \\ 0 & 0 & \varepsilon^3 \end{pmatrix} : \varepsilon^5 = 1 \right\rangle$ giving a three dimensional quotient singularity. The algebra to consider is



$$\begin{array}{lll} x_1y_2 = y_1x_3 & x_1z_2 = z_1x_4 & y_1z_3 = z_1y_4 \\ x_2y_3 = y_2x_4 & x_2z_3 = z_2x_5 & y_2z_4 = z_2y_5 \\ x_3y_4 = y_3x_5 & x_3z_4 = z_3x_1 & y_3z_5 = z_3y_1 \\ x_4y_5 = y_4x_1 & x_4z_5 = z_4x_2 & y_4z_1 = z_4y_2 \\ x_5y_1 = y_5x_2 & x_5z_1 = z_5x_3 & y_5z_2 = z_5y_3 \end{array}$$

Consider $\alpha = (1,1,1,1,1)$ with stability $\theta = (-4,1,1,1,1)$. Consider the open set given by $x_1 \neq 0$, $y_1 \neq 0$, $y_3 \neq 0$ and $z_1 \neq 0$. After changing basis so that these are the identity we have



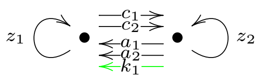
$$\begin{array}{ccc} y_2 = x_3 & z_2 = x_4 & z_3 = y_4 \\ x_2 = y_2x_4 & x_2z_3 = z_2x_5 & y_2z_4 = z_2y_5 \\ x_3y_4 = x_5 & x_3z_4 = z_3 & z_5 = z_3 \\ x_4y_5 = y_4 & x_4z_5 = z_4x_2 & y_4 = z_4y_2 \\ x_5 = y_5x_2 & x_5 = z_5x_3 & y_5z_2 = z_5 \end{array}$$

from which elimination of variables gives that this open set is parameterized by $a = y_5$, $b = x_3$, $c = z_4$ and $d = x_4$ subject to the one relation $ad = bc$. This is singular at the origin and so consequently the moduli space is singular. In fact in this example it is also irreducible.

Example 4.6. Consider the group $\frac{1}{3}(1,1,0)$ giving the three dimensional singularity

$$\mathbb{C}[x,y,z]^{\frac{1}{3}(1,1,0)} = \mathbb{C}[x,y]^{\frac{1}{3}(1,1)} \otimes_{\mathbb{C}} \mathbb{C}[z]$$

i.e. really just a surface crossed with $\mathbb{C}$. In this case the algebra to consider is the higher-dimensional reconstruction algebra



$$\begin{array}{llll} c_1a_2 = c_2a_1 & a_2c_1 = a_1c_2 & z_1c_1 = c_1z_2 & z_2a_1 = a_1z_1 \\ c_1k_1 = c_2a_2 & k_1c_1 = a_2c_2 & z_1c_2 = c_2z_2 & z_2a_2 = a_2z_1 \\ & & & z_2k_1 = k_1z_1 \end{array}$$

An easy calculation shows that for $\alpha = (1,1)$ and $\theta_0 = (-1,1)$ we resolve the singularity; unsurprisingly its just the minimal resolution crossed with $\mathbb{C}$. Again the same is true for $\theta_1 = (1,-1)$ but again we have to pass to components. The point in this example is that although for $\theta_0 = (-1,1)$ the moduli space is projective over $\mathbb{C}[x,y,z]^{\frac{1}{3}(1,1,0)}$, the zeroth part of the graded ring which we take the Proj of (i.e. the invariants $k[\mathscr{R}]^G$) is *not* $\mathbb{C}[x,y,z]^{\frac{1}{3}(1,1,0)}$, so a little care should be taken. The reason for this is that both z_1 and z_2 belong to $k[\mathscr{R}]^G$, and there is no relation which tells us they are the same (they are however the same as soon as $c_1 \neq 0$ or $c_2 \neq 0$). Thus $k[\mathscr{R}]^G$ has an 'extra' z.

One of the advantages of quivers is that they allow you to resolve singularities explictly in examples you wouldn't be able to do otherwise, especially in the case of quotients by a non-abelian group: we will illustrate this principle in more complicated examples in a future lecture.

LECTURES ON RECONSTRUCTION ALGEBRAS II

MICHAEL WEMYSS

1. Introduction

Last lecture I introduced quivers with relations. Then after choosing a dimension vector α and character θ such that $\sum_{i\in Q_0}\alpha_i\theta_i = 0$ we constructed what we called a moduli space. I again emphasize that, given kQ/R, we need to make *two* choices to define the moduli space.

This seminar is aiming to resolve singularities (in particular rational surfaces) so today I'm going to start to go in that direction. First though I have one thing to finish from last time, namely to prove that the spaces we introduced are actually moduli spaces in the strict sense of the word. I'll do this in the first section. I will then spend the rest of the lecture giving the geometric motivation of a reconstruction algebra, and I will highlight many of the subtleties and technicalities we will need to overcome in future.

2. Why its a Moduli Space

In the last lecture, given a dimension vector α and stability θ such that $\sum_{i\in Q_0}\alpha_i\theta_i = 0$ we constructed a space $\mathfrak{M}^{ss}_\theta(A,\alpha) = \mathfrak{M}^{ss}_\theta$ and called it a moduli space. In this section we justify the name: we are going to rigorously define what a 'moduli space' is and then apply it to quivers. Proving that the 'moduli spaces' from the last lecture are actually moduli spaces is important since (in some circumstances) it gives us the existence of a *universal bundle* on the space.

First some motivation: in what follows, 'moduli set' means a set of things we would *like* to parameterize by a geometric object. The natural question to ask is

Q1: Does there exist a scheme X whose closed points are the objects in the 'moduli set'
A: Usually no.

Thus we ask

Q2: Does there exist a scheme X whose closed points are 'some' of the objects in the 'moduli set'
A: More often

We clearly have to make this more precise. To do this, for any category $\mathcal{C}$ define $[\mathcal{C}, \texttt{Set}]$ to be the category of contravariant functors from $\mathcal{C}$ to $\texttt{Set}$. For any object X in $\mathcal{C}$ define

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}}(-,X): \quad & \mathcal{C} \to \texttt{Set} \\ & C \mapsto \mathrm{Hom}_{\mathcal{C}}(C,X) \end{aligned}$$

in the obvious way, so $\mathrm{Hom}_{\mathcal{C}}(-,X) \in [\mathcal{C}, \texttt{Set}]$. We call this *the functor of points of* X since in many examples (but not all!) there exists an object Z of $\mathcal{C}$ with the property that $\mathrm{Hom}_{\mathcal{C}}(Z,X)$ is the set of points of X. For example in the category of groups $\texttt{Gp}$, $\mathbb{Z}$ is an object for which $\mathrm{Hom}_{\texttt{Gp}}(\mathbb{Z},G) = |G|$ as sets, for any group G. Another example would be $\mathbb{Z}[X]$ in the category of rings.

Now Yoneda's Lemma tells us that

$$\begin{array}{ccc} \mathcal{C} & \to & [\mathcal{C}, \texttt{Set}] \\ C & \mapsto & \mathrm{Hom}_{\mathcal{C}}(-,C) \end{array}$$

is an embedding, so we can view $\mathcal{C}$ inside the category $[\mathcal{C}, \texttt{Set}]$. This may look like we've made things more difficult but in fact it may be the case that in the larger category $[\mathcal{C}, \texttt{Set}]$ some constructions are much easier. Anyway,

Definition 2.1. *We call $F \in [\mathcal{C}, \mathtt{Set}]$ representable if F is naturally isomorphic to* $\mathrm{Hom}_{\mathcal{C}}(-, A)$ *for some object A of $\mathcal{C}$.*

Denote the category of affine varieties by $\mathtt{AfVar}$ then by Yoneda affine varieties are precisely those functors $\mathtt{AfVar} \to \mathtt{Set}$ which are representable. This is all very tautological. Note that affine algebraic groups are (by definition) those representable functors $\mathtt{AfVar} \to \mathtt{Set}$ which take values in the category $\mathtt{Gp}$ (instead of $\mathtt{Set}$).

Now a moduli problem for some class of objects in algebraic geometry consists of

- for every scheme X, a notion of a family parameterized by the scheme X.

We call this a family over X. Note at this stage this is inprecise, but the point is that we specialize this general framework to a precise meaning of 'family over X' whenever we want to do anything. Now the moduli problem is considered solved if there exists a single scheme Y such that the family over Y is universal, in the sense that given any other X, every member of the family over X is uniquely induced by a morphism $X \to Y$.

Denoting the category of schemes by $\mathtt{Sch}$, more formally the moduli problem is a contravariant functor

$$\begin{array}{rcl} F : \mathtt{Sch} & \to & \mathtt{Set} \\ S & \mapsto & \text{the set } \{\text{members of the family over S}\} \end{array}$$

and the moduli problem is considered solved if F is representable. This is again tautological: if $F \cong \mathrm{Hom}(-, Y)$ then

$$\{\text{members of the family over X}\} = FX \cong \mathrm{Hom}(X, Y).$$

This leads to the following definition

Definition 2.2. *If a contravariant functor $F : \mathtt{Sch} \to \mathtt{Set}$ is represented by a scheme Y, we call Y the fine moduli space of F.*

This is normally too strong since many moduli problems don't have representable functors. So we compromise:

Definition 2.3. *Given $F \in [\mathtt{Sch}, \mathtt{Set}]$, a scheme Y is said to be a best approximation to F (or sometimes Y corepresents F) is there is a natural transformation*

$$\alpha : F \to \mathrm{Hom}(-, Y)$$

which is universal amongst the natural transformations from F to schemes, i.e. given any other $\beta : F \to \mathrm{Hom}(-, Z)$, there exists a unique natural transformation

$$\begin{tikzcd} F \arrow[r, "\alpha"] \arrow[dr, "\beta"'] & \mathrm{Hom}(-, Y) \arrow[d, dotted, "\exists!"] \\ & \mathrm{Hom}(-, Z) \end{tikzcd}$$

such that the diagram commutes. If F is a moduli functor, we cal (Y, α) the moduli space of F. If further (Y, α) satisfies

$$\alpha_{\mathrm{Spec}\mathbb{C}} : F(\mathrm{Spec}\mathbb{C}) \to \mathrm{Hom}(\mathrm{Spec}\mathbb{C}, Y)$$

is bijective, we call (Y, α) a coarse moduli space.

We now apply this to quivers. To begin we define the notion of a family over X:

Definition 2.4. *A family of kQ-modules with dimension vector $\alpha = (\alpha_i)$ over a scheme X is an assignment, for each vertex i, of a vector bundle $\mathscr{V}_i$ of rank α_i, and for every arrow in Q a corresponding morphism of vector bundles.*

If you like, you can think of this as specifying a map $kQ/R \to \mathrm{End}(\oplus_{i \in Q_0} \mathscr{V}_i)$. Or you can also view it as a representation in the category of vector bundles $\mathtt{Vb}X$. The above definition really is a family of representations over X in the obvious way: for any point $x \in X$ if we take the stalk of the bundles (=the fibre) at x then each vertex just becomes a finite dimensional vector space and the morphisms become linear maps such that the relations still hold. This isn't saying anything other than a vector bundle is locally trivial. Thus for every point $x \in X$ we get an actual representation of kQ/R.

We now make our moduli problem precise by defining the families we would like to classify:

Definition 2.5. *A family of semistable kQ/R-modules with dimension vector α over a scheme X is just a family of kQ-modules with dimension vector $\alpha = (\alpha_i)$ as above, in which all members in the family are θ-semistable. We have the similar notion for θ-stability.*

This just means that for every point $x \in X$, the associated stalk (i.e. actual representation) is θ-semistable.

Now every θ-semistable M has a Jordan-Hölder filtration

$$0 = M_0 \subset M_1 \subset \ldots \subset M_{n-1} \subset M_n = M$$

in which every subobject M_i is θ-semistable, and every factor M_i/M_{i-1} is simple (in the category of θ-semistable modules). This is just constructed in the standard way; since M is finite dimensional the process must eventually finish.

It is clear that if M is θ-stable then the JH filtration is just $0 \subset M$ (since by definition M has no θ-semistable subobjects). In more fancy language the θ-stable objects are precisely the simple objects in the category of θ-semistable objects.

Definition 2.6. *Two θ-semistable objects are called S-equivalent (with respect to θ) if their Jordan-Hölder filtrations have isomorphic composition factors*

By the above discussion this collapses in the case of stability: two θ-stable modules M and N are S-equivalent if and only if they are isomorphic, since their JH filtrations are just $0 \subset M$ and $0 \subset N$.

The quotient (=moduli space) defined last time $\mathfrak{M}^{ss}_\theta := \mathscr{R}/\!\!/_\chi G$ parameterizes the θ-semistable representations *up to S-equivalence.* I'm not going to explain why this is true, since it involves more GIT than I want to get into. The open set of the quotient which corresponds to the stable points thus parameterizes the θ-stable representations *up to isomorphism.* This answers a question Osamu asked last time.

If θ is generic then stability and semistability coincide (by definition), thus in these cases we are always classifying up to isomorphism. In practice we're only going to be dealing with generic stability conditions.

Now we have defined the moduli problem, so we get the moduli functors

$$\begin{array}{rcl} \mathcal{M}^{ss}_{kQ,\alpha,\theta} : \mathtt{Sch} & \to & \mathtt{Set} \\ X & \mapsto & \text{the set } \{\text{families of } \theta\text{-semistable } kQ/R \text{ modules with dim } \alpha \text{ over } X\}/S\text{-equiv} \\ \mathcal{M}^{s}_{kQ,\alpha,\theta} : \mathtt{Sch} & \to & \mathtt{Set} \\ X & \mapsto & \text{the set } \{\text{families of } \theta\text{-stable } kQ/R \text{ modules with dim } \alpha \text{ over } X\}/\cong \end{array}$$

Theorem 2.7 (King 5.2)**.** *$\mathfrak{M}^{ss}_\theta$ is a coarse moduli space for the functor $\mathcal{M}^{ss}_{kQ,\alpha,\theta}$.*

Denote the stable points in $\mathfrak{M}^{ss}_\theta$ by $\mathfrak{M}^{s}_\theta$, then

Theorem 2.8 (King, 5.3)**.** *If α is indivisible, $\mathfrak{M}^{s}_\theta$ represents the functor $\mathcal{M}^{s}_{kQ,\alpha,\theta}$, i.e. $\mathfrak{M}^{s}_\theta$ is a fine moduli space.*

Thus for generic θ and indivisible α, $\mathfrak{M}^{ss}_\theta$ is a fine moduli space. This is important as it means we have a universal bundle[1]: since for generic θ and indivisible α

$$\mathcal{M}^{s}_{kQ,\alpha,\theta} \cong \mathrm{Hom}(-, \mathfrak{M}^{ss}_\theta)$$

as functors from schemes to sets, apply both sides to the scheme $\mathfrak{M}^{ss}_\theta$. Then

$$1 \in \mathrm{Hom}(\mathfrak{M}^{ss}_\theta, \mathfrak{M}^{ss}_\theta) \cong \mathcal{M}^{s}_{kQ,\alpha,\theta}(\mathfrak{M}^{ss}_\theta)$$

[1]this is backwards: the theorem is proved by exhibiting such a bundle!

so we have a family of θ-stable kQ/R modules with dimension vector α over $\mathfrak{M}_\theta^{ss}$ corresponding to the identity map. This just means that for every point $x \in \mathfrak{M}_\theta^{ss}$, the representation in this family corresponding to x is just x. We call this family the universal family.

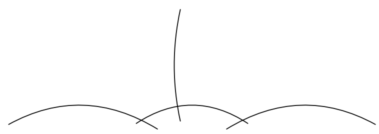
3. Geometric Motivation of Reconstruction Algebras

Before talking about the $\mathrm{SL}(2,\mathbb{C})$ McKay correspondence and its generalization to $\mathrm{GL}(2,\mathbb{C})$ I'll first give some motivation as to what we might regard as being the 'best' possible answer.

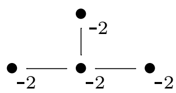
In this section consider a rational normal surface singularity $X = \mathrm{Spec}R$ with minimal resolution $\widetilde{X} \xrightarrow{\pi} X$. From this we have the dual graph, which you should view as a simplified picture of the resolution:

Definition 3.1. *Denote by $\{E_i\}$ the exceptional collection of $\mathbb{P}^1$s. Define the (labelled) dual graph as follows: for every E_i draw a dot, and join two dots if the corresponding $\mathbb{P}^1$'s intersect. Additionally, decorate each vertex with the self-intersection number corresponding to the curve at that vertex.*

In practice what this means is that if we have a collection of $\mathbb{P}^1$'s (which are one-dimensional, so we draw as lines) intersecting as follows:



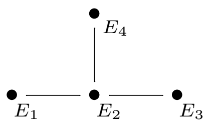
with all curves having self-intersection number (-2), then the dual graph is



The theory of rational normal surfaces is in many ways dictated by the following piece of combinatorial data (the fundamental cycle Z_f) which we can associate to the dual graph:

Definition 3.2 (Artin)**.** *For the dual graph $\{E_i\}$, define the fundamental cycle $Z_f = \sum_i r_i E_i$ (with each $r_i \geq 1$) to be the unique smallest element such that $Z_f \cdot E_i \leq 0$ for all vertices i.*

What this means in practice: for the dual graph



$$E_1 \cdot E_1 = -2 \quad E_2 \cdot E_2 = -2 \quad E_3 \cdot E_3 = -2 \quad E_4 \cdot E_4 = -2$$

first try the smallest element $Z_r = E_1 + E_2 + E_3 + E_4$:

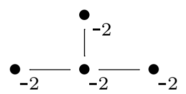
$$\begin{aligned}
Z_r \cdot E_1 &= E_1 \cdot E_1 + E_2 \cdot E_1 + E_3 \cdot E_1 + E_4 \cdot E_1 = (-2) + 1 + 0 + 0 = -1 \leq 0 \\
Z_r \cdot E_2 &= E_1 \cdot E_2 + E_2 \cdot E_2 + E_3 \cdot E_2 + E_4 \cdot E_2 = 1 + (-2) + 1 + 1 = 1 \nleq 0 \\
Z_r \cdot E_3 &= E_1 \cdot E_3 + E_2 \cdot E_3 + E_3 \cdot E_3 + E_4 \cdot E_3 = 0 + 1 + (-2) + 0 = -1 \leq 0 \\
Z_r \cdot E_4 &= E_1 \cdot E_4 + E_2 \cdot E_4 + E_3 \cdot E_4 + E_4 \cdot E_4 = 0 + 1 + 0 + (-2) = -1 \leq 0
\end{aligned}$$

Since it fails against E_2, try $Z_2 = E_1 + 2E_2 + E_3 + E_4$. A similar calculation shows that $Z_2 \cdot E_i \leq 0$ for all curves E_i. Consequently $Z_f = Z_2$, and we write this as $Z_f = \begin{smallmatrix} & 1 & \\ 1 & 2 & 1 \end{smallmatrix}$.

Observe that changing the middle curve in the above example changes the fundamental cycle to be $Z_f = \begin{smallmatrix} & 1 & \\ 1 & 1 & 1 \end{smallmatrix}$, but keeping the middle curve the same and changing any other curve results in the same $Z_f = \begin{smallmatrix} & 1 & \\ 1 & 2 & 1 \end{smallmatrix}$.

I emphasize that Z_f is defined *entirely* in terms of the dual graph. Consequently given a dual graph you can (if you wish) think of Z_f as a purely combinatorial piece of data which we can associate to it, but it is perhaps best to think a little more geometrically.

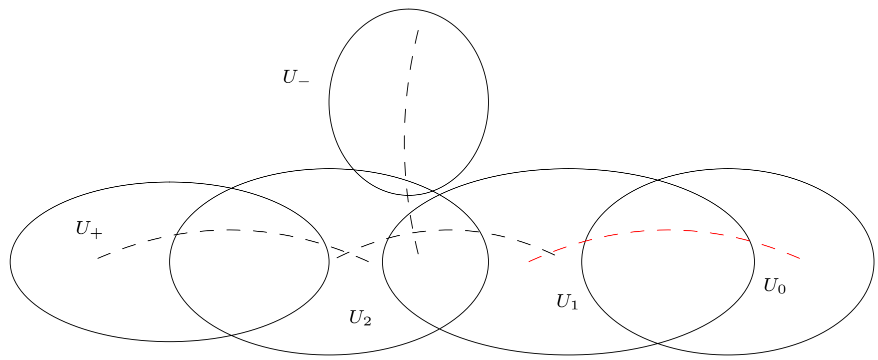
Now in fact



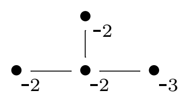
in the above discussion is the dual graph of the minimal resolution of $\mathbb{C}^2/BD_{4\cdot 2}$ where $BD_{4\cdot 2}$ is the binary dihedral group of order 8 inside $\mathrm{SL}(2,\mathbb{C})$:

$$BD_{4.2} := \left\langle \begin{pmatrix} \varepsilon_4 & 0 \\ 0 & \varepsilon_4^3 \end{pmatrix}, \begin{pmatrix} 0 & \varepsilon_4 \\ \varepsilon_4 & 0 \end{pmatrix} \right\rangle$$

This has been extensively studied by many people. Say we have an open cover of the minimal resolution looking something like:



Now say we want to change the red curve in the minimal resolution into a (-3) curve, i.e. we want the dual graph[2] to become



How should we go about doing this? Note first that the fundamental cycle is still $Z_f = \begin{smallmatrix} & 1 & \\ 1 & 2 & 1 \end{smallmatrix}$. We want to change the original space as little as possible to achieve our goal, so it would appear sensible to suggest that we only (at worst) change the equation of the open set U_0, and also change how U_0 glues to U_1. The change in glue will give the change in self-intersection number. The rest of the open sets (and their glues) will remain the same, and so we will have the desired configuration of $\mathbb{P}^1$s. I'm actually glossing over the fact that our map down to the singularity also changes, but the quiver takes care of this too so we shouldn't worry.

Here comes the key point:

Remark 3.3. If we change the geometry to accommodate a different self-intesection number, then *provided* Z_f *does not change* the new geometry will be *very* similar to the old geometry.

This is a subtle change in approach, so I'll emphasize it again. If you are given a group G inside $\mathrm{GL}(2,\mathbb{C})$ then instead of trying to resolve it using the G-Hilbert scheme (which we view as a 'new' space dependent on the group G), we should instead view the resolution as being a very small modification of a space we already understand. It is the (yet to be defined) reconstruction algebra which encodes the difference. Of course at this stage we don't know what space the resolution will be similar to, but the reconstruction algebra will tell us this.

The G-Hilbert scheme turns out to give the minimal resolution, but I do not know of any conceptual reason why this should be true. The groups under consideration can become very large and complicated, but the geometry stays quite simple.

Another point: in the above example if we had changed the middle curve instead of the red curve, you might think it would be more complicated as lots of things would have to

[2]In fact this new dual graph corresponds to the non-abelian group $\mathbb{D}_{5,3}$ of order 24

change. However I contest that this is actually the easiest case, since the fundamental cycle has decreased. Since Z_f can only decrease (i.e. improve) or remain the same under changing a self-intersection number, you should view this as saying that the difficulty in the geometry either

(A) remains the same (when Z_f stays the same)
(B) becomes easier (when Z_f changes, i.e. decreases)

As we shall see this is very important, since in many cases for non-abelian subgroups of $\mathrm{GL}(2,\mathbb{C})$ to extract the geometry explicitly from the reconstruction algebra is *precisely* the same level of difficultly as the toric case. The slogan is

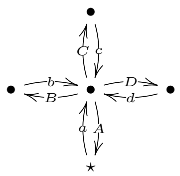
Slogan 3.4. *Take any non-abelian subgroup of* $\mathrm{GL}(2,\mathbb{C})$*. Then if the fundamental cycle* Z_f *is reduced (i.e. consists only of* 1*'s), the geometry is not toric, but it may as well be.*

In practice Z_f is reduced almost all of the time. I'll show how the above slogan works in my next lecture, but for now I'll illustrate case (A) with an example.

4. A computation

Earlier I promised to give an example of explicitly resolving a singularity which would be very difficult to do without quivers, and also I promised to give an example of a computation of a non-abelian group action. I'll now do this, and in the process I'll be able to illustrate some of the points I raised in the previous section. At the moment you should view the NC rings that I use in this section as being constructed by magic, but I'll explain in my next lecture where I get them from.

Example 4.1. Consider the group $BD_{4\cdot 2}$ of order 8. This is classical McKay Correspondence territory, so the algebra to consider is the preprojective algebra



$$aA = bB = cC = dD = 0$$
$$Aa + Bb + Cc + Dd = 0$$

This is Morita equivalent to the skew group ring, if you know about these things. We choose dimension vector and stability

$$\alpha = \begin{smallmatrix} & 1 & \\ 1 & 2 & 1 \\ & 1 & \end{smallmatrix} \qquad \theta = \begin{smallmatrix} & 1 & \\ 1 & 1 & 1 \\ & \text{-5} & \end{smallmatrix}$$

Notice that $\sum_{i\in Q_0} \alpha_i\theta_i = 0$ so we can form the moduli space. With these choices the computation becomes more complicated than the ones we did before, but not massively so; to now specify an open set we must

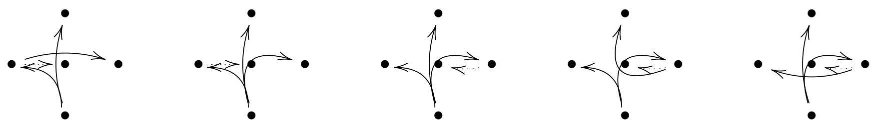
- specify, for each one-dimensional irreducible representation ρ, a non-zero path (which we can change basis to assume to be the identity) from the trivial representation to the vertex ρ.
- specify paths $(0\ 1)$ and $(1\ 0)$ from the trivial representation to the 2-dimensional representation.

Different choices in the above lead to different open sets. Note that we must be able to make such choices for any θ-stable module M since by definition M is $\star$-generated and so paths leaving the trivial vertex must generate the vector spaces at all other vertices. For a stable M, it must be true that $a \neq 0$ and so after changing basis we can (and will) always assume that $a = (1\ 0)$.

Define the open sets U_0, U_1, U_2, U_+ and U_- by the following conditions:

U_0	$aB = 1$	$aC = 1$	$aBbD = 1$	$a = (1\ 0)$	$b = (0\ 1)$
U_1	$aB = 1$	$aC = 1$	$aD = 1$	$a = (1\ 0)$	$b = (0\ 1)$
U_2	$aB = 1$	$aC = 1$	$aD = 1$	$a = (1\ 0)$	$d = (0\ 1)$
U_+	$aB = 1$	$aDdC = 1$	$aD = 1$	$a = (1\ 0)$	$d = (0\ 1)$
U_-	$aDdB = 1$	$aC = 1$	$aD = 1$	$a = (1\ 0)$	$d = (0\ 1)$

Pictorially we draw this as follows:



U_0 U_1 U_2 U_+ U_-

where the solid black lines correspond to the identity, and the dotted arrow corresponds to the choice of vector $(0\ 1)$. These actually cover the moduli, but the proof is a bit messy. Note that there are *lots* of other open covers we could take.

We do the U_0 calculation in full, and just summarize the others. Any stable module in U_0 looks like

$$\begin{tikzcd}
 & \mathbb{C} \arrow[d, shift left, "{(c_1\ c_2)}"] & \\
\mathbb{C} \arrow[r, shift left, "{(0\ 1)}"] & \mathbb{C}^2 \arrow[l, shift left, "{\left(\begin{smallmatrix}1\\ B_2\end{smallmatrix}\right)}"] \arrow[u, shift left, "{\left(\begin{smallmatrix}1\\ C_2\end{smallmatrix}\right)}"] \arrow[r, shift left, "{\left(\begin{smallmatrix}D_1\\ 1\end{smallmatrix}\right)}"] \arrow[d, shift left, "{\left(\begin{smallmatrix}A_1\\ A_2\end{smallmatrix}\right)}"] & \mathbb{C} \arrow[l, shift left, "{(d_1\ d_2)}"] \\
 & \mathbb{C} \arrow[u, shift left, "{(1\ 0)}"] &
\end{tikzcd}$$

where the variables are scalars, subject only to the quiver relations. Now

- $aA = 0$ implies $A_1 = 0$
- $bB = 0$ implies $B_2 = 0$
- $cC = 0$ implies $c_1 = -c_2C_2$
- $dD = 0$ implies $d_2 = -d_1D_1$

and so plugging this in our module becomes

$$\begin{tikzcd}
 & \mathbb{C} \arrow[d, shift left, "{(-c_2C_2\ c_2)}"] & \\
\mathbb{C} \arrow[r, shift left, "{(0\ 1)}"] & \mathbb{C}^2 \arrow[l, shift left, "{\left(\begin{smallmatrix}1\\ 0\end{smallmatrix}\right)}"] \arrow[u, shift left, "{\left(\begin{smallmatrix}1\\ C_2\end{smallmatrix}\right)}"] \arrow[r, shift left, "{\left(\begin{smallmatrix}D_1\\ 1\end{smallmatrix}\right)}"] \arrow[d, shift left, "{\left(\begin{smallmatrix}0\\ A_2\end{smallmatrix}\right)}"] & \mathbb{C} \arrow[l, shift left, "{(d_1\ -d_1D_1)}"] \\
 & \mathbb{C} \arrow[u, shift left, "{(1\ 0)}"] &
\end{tikzcd}$$

But now there is only one relation left, namely $Aa + Bb + Cc + Dd = 0$. This gives

$$\begin{pmatrix} 0 & 0 \\ A_2 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} -c_2C_2 & c_2 \\ -c_2C_2^2 & c_2C_2 \end{pmatrix} + \begin{pmatrix} d_1D_1 & -d_1D_1^2 \\ d_1 & -d_1D_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

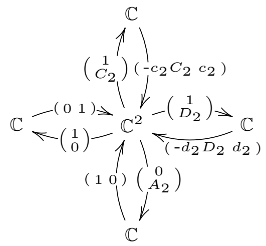
which yields the four conditions

$$\begin{aligned} c_2C_2 &= d_1D_1 \\ c_2 &= d_1D_1^2 - 1 \\ A_2 &= c_2C_2^2 - d_1 \\ c_2C_2 &= d_1D_1 \end{aligned}$$

The second and third conditions eliminate the variables c_2 and A_2, whereas the first and last conditions are the same. Substituting the second condition into the first we see that this open set is completely parameterized by d_1, D_1 and C_2 subject to the one relation $d_1D_1 = (d_1D_1^2 - 1)C_2$, so U_0 is a smooth hypersurface in $\mathbb{C}^3$.

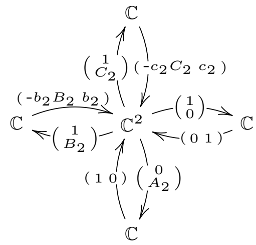
Similarly we have

U_1



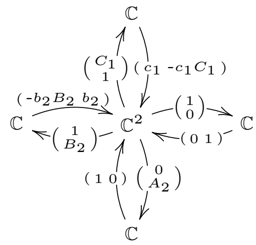
$$\begin{gathered} c_2C_2 = \text{-}d_2D_2 \\ 1 + c_2 + d_2 = 0 \\ A_2 = c_2C_2^2 + d_2D_2^2 \\ c_2C_2 = \text{-}d_1D_1 \end{gathered} \qquad \mathbb{C}^3_{d_2,D_2,C_2}/(1+d_2)C_2 = d_2D_2$$

U_2



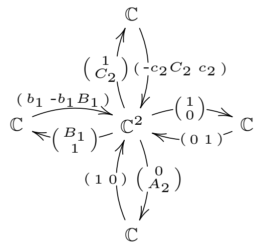
$$\begin{gathered} b_2B_2 = \text{-}c_2C_2 \\ 1 + b_2 + c_2 = 0 \\ A_2 = b_2B_2^2 + c_2C_2^2 \\ b_2B_2 = \text{-}c_2C_2 \end{gathered} \qquad \mathbb{C}^3_{b_2,B_2,C_2}/(1+b_2)C_2 = b_2B_2$$

U_+



$$\begin{gathered} b_2B_2 = c_1C_1 \\ b_2 = c_1C_1^2 - 1 \\ A_2 = b_2B_2^2 - c_1 \\ b_2B_2 = c_1C_1 \end{gathered} \qquad \mathbb{C}^3_{c_1,B_2,C_1}/(c_1C_1^2-1)B_2 = c_1C_1$$

U_-



$$\begin{gathered} b_1B_1 = c_2C_2 \\ c_2 = b_1B_1^2 - 1 \\ A_2 = c_2C_2^2 - b_1 \\ b_1B_1 = c_2C_2 \end{gathered} \qquad \mathbb{C}^3_{b_1,B_1,C_2}/(b_1B_1^2-1)C_2 = b_1B_1$$

Note in U_2 above the equation $1+b_2+c_2=0$ really means that we have a choice of co-ordinate between b_2 and c_2; thus we could equally well parameterize U_2 as $\mathbb{C}^3_{c_2,B_2,C_2}/c_2C_2=(1+c_2)B_2$.

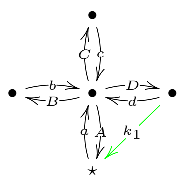
Hence we see that the space is covered by 5 open sets, each a smooth hypersurface in $\mathbb{C}^3$. It is also quite easy to write down the glues (I don't have time), and just see the configuration of $\mathbb{P}^1$'s: for example the gluing between U_0 and U_1 is

$$U_0 \ni (d_1, D_1, C_2) \xleftrightarrow{D_1\neq 0} (\text{-}d_1D_1^2, D_1^{-1}, C_2) \in U_1$$

The picture of the glues should (roughly) coincide with the picture I drew earlier.

The next example explains how to change the red $\mathbb{P}^1$ in the previous picture into a (-3)-curve.

Example 4.2. Consider the reconstruction algebra

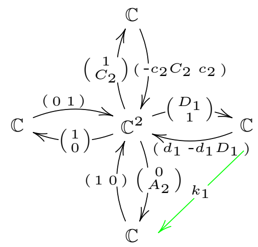


$$\begin{gathered} aA = bB = cC = dD = 0 \\ Aa + Bb + Cc + Dd = 0 \\ k_1aD = dBbD \\ aDk_1 = aCcA \end{gathered}$$

Choose dimension vector and stability as in the previous example. Now notice that *the same* conditions that defined an open cover in the previous example give an open cover here (since the stability cannot 'see' the extra arrows).

Now our old calculation tells us almost everything, except now we have a new variable k_1 inside every open set. The point is that the *only* open set which changes is U_0. The reason for this is quite simple: in the relations $k_1aD = dBbD$ and notice that $aD = 1$ in every open set except U_0. Thus $k_1 = dBbD$ in every open set except U_0 and consequently we can put k_1 in terms of the other variables. Hence k_1 isn't really an extra variable in these open sets, so they do not change.

What happens to U_0? Well by the previous calculation we have

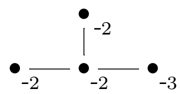


$$\begin{aligned} c_2C_2 &= d_1D_1 \\ c_2 &= d_1D_1^2 - 1 \\ A_2 &= c_2C_2^2 - d_1 \\ c_2C_2 &= d_1D_1 \\ \hline k_1D_1 &= d_1 \\ D_1k_1 &= c_2A_2 \end{aligned}$$

Since $d_1 = k_1D_1$, instead of being given by d_1, D_1, C_2 subject to $d_1D_1 = (d_1D_1^2 - 1)C_2$, the open set is now given by k_1, D_1, C_2 subject to $k_1D_1^2 = (k_1D_1^3 - 1)C_2$. Also, the gluing between U_0 and U_1 has changed to

$$U_0 \ni (k_1, D_1, C_2) \xleftrightarrow{D_1 \neq 0} (\text{-}(k_1D_1)D_1^2, D_1^{-1}, C_2) = (\text{-}k_1D_1^3, D_1^{-1}, C_2) \in U_1$$

Thus we see that the red curve has changed into a (-3)-curve, nothing else in the open cover has changed and so the dual graph is now



LECTURES ON RECONSTRUCTION ALGEBRAS III

MICHAEL WEMYSS

1. Introduction

Last lecture I gave lots of geometric motivation behind the idea of a reconstruction algebra. I said that instead of viewing the minimal resolution $\widetilde{X}$ of a quotient singularity $\mathbb{C}^2/G$ as G-Hilb (which a priori has nothing to do with other resolutions) we should instead view $\widetilde{X}$ as being very similar to a space we already understand. The reconstruction algebra encodes the difference. There are two main problems with these statements:

(1) we don't yet know what space to compare $\widetilde{X}$ too!
(2) we haven't defined the reconstruction algebra yet.

In fact it turns out, after we define the reconstruction algebra, that its underlying quiver tells us the answer to (1).

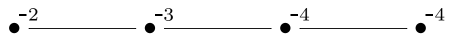
So today I'm going to lead up to the definition of the reconstruction algebra, and consequently I'm going to have to change perspective slightly and become more algebraic. I'll try and give some motivation from the world of commutative ring theory (=CM modules here) and also from representation theory.

First though I'll stay geometrical and illustrate the slogan I stated last time.

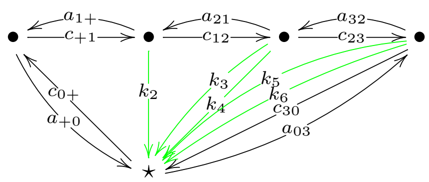
2. Non-toric Toric Geometry

Last time I made the somewhat counter-intuitive statement that for most non-abelian finite subgroups $G \leq \mathrm{GL}(2,\mathbb{C})$ (namely those with reduced fundamental cycle), to resolve the singularity explicitly is the same level of difficulty as toric geometry.

Example 2.1. I'm going to start by computing the geometry in a toric example. This also illustrates the pattern in the reconstruction relations. Consider the group $\frac{1}{67}(1,41)$. The continued fraction expansion $\frac{67}{41} = [2,3,4,4]$ and so the dual graph of the minimal resolution of $\mathbb{C}^2/\frac{1}{67}(1,41)$ is

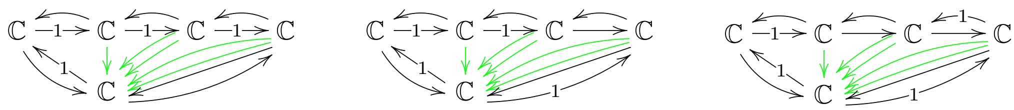


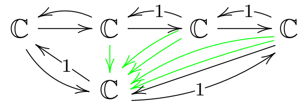
The reconstruction algebra of Type A in this example is

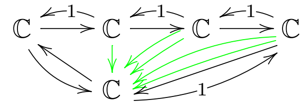


$$a_{+0}c_{0+} = c_{+1}a_{1+}$$
$$a_{1+}c_{+1} = k_2a_{03}a_{32}a_{21} \qquad a_{03}a_{32}a_{21}k_2 = c_{0+}a_{+0}$$
$$k_2c_{0+}c_{+1} = c_{12}a_{21}$$
$$a_{21}c_{12} = k_3a_{03}a_{32} \qquad a_{03}a_{32}k_3 = c_{0+}c_{+1}k_2$$
$$k_3c_{0+}c_{+1}c_{12} = k_4a_{03}a_{32} \qquad c_{0+}c_{+1}c_{12}k_3 = a_{03}a_{32}k_4$$
$$k_4c_{0+}c_{+1}c_{12} = c_{23}a_{32}$$
$$a_{32}c_{23} = k_5a_{03} \qquad a_{03}k_5 = c_{0+}c_{+1}c_{12}k_4$$
$$k_5c_{0+}c_{+1}c_{12}c_{23} = k_6a_{03} \qquad c_{0+}c_{+1}c_{12}c_{23}k_5 = a_{03}k_6$$
$$k_6c_{0+}c_{+1}c_{12}c_{23} = c_{30}a_{03} \qquad c_{0+}c_{+1}c_{12}c_{23}k_6 = a_{03}c_{30}$$

where the bizarre labeling will soon become clear. Pick dimension vector $(1,1,1,1,1)$ and stability condition $(-4,1,1,1,1)$, where the -4 is in the position of $\star$. We are used to this now - it means that a module M of dimension vector $(1,1,1,1,1)$ is θ-stable if and only if for every vertex in the quiver there is a non-zero path from $\star$ to that vertex. Consequently we have the following five open sets:





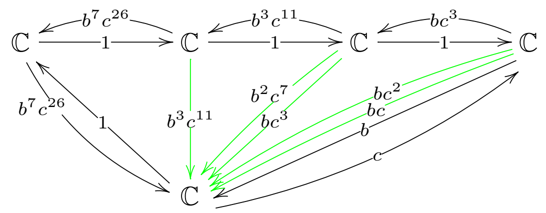


This is why we choose the stability $(-4,1,1,1,1)$, since it is 'blind' to the extra green arrows. Notice no matter how many extra green arrows we add to the above quiver, picking the dimension and stability as above the moduli is still covered by five open sets. Call the open set in the top left U_0. I shall now show that $U_0 \cong \mathbb{C}^2$, i.e. the open set U_0 is parameterized by two variables b and c subject to no relation.

Place b in the position of c_{30} and c in the position of a_{03}. I claim that every other arrow is determined by these. Well

$$\begin{aligned} k_6c_{0+}c_{+1}c_{12}c_{23} = c_{30}a_{03} \quad &\Rightarrow \quad k_6 = bc \\ k_5c_{0+}c_{+1}c_{12}c_{23} = k_6a_{03} \quad &\Rightarrow \quad k_5 = k_6c = bc^2 \\ a_{32}c_{23} = k_5a_{03} \quad &\Rightarrow \quad a_{32} = k_5c = bc^3 \\ &\vdots \end{aligned}$$

Continuing in this fashion (it is best done visually; I will explain this in the lecture), we get



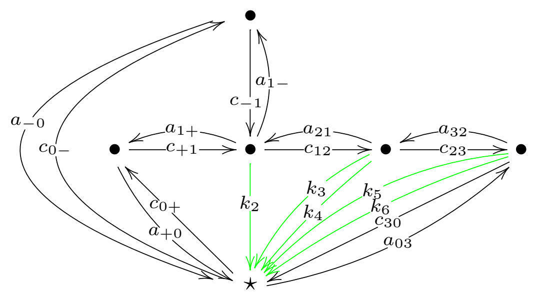
and so indeed this open set is just $\mathbb{C}^2$. Now the next open set U_1 is also just $\mathbb{C}^2$, and is parameterized by the variables in the c_{23} and a_{32} positions. By changing basis at the vertex 3 it immediate from the above picture (multiply all arrows out of vertex 3 by c, divide all arrows into vertex 3 by c) that the glue is

$$U_0 \ni (b,c) \leftrightarrow (b^{-1}, bc^4) \in U_1$$

Example 2.2. We are now going to explicitly resolve the singularity $\mathbb{C}^2/\mathbb{D}_{56,15}$, where

$$\mathbb{D}_{56,15} := \left\langle \begin{pmatrix} \varepsilon_{30} & 0 \\ 0 & \varepsilon_{30}^{-1} \end{pmatrix}, \begin{pmatrix} 0 & \varepsilon_4 \\ \varepsilon_4 & 0 \end{pmatrix}, \begin{pmatrix} \varepsilon_{82} & 0 \\ 0 & \varepsilon_{82} \end{pmatrix} \right\rangle$$

is a non-abelian group of order 2460. I claim this is really easy, once you know the reconstruction algebra. In this case it is

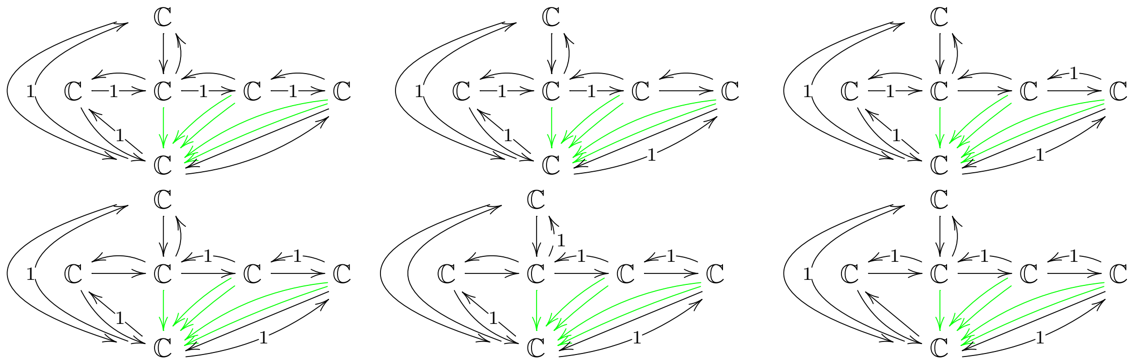


$$\begin{gathered} c_{0+}c_{+1} - c_{0-}c_{-1} = 4A_{01} \\ c_{0+}a_{+0} = c_{0-}a_{-0} \\ a_{-0}c_{0-} = c_{-1}a_{1-} \\ a_{1+}c_{+1} = a_{1-}c_{-1} \\ \hline a_{+0}c_{0+} = c_{+1}a_{1+} \\ a_{1+}c_{+1} = k_2a_{03}a_{32}a_{21} \qquad a_{03}a_{32}a_{21}k_2 = c_{0+}a_{+0} \\ k_2c_{0+}c_{+1} = c_{12}a_{21} \\ a_{21}c_{12} = k_3a_{03}a_{32} \qquad a_{03}a_{32}k_3 = c_{0+}c_{+1}k_2 \\ k_3c_{0+}c_{+1}c_{12} = k_4a_{03}a_{32} \qquad c_{0+}c_{+1}c_{12}k_3 = a_{03}a_{32}k_4 \\ k_4c_{0+}c_{+1}c_{12} = c_{23}a_{32} \\ a_{32}c_{23} = k_5a_{03} \qquad a_{03}k_5 = c_{0+}c_{+1}c_{12}k_4 \\ k_5c_{0+}c_{+1}c_{12}c_{23} = k_6a_{03} \qquad c_{0+}c_{+1}c_{12}c_{23}k_5 = a_{03}k_6 \\ k_6c_{0+}c_{+1}c_{12}c_{23} = c_{30}a_{03} \qquad c_{0+}c_{+1}c_{12}c_{23}k_6 = a_{03}c_{30} \end{gathered}$$

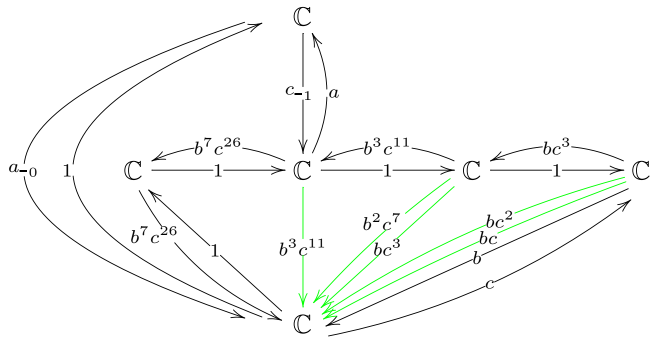
Note that the relations below the horizontal line are exactly the same as for the toric example we did earlier. Choose dimension vector $(1,1,1,1,1,1)$ and stability $(-5,1,1,1,1,1)$ where the -5 corresponds to the position $\star$. Its not too hard to see[1] that the moduli is covered by

[1]If you get stuck, use the only non-monomial relation. If you're still stuck, try using the non-monomial relation again.

the following six open sets



Denote these by U_0, U_1, U_2, U_3, U_+ and U_- respectively. Lets look at U_0. Setting $a = a_{1-}$, $b = c_{30}$ and $c = a_{03}$ then using *exactly* the same calculation as the toric example earlier, U_0 looks like



now subject to the 4 new relations above the horizontal line. But these give that $a_{-0} = b^7c^{26}$, $c_{-1} = 1 - 4b^4c^{15}$ and $a(1 - 4b^4c^{15}) = b^7c^{26}$. Thus our open set is $\mathbb{C}^3_{a,b,c}$ subject to the one equation $a(1 - 4b^4c^{15}) = b^7c^{26}$. Note that basically everything in this calculation is the same as the toric case, except the one non-monomial relation ends up giving us a hypersurface in $\mathbb{C}^3$.

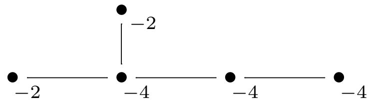
The other open sets are done similarly, and all follow very quickly from the toric case. We find that we can read off the co-ordinates in the following positions (I've also stated their abstract equations):

$$\begin{array}{ccc}
U_0 & (a_{1-}, c_{30}, a_{03}) & a(1 - 4b^4c^{15}) = b^7c^{26} \\
U_1 & (a_{1-}, c_{23}, a_{32}) & a(1 - 4bc^4) = b^2c^7 \\
U_2 & (a_{1-}, c_{12}, a_{21}) & a(1 - 4c) = bc^2 \\
U_3 & (a_{1-}, a_{1+}, c_{+1}) & a(c - 4) = bc \\
U_+ & (c_{0-}, a_{1+}, a_{-0}) & b(a^2c + 4) = ac \\
U_- & (c_{0+}, a_{1-}, a_{+0}) & b(a^2c - 4) = ac
\end{array}$$

Actually there is a choice of coordinate in U_3 above since we can pick the position c_{-1} instead of c_{+1}; denoting d as this new third coordinate changes the abstract equation to a, b, d subject to $ad = b(4 - d)$. With respect to the above ordering, the gluing of these open sets is:

$$\begin{array}{ccc}
U_0 \ni (a, b, c) & \leftrightarrow & (a, c^{-1}, c^4b) \in U_1 \\
U_1 \ni (a, b, c) & \leftrightarrow & (a, c^{-1}, c^4b) \in U_2 \\
U_2 \ni (a, b, c) & \leftrightarrow & (ca, c^3b, c^{-1}) \in U_3 \\
U_3 \ni (a, b, d) & \leftrightarrow & (a^{-1}, b, a^2d) \in U_+ \\
U_3 \ni (a, b, c) & \leftrightarrow & (b^{-1}, a, b^2c) \in U_-
\end{array}$$

The dual graph in this example is



3. The $SL(2)$ McKay Correspondence: Preliminaries

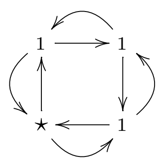
The last section was very geometrical; I'll now come back and motivate the algebraic side. If we take a finite subgroup G of $\mathrm{GL}(2,\mathbb{C})$ we would like to use the representation theory of G to resolve the singularity $\mathbb{C}^2/G$. In this section I'll introduce the notions needed to explain the classical McKay correspondence (i.e. when $G \leq \mathrm{SL}(2,\mathbb{C})$), but I'll define everything when $G \leq GL(2,\mathbb{C})$.

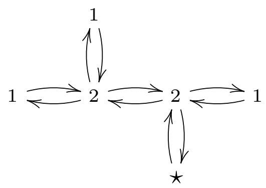
The geometry of $\mathbb{C}^2/G$ is a function of two variables, the group G and the natural representation $V = \mathbb{C}^2$. Changing either may change the geometry. Consequently the representation theory by itself will tell us nothing about the geometry (since it is only a function of one variable, namely the group G), so we have to enrich the representations with the action of G on V. We will do this in three ways: the first is as follows

Definition 3.1. *For given finite G acting on $\mathbb{C}^2 = V$, the McKay quiver is defined to be the quiver with vertices corresponding to the isomorphism classes of indecomposable representations, and the number of arrows from ρ_1 to ρ_2 is defined to be*

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}G}(\rho_1, \rho_2 \otimes V)$$

Example 3.2. For the groups $\frac{1}{4}(1,3)$ and $BD_{4\cdot 3}$ inside $\mathrm{SL}(2,\mathbb{C})$ the McKay quivers are





respectively, where the number on a vertex is the dimension of the representation at that vertex.

Beware that sometimes the McKay quiver is defined with the arrows reversed, i.e. the number of arrows from ρ_1 to ρ_2 is $\dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}G}(\rho_2, \rho_1 \otimes V)$. This is just a convention, so it doesn't really matter.

The second way we are going to encode the geometry into the representation theory is to treat every representation as a semi-invariant, and take the corresponding endomorphism ring:

Definition 3.3. *For a representation ρ, form $\rho \otimes_{\mathbb{C}} \mathbb{C}[x,y]$. Now G acts on both sides of the tensor, so we can form $(\rho \otimes \mathbb{C}[x,y])^G$, which is a CM module over the invariant ring $\mathbb{C}[x,y]^G = (\rho_0 \otimes \mathbb{C}[x,y])^G$ where ρ_0 is the trivial representation. We denote $S_\rho := (\rho \otimes \mathbb{C}[x,y])^G$ and call it the CM module associated to ρ. Denote $A := \mathrm{End}_{\mathbb{C}[x,y]^G}\left(\oplus_{\rho \in \mathrm{Irr} G} (\rho \otimes \mathbb{C}[x,y])^G\right)$*

In fact the above gives a 1-1 correspondence between the representations and the CM modules. You should perhaps view the CM modules as being 'better' than the representations, since they generalize to the non-quotient singularity case.

The third way to encode the geometry into the representation theory is done as follows:

Definition 3.4. *Define the skew group ring $\mathbb{C}[x,y]\#G$ to be the vector space $\mathbb{C}[x,y] \otimes_{\mathbb{C}} \mathbb{C}G$ with multiplication given by*

$$(f_1 \otimes g_1) \cdot (f_2 \otimes g_2) := (f_1(g_1 \cdot f_2)) \otimes g_1 g_2$$

You should view this as the algebra put together from $\mathbb{C}G$ and $\mathbb{C}[x,y]$ in a natural way, and it is the twist in the multiplication which is encoding the action of G on V. Note that a $\mathbb{C}[x,y]\#G$ module is exactly the same thing as $\mathbb{C}[x,y]$ module M (=coherent sheaf on $\mathbb{C}^2$) with a compatible G-action; i.e. a G-action such that

$$g(f\cdot m) = g(f)\cdot g(m) \quad \text{for all } f\in\mathbb{C}[x,y], g\in G, m\in M$$

Thus we can think of modules for $\mathbb{C}[x,y]\#G$ a little more geometrically as G-equivariant sheaves on $\mathbb{C}^2$.

The following theorem due to Auslander tells us that our two naturally defined algebras give us the same answer:

Theorem 3.5. *If $G \leq GL(2,\mathbb{C})$ is small (i.e. contains no pseudoreflections) then*

$$\mathbb{C}[x,y]\#G \cong \mathrm{End}_{\mathbb{C}[x,y]^G}\left(\oplus_{\rho\in\mathrm{Irr}G} S_\rho^{\oplus \dim\rho}\right).$$

Consequently (killing multiplicity) $A = \mathrm{End}_{\mathbb{C}[x,y]^G}(\oplus_{\rho\in\mathrm{Irr}G} S_\rho)$ is Morita equivalent to the skew group ring $\mathbb{C}[x,y]\#G$.

Actually the three ways of encoding the geometry onto the representation theory give us the same answer:

Lemma 3.6. *The underlying quiver of $\mathbb{C}[x,y]\#G$ (and thus $A = \mathrm{End}_{\mathbb{C}[x,y]^G}(\oplus_{\rho\in\mathrm{Irr}G} S_\rho)$ when the group is small) is the McKay quiver.*

The relations on the McKay quiver that give the Morita equivalence with the skew group ring (at least in the case when G is small) are known as the mesh relations from AR theory. Perhaps more will be said about this later.

4. The $SL(2)$ McKay Correspondence

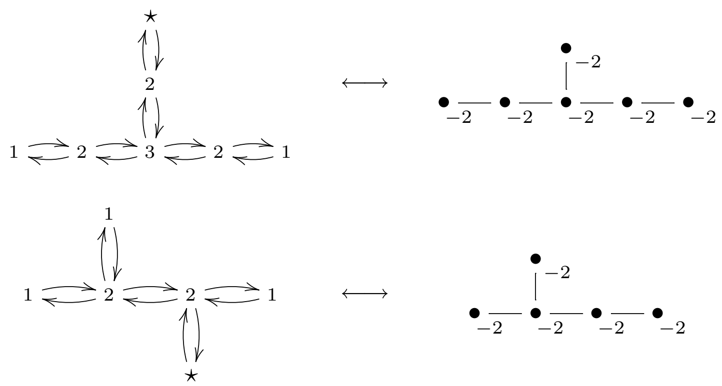
The last section introduced the algebra and notation, in this section we get to the point. Let $G \leq \mathrm{SL}(2,\mathbb{C})$ and denote by $\widetilde{X} \to \mathbb{C}^2/G$ the minimal resolution. Firstly, there is a 1-1 correspondence

$$\{\text{exceptional curves}\} \leftrightarrow \{\text{non-trivial irreducible representations}\}$$

where recall that the right hand side is in 1-1 correspondence with the non-free CM modules. I emphasize that so far this is a numerical correspondence, we want more structure. McKay observed that

$$\{\text{dual graph}\} \quad \rightleftarrows \quad \text{McKay quiver}$$

where we go from one side to the other by deleting (or adding) the vertex corresponding to the trivial representation. For example



If we consider an algebra instead of just a quiver (by adding relations, which are the preprojective relations if you know about these things) we can say more. Firstly the above correspondence becomes

$$\{\text{dual graph}\} \quad \rightleftarrows \quad \text{quiver of } \mathrm{End}_{\mathbb{C}[x,y]^G}(\oplus_{\rho\in\mathrm{Irr}G} S_\rho)$$

In fact there are also statements about the derived category and quiver GIT. I will now summarize all this into one theorem. By Theorem 3.5 we can replace $\mathbb{C}[x,y]\#G$ by $\mathrm{End}_R(\oplus_{\rho\in\mathrm{Irr}G} S_\rho)$ throughout.

Theorem 4.1 (SL$(2,\mathbb{C})$ McKay Correspondence). *Let G be a finite subgroup of* SL$(2,\mathbb{C})$, *denote $R = \mathbb{C}[x,y]^G$, $X = \mathrm{Spec}R$ and $\widetilde{X} \xrightarrow{\pi} X$ the minimal resolution. Then*

(i) *There is a 1-1 correspondence*

$$\{\text{exceptional curves}\} \leftrightarrow \{\text{indecomposable non-free CM modules}\}$$

(ii) *The McKay quiver gives the dual graph $\widetilde{X}$ after we delete the trivial vertex. The only possibilities are the Dynkin diagrams of type ADE.*

(iii) *The co-efficients in Z_f correspond to the dimensions of the representations associated to the vertices.*

(iv) $D^b(\mathfrak{mod}\mathbb{C}[x,y]\#G) \approx D^b(\mathfrak{coh}\widetilde{X})$.

(v) *Considering $\mathbb{C}[x,y]\#G$, take the dimension vector α given by the co-efficients in Z_f. Then for* any *generic stability condition θ,*

$$\mathfrak{M}^s_\theta \xrightarrow{\pi} X$$

is the minimal resolution.

I should add some remarks. View (i) as a numerical correspondence, to which (ii) and (iii) adds more structure. To improve this we have to add in relations (i.e. we need to consider an algebra, not just a quiver) and as soon as we do this we can start talking about modules, and so consider (iv) and (v). The derived equivalence in (iv) can be seen using either Fourier-Mukai transforms or tilting. Perhaps (v) is the strongest statement.

I should also say that via Artin-Verdier we can view this correspondence geometrically on the minimal resolution in terms of full sheaves and their Chern classes. This is important when considering Wunram's generalization later.

The above theorem fails for GL$(2,\mathbb{C})$ but I shall explain how to modify the above so that properties (i)-(iv) hold. Property (v) the way it is stated will turn out to be false even after the modification, however there will be one particular stability condition which will work.

LECTURES ON RECONSTRUCTION ALGEBRAS IV

MICHAEL WEMYSS

1. Introduction

Last lecture Osamu introduced the notion of a special CM module and gave some of their properties. Here I briefly recap some of his lecture and also add the geometric part of the definition.

Let $X = \mathrm{Spec}R$ be an affine complete rational surface singularity, denote the minimal resolution by $f : \widetilde{X} \to \mathrm{Spec}R$ and the exceptional curves by $\{E_i\}$. Also, for a given CM module M of R, denote by $\mathcal{M} := \pi^*M/\mathrm{torsion}$ the corresponding vector bundle on $\widetilde{X}$.

Definition-Proposition 1.1. *A CM module M is called special if one of the following equivalent conditions hold*
1. $H^1(\mathcal{M}^\vee) \cong \mathrm{Ext}^1_{\widetilde{X}}(\mathcal{M}, \mathscr{O}) = 0$
2. $M \otimes_R \omega_R/\mathrm{torsion}$ *is CM*
3. $\mathrm{Ext}^1_R(M, R) = 0$
4. $\mathrm{Hom}_R(M, R)$ *is the first syzygy of some CM module.*
5. $\Omega M \cong \mathrm{Hom}_R(M, R)$.

How to interpret this: 1 $\iff$ 2 is due to Wunram, and links the geometric notion of 1 (which involves the minimal resolution) to the more algebraic notion of 2 (which does not involve the minimal resolution). Condition 3 says that we can deduce the vanishing of the ext group upstairs (i.e. 1) by deducing the vanishing of the ext group downstairs on the singularity. This is *very* useful, but note that such a phenomenon is very rare! It is still not clear from conditions 1,2 or 3 how to obtain special CM modules - it is 4 which now helps since just taking the syzygy of your favorite CM module (and then taking the dual) gives you a special CM module. Condition 5 is a refinement of condition 4 (for example when $G \leq \mathrm{SL}(2, \mathbb{C})$ is gives an alternative proof that $\Omega^2 = id$) and is useful in proving homological statements.

We arrive at the definition:

Definition 1.2. *The ring $\mathrm{End}_R(\oplus M)$, where the sum is over all indecomposable special CM modules, is called the reconstruction algebra.*

A long time ago I said that instead of viewing the minimal resolution as G-Hilb (which you can do), the new idea is to instead view the minimal resolution as being very similar to a space we already understand. It is the reconstruction algebra which tells us which space to compare to, and it is the reconstruction algebra which encodes the difference. This is related to why I call $\mathrm{End}_R(\oplus M)$ the reconstruction algebra, which I shall now explain in the next section.

2. The Correspondence

Recall that given the data of a dual graph, simple combinatorics give us Artin's fundamental cycle Z_f. In the case of finite subgroups of $\mathrm{SL}(2, \mathbb{C})$ these numbers are what you expect. I need one further piece of combinatorial data, since now the canonical sheaf need not be trivial and so we need to encode this combinatorially. It is already known how to do this: use the canonical cycle Z_K. It is the rational cycle defined by the condition

$$Z_K \cdot E_i = -K_{\widetilde{X}} \cdot E_i$$

for all i. By adjunction this means that

$$Z_K \cdot E_i = E_i^2 + 2$$

for all i. Note that on the minimal resolution the self-intersection number of every curve is ≤ -2 and so consequently $Z_K \cdot E_i \leq 0$ for all i.

The canonical cycle appears in the theorem below since at some point in the proof Serre duality is envoked.

Theorem 2.1. *Let $\widetilde{X} \to \operatorname{Spec} R$ be the minimal resolution of some affine complete rational surface singularity. Then $\operatorname{End}_R(\oplus M)$ can be written as a quiver with relations as follows: for every exceptional curve E_i associate a vertex labelled i, and also associate a vertex $\star$ corresponding to the free module. Then the number of arrows and relations between the vertices is given as follows:*

	Number of arrows	Number of relations
$i \to j$	$(E_i \cdot E_j)_+$	$(-1 - E_i \cdot E_j)_+$
$\star \to \star$	0	$-Z_K \cdot Z_f + 1 = -1 - Z_f \cdot Z_f$
$i \to \star$	$-E_i \cdot Z_f$	0
$\star \to i$	$((Z_K - Z_f) \cdot E_i)_+$	$((Z_K - Z_f) \cdot E_i)_-$

From this I should make some remarks

- We call $\operatorname{End}_R(\oplus M)$ the reconstruction algebra since it can be reconstructed from the dual graph of the minimal resolution. Although it looks quite complicated, the combinatorics are actually very easy (see lemma below).
- If you already know the dual graph (e.g. through the Brieskorn classification for quotient singularities) to obtain the quiver is very quick. If you don't know the dual graph then at least in the case of quotient singularities there is another way to build the reconstruction algebra, using the AR quiver. If you like, you can view this AR quiver method as another (but not so good) proof of the Brieskorn classification.
- Some version of the above theorem holds for non-minimal resolutions too, but the quiver and relations are sightly different.

Ideally we don't want to compute all the combinatorics in all examples, so the next lemma is useful since it reduces the calculation of the quiver to simply adding arrows to a certain base quiver[1]. This also tells you which space to compare to!

Key Lemma 2.2. *Suppose two curve systems $E = \{E_i\}$ and $F = \{F_i\}$ have the same dual graph **and** fundamental cycle, such that $-F_i^2 \leq -E_i^2$ for all i. Then the quiver for the curve system E is obtained from the quiver of the curve system F by adding $-E_i^2 + F_i^2$ extra arrows $i \to \star$ for every curve E_i.*

Thus if you have a dual graph and you want to compute the corresponding quiver, just reduce the self-intersction numbers (i.e. make them closer to -2) in such a way that the fundamental cycle does not change. Calculate this base quiver. Then just add extra arrows as in the Lemma. This will make more sense after some examples.

3. Some examples

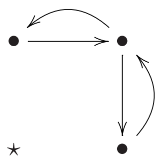
I'll start with type A, i.e. cyclic groups. Since its hard to draw n vertices, consider only the case of A_3.

Example 3.1. Consider the group $\frac{1}{4}(1,3)$. For this example the dual graph is

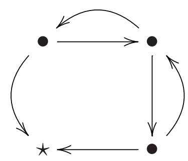
$$\underset{-2}{\bullet} \text{---} \underset{-2}{\bullet} \text{---} \underset{-2}{\bullet}$$

[1]If you were in my talk last week in Kyoto and were wondering why everything didn't make sense after some point, it is because I forgot to say the Key Lemma. Oops.

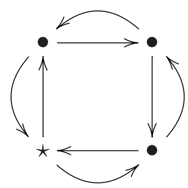
After the $i \to j$ and $\star \to \star$ steps in the theorem, we have the following picture



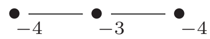
Now to calculate how to connect $\star$, we need to know the fundamental cycle. But here $Z_f = \begin{smallmatrix} 1 & 1 & 1 \end{smallmatrix}$ and so in matrix from $(-E_i \cdot Z_f)_{i \in I} = \begin{smallmatrix} 1 & 0 & 1 \end{smallmatrix}$. Thus after the $i \to \star$ step:



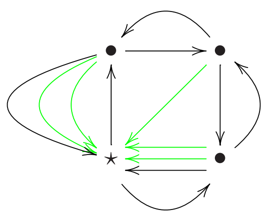
For the $\star \to i$ step notice that since all curves are (-2)-curves the canonical cycle is trivial, thus the number of arrows $\star \to i$ is equal to the number of arrows $i \to \star$. Consequently the quiver of the reconstruction algebra is



Example 3.2. Consider now the dual graph

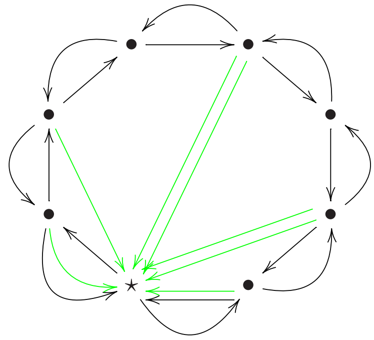


corresponding to the group $\frac{1}{40}(1, 11)$. Now the Z_f is the same as the previous example, so by Lemma 2.2 we just have to add extra arrows to the above; we thus deduce that the reconstruction algebra is

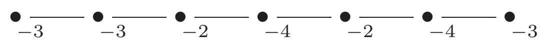


All other cyclic group cases are identical, and follow easily. For example

Example 3.3. For the group $\frac{1}{693}(1, 256)$, the reconstruction algebra is

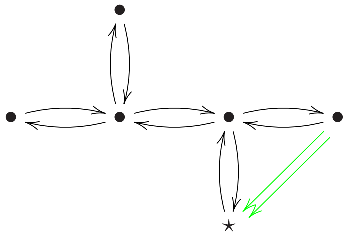
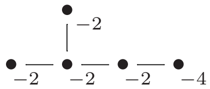
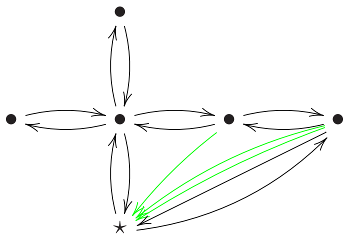
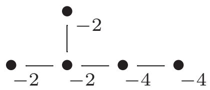
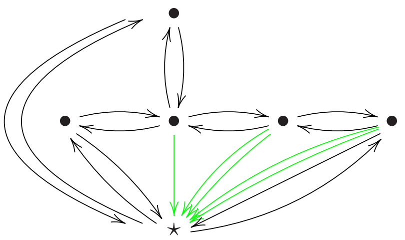
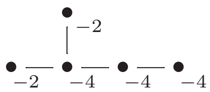


corresponding to the dual graph



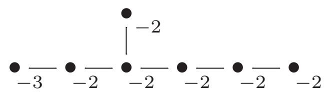
We now venture into some non-abelian groups. Right at the end I'll do some crazy non-quotient singularities.

Example 3.4. Some dihedral groups. I restrict to only 3 examples; notice that we have already seen the group $\mathbb{D}_{56,15}$ in Lecture 3.

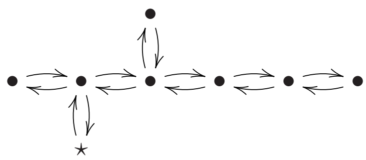
Reconstruction Algebra	dual graph	Z_f	group
		$\begin{smallmatrix} & 1 & & \\ 1 & 2 & 2 & 1 \end{smallmatrix}$	$\mathbb{D}_{10,7}$
		$\begin{smallmatrix} & 1 & & \\ 1 & 2 & 1 & 1 \end{smallmatrix}$	$\mathbb{D}_{26,15}$
		$\begin{smallmatrix} & 1 & & \\ 1 & 1 & 1 & 1 \end{smallmatrix}$	$\mathbb{D}_{56,15}$

Now for some examples of non-quotient singularities:

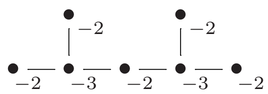
Example 3.5. Consider the dual graph



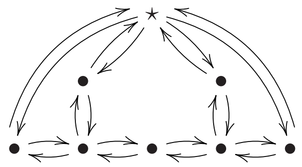
This is one of Artin's rational triple points; there are combinatorics which tell us that this corresponds to some rational singularity. It is not a quotient singularity by Brieskorn's classification, but it does look quite similar to the dual graph corresponding to the group E_7. Now here the fundamental cycle $Z_f = \begin{smallmatrix} & & 2 & & & \\ 1 & 3 & 4 & 3 & 2 & 1 \end{smallmatrix}$ (compare to $\begin{smallmatrix} & & 2 & & & \\ 2 & 3 & 4 & 3 & 2 & 1 \end{smallmatrix}$ for E_7) which makes the reconstruction algebra in this case



Example 3.6. Consider the dual graph



The fundamental cycle is reduced, i.e. $Z_f = \begin{smallmatrix} & 1 & & 1 & \\ 1 & 1 & 1 & 1 & 1 \end{smallmatrix}$ Hence the reconstruction algebra is



NONCOMMUTATIVE RESOLUTION VIA FROBENIUS MORPHISMS AND D-MODULES

TAKEHIKO YASUDA

This is a note of my talk at Nagoya University in November, 2008. The aim of the talk was to outline my joint work with Yukinobu Toda [6].

1. Motivation

Throughout the note, we work over an algebraically closed field k of characteristic $p > 0$.

Consider a finite group $G \subset GL_d(k)$ of order prime to p. Then the associated G-Hilbert scheme $\mathrm{Hilb}^G(\mathbb{A}^d)$ is a blowup of the quotient variety $X := \mathbb{A}^d/G$. Namely we have a natural projective and birational morphism

$$\mathrm{Hilb}^G(\mathbb{A}^d) \to X.$$

This is a restriction of the Hilbert-Chow morphism.

On the other hand, for each $e \in \mathbb{Z}_{\geq 0}$, the e-th F-blowup of X, $\mathrm{FB}_e(X)$, is defined as the universal flattening of the e-th k-linear Frobenius

$$F^e : X_e \to X.$$

It was found [9] that for sufficiently large e, $\mathrm{Hilb}^G(\mathbb{A}^d)$ and $\mathrm{FB}_e(X)$ are isomorphic. The isomorphism was constructed as follows: Indeed there is a canonical morphism

$$\mathrm{Hilb}^G(\mathbb{A}^d) \to \mathrm{FB}_e(X)$$

for any e. A point of $\mathrm{Hilb}^G(\mathbb{A}^d)$ is identified with a 0-dimensional subscheme of $\mathbb{A}^d$ of length $l := \sharp G$, write $Z \subset \mathbb{A}^d$. Then pull it back by the Frobenius $\mathbb{A}^d_e \to \mathbb{A}^d$ and obtain a subscheme $\tilde{Z} \subset \mathbb{A}^d_e$ of length lp^{de}. Then take the quotient scheme $\tilde{Z}/G$, which is a subscheme of X and corresponds to a point of $\mathrm{FB}_e(X)$. It defines the map $\mathrm{Hilb}^G(\mathbb{A}^d) \to \mathrm{FB}_e(X)$, which coincides with the morphism $\mathrm{Hilb}^G(\mathbb{A}^d) \to X$ if $e = 0$.

One of the motivations of our work is to understand the isomorphism from the viewpoint of noncommutative geometry.

Bridgeland-King-Reid [2] proved that if $G \subset SL_d(k)$, $d = 2, 3$, then the derived category of coherent sheaves on $\mathrm{Hilb}^G(\mathbb{A}^d)$ is equivalent to that of G-equivariant ones on $\mathbb{A}^d$.

$$D(\mathrm{Hilb}^G(\mathbb{A}^d)) \cong D^G(\mathbb{A}^d)$$

The equivalence is obtained as the Fourier-Mukai transform associated to the universal family in the following diagram,

$$\begin{tikzcd} \text{Univ. fam.} \arrow[r] \arrow[d] & \mathbb{A}^d \arrow[d] \\ \mathrm{Hilb}^G(\mathbb{A}^d) \arrow[r] & X \end{tikzcd}$$

But we have a similar diagram associated to the F-blowup,

$$\begin{tikzcd} \text{Univ. fam.} \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{FB}_e(X) \arrow[r] & X \end{tikzcd}$$

The other motivation of our work is to obtain a similar result as the Bridgeland-King-Reid's one for the F-blowup.

We look at the problems from the viewpoint of the noncommutative resolution after Van den Bergh [7].

2. Morita equivalence of the skew group ring and a ring of differential operators.

Let $G \subset GL_d(k)$ be as before, $S := k[x_1, \dots, x_d]$ with the natural G-action and $R := S^G$ the ring of invariants. Thus we have $\mathbb{A}^d = \operatorname{Spec} S$ and $X = \operatorname{Spec} R$. Moreover we assume that G is small, that is, there is no reflection. Then the skew group ring $S * G$ is canonically isomorphic to the endomorphism ring $\mathrm{End}_R(S)$ of S as an R-module, and then we write $A := S * G = \mathrm{End}_R(S)$. This ring is a noncommutative resolution of R in the sense that they have finite global dimension. Using the noncommutative ring A, Van den Bergh translated the Bridgeland-King-Reid equivalence into his framework of the noncommutative resolution [7] as follows: A G-equivariant coherent sheaf on $\mathbb{A}^d$ is nothing but a (left) A-module. On the other hand, $\mathrm{Hilb}^G(\mathbb{A}^d)$ is (an irreducible component of) the moduli space of stable A-modules. Thus the equivalence is interpreted as a derived equivalence between the abelian category of A-modules and that of coherent sheaves on the moduli space of A-modules.

$$D(\text{the moduli space of } A\text{-modules}) \cong D(A).$$

Now the translated statement is applicable to a broader range of issues.

Pursuing the analogy between the G-Hilbert scheme and the F-blowup, we shall consinder the ring $D_e := \mathrm{End}_R(R^{1/q})$, $q = p^e$, in place of $A = \mathrm{End}_R(S)$.

$$\begin{tikzcd} S \arrow[r, phantom, ""] & \\ & A = \mathrm{End}_R(S) = S * G \\ R \arrow[uu, hook, "\text{Galois}"] \arrow[ur, phantom, "\rightsquigarrow"] & \end{tikzcd}$$

$$\begin{tikzcd} R^{1/q} & \\ & D_e = \mathrm{End}_R(R^{1/q}) \\ R \arrow[uu, hook, "\text{purely inseparable}"] \arrow[ur, phantom, "\rightsquigarrow"] & \end{tikzcd}$$

Proposition 2.1. *For sufficiently large e, A and D_e are Morita equivalent.*

Outline of the proof. The proposition follows from the fact that as R-modules, S is a direct summand of $(R^{1/q})^{\oplus l}$ for some l, and vice versa. Indeed we saw [6] that S and $R^{1/q}$ are full modules of covariants, that is, they contain as a a summand every indecomposable module of covariant. $\square$

Since Morita equivalent rings have the same global dimension, we obtain:

Corollary 2.2. *D_e is a noncommutative resolution of R in the sense that it has a finite global dimension.*

We can see that the F-blowup is an irreducible component of the moduli space of stable D_e-modules. But for $e \gg 0$, since the categories of A-modules and D_e-modules are equivalent, we obtain isomorphic moduli spaces. Thus the isomorphism $\mathrm{Hilb}^G(\mathbb{A}^d) \cong \mathrm{FB}_e(\mathbb{A}^d/G)$ is now a direct consequence of the Morita equivalence. Moreover applying Van den Bergh's interpretation of the Bridgeland-King-Reid derived equivalence to the F-blowup and the ring D_e, we see that if $G \subset SL_d(k)$, $d = 2, 3$, then for $e \gg 0$, $\mathrm{FB}_e(X)$ and D_e are derived equivalent.

$$D(\mathrm{FB}_e(X)) \cong D(D_e)$$

Remark 2.3. The stability of modules, in fact, depends on a parameter called the stability condition. For a general stability condition, the stable $S * G$-modules are called G-constellations [3].

Remark 2.4. If G is abelian and if $q > \sharp G$, then the assertion of the proposition holds. But in the non-abelian case, we have not obtained such an effective estimate on how large e is enough.

Remark 2.5. Each element of D_e is a differential operator on $R^{1/q}$. Moreover the ring
$$\bigcup_{e \geq 0} \mathrm{End}_{R^{p^e}}(R)$$
is the ring of all differential operators on R.

In the Galois theory for purely inseparable extensions, derivations play a role of automorphisms in the Galois theory of normal extensions (see [4]). Hence it seems natural that differential operators appear instead of the group G of automorphisms.

3. D_e for some other singularities

Let now R be a Noetherian complete local domain over k. The ring $D_e := \mathrm{End}_R(R^{1/q})$ is well-defined for such R, not only in the case of quotient singularities. Therefore it is natural to ask

Problem 3.1. When is D_e a noncommutative resolution?

We have proved that the answer is affirmative in the following cases:

(1) the 1-dimensional case
(2) the singularity of type A_1 (in odd characteristic), that is, R is of the form
$$k[[x_0, x_1, \dots, x_d]]/(x_0^2 + x_1^2 + \cdots + x_d^2).$$

In the first case, for sufficiently large q, we have that $\bar{R} \subset R^{1/q}$, where $\bar{R}$ is the normalization. For such q, indeed D_e is Morita equivalent to $\bar{R}$.

In the second case, we can see that for $e > 0$, $R^{1/q}$ is a representation generator, that is, contains as a summand every indecomposable maximal Cohen-Macaulay module. Then from a theorem of Auslander [1] (see also [5]), D_e has finite global dimension.

Problem 3.2. Suppose that R has finite representation type, that is, there are only finitely many indecomposable maximal Cohen-Macaulay modules up to isomorphisms. Then for sufficiently large q, is $R^{1/q}$ a representation generator? In particular, what about the case of simple singularities?

4. F-blowup as the moduli space

We saw that in the case of quotient singularities, the F-blowup is the moduli space of D_e-modules. We can also show that this holds for F-pure singularities. A k-algebra R is F-pure if the inclusion map $R \hookrightarrow R^{1/p}$ splits as an R-module map. Then for any e, we can write $R^{1/q} \cong R \oplus M$ for some R-module M. From this decomposition and some parameters, we obtain the stability condition of D_e-modules [7].

Proposition 4.1. *Let $X := \operatorname{Spec} R$. Then $\mathrm{FB}_e(X)$ is canonically identified with the irreducible component dominating X of stable D_e-module with respect to the mentioned stability condition.*

Remark 4.2. When X is F-pure, the sequence of F-blowups is monotone, that is, $\mathrm{FB}_{e+1}(X)$ dominates $\mathrm{FB}_e(X)$ for every e (see [8]).

References

[1] M. Auslander, *Isolated singularities and existence of almost split sequences*, in Representation theory, II (Ottawa, Ont., 1984), vol. 1178 of Lecture Notes in Math., Springer, Berlin, 1986, pp. 194–242.
[2] T. Bridgeland, A. King, and M. Reid, *The McKay correspondence as an equivalence of derived categories*, J. Amer. Math. Soc., 14 (2001), pp. 535–554 (electronic).
[3] A. Craw and A. Ishii, *Flops of G-Hilb and equivalences of derived categories by variation of GIT quotient*, Duke Math. J., 124 (2004), pp. 259–307.
[4] N. Jacobson, *Lectures in abstract algebra. III*, Springer-Verlag, New York, 1975. Theory of fields and Galois theory, Second corrected printing, Graduate Texts in Mathematics, No. 32.
[5] G. J. Leuschke, *Endomorphism rings of finite global dimension*, Canad. J. Math., 59 (2007), pp. 332–342.
[6] Y. Toda and T. Yasuda, *Noncommutative resolution, F-blowups and D-modules*. arXiv:0810.1804v1.
[7] M. Van den Bergh, *Non-commutative crepant resolutions*, in The legacy of Niels Henrik Abel, Springer, Berlin, 2004, pp. 749–770.
[8] T. Yasuda, *On monotonicity of F-blowup sequences*, arXiv:0803.3373v2.
[9] ——, *Universal flattening of Frobenius*. arXiv:0706.2700v3.

Department of Mathematics and Computer Science, Faculty of Science, Kagoshima University, 1-21-35 Korimoto, Kagoshima 890-0065, Japan

E-mail address: yasuda@sci.kagoshima-u.ac.jp

Mutations and noncommutative Donaldson-Thomas theory

Kentaro Nagao

February 15, 2009

Abstract

Given a quiver with a potential, we can define counting invariants so called noncommutative Donaldson-Thomas invariants. In this note, we study how the generating function of the invariants changes under mutations of the quiver.

1 Quiver with potentials

Let $Q = (I, H)$ be a quiver and ω be a potential which is homogeneous with respect to a degree $H \to \mathbb{Z}_{>0}$. Assume that $A = (Q, \omega)$ is 3-dimensional Calabi-Yau in the sense of Bocklandt [Boc08]. In this section we give some examples of such quivers.

1.1 Conifold

Let $Y_{(-1,-1)} = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ be the resolved conifold and $\pi \colon Y_{(-1,-1)} \to \mathbb{P}^1$ be the projection. The vector bundle $\mathcal{P}_{(-1,-1)} := \mathcal{O}_{Y_{(-1,-1)}} \oplus \pi^* \mathcal{O}_{\mathbb{P}^1}(1)$ is a tilting generator of the derived category $D^b\left(\mathrm{Coh}\, Y_{(-1,-1)}\right)$ and we have the derived equivalence

$$\mathbb{R}\mathrm{Hom}\left(\mathcal{P}_{(-1,-1)}, -\right) : D^b\left(\mathrm{Coh}\, Y_{(-1,-1)}\right) \xrightarrow{\sim} D^b\left(A_{(-1,-1)}\text{-mod}\right),$$

where $A_{(-1,-1)} = \mathrm{End}_Y\left(\mathcal{P}_{(-1,-1)}\right)$. Let $Q_{(-1,-1)}$ be the quiver in Figure 1 and $\omega_{(-1,-1)} = a_1 b_1 a_2 b_2 - a_1 b_2 a_2 b_1$. Then we have $A_{(-1,-1)} \simeq (Q_{(-1,-1)}, \omega_{(-1,-1)})$.

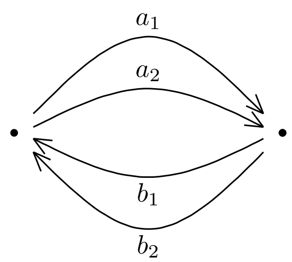


Figure 1: $Q_{(-1,-1)}$

1.2 Local $\mathbb{P}^2$

The next example have been studied carefully by T. Bridgeland ([Bri], [Bri06]). Let $Y_{-3} := \mathcal{O}_{\mathbb{P}^2}(-3)$ be the total space of the canonical bundle on $\mathbb{P}^2$ and $\pi\colon Y_{(-3)} \to \mathbb{P}^2$ be the projection. The vector bundle

$$\mathcal{P}_{(-3)} := \mathcal{O}_{Y_{(-3)}} \oplus \pi^*\mathcal{O}_{\mathbb{P}^2}(1) \oplus \pi^*\mathcal{O}_{\mathbb{P}^2}(2)$$

is a tilting generator of the derived category $D^b\left(\operatorname{Coh} Y_{(-3)}\right)$ and we have the derived equivalence

$$\mathbb{R}\mathrm{Hom}\left(\mathcal{P}_{(-3)}, -\right) : D^b\left(\operatorname{Coh} Y_{(-3)}\right) \xrightarrow{\sim} D^b\left(A_{(-3)}\text{-mod}\right),$$

where $A_{(-3)} = \operatorname{End}_Y\left(\mathcal{P}_{(-3)}\right)$. Let $Q_{(-3)}$ be the quiver in Figure 2 and

$$\omega_{(-3)} = \sum_{\sigma \in \mathfrak{S}_3} \varepsilon(\sigma) c_{\sigma(1)} b_{\sigma(2)} a_{\sigma(3)}.$$

Then we have $A_{(-3)} \simeq \left(Q_{(-3)}, \omega_{(-3)}\right)$.

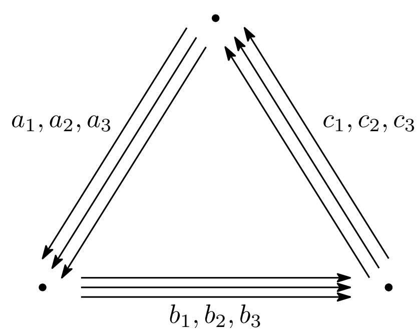


Figure 2: $Q_{(-3)}$

1.3 Geometric engineering

Let Γ be a finite subgroup of $\mathrm{SL}(2, \mathbb{C})$, which acts on the resolved conifold $Y_{(-1,-1)}$ fiber-wisely, and $Y_\Gamma \to X_\Gamma = Y_{(-1,-1)}/\Gamma$ be the crepant resolution (see Figure 3). By the derived McKay correspondence we have

$$D^b\left(\operatorname{Coh} Y_\Gamma\right) \simeq D^b\left(\operatorname{Coh}^\Gamma Y_{(-1,-1)}\right).$$

The Γ-equivariant vector bundle

$$\mathcal{P}_\Gamma := \bigoplus_{\rho \in \operatorname{Irr}(\Gamma)} \mathcal{P} \otimes \rho$$

is a tilting generator of $D^b\left(\operatorname{Coh}^\Gamma Y_{(-1,-1)}\right)$ and the endomorphism algebra $A_\Gamma := \operatorname{End}\left(\mathcal{P}_\Gamma\right)$ can be described as follows: the vertex set of Q_Γ is

$$\{(\epsilon, \rho) \mid \epsilon \in \{0, 1\}, \rho \in \operatorname{Irr}(\Gamma)\}.$$

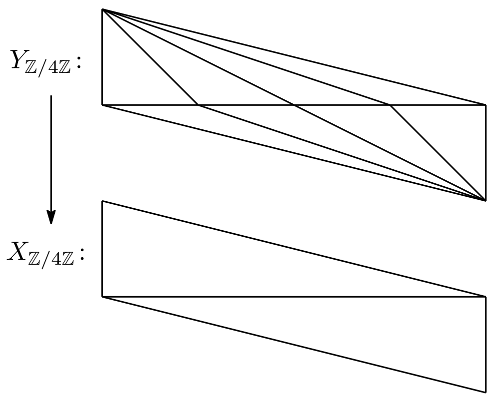


Figure 3: the crepant resolution $Y_{\mathbb{Z}/4\mathbb{Z}} \to X_{\mathbb{Z}/4\mathbb{Z}}$

We have two arrows a^1_ρ and a^2_ρ from $(0,\rho)$ to $(1,\rho)$ for each ρ. If ρ and ρ' are connected by edges in the McKay quiver, we have one arrow from $(1,\rho)$ to $(0,\rho')$ and one arrow from from $(1,\rho')$ to $(0,\rho)$. Let $b_{\rho,\rho'}$ and $b_{\rho',\rho}$ denote these arrows respectively. The potential ω_Γ is the sum of the following elements:

$$a^0_\rho \circ b_{\rho',\rho} \circ a^1_{\rho'} \circ b_{\rho,\rho'}.$$

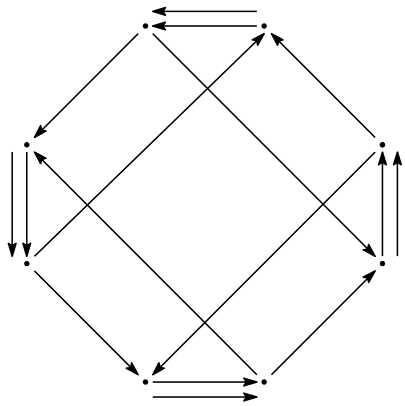


Figure 4: the quiver $Q_{\mathbb{Z}/4\mathbb{Z}}$

1.4 Small crepant resolutions of toric CY 3-folds

Let $X_{a,b}$ be the affine toric Calabi-Yau 3-fold associated with the trapezoid (or possibly triangle) with height 1 and with parallel edges of length a and b. Let σ be a partition of the trapezoid into triangles with areas $1/2$ and $Y^\sigma_{a,b} \to X_{a,b}$ be the associated crepant resolution. The inverse image of $0 \in X_{a,b}$ is the A_{a+b-1} configuration of $(-1,-1)$ or $(0,-2)$-curves. In [Nag], using the result of M. Van den Bergh ([VdB04]), the author constructed a tilting vector bundle with endomorphism algebra $A_\sigma = (Q_\sigma, \omega_\sigma)$. The Q_σ is given by adding some loops to the affine Dynkin quiver of type A_{a+b-1}. Roughly speaking, a vertex with a loop corresponds to a $(0,-2)$-curve. See [Nag] for details.

Example 1.1. *Let $a = 2$, $b = 4$ and σ be the partition in Figure 5. Then the*

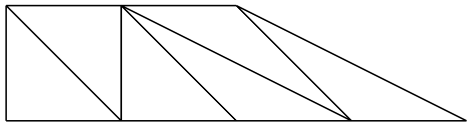


Figure 5: a partition σ

quiver Q_σ is given as in Figure 6.

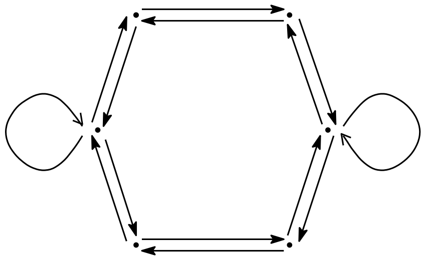


Figure 6: the quiver Q_σ

1.5 Non-toric case: obstructed $(0,-2)$-curve

For $n \geq 2$ we patch two $\mathbb{C}^3$ with coordinates $\{(x, y, z)\}$ and $\{X, Y, Z\}$ respectively by the following transition functions to construct the Calabi-Yau 3-fold $Y^n_{(0,-2)}$:

$$X = x^{-1}, \quad Y = x^2 y + x z^n, \quad Z = z.$$

The subvariety $\{y \equiv z \equiv 0\} \cup \{Y \equiv Z \equiv 0\}$ is an obstructed $(0,-2)$-curve. In [AK06] the endomorphism algebra of a tilting vector bundle is computed: the quiver $Q^n_{(0,-2)}$ is given by adding two loops l_0 and l_1 for each vertex to the quiver in Figure 1.

2 Mutations

Let P_k be the projective A-module associated with a vertex $k \in I$ and we set $P := \bigoplus_k P_k (= A)$. We define the new A-module

$$P'_k := \operatorname{coker}\left(P_k \to \bigoplus_{h \in H;\, \mathrm{out}(h) = k} P_{\mathrm{in}(h)}\right).$$

The object $\mu_k(P) = \bigoplus_{l \neq k} P_l \oplus P'_k$ is a tilting generator in $D^b(A\text{-mod})$. Let $\mu_k(A)$ denote the endomorphism algebra $\operatorname{End}(\mu_k(P))$.

Example 2.1. *Recall that we take the tilting vector bundle*

$$\mathcal{P}_{(-1,-1)} := \mathcal{O}_{Y_{(-1,-1)}} \oplus \pi^* \mathcal{O}_{\mathbb{P}^1}(1)$$

on the resolved conifold $Y_{(-1,-1)}$ and get the derived equivalence

$$D^b(\operatorname{Coh} Y_{(-1,-1)}) \xrightarrow{\sim} D^b(\operatorname{mod} A_{(-1,-1)}).$$

We identify objects in the two categories under the derived equivalence. Let P_0 and P_1 denote the projective $A_{(-1,-1)}$-modules $\mathcal{O}_{Y_{(-1,-1)}}$ and $\mathcal{L} := \pi^*\mathcal{O}_{\mathbb{P}^1}(1)$ respectively. We mutate the quiver $A_{(-1,-1)}$ at the vertex 0, then we have

$$P_0' = \operatorname{coker}\left(\mathcal{O}_{Y_{(-1,-1)}} \to \mathcal{L} \oplus \mathcal{L}\right) \simeq \mathcal{L}^2.$$

Hence we have

$$\mu_0(P) = P \otimes \mathcal{L}, \quad \mu_0\left(A_{(-1,-1)}\right)\text{-mod} = A_{(-1,-1)}\text{-mod} \otimes \mathcal{L}.$$

and

$$\mu_0\left(A_{(-1,-1)}\right) \simeq A_{(-1,-1)}.$$

In general, if Q does not have any 1-cycles nor 2-cycles, then $\mu_k(A)$ is given by the mutation of the original quiver with the potential $A = (Q, \omega)$ in the sense of [FZ02] and [DWZ].

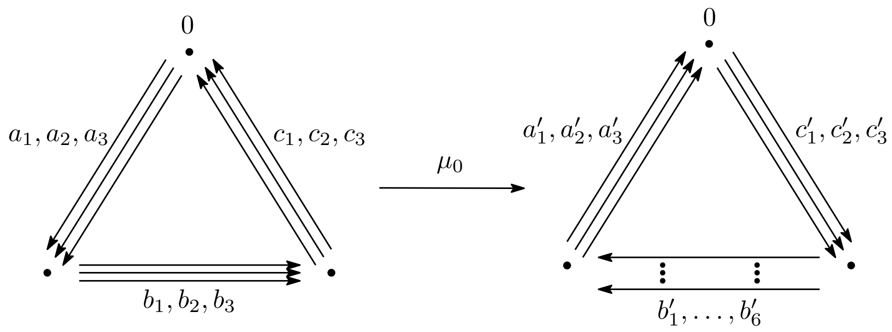


Figure 7: a mutation for $Q_{(-3)}$

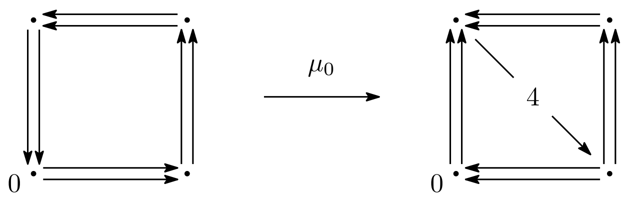


Figure 8: a mutation for $Q_{\mathbb{Z}/2\mathbb{Z}}$

3 Moduli spaces and counting invariants

Let (F, s) be a pair of a finite dimensional A-modules and a map $s\colon P_k \to F$. For a pair of real numbers $\zeta \in \mathbb{R}^I$, a pair (F, s) said to be ζ-(semi)stable if and only if the following two conditions satisfied:

- for any nonzero subobject $0 \neq F' \subseteq F \in A$-mod, we have
$$\underline{\dim}^A V' \cdot \zeta \,(\leq)\, 0,$$
- for any proper subobject $F' \subsetneq F \in A$-mod through which s factors, we have
$$\underline{\dim}^A V' \cdot \zeta \,(\leq)\, \underline{\dim}^A V \cdot \zeta,$$

where $\underline{\dim}^A$ is the composition of the canonical map
$$\mathrm{Obj}(D^b(A\text{-mod})) \to K(A\text{-mod})$$
and the linear map
$$K(A\text{-mod}) \to \mathbb{Z}^I$$
such that $(\underline{\dim}^A V)_i = \dim V_i$ for an A-module V.

By the result of A. King ([Kin94]), for $\mathrm{v} \in \mathbb{Z}^I$ we can construct the moduli space
$$\mathfrak{M}^k_\zeta(\mathrm{v}) := \{(V, s) \mid \underline{\dim}^A V = \mathrm{v},\ \zeta\text{-stable}\}.$$
by geometric invariant theory. We define the counting invariants
$$D^{\mathrm{eu}}_{\zeta,k}(\mathrm{v}) := \chi\left(\mathfrak{M}^k_\zeta(\mathrm{v})\right)$$
as the Euler characteristics of the moduli spaces and their generating function
$$\mathcal{Z}^{\mathrm{eu}}_{\zeta,k}(\mathbf{q}) := \sum_{\mathrm{v}} D^{\mathrm{eu}}_{\zeta,k}(\mathrm{v}) \cdot \mathbf{q}^{\mathrm{v}}.$$

Example 3.1. *In the conifold case*

(1) For ζ_{triv} such that $\zeta^0_{\mathrm{triv}}, \zeta^1_{\mathrm{triv}} > 0$ then
$$\mathfrak{M}^k_{\zeta_{\mathrm{triv}}}(\mathrm{v}) = \begin{cases} \emptyset & \mathrm{v} = 0, \\ \mathrm{pt} & \mathrm{v} \neq 0 \end{cases}$$
and hence $\mathcal{Z}^{\mathrm{eu}}_{\zeta,k} = 1$.

(2) For ζ_{cyclic} such that $\zeta^0_{\mathrm{cyclic}}, \zeta^1_{\mathrm{cyclic}} < 0$ then a pair (F, s) is ζ_{cyclic}-stable if and only if s is surjective. The moduli space have been studied in **non-commutative Donaldson-Thomas theory** *by B. Szendroi ([Sze]). Let $\mathcal{Z}^{\mathrm{eu}}_{\mathrm{NCDT},k}(\mathbf{q})$ denote the generating function of the counting invariants.*

3.1 Remark on virtual counting

Let $\mathcal{M} = \mathcal{M}^k_\zeta(\mathrm{v})$ be the moduli stack of framed representations of the quiver (*without relation*) Q. Taking the trace of the potential, we can define the function on $\mathcal{M}$. Then the moduli space $\mathfrak{M} := \mathfrak{M}^k_{\zeta_{\mathrm{triv}}}(\mathrm{v})$ is the critical locus of this function. We take the Euler characteristic of the Milnor fiber around each critical point to get the constructible function $\nu\colon \mathfrak{M} \to \mathbb{Z}$. The virtual counting of the moduli space is given as the weighted Euler characteristic:
$$D_{\zeta,k}(\mathrm{v}) := \sum_n \chi(\nu^{-1}(n)).$$

The function ν is called *Behrend's constructible function* (or χ*-function.* When the moduli space is compact, the weighted Euler characteristic coincides with the virtual counting ([Beh]), which is defined by integrating the constant function 1 over the virtual fundamental cycle $[\mathfrak{M}]^{\mathrm{vir}}$ ([BF]).

The virtual counting is believe to be the *correct* invariant rather than the Euler characteristic.

One of the reasons is its "deformation invariance". For example, the Donaldson-Thomas invariants of a smooth projective Calabi-Yau 3-fold Y, which are defined as virtual countings of Hilbert schemes of curves, are invariant under the deformation of Y. Though, in our setting deformation invariance is a subtle problem since the moduli is not compact.

Another reason is that the (conjectural) "rationality property" of the generating function (see [PT], [MR]).

Example 3.2. *In the example in §1.5, the generating function of the virtual counting is given by*

$$\begin{aligned}&\mathcal{Z}_{\zeta_{\mathrm{cyclic}},0}(q_0,q_1)\\&=\prod_i\left(1-(-q_0)^iq_1^{i-1}\right)^{ni}\cdot\prod_i\left(1-(-q_0q_1)^i\right)^{-2i}\cdot\prod_i\left(1-(-q_0)^iq_1^{i+1}\right)^{ni},\end{aligned}$$

and the generating function of the Euler characteristics is given by

$$\begin{aligned}\mathcal{Z}^{\mathrm{eu}}_{\zeta_{\mathrm{cyclic}},0}(q_0,q_1)=&\prod_i\left(1+q_0^iq_1^{i-1}+\cdots+q_0^{ni}q_1^{n(i-1)}\right)^i\\&\cdot\prod_i\left(1-q_0q_1^i\right)^{-2i}\cdot\prod_i\left(1+q_0^iq_1^{i+1}+\cdots+q_0^{ni}q_1^{n(i+1)}\right)^i.\end{aligned}$$

When the 3-dimensional Calabi-Yau quiver is derived from a *brane tiling*, then the virtual counting coincides with the Euler characteristic up to sign (see [MR]).

4 Results

For $k\in I$ we define the map $\mu_k\colon\mathbb{Z}^I\to\mathbb{Z}^I$ by

$$(\mu_k(\mathbf{v}))_l=\begin{cases}-\mathbf{v}_k+\sum_{h;\mathrm{out}(h)=k}\mathbf{v}_{\mathrm{in}(h)} & l=k,\\ \mathbf{v}_l & \text{otherwise}\end{cases}$$

for $\mathbf{v}\in\mathbb{Z}^I$. We also define $\mu_k\colon\mathbb{R}^I\to\mathbb{R}^I$ by

$$\mathbf{v}\cdot\zeta=\mu_k(\mathbf{v})\cdot\mu_k(\zeta)$$

for any $\mathbf{v}$ and ζ.

Let $\zeta\in\mathbb{R}^I$ be a generic stability parameter satisfying the following conditions:

- there exists $\eta\in\mathbb{R}^I_{>0}$ such that $\zeta+d\cdot\eta$ on an intersection of two walls for any $d\geq 0$.
- We have the sequence $C_0,\dots,C_r$ of chambers such that

- $\zeta - d \cdot \eta \in \cup \overline{C_s}$ for any $d \geq 0$,
- for any C_s, there exists some $d \geq 0$ such that $\zeta - d \cdot \eta \in C_s$, and
- suppose $\zeta - d \cdot \eta \in C_s$, $\zeta - d' \cdot \eta \in C_{s'}$ and $s < s'$, then $d > d'$.

- we have the sequence $k_1, \dots, k_r$ of elements in I such that

$$\overline{C_{s-1}} \cap \overline{C_s} \subset W_{\alpha^s} \quad (\alpha^s = \mu_{k_{s-1}} \circ \cdots \circ \mu_{k_1}(\alpha_{k_s})),$$

where α_k denote the simple root vector.

We denote $\mu_s := \mu_{k_s} \circ \cdots \circ \mu_{k_1}$, $\Psi_s := \Psi_{k_s} \circ \cdots \circ \Psi_{k_1}$ and $\mu_\zeta := \mu_r$, $\Psi_\zeta := \Psi_r$.

We set $\mathcal{P} := A$-mod and denote by $\mathcal{P}_\zeta$ the image of the Abelian category $\mu_\zeta(A)$-mod under the equivalence Ψ_ζ^{-1}.

Definition 4.1. *Let (V, s) be a pair of an element $V \in \mathcal{P}_\zeta$ and a map $s\colon P_k \to V$. For $\xi \in \mathbb{R}^{\hat{I}}$, we say (V, s) is $(\xi, \mathcal{P}_\zeta)$-(semi)stable if the following conditions are satisfied:*

(A) for any nonzero subobject $0 \neq S \subseteq V$ in $\mathcal{P}_\zeta$, we have

$$\xi \cdot \underline{\dim}\, S \,(\leq)\, 0,$$

(B) for any proper subobject $T \subsetneq V$ in $\mathcal{P}_\zeta$ which s factors through, we have

$$\xi \cdot \underline{\dim}\, T \,(\leq)\, \xi \cdot \underline{\dim}\, V.$$

From now on, the ζ-(semi)stability for a pair (F, s) with $F \in \mathcal{P} = A$-mod is written as the $(\zeta, \mathcal{P})$-(semi)stability. We set $\xi_{\text{cyclic}} := \mu_\zeta(\zeta)$. Note that $(\xi_{\text{cyclic}})_l < 0$ for any $l \in I$.

Theorem 4.2. (1) [Nag, Lemma 3.5] *Let (F, s) be a $(\zeta, \mathcal{P})$-stable, then $F \in \mathcal{P}_\zeta$.*

(2) [Nag, Proposition 3.6] *Let (F, s) be a $(\zeta, \mathcal{P})$-stable, then (F, s) is $(\xi_{\text{cyclic}}, \mathcal{P}_\zeta)$-stable.*

(3) [Nag, Lemma 3.7] *Let (F, s) be a $(\xi_{\text{cyclic}}, \mathcal{P}_\zeta)$-stable, then $F \in \mathcal{P}$.*

(4) [Nag, Proposition 3.8] *Let (F, s) be a $(\xi_{\text{cyclic}}, \mathcal{P}_\zeta)$-stable, then (F, s) is $(\zeta, \mathcal{P})$-stable.*

This theorem claims that replacing t-structures corresponds to replacing stability conditions. In particular, we can define

$$\mathfrak{M}^k_{\mathcal{P}_\zeta, \zeta_{\text{cyclic}}}(\mathrm{v}) := \{(V, s) \mid V \in \mathcal{P}_\zeta,\ \underline{\dim}^A V = \mathrm{v},\ (\zeta_{\text{cyclic}}, \mathcal{P}_\zeta)\text{-stable}\}$$

which is isomorphic to $\mathfrak{M}^k_{\mathcal{P},\zeta}(\mathrm{v}) = \mathfrak{M}^k_\zeta(\mathrm{v})$. We can also define the generating function

$$\mathcal{Z}^{\text{eu}}_{\mathcal{P}_\zeta, k}(\mathbf{q}) = \sum_v \chi(\mathfrak{M}^k_{\mathcal{P}_\zeta, \zeta_{\text{cyclic}}}(\mathrm{v})) \cdot \mathbf{q}^{\mathrm{v}}.$$

of the counting invariants.

In [NN] and [Nag], we study how the generating function changes when we replace the stability condition. Now we get the following formula describing how the generating function changes when we mutate the quiver.

Theorem 4.3.

$$\mathcal{Z}^{\mathrm{eu}}_{\mathcal{P}_\varsigma,k}(\mathbf{q}) = \left(1+\mathbf{q}^{\alpha^r}\right)^{(\alpha^r)_k} \mathcal{Z}^{\mathrm{eu}}_{\mathcal{P}_{r-1},k}(\mu_{k_r}(\mathbf{q})),$$

where the left hand side is given by substituting q_k with

$$\left(1+\mathbf{q}^{\alpha^r}\right)^{\langle \alpha_k,\alpha^r\rangle} q_k.$$

Here $\langle -,-\rangle$ is the Euler pairing on $K(D^b(\mathrm{Coh}(Y)))$.

Example 4.4. *In the conifold case, the Euler pairing is trivial and the theorem provides a conceptual interpretation of Young's combinatorial formula ([You]).*

Acknowledgement

The author would like to thank Hiraku Nakajima for collaborating in the paper [NN] and for many valuable discussion. He is also grateful to Yukari Ito for the invitation to Nagoya.

References

[AK06] P.S. Aspinwall and S. Katz, *Computation of superpotentials for D-branes*, Comm. Math. Phys. **264** (2006), no. 1, 227–253.

[Beh] K. Behrend, *Donaldson-Thomas invariants via microlocal geometry*, math.AG/0507523.

[BF] K. Behrend and B. Fantechi, *Symmetric obstruction theories and Hilbert schemes of points on threefolds*, math.AG/0512556.

[Boc08] R. Bocklandt, *Graded Calabi Yau algebras of dimension* 3, J. Pure Appl. Algebra **212** (2008), no. 1, 14–32.

[Bri] T. Bridgeland, *T-structures on some local Calabi-Yau varieties*, math.AG/0502050.

[Bri06] ———, *Stability conditions on a non-compact Calabi-Yau threefold*, Comm. Math. Phys. **266** (2006), no. 3, 715–733.

[DWZ] H. Derksen, J. Weyman, and A. Zelevinsky, *Quivers with potentials and their representations I: Mutations*, arXiv:0704.0649.

[FZ02] S. Fomin and A. Zelevinsky, *Cluster Algebra I: Foundations*, J. Amer. Math. Soc. **15** (2002), no. 2, 497–529.

[Kin94] A.D. King, *Moduli of renresentations of finite dimensional algebras*, J. Algebra. **45** (1994), no. 4, 515–530.

[MR] S. Mozgovoy and M. Reineke, *On the noncommutative donaldson-thomas invariants arising from brane tilings*, arXiv:0809.0117.

[Nag] K. Nagao, *Derived categories of small* 3*-dimensional toric Calabi-Yau varieties and curve counting invariants*, arXiv:0809.2994.

[NN] K. Nagao and H. Nakajima, *Counting invarinats of perverse coherent systems on 3-folds and their wall-crossings*, arXiv:0809.2992.

[PT] R. Pandharipande and R.P. Thomas, *Curve counting via stable pairs in the derived category*, arXiv:0707.2348.

[Sze] B. Szendroi, *Non-commutative Donaldson-Thomas theory and the conifold*, arXiv:0705.3419v3.

[VdB04] M. Van den Bergh, *Three-dimensional flops and noncommutative rings*, Duke Math. J. **122** (2004), no. 3, 423–455.

[You] B. Young, *Computing a pyramid partition generating function with dimer shuffling*, arXiv:0709.3079.

Dihedral Groups, G-Hilb and $\mathcal{M}_\theta(Q,R)$

– Álvaro Nolla de Celis – Japan Dec.'08

$\mathbb{C}^2 \curvearrowleft G \subset Gl(2,\mathbb{C})$ small finite

$$\begin{array}{c} \downarrow \end{array}$$

minimal resolution $Y \longrightarrow \mathbb{C}^2/G$

$E = \bigcup E_i$ excl. divisor

<u>Special McKay Correspondence</u>: Let $\mathrm{Irr}\,G = \{\rho_0, \rho_1, \ldots, \rho_n\}$ irreducible representations of G. Then

$$\left\{\begin{array}{c}\text{Exceptional Curves}\\ E_i\end{array}\right\} \xleftrightarrow{\text{1-to-1}} \left\{\begin{array}{c}\text{Irreducible \underline{Special} Representations}\\ \rho_i\end{array}\right\}$$

<u>Example-Notation</u>: $G = \left\langle \frac{1}{12}(1,7) \right\rangle := \left\langle \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^7 \end{pmatrix} \;\middle|\; \begin{array}{c}\varepsilon^{12}=1\\ \text{primitive}\end{array} \right\rangle$

$$-2 \;\; -4 \;\; -2 \longrightarrow \mathbb{C}^2/G \quad \text{toric} \qquad \frac{12}{7} = [2,4,2]$$

but G cyclic order 12 $\Rightarrow$ G has 12 irreducible repr.

<u>Goal</u>: To construct <u>explicitly</u> Y for $G \subset GL(2,\mathbb{C})$ small binary dihedral by giving an affine open cover of it.

$$\boxed{Y = G\text{-Hilb}\,\mathbb{C}^2 = \mathcal{M}_\theta(Q,R)}$$

G-Hilb

G-Hilb $\mathbb{C}^2$ = Moduli space param. G-clusters

A G-cluster is a G-invariant 0-dimensional subscheme $Z \subset \mathbb{C}^2$ such that

$$\mathcal{O}_Z = \mathbb{C}[x,y]/I_Z \underset{\mathbb{C}[G]}{\simeq} \mathbb{C}[G]$$ the regular representation

$$\mathbb{C}[G] = \bigoplus_{\rho_i \in \operatorname{Irr} G} \rho_i^{\dim \rho_i}$$

$\Rightarrow$ G-Hilb $\mathbb{C}^2$ parametrizes ideals $I \subset \mathbb{C}[x,y]$ such that $\mathbb{C}[x,y]/I$, as a vector space, has in its basis 1 element in each 1-dim representation

2 " " " 2-dim "

⋮

and so on

$GL(2,\mathbb{C})$

Example $\frac{1}{5}(1,2)$; $G = \left\langle \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^2 \end{pmatrix} \,\middle|\, \varepsilon^5 = 1 \text{ prim.} \right\rangle \subset GL(2,\mathbb{C})$, $\alpha = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^2 \end{pmatrix}$, $\begin{cases} x \longmapsto \varepsilon x \\ y \longmapsto \varepsilon^2 y \end{cases}$

	α	
ρ_0	1	$1, x^5, y^5, x^3y, xy^2, \dots$
→ ρ_1	ε	$x, y^3, x^4y, \dots$
→ ρ_2	ε^2	$x^2, y, x^5y, \dots$
ρ_3	ε^3	$x^3, y^4, xy, \dots$
ρ_4	ε^4	$x^4, y^2, x^2y, \dots$

0
3
1 2
4 0 1
2 3 4 0
0 1 2 3 4 0

Possible bases of $\mathbb{C}[x,y]/_I = \bigoplus \rho_i^{\dim \rho_i}$?

$y \notin$ base $\Rightarrow y^i \notin$ base

$y \in$ base $\Rightarrow x^2 \notin$ base

$x \notin$ base $\Rightarrow x^i \notin$ base

G-graphs:

$\boxed{1 \;\; x \;\; x^2 \;\; x^3 \;\; x^4}$ x^5, y

$\begin{array}{ll} & y^3 \\ y^2 & xy^2 \\ y & xy \\ 1 & x \end{array}$ x^2

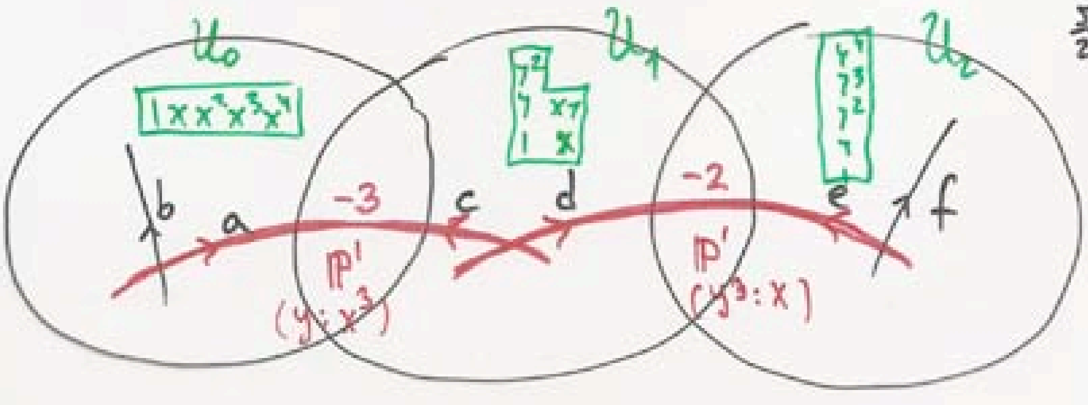
$\begin{array}{l} y^5 \\ y^4 \\ y^3 \\ y^2 \\ y \\ 1 \end{array}$ x

$I_{a,b} = \begin{pmatrix} y = ax^2 \\ x^5 = b\cdot 1 \end{pmatrix} \rightsquigarrow \mathbb{C}^2_{a,b}$ $\quad a, b \in \mathbb{C}$

$I_{c,d} = \begin{pmatrix} x^2 = cy \\ y^3 = dx \\ xy^2 = cd \end{pmatrix} \rightsquigarrow \mathbb{C}^2_{c,d}$ $\quad c, d \in \mathbb{C}$

$I_{e,f} = \begin{pmatrix} x = ey^3 \\ y^5 = f\cdot 1 \end{pmatrix} \rightsquigarrow \mathbb{C}^2_{e,f}$ $\quad e, f \in \mathbb{C}$

$\Rightarrow G\text{-}\mathcal{H}ilb\,\mathbb{C}^2 = \mathbb{C}^2_{a,b} \cup \mathbb{C}^2_{c,d} \cup \mathbb{C}^2_{e,f}$



$\frac{5}{2} = [3,2]$

$\mathcal{M}_\theta(Q,R)$

$\mathcal{M}_\theta(Q,R)$ = Moduli space of θ-stable quiver representations of the McKay quiver (Q,R)

Q = McKay Quiver : V = natural repr. $(G \hookrightarrow GL(2,\mathbb{C}))$

(R = Relations) Form $V \otimes \rho_i = \sum a_{ij}\rho_j \rightsquigarrow \bullet = \rho_i$

$\underset{\rho_i}{\bullet} \!\!\rightarrow\!\! \underset{\rho_j}{\bullet} \iff a_{ij} \neq 0$

A Representation of Q is $W = (\underbrace{W_i}_{\text{v.s. dim } d_i}, \overbrace{\varphi_a}^{\text{linear map}})_{\underset{\text{vertex}}{i \in Q_0},\ \underset{\text{arrow}}{a \in Q_1}}$

Example : $\frac{1}{3}(1,2)$  Q  W

Dimension vector : $\underline{d} = (\dim W_i)$

Isomorphism classes of quiver representations are orbits under $G = \prod GL(d_i)$ (change of basis)

$$\mathcal{M}_\theta(Q,R) = \mathbb{V}(I_R) /\!\!/_\theta\, G$$

I_R = ideal of relations

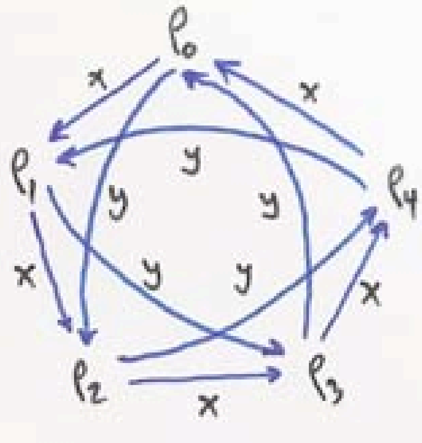
θ generic (semistable $\Rightarrow$ stable)

TAKE $\underline{d} = (\dim \rho_i)$ and $\theta = (-\sum \dim \rho_i, 1, 1, \dots, 1)$

Then, Stable $\Rightarrow \exists$ $(\dim \rho_i)$ non-zero paths from ρ_0 to any other vertex ρ_i (1-generated)

Ejemplo $\frac{1}{5}(1,2)$

ρ_0	$1,\ x^5,\ y^5,\ x^3y,\ xy^2, \ldots$
ρ_1	$x,\ y^3,\ x^4y$
ρ_2	$x^2,\ y,\ x^5y \ldots$
ρ_3	$x^3,\ y^4,\ xy, \ldots$
ρ_4	$x^4,\ y^2,\ x^2y, \ldots$

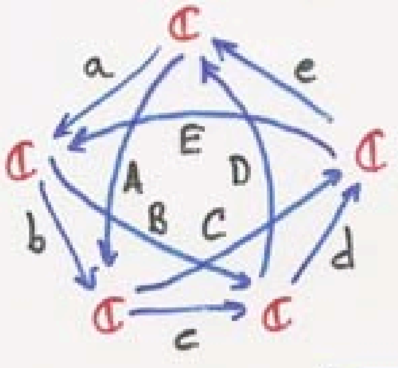


McKay Quiver Q

Representation of Q:

$\underline{d} = (1,1,1,1,1)$

$\theta = (-4,1,1,1,1)$



Relations: "$xy = yx$"

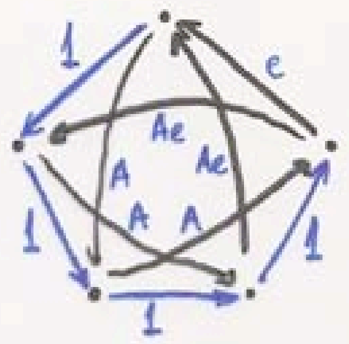
$$\boxed{\begin{gathered} aB = Ac \\ bC = Bd \\ cD = Ce \\ dE = Da \\ eA = Eb \end{gathered}}$$

$U_0 \subset \mathcal{M}_\theta(Q,R)$ given by

$a \neq 0,\ b \neq 0,\ c \neq 0,\ d \neq 0$

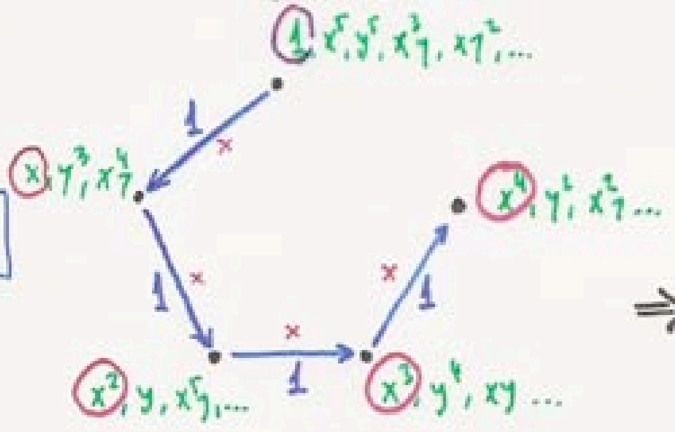
change of basis
+
relations

$\boxed{\mathcal{M}_\theta(Q,R)}$



$\Rightarrow \quad U_0 = \mathbb{C}^2_{A,e}$

$\boxed{G\text{-Hilb}}$

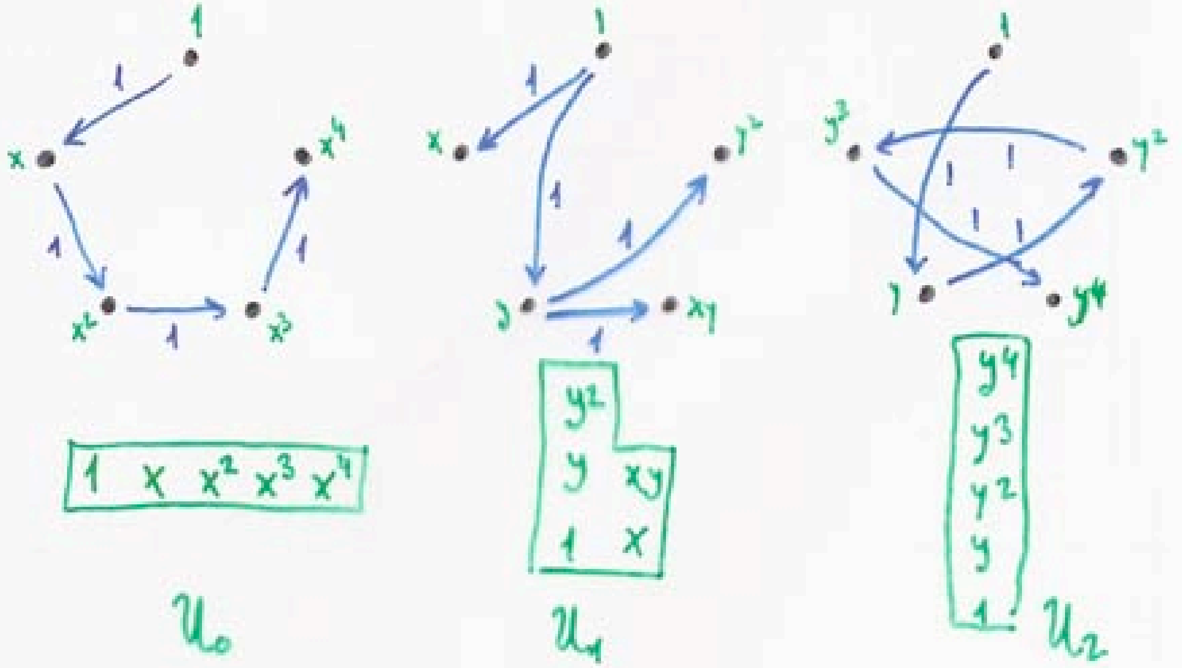


$$\Rightarrow \quad \begin{array}{l} y \\ \boxed{1 \quad x \quad x^2 \quad x^3 \quad x^4} \quad x^5 \end{array}$$

$$I_{A,e} = \begin{pmatrix} x^5 - e \\ y - Ax^2 \end{pmatrix}$$

$$\Rightarrow \quad U_0 = \mathbb{C}^2_{A,e} \subset G\text{-Hilb}\ \mathbb{C}^2$$

$\Rightarrow$ $\mathcal{M}_\theta(Q,R)$ for $\frac{1}{5}(1,2)$ is covered by the following open sets:



$\mathcal{U}_0$ $\mathcal{U}_1$ $\mathcal{U}_2$

<u>Note</u>: There are lot of possibilities !! For example:

... etc

all of them give open sets $U \subset \mathcal{M}_\theta(Q,R)$
$\Rightarrow$ lot of choice in the way of covering $\mathcal{M}_\theta(Q,R)$

$\Longrightarrow$ The G-graphs (G-Hilb) determine a particular covering for $\mathcal{M}_\theta(Q,R)$ with the smallest number of open sets.

Binary Dihedral Groups $G \subset GL(2,\mathbb{C})$

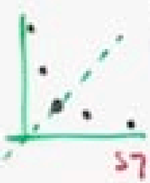
$$G = BD_{2n}(a) := \Big\langle \underbrace{\tfrac{1}{2n}(1,a)}_{\alpha},\ \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}}_{\beta} \ \Big|\ (2n,a)=1,\ \underbrace{a^2 \equiv 1\ (2n)}_{(*)} \Big\rangle$$

$$\alpha: \begin{cases} x \mapsto \varepsilon x \\ y \mapsto \varepsilon^a y \end{cases} \qquad \beta: \begin{cases} x \mapsto y \\ y \mapsto -x \end{cases}$$

$|BD_{2n}(a)| = 4n$, $\quad A = \langle \frac{1}{2n}(1,a) \rangle \trianglelefteq G$ index 2 $\quad (\beta^2 \in A)$

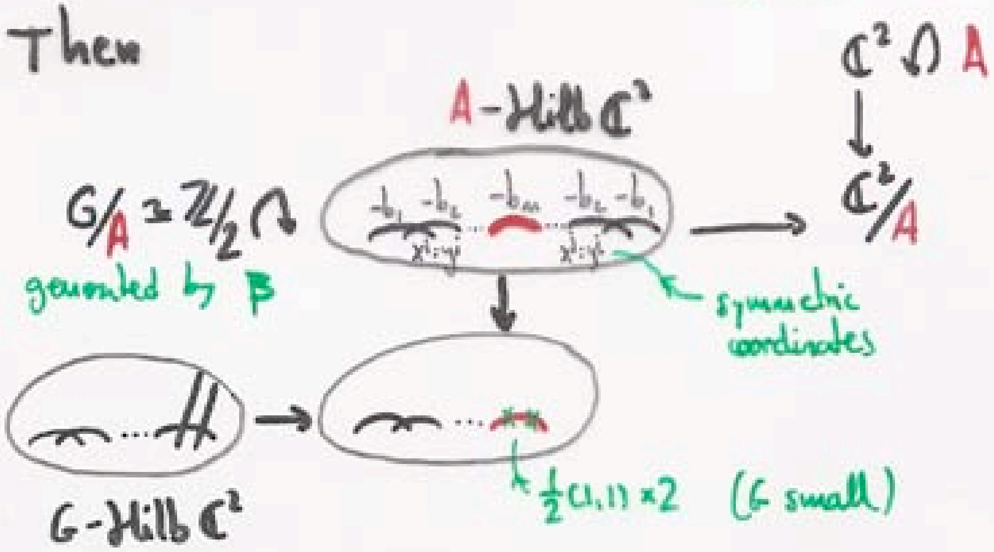
- Condition $(*)$ creates lots of symmetry:

$x^i y^j$ A-inv $\Rightarrow i + aj \equiv 0\ (2n) \Rightarrow ai + j \equiv 0\ (2n) \Rightarrow x^j y^i$ A-inv

$\Rightarrow$  M lattice of monomials, symmetric $\Rightarrow$  $L = M^\vee$, symmetric

$$\Rightarrow \frac{2n}{a} = \underbrace{[b_1, b_2, \dots, b_{m-1}, b_m, b_{m-1}, \dots, b_2, b_1]}_{\text{symmetric}}$$

Then



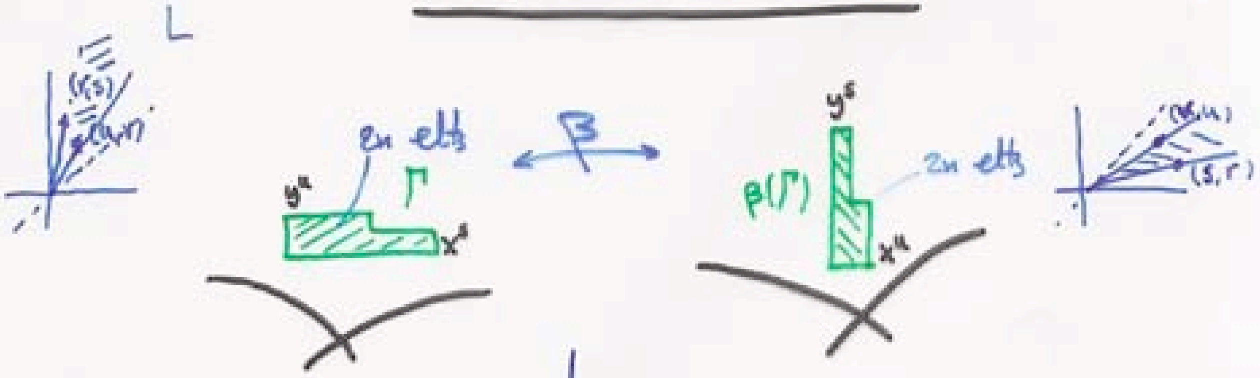
- G-Hilb $\mathbb{C}^2$ = G/A - Hilb (A-Hilb $\mathbb{C}^2$)

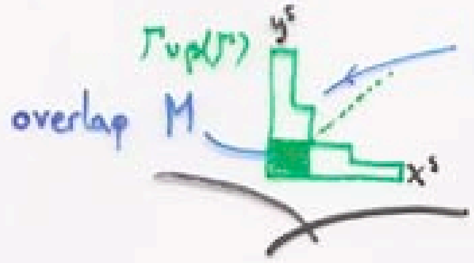
- <u>Thm (Ishii)</u>: G-Hilb is the minimal resolution of $\mathbb{C}^2/G$ for $G \subset GL(2,\mathbb{C})$ small finite

Plan of attack :

$$
\begin{tikzcd}
G \subset GL(2,\mathbb{C}) \text{ binary small} \arrow[d] & \\
G\text{-graphs } \Gamma_i \arrow[r, "\text{associate}"] \arrow[d] & \text{Corresponding } \theta\text{-stable repr. of } (Q,R) \arrow[d] \\
\text{Open sets } U_i \subset G\text{-Hilb} \arrow[dr, dash] & \text{Equations for } U_i \arrow[d, dash] \\
& \text{explicit open cover}
\end{tikzcd}
$$

G-Graphs for $BD_{2n}(a)$

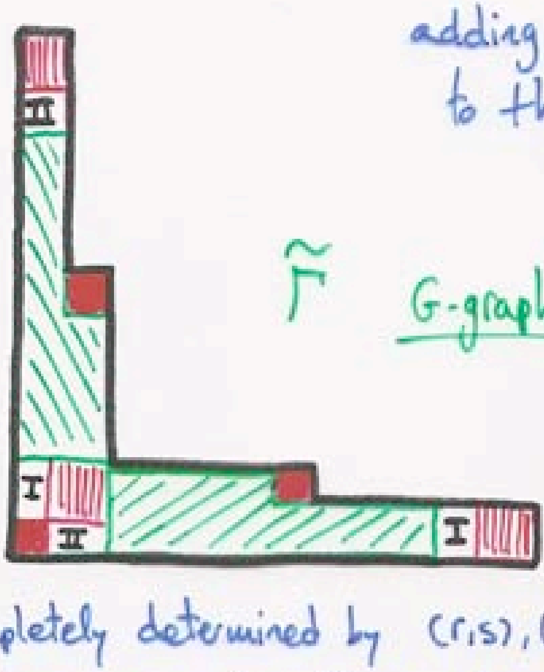




less than 2n elts!

$\Gamma\cup\beta(\Gamma)$ not yet a G-graph

$\exists\,!$ way of extending $\Gamma\cup\beta(\Gamma)$ to a G-graph $\tilde{\Gamma}$ by adding new elements belonging to the representations in M



$\tilde{\Gamma}$ G-graph

$\tilde{\Gamma}$ is completely determined by $(r,s),(u,v)$

$\Rightarrow$ G-graph are given by the continued fraction $\frac{2n}{a}$

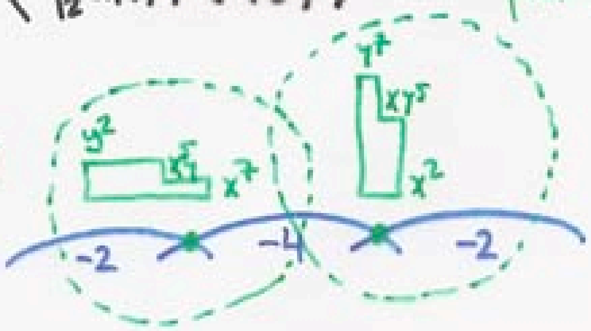
$$BD_{12}(7) = \left\langle \alpha = \frac{1}{12}(1,7),\ \beta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle$$

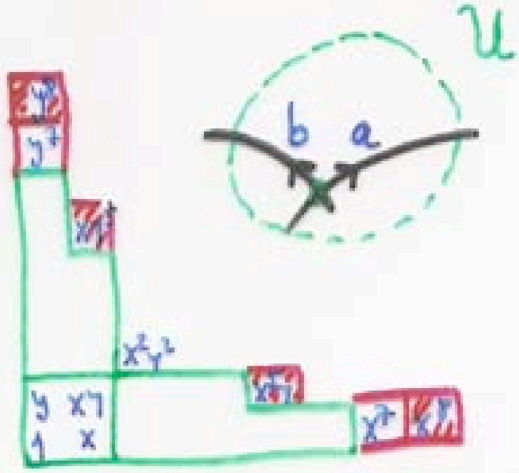
	α	β	
ρ_0^+	1	1	$1,\ x^{12}+y^{12},\ x^5y-xy^5,\ x^6y^6,\ x^8y^4+x^4y^8$
ρ_0^-	1	-1	$x^{12}-y^{12},\ x^5y+xy^5,\ x^3y^3$
ρ_1^+	ε^2	1	$x^2+y^2,\ x^7y-xy^7$
ρ_1^-	ε^2	-1	$x^2-y^2,\ x^7y+xy^7$
ρ_2^+	ε^4	1	$x^4+y^4,\ x^9y-xy^9,\ x^2y^2$
ρ_2^-	ε^4	-1	$x^4-y^4,\ x^9y+xy^9,\ x^5y^5$
ρ_3^+	-1	1	$x^6+y^6,\ x^{11}y-xy^{11},\ x^4y^2+x^2y^4$
ρ_3^-	-1	-1	$x^6-y^6,\ x^{11}y+xy^{11},\ x^4y^2-x^2y^4$
ρ_4^+	ε^8	1	$x^8+y^8,\ x^6y^2+x^2y^6,\ x^4y^4$
ρ_4^-	ε^8	-1	$x^8-y^8,\ xy,\ x^6y^2-x^2y^6$
ρ_5^+	ε^{10}	1	$x^{10}+y^{10},\ x^3y-xy^3$
ρ_5^-	ε^{10}	1	$x^{10}-y^{10},\ x^3y+xy^3$
Q_1	$\begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^7 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	$(x,+y),(y^7,-x^7),(x^6y,-xy^6),(x^2y^5,-x^5y^2)$
Q_2	$\begin{pmatrix} \varepsilon^3 & 0 \\ 0 & \varepsilon^9 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	$(x^3,y^3),(y^9,-x^9),(xy^2,x^2y),(x^8y,-xy^8)$
Q_3	$\begin{pmatrix} \varepsilon^5 & 0 \\ 0 & \varepsilon^{11} \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	$(x^5,y^5),(y^{11},x^{11}),(xy^4,x^4y),(x^{10}y,-xy^{10})$

Example of extension

$BD_{12}(7) = \langle \frac{1}{12}(1,7),\ \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} \rangle$ $\qquad \rho(I) = (x^2, y^7, xy^5)$

$I = (y^2, x^7, x^5 y)$ $\qquad \frac{12}{7} = [2,4,2]$





$\rightsquigarrow \tilde{I} = (x^2y^2,\ x^8 - y^8,\ x^5y - xy^5)$

$x^2y^2,\ x^4+y^4 \in \rho_4^+$; $xy,\ x^8 - y^8 \in \rho_8^-$; $1,\ x^5y - xy^5 \in \rho_0^+$

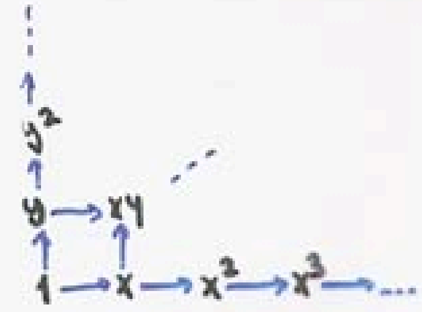
basis $\qquad$ basis $\qquad$ basis

$\Rightarrow \tilde{I} = (x^2y^2 - a(x^4+y^4),\ x^8 - y^8 - b\,xy \quad x^5y - xy^5 - c)$

$\rightarrow$ Equation of $\mathcal{U}$? $\exists\ f(a,b,c) = 0$ hypersurface in $\mathbb{C}^3_{b,c,e}$ non-singular

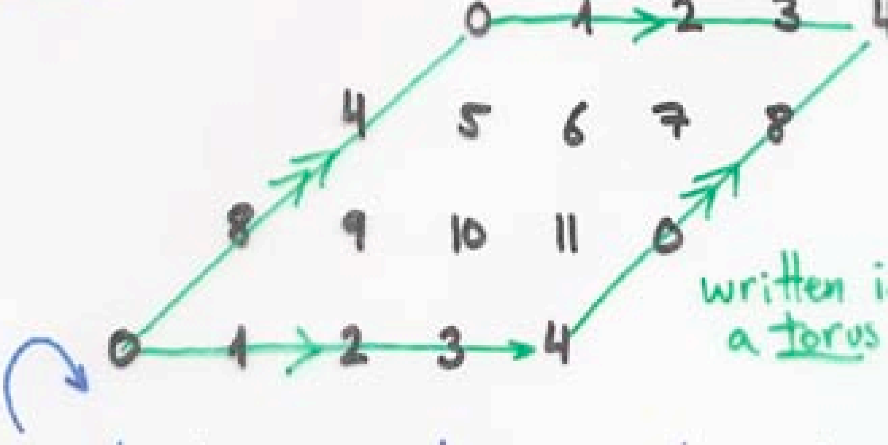
Problems:
- The calculation of f is complicated in general.
- Lots of choices in the selection of the basis elements, e.g. $x^2y^2,\ x^4+y^4,\ (x^2+y^2)^2,\ (x^2-y^2)^2 \in \rho_4^+$ Which one should we choose?

Soln: $\mathcal{M}_\theta(Q,R)$



$$BD_{12}(7) = \left\langle \tfrac{1}{12}(1,7),\ \beta \right\rangle$$

McKay Quiver
for $A = \left\langle \tfrac{1}{12}(1,7) \right\rangle$

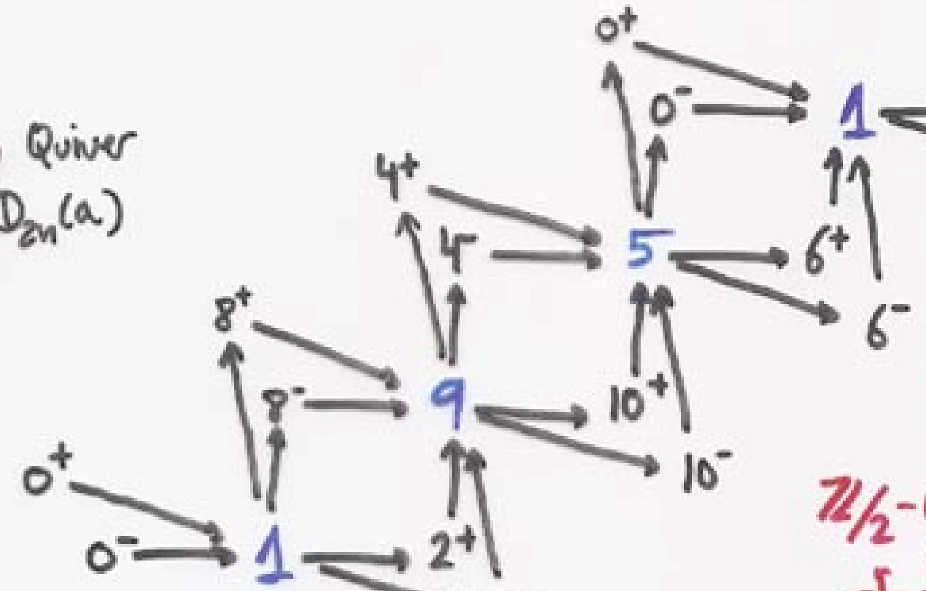


β acts by conjugation on McKay Q(A)

Fixed : ρ_0 ρ_2 ρ_4 ρ_6 ρ_8 ρ_{10}

Free orbits : $\rho_1 \rightleftarrows \rho_7$ $\rho_3 \rightleftarrows \rho_9$ $\rho_5 \rightleftarrows \rho_{11}$

McKay Quiver
for $BD_{2n}(a)$



$\mathbb{Z}/2$-orbifold quotient
of the McKay Quiver
of $\tfrac{1}{2n}(1,a)$

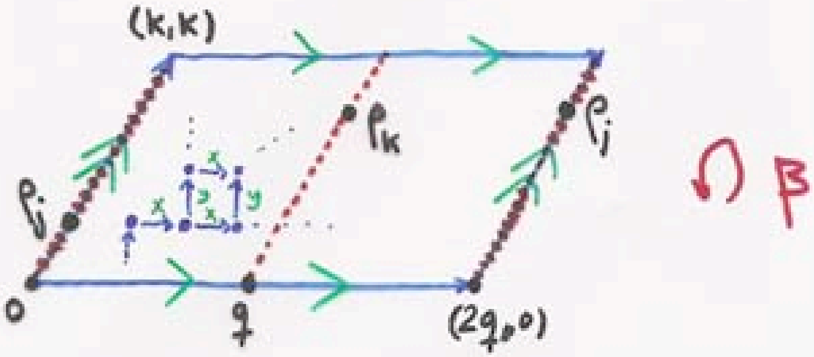
Fixed points ρ_i by β $\rightsquigarrow$ two 1-dim repr. ρ_i^+, ρ_i^-

Free orbits $\rho_i \rightleftarrows \rho_j$ $\rightsquigarrow$ one 2-dim repr V_i

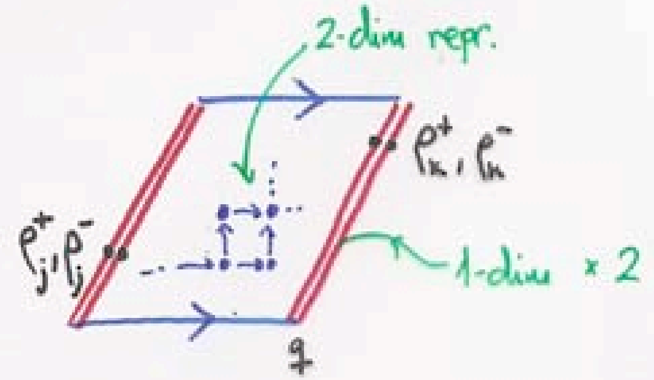
In general,

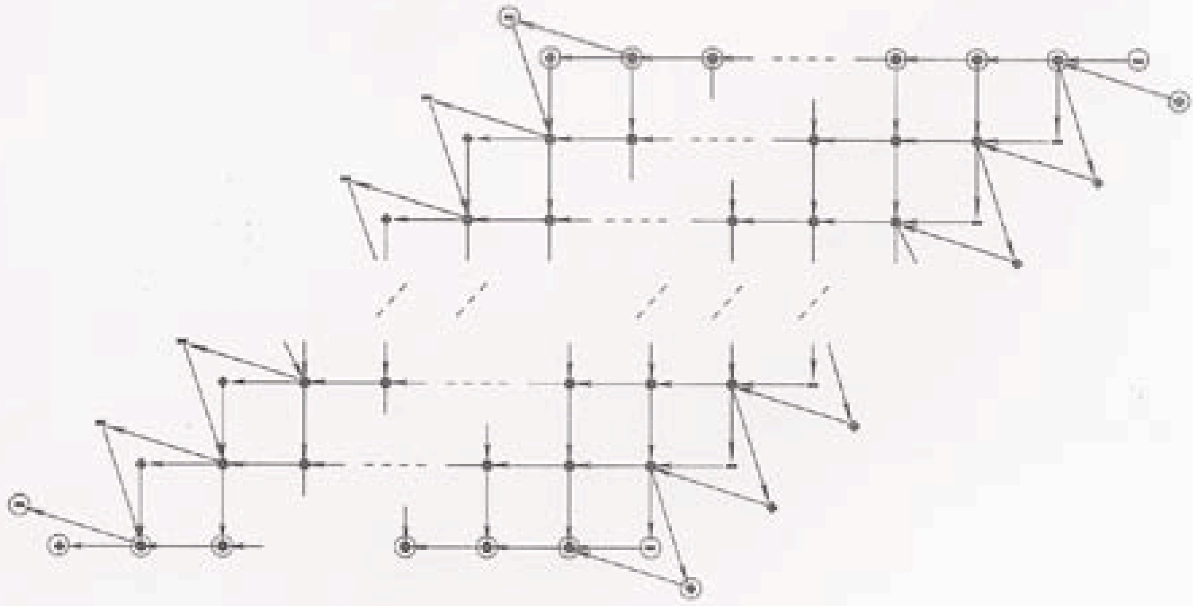
····· = fixed locus

McKay Quiver for $\frac{1}{2n}(1,a)$

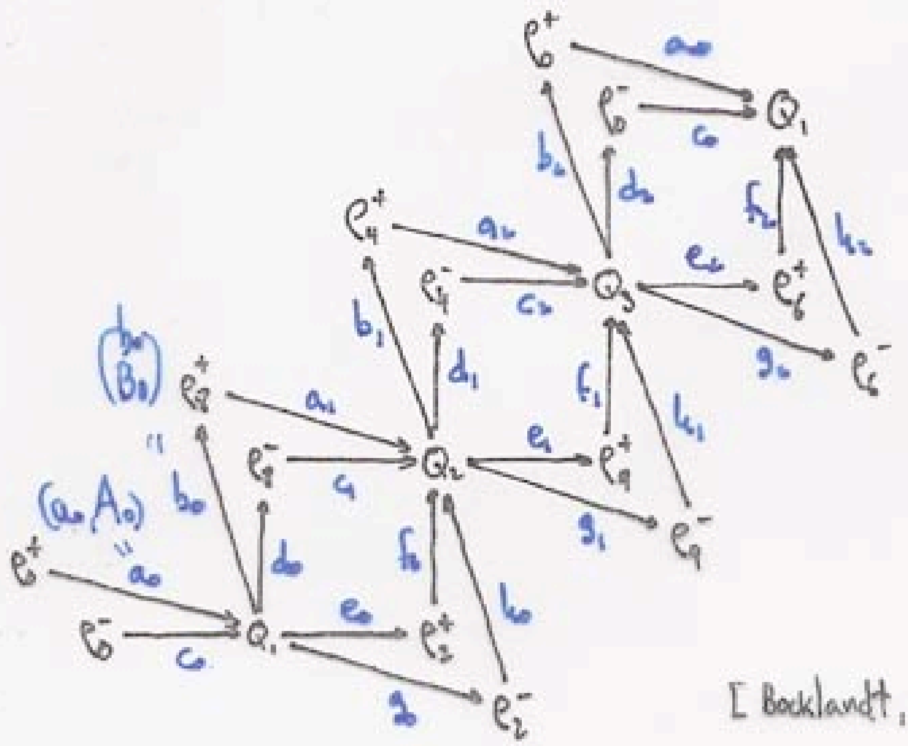


McKay Quiver for $\langle \frac{1}{2n}(1,a), \beta \rangle$





McKay quiver & Relations for $BD_{12}(7)$

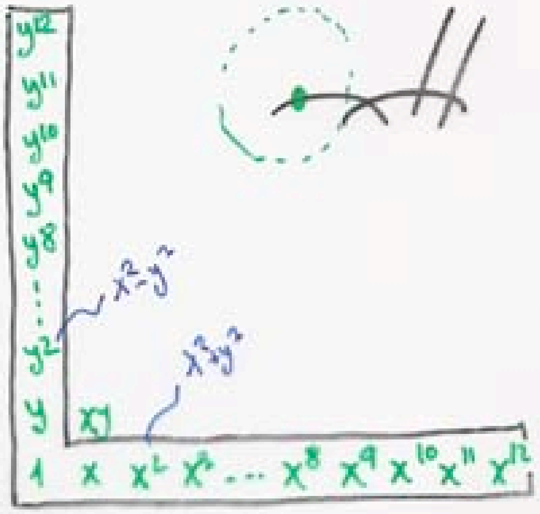


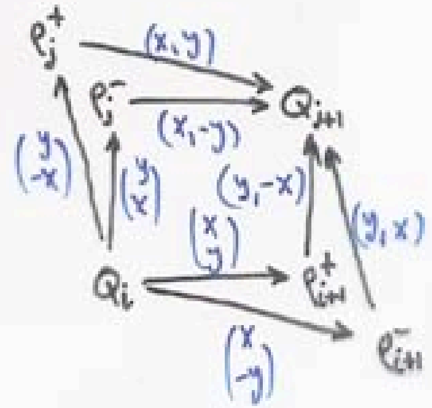
[Bocklandt, Schedler, Wemyss '08]

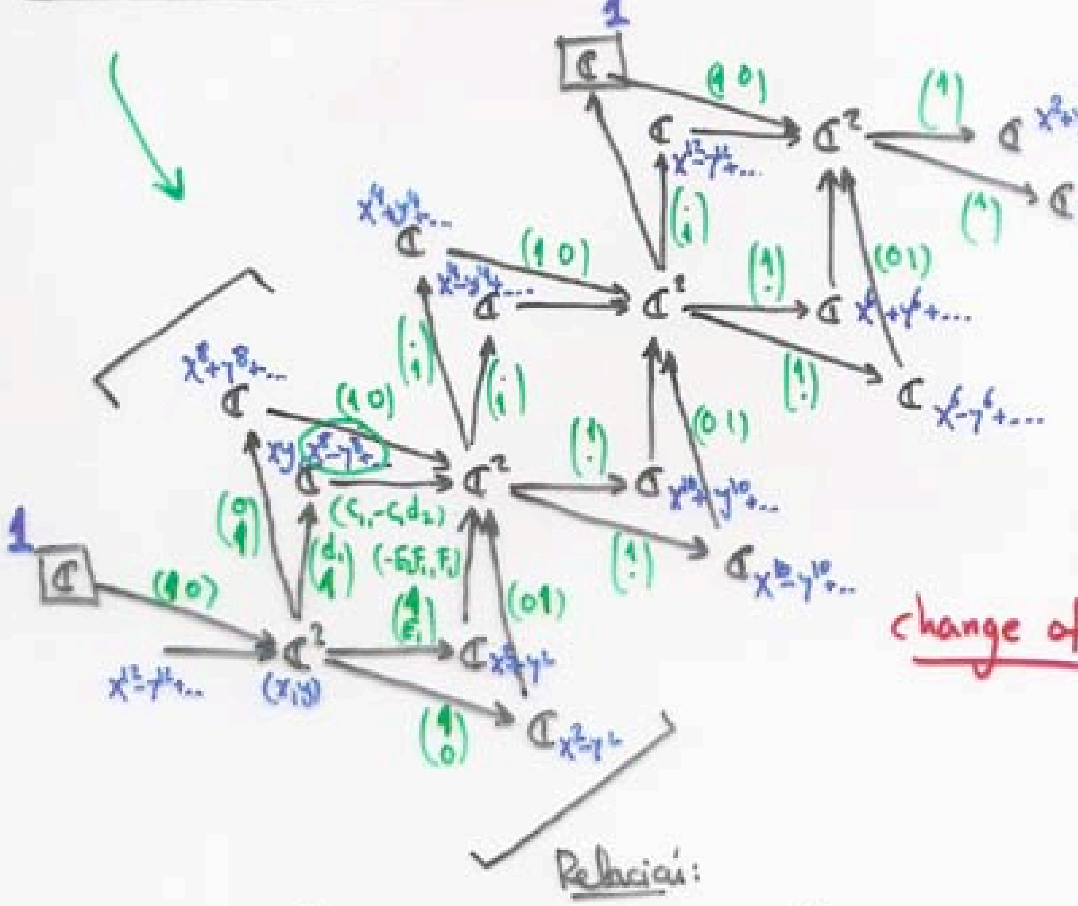
$$a_0 b_0 = 0 \qquad c_0 d_0 = 0 \qquad f_0 e_1 = 0 \qquad h_0 g_1 = 0$$
$$a_1 b_1 = 0 \qquad c_1 d_1 = 0 \qquad f_1 e_2 = 0 \qquad h_1 g_2 = 0$$
$$a_2 b_2 = 0 \qquad c_2 d_2 = 0 \qquad f_2 e_0 = 0 \qquad h_2 g_0 = 0$$

$$b_0 a_1 + d_0 c_1 = e_0 f_0 + g_0 h_0$$
$$b_1 a_2 + d_1 c_2 = e_1 f_1 + g_1 h_1$$
$$b_2 a_0 + d_2 c_0 = e_2 f_2 + g_2 h_2$$

"mesh relations"







change of basis

Relació:

Relations:

$$\begin{pmatrix}0\\1\end{pmatrix}(1\,0) + \begin{pmatrix}d_1\\1\end{pmatrix}(c_1, -c_1 d_2) = \begin{pmatrix}1\\E_1\end{pmatrix}(-E_2F_1, F_1) + \begin{pmatrix}1\\0\end{pmatrix}(0\,1)$$

$$\begin{pmatrix}0&0\\1&0\end{pmatrix} + \begin{pmatrix}d_1c_1 & -c_1d_1d_2\\ c_1 & -c_1d_2\end{pmatrix} = \begin{pmatrix}-E_2F_1 & F_1\\ -E_1E_2F_1 & E_1F_1\end{pmatrix} + \begin{pmatrix}0&1\\0&0\end{pmatrix}$$

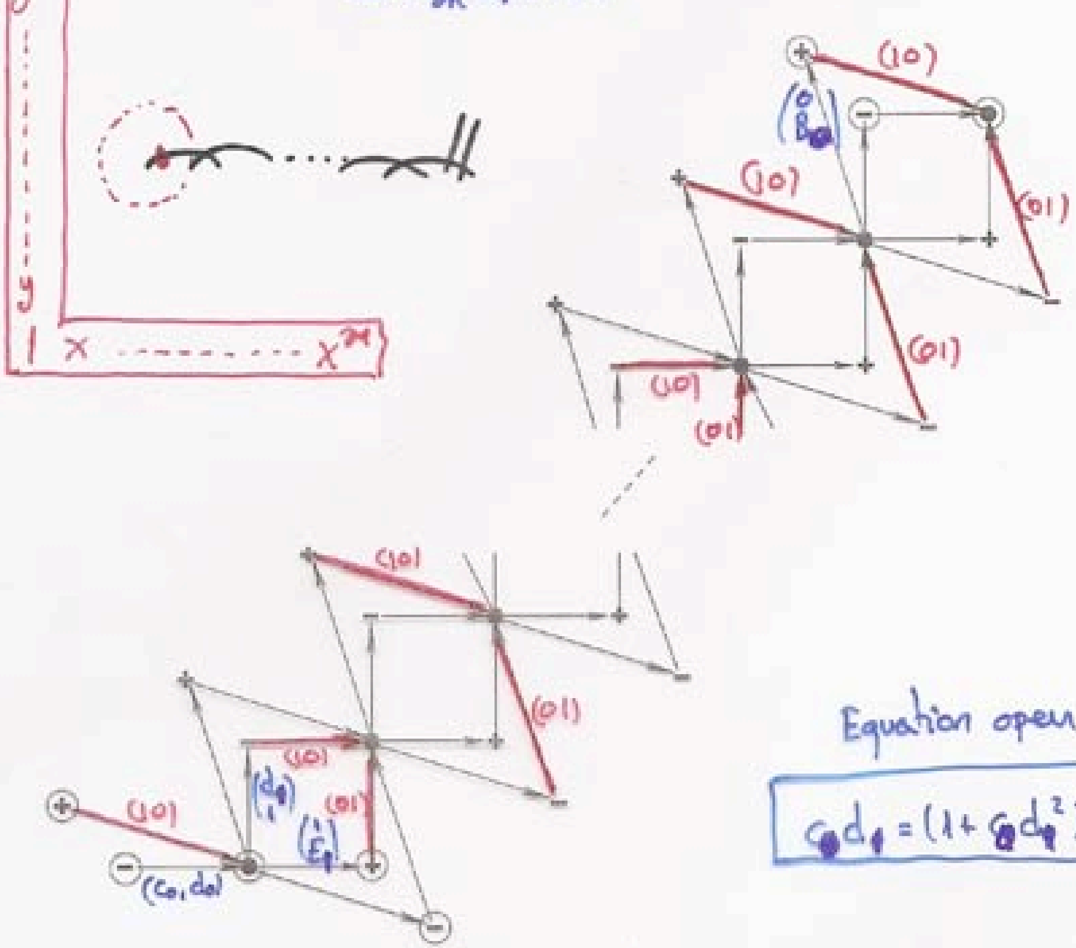
$\Longrightarrow$ $c_1d_1 = -E_1F_1$ $\qquad$ $\underline{F_1 = -1 - c_1d_1d_2}$

$\underline{c_1 = -1 - E_1E_2F_1}$ $\qquad$ $c_1d_2 = -E_1F_1$

$\vdots$

$\longrightarrow$ $\boxed{c_0d_1 = E_1(1+c_0d_1^2)} \subset \mathcal{M}_\theta(Q,R)$

$BD_n(n+1)$



Equation open set:

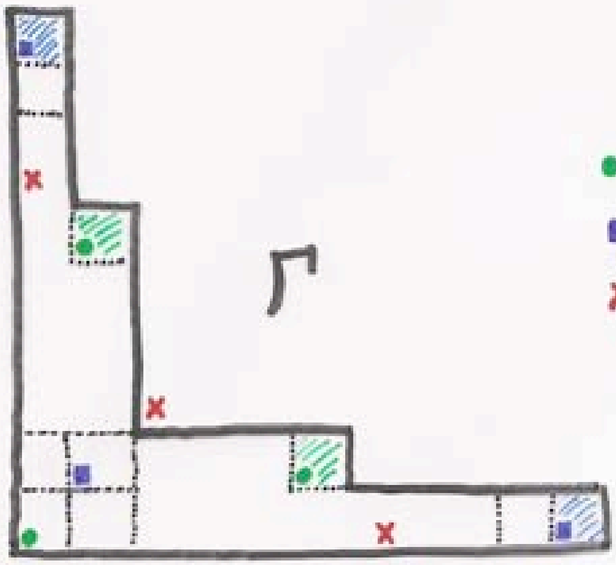
$$\boxed{c_\varphi d_\varphi = (1 + c_\varphi d_\varphi^2)\, E_\varphi}$$

Ideals:

$$I_{c_0, d_\varphi, E_\varphi} = \left(\begin{array}{l} 2xy - d_\varphi (x^2+y^2)^k (x^2-y^2) \\ 2xy(x^2+y^2)^{k-1}(x^2-y^2) - E_\varphi (x^2+y^2) \\ x(x^2+y^2)^{2k-1}(x^2-y^2) - c_0 x - c_0 d_\varphi\, y (x^2+y^2)^{k-1}(x^2-y^2) \\ -y(x^2+y^2)^{2k-1}(x^2-y^2) - c_0 y - c_0 d_\varphi\, x(x^2+y^2)^{k-1}(x^2-y^2) \\ (x^2+y^2)^{2k-1}(x^2-y^2)^2 - B_\varphi \end{array} \right)$$

Special Representations

- In general,



● elts in ρ_0
■ elts in ρ_A
× elts in ρ_B

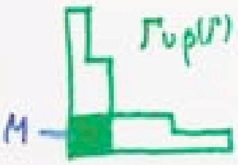
Ishii: $y \in E \Rightarrow I_y / \mathfrak{m} I_y \cong \begin{cases} \rho_i \oplus \rho_0 & \text{if} \\ \rho_i \oplus \rho_j \oplus \rho_0 & \text{if} \end{cases}$

$E_i \rightsquigarrow \rho_i$
excl special

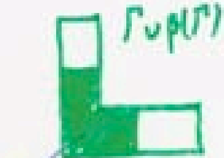
$\Rightarrow$

$\Rightarrow$ Special representations are given by cont. frac. $\frac{2n}{a}$

- <u>Dimension</u> of the special representations :



Type A $\rightsquigarrow$ 1-dim special repr.



Type B $\rightsquigarrow$ 2-dim special repr.

$\exists$ Only two types of "gluing"

SPECIAL MCKAY CORRESPONDENCE

YUKARI ITO

1. Introduction

This note is based on the paper "Special McKay correspondence " [6] by the author.

The McKay correspondence is originally a correspondence between the topology of the minimal resolution of a 2-dimensional rational double point, which is a quotient singularity by a finite group G of $SL(2, \mathbb{C})$, and the representation theory (irreducible representations or conjugacy classes) of the group G. We can see the correspondence via Dynkin diagrams, which came from McKay's observation in 1979 [10].

Let G be a finite subgroup of $SL(2, \mathbb{C})$, then the quotient space $X := \mathbb{C}^2/G$ has a rational double point at the origin. As there exists the minimal resolution $\widetilde{X}$ of the singularity, we have the exceptional divisors E_i. The dual graph of the configuration of the exceptional divisors is just the Dynkin diagram of type A_n, D_n, E_6, E_7 or E_8.

On the other hand, we have the set of the irreducible representations ρ_i of the group G up to isomorphism and let ρ be the natural representation in $SL(2, \mathbb{C})$. The tensor product of these representations

$$\rho_i \otimes \rho = \Sigma_{j=0}^{r} a_{ij} \rho_j,$$

where r is the number of the non-trivial irreducible representations, gives a set of integers a_{ij} and it determines the Cartan matrix which defines the Dynkin diagram. [1]

Then we have a one-to-one numerical correspondence between non-trivial irreducible representations $\{\rho_i\}$ and irreducible exceptional curves $\{E_i\}$, that is, the intersention matrix of the exceptional divisors can be written as $(-1)\times$ Cartan matrix.

This phenomenon was explained geometrically in terms of vector bundles on the minimal resolution by Gonzalez-Sprinberg and Verdier ([4]) by case-by-case computations in 1983. In 1985, Artin and Verdier [1] proved this more generally with relexive modules and this theory was developed by Esnault and Knörrer ([2], [3]) for more general quotient

[1] More precisely, the Cartan matrix is defined as the matrix $2E - A$, where E is the $(r-1) \times (r-1)$ identity matrix and $A = \{a_{ij}\}$ $(i, j \neq 0)$.

surface singularities. After Wunram [14] constructed a nice generalized McKay correspondence for any quotient surface singularities in 1986 in his dissertation, Riemenschneider intoruduced the notion of “special representation etc.” and made his propaganda for the more generalized McKay correspondence [11].

In particular, we would like to discuss special representations and the minimal resolution for quotient surface singularities from now on. Around 1996, Nakamura and the author showed another way to the McKay correspondence with the help of the G-Hilbert scheme, which is a 2-dimensional G-fixed set of the usual Hilbert scheme of $|G|$-points on $\mathbb{C}^2$ and isomorphic to the minimal resolution. Kidoh [9] proved that the G-Hilbert scheme for general cyclic surface singularities is the minimal resolution. Then Riemenschneider checked the cyclic case and conjectured that the representations which are given by the Ito-Nakamura type McKay correspondence via G-Hilbert scheme are just special representations in 1999 ([12]) and this conjecture was proved by A. Ishii ([5]). In this paper, we will give another characterization of the special representations by combinatorics for the cyclic quotient case using results on the G-Hilbert schemes.

2. Special representations

In this section, we will discuss the special representations. Let G be a finite small subgroup of $GL(2,\mathbb{C})$, that is, the action of the group G is free outside the origin, and ρ be a representation of G on V. G acts on $\mathbb{C}^2 \times V$ and the quotient is a vector bundle on $(\mathbb{C}^2 \setminus \{0\})/G$ which can be extended to a reflexive sheaf $\mathcal{F}$ on $X\colon = \mathbb{C}^2/G$.

For any reflexive sheaf $\mathcal{F}$ on a rational surface singularity X and the minimal resolution $\pi\colon \tilde{X} \to X$. We define a sheaf $\widetilde{\mathcal{F}}\colon = \pi^*\mathcal{F}/\text{torsion}$.

Definition 2.1. ([2]) The sheaf $\widetilde{\mathcal{F}}$ is called a *full sheaf* on $\tilde{X}$.

Theorem 2.2. ([2]) *A sheaf $\widetilde{\mathcal{F}}$ on $\tilde{X}$ is a full sheaf if the following conditions are fulfilled:*

1. *$\widetilde{\mathcal{F}}$ is locally free,*
2. *$\widetilde{\mathcal{F}}$ is generated by global sections,*
3. *$H^1(\tilde{X}, \widetilde{\mathcal{F}}^\vee \otimes \omega_{\tilde{X}}) = 0$, where $\vee$ means the dual.*

Note that a sheaf $\widetilde{\mathcal{F}}$ is indecomposable if and only if the corresponding representation ρ is irreducible. Therefore we obtain an indecomposable full sheaf $\widetilde{\mathcal{F}}_i$ on $\tilde{X}$ for each irreducible representation ρ_i, but in general, the number of the irreducible representations is larger than that of irreducible exceptional components. Therefore Wunram and Riemenschneider inroduced the notion of a speciality for full sheaves:

Definition 2.3. ([11]) A full sheaf is called *special* if and only if

$$H^1(\tilde{X}, \widetilde{\mathcal{F}}^{\vee}) = 0.$$

A reflexive sheaf $\mathcal{F}$ on X is *special* if $\widetilde{\mathcal{F}}$ is so.

A representation ρ is *special* if the associated reflexive sheaf $\mathcal{F}$ on X is special.

With these definitions, following equivalent conditions for the speciality hold:

Theorem 2.4. ([11], [14])

1. $\widetilde{\mathcal{F}}$ is special $\Longleftrightarrow$ $\widetilde{\mathcal{F}} \otimes \omega_{\tilde{X}} \to [(\mathcal{F} \otimes \omega_{\tilde{X}})^{\vee\vee}]^{\sim}$ is an isomorphism,

2. $\mathcal{F}$ is special $\Longleftrightarrow$ $\mathcal{F} \otimes \omega_{\tilde{X}}/torsion)$ is reflexive,

3. ρ is a special representation $\Longleftrightarrow$ $(\Omega^2_{\mathbb{C}^2})^G \otimes (\mathcal{O}_{\mathbb{C}^2} \otimes V)^G \to (\Omega^2_{\mathbb{C}^2} \otimes V)^G$ is surjective.

Then we have following nice generalized McKay correspondence for quotient surface singularities:

Theorem 2.5. ([14]) *There is a bijection between the set of special non-trivial indecomposable reflexive modules $\mathcal{F}_i$ and the set of irreducible components E_i via $c_1(\widetilde{\mathcal{F}}_i)E_j = \delta_{ij}$ where c_1 is the first Chern class, and also a one-to-one correspondence with the set of special non-trivial irreducible representations.*

As a corollary of this theorem, we get the original McKay correspondence for finite subgroups in $SL(2, \mathbb{C})$ back because in thsi case all irreducible representations are special.

3. G-Hilbert schemes and combinatorics

In this section, we will discuss G-Hilbert schemes and a new way to find the special representations for cyclic quotient singularities by combinatorics.

Hilbert scheme of n-points on $\mathbb{C}^2$ can be described as a set of ideals:

$$\mathrm{Hilb}^n(\mathbb{C}^2) = \{I \subset \mathbb{C}[x, y] \mid I : \text{ideal}, \dim \mathbb{C}[x, y]/I = n\}.$$

It is a $2n$-dimensional smooth projective variety. The G-Hilbert scheme $\mathrm{Hilb}^G(\mathbb{C}^2)$ was introduced in the paper by Nakamura and the author ([7]) as follows:

$$\mathrm{Hilb}^G(\mathbb{C}^2) = \{I \subset \mathbb{C}[x, y] \mid I : G\text{-invariant ideal}, \mathbb{C}[x, y]/I \cong \mathbb{C}[G]\},$$

where $|G| = n$. This is a union of components of fixed points of G-action on $\mathrm{Hilb}^n(\mathbb{C}^2)$ and in fact it is just the minimal resolution of the quotient singularity $\mathbb{C}^2/G$. It was proved for $G \in SL(2, \mathbb{C})$ in [7] first by the properties of $\mathrm{Hilb}^n(\mathbb{C}^2)$ and finite group action of G and they state a McKay correspondence in terms of ideals of G-Hilbert schemes.

Later Kidoh ([9]) proved that the G-Hilbert scheme for any small cyclic subgroup in $GL(2, \mathbb{C})$ is also the minimal resolution of the corresponding cyclic quotient singularities and Riemenschneider conjectured that the G-Hilbert scheme for any $G \subset GL(2, \mathbb{C})$ is the minimal resolution of the quotient singularity $\mathbb{C}^2/G$ and it was based on his result. That is, he checked the irreducible representation which are given by the ideals of G-Hilbert scheme, so-called Ito-Nakamura type McKay correspondence, are just the same as the special representations defined by himself [12], see also [11] A. Ishii ([5]) proved more generally that the G-Hilbert scheme for any small $G \subset GL(2, \mathbb{C})$ is always isomorphic to the minimal resolution of the singularity $\mathbb{C}^2/G$ and the conjecture is true:

Theorem 3.1. ([5]) *Let G be a finite small subgroup of $GL(2, \mathbb{C})$.*

(i) G-Hilbert scheme $Hilb^G(\mathbb{C}^2)$ is the minimal resolution of $\mathbb{C}^2/G$.

(ii) For $y \in Hilb^G(\mathbb{C}^2)$, denote by I_y the ideal corresponding to y and let m be the maximal ideal of $\mathcal{O}_{\mathbb{C}^2}$ corresponding to the origin 0. If y is in the exceptional locus, then, as representations of G, we have

$$I_y/mI_y \cong \begin{cases} \rho_i \oplus \rho_0 & \text{if } y \in E_i \text{ and } y \notin E_j \text{ for } j \neq i, \\ \rho_i \oplus \rho_j \oplus \rho_0 & \text{if } y \in E_i \cap E_j, \end{cases} \tag{3.2}$$

where ρ_i is the special representation associated with the ireducible exceptional curve E_i.

Remark 3.3. In dimension two, we can say that G-Hilbert scheme is the same as a 2-dimensional irreducible component of the G-fixed set of $\mathrm{Hilb}^n(\mathbb{C}^2)$. A similar statement holds for $G \subset SL(3, \mathbb{C})$ in dimension three, that is, the G-Hilbert scheme is a 3-dimensional irreducible component of the G-fixed set of $\mathrm{Hilb}^n(\mathbb{C}^3)$ and a crepant resolution of the quotient singularity $\mathbb{C}^3/G$. In this case note that $\mathrm{Hilb}^n(\mathbb{C}^3)$ is not smooth.

Moreover, Haiman proved that $\mathcal{S}_n$-Hilbert scheme $\mathrm{Hilb}^{\mathcal{S}_n}(\mathbb{C}^{2n})$ is a crepant resolution of $\mathbb{C}^{2n}/\mathcal{S}_n = n$-th symmetric product of $\mathbb{C}^2$, i.e.,

$$\mathrm{Hilb}^{\mathcal{S}_n}(\mathbb{C}^{2n}) \cong \mathrm{Hilb}^n(\mathbb{C}^2)$$

in process of the proof of $n!$ conjecture. (cf. [8])

From now on, we restrict our considerations to $G \subset GL(2, \mathbb{C})$ cyclic. Wunram constructed the generalized McKay correspondence for cyclic surface singularities in the paper [13] and we have to consider the corresponding geometrical informations (the minimal resolution, reflexive sheaves and so on) to obtain the special representations. Here we would like to give a new characterization of the special representations in terms of combinatorics. It is much easier to find the special representation because we don't need any geometrical objects, but based on the result of G-Hilbert schemes.

Let us discuss the new characterization of the special representations in terms of combinatorics. Let G be a cyclic group $C_{r,a}$ which is generated by a matrix $\begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^a \end{pmatrix}$ where $\epsilon^r = 1$ and $\gcd(r, a) = 1$ and consider a character map $\mathbb{C}[x, y] \longrightarrow \mathbb{C}[t]/t^r$ as $x \mapsto t$ and $y \mapsto t^a$, then we have a corresponding characters for each monomials in $\mathbb{C}[x, y]$.

Let I_p be the ideal of the G-fixed point p in the G-Hilbert scheme, then we can define the following sets.

Consider a G-invariant subscheme $Z_p \subset \mathbb{C}^2$ for which $H^0(Z_p, \mathcal{O}_{Z_p}) = \mathcal{O}_{\mathbb{C}^2}/I_p$ is the regular representation of G. Then the G-Hilbert scheme can be regarded as a moduli space of such Z_p.

Definition 3.4. The set of monomials in $\mathbb{C}[x, y]$ $Y(Z_p)$ is called *G-cluster* if all monomials on $Y(Z_p)$ are not in I_p and it can be drawn as a Young diagram of $|G|$ boxes.

Definition 3.5. For any small cyclic group G, let $B(G)$ be the set of monomials which are not divisible, by any G-invariant monomial and call it G-basis.

Definition 3.6. If $|G| = r$, then let $L(G)$ be $\{1, x, \cdots, x^{r-1}, y, \cdots, y^{r-1}\}$, i.e., the set of monomials which cannot be devided by x^r, y^r or xy. We call it L-space for G because the shape of this diagram looks as the chapital "L."

Definition 3.7. The monomial $x^m y^n$ is of *weight* k if $m + an = k$.

Let us describe the method to find the special representations of G with these diagrams:

Theorem 3.8. *For a small finite cyclic subgroup of $GL(2, \mathbb{C})$, the irreducible representation ρ_i is special if and only if the corresponding monomial in $B(G)$ are not contained in the set of monomials $B(G) \setminus L(G)$.*

Proof. In Theorem 2.4 (3), we have the definition of the special representation, and it is not easy to compute all special representations. However look at the behavior of the monomials in $\mathbb{C}[x, y]$ under the map $\Phi_i\, (\Omega^2_{\mathbb{C}^2})^G \otimes (\mathcal{O}_{\mathbb{C}^2} \otimes V_i)^G \to (\Omega^2_{\mathbb{C}^2} \otimes V_i)^G$ for each representation ρ_i:

First, let us consider the monomial bases of each set. Let $V_i = \mathbb{C}e_i$ and $\rho(g)e_i = \epsilon^{-i}$. An element $f(x, y)dx \wedge dy \otimes \rho_i$ is in $(\Omega^2_{\mathbb{C}^2} \otimes V_i)^G$ if and only if

$$g^* f(x,y)dx \wedge dy \cdot \epsilon^{1+a} \otimes \epsilon^{-i} = f(x,y)dx \wedge dy,$$

that is,

$$g^*(f(x,y)dx \wedge dy) = \epsilon^{i-(a+1)}(f(x,y)dx \wedge dy).$$

Therefore the monomial base for $(\Omega^2_{\mathbb{C}^2} \otimes V_i)^G$ is a set of monomials $f(x,y)$ such that

$$g : f(x,y) \mapsto \epsilon^{i-(a+1)} f(x,y)$$

under the action of G, that is, monomials of weight $i-(a+1)$.

Similarly, we have the monomial bases for $(\Omega^2_{\mathbb{C}^2})^G$ as the set of monomials $f(x.y)$ of weight $r-(a+1)$.

The monomial bases for $(\mathcal{O}_{\mathbb{C}^2} \otimes V_i)^G$ is given as a set of monomials $f(x,y)$ of weight i.

Let us check the surjectivity of the map Φ_i. If Φ_i is surjective, then all the monomial bases in $(\Omega^2_{\mathbb{C}^2} \otimes V_i)^G$ can be obtained as a product of the monomial basis of two other sets. Therefore the degree of the monomials in $(\Omega^2_{\mathbb{C}^2} \otimes V_i)^G$ must be higher than the degree of the monomials in $(\mathcal{O}_{\mathbb{C}^2} \otimes V_i)^G$.

Now look at the map Φ_{a+1}. The vector space $(\mathcal{O}_{\mathbb{C}^2} \otimes V_{a+1})^G$ is generated by the monomials of weight $a+1$, i.e., $x^{a+1}, xy, \cdots, y^b$ where $ab = a+1 \mod r$. On the other hand, $(\Omega^2_{\mathbb{C}^2} \otimes V_{a+1})^G$ is generated by the degree 0 monomial 1. Then the map Φ_{a+1} is not surjective.

By this, if a monomial of type $x^m y^n$, where $mn \neq 0$, is a base of $(\mathcal{O}_{\mathbb{C}^2} \otimes V_i)^G$, then there exists a monomial $x^{m-1}y^{n-1}$ in $(\Omega^2_{\mathbb{C}^2} \otimes V_i)^G$ and the degree become smaller under tha map Φ_i. This means Φ_i is not surjective.

Moreover, if the bases of $(\mathcal{O}_{\mathbb{C}^2} \otimes V_i)^G$ is generated only by x^i and y^j where $aj = i \mod r$, then the degrees of the monomials in $(\Omega^2_{\mathbb{C}^2} \otimes V_i)^G$ is bigger and Φ_i is surjective. Thus we have the assertion. $\square$

Remark 3.9. From this theorem, we can also say that a representation ρ_i is special if and only if the number of the generators of the space $(\mathcal{O}_{\mathbb{C}^2} \otimes V_i)^G$ is 2.

Theorem 3.10. *Let p be a fixed point by G-action, then we can define an ideal I_p by the G-cluster and the configuration of the exceptional locus can be described by these data.*

Proof. The defining equation of the ideal I_p is given by

$$\begin{cases} x^a = \alpha y^c, \\ y^b = \beta x^d, \\ x^{a-d} y^{b-c} = \alpha\beta, \end{cases}$$

where α and β are complex numbers and both x^a and y^c (resp. y^b and x^d) correspond the same representation (or character).

The pair (α, β) is a local affine coorinate near the fixed point p and it is also obtained from the calculation with toric geometry. Moreover each axis of the affine chart is just a exceptional curve or the original axis of $\mathbb{C}^2$. The exceptional curve is isomorphic to a $\mathbb{P}^1$ and the points on it is written by the ratio like $[x^a : y^b]$ (resp. $[x^d : y^c]$) which is corresponding to a special representation ρ_a (resp. ρ_d). The fixed point p is the intersection point of 2 exceptional curves E_a and E_d.

Thus we can get the whole space of exceptional locus by deformation of the point p and patching the affine pieces. $\square$

We will see a concrete example in the following section. Here we would like to make one remark as a corollary:

Corollary 3.11. *For A_n-type simple singularities, all $n+1$ affine charts can be described by $n+1$ Young diagrams of type $(1, \cdots, 1, k)$.*

Proof. In A_n case, xy is always G-invariant, hence $B(G) = L(G)$. Therefore we have $n+1$ G-clusters and each of them corresponds to the monomial ideal (x^k, y^{n-k+2}, xy). $\square$

4. Example

First, we recall the toric resolution of cyclic quotient singularities because the quotient space $\mathbb{C}^2/G$ is a toric variety.

Let $\mathbb{R}^2$ be the 2-dimensional real vector space, $\{e^i | i = 1, 2\}$ its standard base, L the lattice generated by e^1 and e^2, $N := L + \sum \mathbb{Z}v$, where the summation runs over all the elements $v = 1/r(1, a) \in G = C_{r,a}$, and

$$\sigma := \left\{ \sum_{i=1}^{2} x_i e^i \in \mathbb{R}^2, \quad x_i \geq 0, \forall i, 1 \leq i \leq 2 \right\}$$

the naturally defined rational convex polyhedral cone in $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$. The corresponding affine torus embedding Y_σ is defined as $\mathrm{Spec}(\mathbb{C}[\check{\sigma} \cap M])$, where M is the dual lattice of N and $\check{\sigma}$ the dual cone of σ in $M_{\mathbb{R}}$ defined as $\check{\sigma} := \{\xi \in M_{\mathbb{R}} | \xi(x) \geq 0, \forall x \in \sigma\}$.

Then $X = \mathbb{C}^2/G$ corresponds to the toric variety which is induced by the cone σ within the lattice N.

Fact 1 We can construct a simplicial decomposition S with the verteces on the Newton Boundary, that is, the convex hull of the lattice points in σ except origin.

Fact 2 If $\widetilde{X} := X_S$ is the corresponding torus embedding, then X_S is non-singular. Thus, we obtain the minimal resolution $\pi = \pi_S$:

$\widetilde{X} = X_S \longrightarrow \mathbb{C}^2/G = Y$. Moreover, each lattice point of the Newton boundary corresponds to an exceptional divisor.

Example Let us look at the example of the cyclic quotient singularity of type $C_{7,3}$ which is generated by the matrix $\begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^3 \end{pmatrix}$ where $\epsilon^7 = 1$. The toric resolution of this quotient singularity is given by the triangulation of a lattice $N\colon = \mathbb{Z}^2 + \frac{1}{7}(1,3)\mathbb{Z}$ with the lattice points: See Figure 4.1.

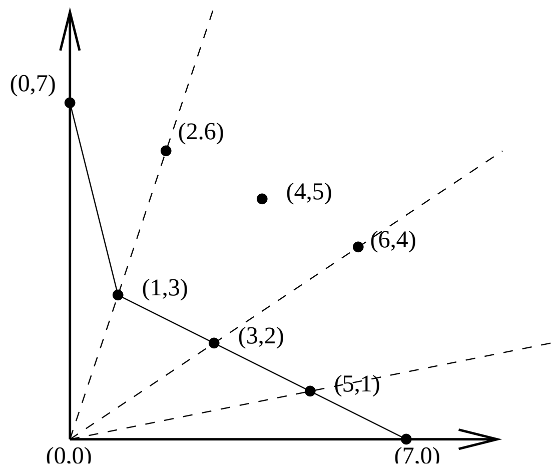


FIGURE 4.1. toric resolution of $\mathbb{C}^2/G$

From this Newton polytope, we can see that there are 3 exceptional divisors and the dual graph gives the configuration of the exceptional components with a deformed coordinate from the original coordinate (x, y) on $\mathbb{C}^2$ as in Figure 4.2.

Therefore we have 4 affine pieces in this example and we have 4 coordinate systems corresponding to each affine piece. In this picture, we will see the corresponding special irreducible representations, but we would like to use our method in the previous section to find the representations.

Let us draw the diagram which corresponds to the G-basis and L-space. First we have the following G-basis $B(G)$ and the corresponding characters in a same diagram. In Figure 4.3 we draw the L-space as shaded part in $B(G)$.

Now we have three monomials xy, x^2y and x^3y in $B(G) \setminus L(G)$ and they correspond to the characters (resp. representations) 4, 5 and 6 (resp. ρ_4, ρ_5 and ρ_6). Therefore we can find a set of special representations, that is, $\{\rho_1, \rho_2, \rho_3\}$, and find the corresponding G-clusters,

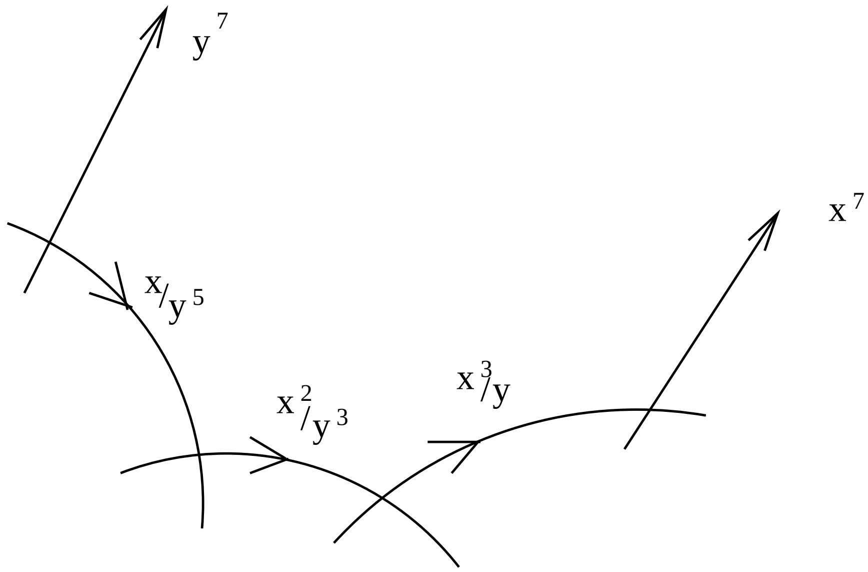


FIGURE 4.2. configuration of $\tilde{X}$

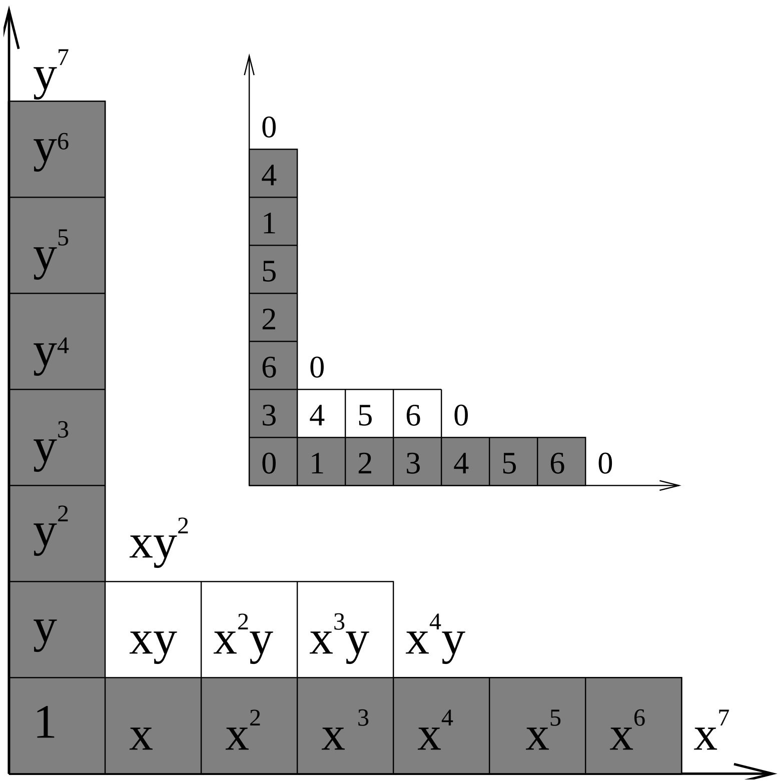


FIGURE 4.3. G-basis $B(G)$ and the characters

representing the origin of the affine charts of the resolution, can be drawn as 4 young diagrams and get the corresponding special representations in this case. See Figure 4.4.

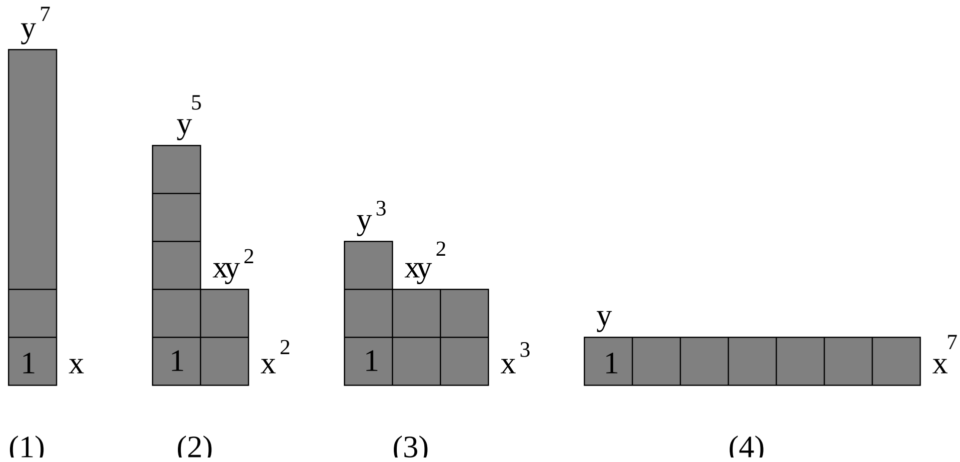


FIGURE 4.4. G-cluster $Y(Z_p)$

Let us see the meanings of the corresponding G-clusters in this case. From $Y(Z_p)$ for (2), we obtain an ideal $I_2 = (y^5, x^2, xy^2)$ for the origin of the affine chart (2) in Figure 4.2, and the corresponding representations are ρ_1, ρ_2 and ρ_0. If we take the maximal ideal m of $\mathcal{O}_{\mathbb{C}^2}$ corresponding to the origin 0, then we have

$$I_2/mI_2 \cong \rho_1 \oplus \rho_2 \oplus \rho_0.$$

Similarly we have the ideal $I_3 = (y^3, x^3, xy^2)$ and

$$I_3/mI_3 \cong \rho_2 \oplus \rho_3 \oplus \rho_0.$$

These descriptions coincide with the results of Theorem 3.1 for an intersecting point at $E_1 \cap E_2$.

For any other points p on the exceptional component E_i, we must have

$$I_p/mI_p \cong \rho_i \oplus \rho_0. \qquad (*)$$

In fact, we can see that on the exceptional divisor E_2 in this example was determined by the ratio $x^2 : y^3$, that is, the corresponing ideal of a point on E_2 can be described as $I_p = (\alpha x^2 - \beta y^3, xy^2 - \gamma)$. Therefore the ratio $(\alpha : \beta)$ gives the coordinate of the exceptional curve ($\cong \mathbb{P}^1$) and we also have $(*)$.

References

[1] M. Artin and J.L. Verdier, *Reflexive modules over rational double points*, Math. Ann. **270** (1985), 79–82.

[2] E. Esnault, *Reflexive modules on quotient surface singularities*, J. Reine Angew. Math. **362** (1985), 63–71.

[3] E. Esnault and H. Knörrer, *Reflexive modules over rational double points*, Math. Ann. **272** (1985), 545–548.

[4] G. Gonzalez-Sprinberg and J.L. Verdier, *Construction géometrique de la correspondance de McKay*, Ann. Sci. École Norm. Sup. **16** (1983), 409–449.

[5] A. Ishii, *On McKay correspondence for a finite small subgroup of $GL(2,\mathbb{C})$*, Journal fur die reine und angewandte Mathematik **549** (2002), 221–233.

[6] Y. Ito, *Special McKay correspondence*, Séminaires et Congrés **6**, SMF (2002), 213–225.

[7] Y. Ito and I. Nakamura, *Hilbert schemes and simple singularities*, in: New trends in Algebraic Geometry (Warwick, June 1996), K. Hulek and others Eds., CUP (1999), 151–233.

[8] M. Haiman, *Hilbert schemes, polygraphs, and the Macdonald positivity conjecture* to appear in J. Amer. Math. Soc.

[9] R. Kidoh, *Hilbert schemes and cyclic quotient singularities*, Hokkaido Mathematical Journal, **30** (2001), 91–103.

[10] J. McKay, *Graphs, singularities and finite groups*, Proc. Symp. Pure Math., **37** (1980) Amer. Math. Soc. 183–186.

[11] O. Riemenschneider, *Characterization and application of special reflexive modules on rational surface singularities*, Institut Mittag-Leffler Report No.3 (1987).

[12] O. Riemenschneider, *On the two dimensional McKay correspondence*, Hamburger Beiträge zur Mathematik aus dem Mathematischen Seminar, Heft 94. Hamburg (2000).

[13] J. Wunram, *Reflexive modules on cyclic quotient surface singularities*, in Singularities, Representations of Algebras, adn Vector Bundles. Greuel, Trautmann (eds.) Lect. Notes math. 1273. Springer (1987).

[14] J. Wunram, *Reflexive modules on quotient surface singularities*, Math Ann. **279** (1988), 583–598.

Graduate school of Mathematics, Nagoya University, Furo-cho, Nagoya 464-8602, Japan

E-mail address: `y-ito@math.nagoya-u.ac.jp`

ALGEBRAS OF FINITE GLOBAL DIMENSION AND SPECIAL COHEN-MACAULAY MODULES

OSAMU IYAMA

In my note I will present some results in joint work [IW] with Michael Wemyss on special Cohen-Macaulay modules. We start with explaining briefly the background in noncommutative algebra. After Auslander [A], algebras of finite global dimension are one of the most important subjects in representation theory. For example, famous Auslander-Reiten theory [Y] is based on certain algebras of global dimension two, called *Auslander algebras* (e.g. see [I]). We will explain the connection to special Cohen-Macaulay modules. I recommend anyone who is interested in non-commutative algebra to learn the work of Auslander (especially [A], which is available in [A2]). Additionally, my recent trial [I] to extend this a little bit.

1. Introduction

Λ any ring, M an arbitrary Λ module. We write pd$M \le n$ if there exists

$$0 \to P_n \to \dots P_0 \to M \to 0$$

such that each P_i is a projective Λ module. We define $\text{gl.dim}\Lambda := \sup\{\text{pd}M : M \in \text{mod}\Lambda\}$. In this note we study rings with finite global dimension.

Example 1.1. (1) Λ is a commutative complete local k-algebra where k is algebraically closed. Cohen's structure theorem says that gl.dimΛ is finite if and only if Λ is power series ring.

(2) Λ finite dimensional k-algebra. Then by Artin-Wedderburn the global dimension of Λ is 0 if and only if Λ is product of matrix rings over k. The global dimension is ≤ 1 if and only if Λ is morita equivalent to the path algebra of some quiver Q.

There are quite a lot of finite dimensional algebra Λ with global dimension 2, which aren't so nice! For instance

$$\bullet \overset{\cdot\cdot}{\rightrightarrows} \bullet \overset{\cdot\cdot}{\rightrightarrows} \bullet$$

divide by arbitrary relations.

2. The setup

Fix R a commutative ring, complete local so that we get Krull-Schmidt in the category of modules. Denote by d the Krull dimension of R.

Question 2.1. (Auslander [A], 1971): (in modified form) Is there an R-algebra Λ such that

(i) global dimension of Λ is finite,
(ii) the center of Λ is R,
(iii) Λ is a finitely generated R-module.

The last two conditions show the relationship between R and Λ. The idea behind (iii) is that Λ is not much bigger than R.

Question 2.2. (Auslander [A]): Is there $M \in \text{mod}R$ such that the global dimension of $\text{End}_R(R \oplus M)$ is finite.

A positive answer to question 2 implies a positive answer to question 1.

In the rest of this note we will discuss this question in a more explicit setting. Note that question 2 is studied in several contexts recently.

(i) Auslander's *representation dimension* [A], which is defined by

$$\inf\{\text{gl.dim}\text{End}_R(R \oplus X \oplus \omega) : X \in \text{CM}(R)\} := \text{rep.dim}R$$

for the canonical module ω. The above questions are variants of this.

(ii) Cluster tilting theory as three dimensional Auslander-Reiten theory. See survey [I].

(iii) Noncommutative resolution of singularities [V].

But in (i) and (ii) these both impose more assumptions on the module M in Q2. The idea in (iii) is to construct M using the usual resolution of R and vice versa.

3. Three results

Theorem 3.1 (Auslander [A])**.** *If $d = 0$ (i.e. an Artin ring), then Q2 is true.*

In fact he showed something much stronger, he doesn't assume commutativity. In this case the module M was constructed explicitly as follows: denote by J_R the Jacobson radical of R. Since Artinian there exists m such that $J_R^m = 0$. Have

$$R = R/J_R^m \twoheadrightarrow R/J_R^{m-1} \twoheadrightarrow \ldots \twoheadrightarrow R/J_R \twoheadrightarrow 0$$

each surjective. Sum them up and take the endomorphism ring.

Theorem 3.2 (König [K], 1991)**.** *If $d = 1$, R is a domain (or reduced, but something similar). Then Q2 is also true.*

In this case let K be the quotient field of R. It is itself infinitely generated, but let $R_0 = R$ and define inductively $R_{i+1} = \{x \in K : xJ_{R_i} \subset J_{R_i}\}$. Get a chain

$$R = R_0 \subset R_1 \subset \ldots \subset R_m$$

which has to stop at the normalization of R, which is R_m (this is where we use the domain bit). Again take the sum of the R_i gives the required module

Want the third case $d = 2$. To do this want to introduce a nice class of modules, the CM modules. Keeping the assumptions on R being commutative complete local,

Definition 3.3. *Let $X \in \text{mod}R$. Define the depth of X to be the maximal length of X-regular sequences. Homolgically, this is equal to*

$$\inf\{i \geq 0 : \text{Ext}^i_R(R/J_R, X) \neq 0\}.$$

We call X a Cohen-Macaulay (CM) R-module if $\text{depth}X = d$

The larger the depth, the nicer the module. We denote $\text{CM}(R)$ the category of CM modules. Some general properties:

(i) $\text{CM}(R)$ is closed under extensions in $\text{mod}R$.

(ii) $\text{CM}(R)$ is closed under kernels of epimorphisms.

(iii) Auslander-Buchsbaum formula: if $\text{pd}X$ is finite then

$$\text{depth}X + \text{pd}X = \text{depth}R.$$

Consequently if R is regular (i.e. global dimension is finite) then $\text{CM}(R)$ is just the projective modules.

This leads to

Question 3.4. Define the *right representation dimension* by

$$\inf\{\text{gl.dim}\text{End}_R(R \oplus X) : X \in \text{CM}(R)\} := \text{r.rep.dim}R.$$

What can we say about $\text{r.rep.dim}R$?

Definition 3.5. *We call R* finite CM type *if there are only finitely many isomorphism classes of indecomposable CM R-modules.*

Example 3.6. (i) So-called simple singularity. A_n is defined as $k[[x_0, x_1, \ldots, x_d]]/f$ where $f = x_0^{n+1} + x_1^2 + x_2^2 + \ldots + x_d^2$. To see this by Knorrer periodicity reduce to either $d = 1$ (easy but a computation) or $d = 2$, which we deal with later. There are also types D_n, E_6, E_7 and E_8.

(ii) 2-dimensional quotient singularities. Take a finite subgroup $G \leq \mathrm{GL}(2, k)$ and consider $R = k[[x, y]]^G$. This has finite CM type. The proof is quite amusing - just show that $\mathrm{CM}(R) = \mathrm{add} k[[x, y]]$.

Now want to go back to Q3: want some nice CM module such that the endomorphism ring has finite global dimension.

Theorem 3.7 (Auslander [A], 1986). *Suppose R is a CM ring of finite CM type. Sum them altogether and take the endomorphism ring. Then this has finite global dimension, and in fact its $\leq max\{2, d\}$.*

4. Krull dimension 2

In the rest of my note, assume the following:

(i) R is still complete local noetherian,
(ii) R is normal domain,
(iii) $d = 2$.

In this setting we have the following:

Remark 4.1. X is CM if and only if it is reflexive i.e. the natural map $X \to X^{**}$ is an isomorphism.

Consequently the functor $*$ is a duality on the category $\mathrm{CM}(R)$.
The key lemma is the following:

Lemma 4.2 (essentially Auslander [A], see also [IW]). *Assume $M \in \mathrm{CM}(R)$ is a generator (i.e. has R as a summand). Then the following two conditions are equivalent, for any $n \geq 0$.*

(1) $\mathrm{gl.dimEnd}_R(M) \leq n + 2$.
(2) *for any $X \in \mathrm{CM}(R)$ there is an exact sequence*

$$0 \to M_n \to \ldots \to M_0 \to X \to 0$$

with each $M_i \in \mathrm{add}M$ such that

$$0 \to (M, M_n) \to \ldots \to (M, M_0) \to (M, X) \to 0$$

is exact.

Proof. Denote $\Lambda = \mathrm{End}_R(M)$. Firstly note that we have an equivalence $(M, -) : \mathrm{add}_R M \to \mathrm{add}_\Lambda \Lambda$.

(2)⇒(1) For all $Y \in \mathrm{mod}\Lambda$ take the first two terms

$$P_1 \to P_0 \to Y \to 0$$

in the projective resolution. We can write the projectives as $P_1 = (M, M_1)$ and $P_0 = (M, M_0)$ and the map $P_1 \to P_0$ comes from $M_1 \to M_0$. Taking the kernel of this map (its CM by the depth lemma) and using (2) we get

$$0 \to M_{n+2} \to \ldots \to M_3 \to M_2 \to M_1 \to M_0$$

such that

$$0 \to (M, M_{n+2}) \to \ldots \to (M, M_3) \to (M, M_2) \to (M, M_1) \to (M, M_0)$$

is exact. But this gives projective resolution of X.
(1)⇒(2) is similar, and uses the fact its a generator. □

Now apply this to a special case

Corollary 4.3 (essentially Auslander [A]). *Let* $M \in \mathrm{CM}(R)$ *generator. Then* $\mathrm{add}M = \mathrm{CM}(R)$ *if and only if* $\mathrm{gl.dimEnd}_R(M) = 2$.

Proof. Apply key lemma to $n = 0$ to get $\mathrm{gl.dimEnd}_R(M) \leq 2$. An easy argument shows that it is in fact equality. □

Remark 4.4.
(1) From the above: $\mathrm{r.rep.dim}R \leq 2$ if and only if R has finite CM type.
(2) Most R is not finite CM type! For $d = 2$ only the quotient singularities are finite CM type.

Thus just summing all the CM isn't going to work, need some new idea. This comes from the special CM modules

Definition 4.5 (Wunram [W]). *We call* $M \in \mathrm{CM}(R)$ *special if and only if* $M \otimes_R \omega / T(M \otimes \omega)$ *is CM, where for* $X \in \mathrm{mod}R$ *define*

$$TX := \{x \in X : \exists 0 \neq r \in R, rx = 0\}.$$

Note that by definition the depth of $M \otimes_R \omega / T(M \otimes \omega)$ is always ≥ 1; the specials are those with depth 2. The problem with the definition is that it is hard to handle. Can show the following:

Lemma 4.6 ([IW]). *For* $M \in \mathrm{CM}(R)$ *the following are equivalent.*

(1) M *is special.*
(2) $\mathrm{Ext}^1_R(M, R) = 0$.
(3) $M^* \in \Omega\mathrm{CM}(R)$*, where* $\Omega\mathrm{CM}(R)$ *is the category of first syzygies of CM modules.*

Remark: denote by $\mathrm{SCM}(R)$ the category of special CM modules. The duality $*$ on the level of $\mathrm{CM}(R)$ induces a duality between $\Omega\mathrm{CM}(R)$ and $\mathrm{SCM}(R)$.

Definition 4.7. *Call* R finite SCM type *if there are only finitely many indecomposable special CM modules.*

Could also state the definition on the level of first syzygies.

Theorem 4.8 (Wunram [W]). *All rational normal surface singularities have finite SCM type.*

Converse holds? Remark: Wunram gives a 1-1 correspondence (for rational singularities) between the exceptional curves on the minimal resolution of $\mathrm{Spec}R$ and the indecomposable non-free special CM modules.

Theorem 4.9 ([IW]). *Let* M *be the sum of all the indecomposable modules in* $\Omega\mathrm{CM}(R)$*, then*

$$\mathrm{gl.dimEnd}_R(M) = \begin{cases} 2 & R\text{Gorenstein} \\ 3 & \textit{else} \end{cases}$$

The idea is quite simple: use $n = 1$ case in the previous key lemma. Thus we need to show that for all $X \in \mathrm{CM}(R)$ there exists

$$0 \to M_1 \to M_0 \to X \to 0$$

such that

$$0 \to (M, M_1) \to (M, M_0) \to (M, X) \to 0$$

is exact. The map $M_0 \to X$ is constructed by taking generators of the module (M, X). These give a map $M_0 = M^n \to X$. Applying $(M, -)$ gives a surjective map, thus the map itself has to be a surjection since M is a generator. Since M_0 is a first syzygy of a CM module, the kernel of the map is also a first syzygy of a CM module, so we are done.

References

[A] M. Auslander, Representation dimension of Artin algebras, Queen Mary Colledge, London, 1971.

[A2] M. Auslander, Selected works of Maurice Auslander. Part 1,2. Edited and with a foreword by Idun Reiten, Sverre O. Smalo, and Oyvind Solberg. American Mathematical Society, Providence, RI, 1999

[I] O. Iyama, Auslander-Reiten theory revisited, Trends in Representation Theory of Algebras and Related Topics, 349–398, European Mathematical Society, 2008. See also arXiv:0803.2841.

[IW] O. Iyama, M. Wemyss, The Classification of Special Cohen-Macaulay Modules, arXiv:0809.1958, to appear in Math. Z.

[K] S. König, Every order is the endomorphism ring of a projective module over a quasi-hereditary order. Comm. Algebra 19 (1991), no. 8, 2395–2401.

[V] M. Van den Bergh, Three-dimensional flops and noncommutative rings. Duke Math. J. 122 (2004), no. 3, 423–455.

[W] J. Wunram, Reflexive modules on quotient surface singularities. Math. Ann. 279 (1988), no. 4, 583–598.

[Y] Y. Yoshino, Cohen-Macaulay modules over Cohen-Macaulay rings. London Mathematical Society Lecture Note Series, 146. Cambridge University Press, Cambridge, 1990.

名古屋大学数学レクチャーノート

No. 1	Selberg 理論入門 久保田富雄	1966 年	50 頁
No. 2	Prolongations of Geometric Structures Akihiko Morimoto	1969 年	193 頁
No. 3	Complex White Noise and Infinite Dimensional Unitary Group Takeyuki Hida	1971 年	53 頁
No. 4	On Some Questions of the Theory of Probability Measures on Banach Spaces N. Vakhania	1973 年	17 頁
No. 5	The Estimation of the Rate of Convergence in the Central Limit Theorem V.V. Sazonov	1975 年	38 頁
No. 6	２階偏微分方程式の幾何学的理論 — Darboux-Goursat 理論の現代的入門 — 土屋昭博（青本和彦記）	1980 年	202 頁
No. 7	Finely Harmonic Functions B. Fuglede	1983 年	124 頁
No. 8	数の幾何学と類体論 (Geometry of numbers and class field theory) 久保田富雄	1985 年	41 頁
No. 9	Gauge 場についてのある考察 —非 Abel Poincaré Lemma— 浅田　明	1985 年	29 頁
No.10	古典２次形式に関する考察 久保田富雄	1992 年	37 頁
No.11	Infinite Dimensional Stochastic Analysis Hui-Hsiung Kuo	1993 年	39 頁

多 元 数 理 講 義 録

Vol. 1	Dedekind 和に関連した話題 —L-関数の特殊値との関係を中心に— 伊藤　博	1998 年	24 頁
Vol. 2	代数的確率論入門 —独立性の諸概念— アカルディ・ルイジ、尾畑伸明	1999 年	193 頁
Vol. 3	Singular Hermitian Metrics and Algebraic Geometry 辻　元 (Appendix —大沢健夫)	2001 年	158 頁
Vol. 4	多変数複素解析三講 多重複素 Green 関数から導入される擬計量 東川和夫 多様体上の Levi 問題—総説と最近の結果— 大沢健夫 $\bar{\partial}$-Neumann 問題 大沢健夫	2002 年	54 頁
Vol. 5	Vortices and Seiberg-Witten Equations (based on lectures at Nagoya University) Armen G. Sergeev (notes taken by Yuuji Tanaka)	2002 年	87 頁
Vol. 6	代数幾何学勉強会講義録	2006 年	84 頁
Vol. 7	Kähler Geometry of Loop Spaces	2009 年	226 頁
Vol. 8	Singularity Seminar 2008 organized by Yukari Ito and Osamu Iyama	2009 年	85 頁

（以下続刊）

Lecture Notes in Mathematics, Nagoya University

No. 1	Introduction to Selberg Theory (in Japanese) T. Kubota	1966	50 pages
No. 2	Prolongations of Geometric Structures A. Morimoto	1969	193 pages
No. 3	Complex White Noise and Infinite Dimensional Unitary Group T. Hida	1971	53 pages
No. 4	On Some Questions of the Theory of Probability Measures on Banach Spaces N. Vakhania	1973	17 pages
No. 5	The Estimation of the Rate of Convergence in the Central Limit Theorem V.V. Sazonov	1975	38 pages
No. 6	Geometric Theory on Partial Differential Equations of Second Order (in Japanese) — A New Introduction to Darboux-Goursat Theory — A. Tsuchiya (Notes by K. Aomoto)	1980	202 pages
No. 7	Finely Harmonic Functions B. Fuglede	1983	124 pages
No. 8	Geometry of Numbers and Class Field Theory (in Japanese) T. Kubota	1985	41 pages
No. 9	An Essay on Gauge Field (in Japanese) —Non-Abelian Poincaré Lemma — A. Asada	1985	29 pages
No.10	An Essay on Quadratic Forms of Classical Type (in Japanese) T. Kubota	1992	37 pages
No.11	Infinite Dimensional Stochastic Analysis Hui-Hsiung Kuo	1993	39 pages

Graduate School Lectures in Mathematics, Nagoya University

Vol.	1	Topics on Dedekind Sums (in Japanese) —Relations to Special Values of L-functions— H. Ito	1998	24 pages
Vol.	2	Introduction to Algebraic Probability Theory (in Japanese) —Concepts of Independence— Luigi Accardi and Nobuaki Obata	1999	193 pages
Vol.	3	Singular Hermitian Metrics and Algebraic Geometry Hajimi Tsuji (Appendix—Takeo Ohsawa)	2001	158 pages
Vol.	4	Three Lectures on Several Complex Variables (in Japanese) A Pseudometric Induced by Pluricomplex Green Function Kazuo Azukawa Levi Problem on Manifolds —A Survey and Recent Results— Takeo Ohsawa $\bar{\partial}$-Neumann Problem Takeo Ohsawa	2002	54 pages
Vol.	5	Vortices and Seiberg-Witten Equations (based on lectures at Nagoya University) Armen G. Sergeev (notes taken by Yuuji Tanaka)	2002	87 pages
Vol.	6	Lectures on Algebraic Geometry	2006	84 pages
Vol.	7	Kähler Geometry of Loop Spaces Armen G. Sergeev	2009	226 pages
Vol.	8	Singularity Seminar 2008 organized by Yukari Ito and Osamu Iyama	2009	85 pages

(to be continued)