LECTURES ON RECONSTRUCTION ALGEBRAS III

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1. Introduction

Last lecture I gave lots of geometric motivation behind the idea of a reconstruction algebra. I said that instead of viewing the minimal resolution $\widetilde{X}$ of a quotient singularity $\mathbb{C}^2/G$ as G-Hilb (which a priori has nothing to do with other resolutions) we should instead view $\widetilde{X}$ as being very similar to a space we already understand. The reconstruction algebra encodes the difference. There are two main problems with these statements:

(1) we don't yet know what space to compare $\widetilde{X}$ too!
(2) we haven't defined the reconstruction algebra yet.

In fact it turns out, after we define the reconstruction algebra, that its underlying quiver tells us the answer to (1).

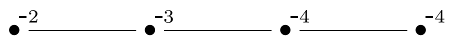
So today I'm going to lead up to the definition of the reconstruction algebra, and consequently I'm going to have to change perspective slightly and become more algebraic. I'll try and give some motivation from the world of commutative ring theory (=CM modules here) and also from representation theory.

First though I'll stay geometrical and illustrate the slogan I stated last time.

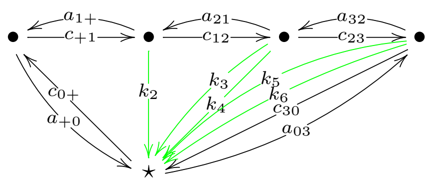
2. Non-toric Toric Geometry

Last time I made the somewhat counter-intuitive statement that for most non-abelian finite subgroups $G \leq \mathrm{GL}(2,\mathbb{C})$ (namely those with reduced fundamental cycle), to resolve the singularity explicitly is the same level of difficulty as toric geometry.

Example 2.1. I'm going to start by computing the geometry in a toric example. This also illustrates the pattern in the reconstruction relations. Consider the group $\frac{1}{67}(1,41)$. The continued fraction expansion $\frac{67}{41} = [2,3,4,4]$ and so the dual graph of the minimal resolution of $\mathbb{C}^2/\frac{1}{67}(1,41)$ is

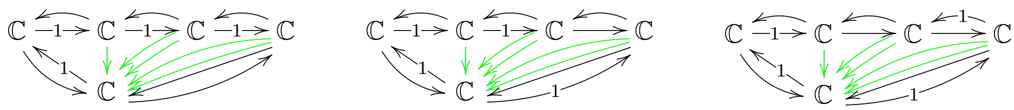


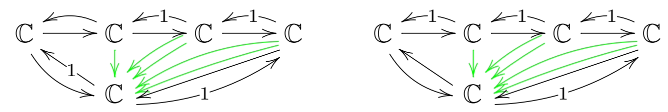
The reconstruction algebra of Type A in this example is



$$a_{+0}c_{0+} = c_{+1}a_{1+}$$
$$a_{1+}c_{+1} = k_2a_{03}a_{32}a_{21} \qquad a_{03}a_{32}a_{21}k_2 = c_{0+}a_{+0}$$
$$k_2c_{0+}c_{+1} = c_{12}a_{21}$$
$$a_{21}c_{12} = k_3a_{03}a_{32} \qquad a_{03}a_{32}k_3 = c_{0+}c_{+1}k_2$$
$$k_3c_{0+}c_{+1}c_{12} = k_4a_{03}a_{32} \qquad c_{0+}c_{+1}c_{12}k_3 = a_{03}a_{32}k_4$$
$$k_4c_{0+}c_{+1}c_{12} = c_{23}a_{32}$$
$$a_{32}c_{23} = k_5a_{03} \qquad a_{03}k_5 = c_{0+}c_{+1}c_{12}k_4$$
$$k_5c_{0+}c_{+1}c_{12}c_{23} = k_6a_{03} \qquad c_{0+}c_{+1}c_{12}c_{23}k_5 = a_{03}k_6$$
$$k_6c_{0+}c_{+1}c_{12}c_{23} = c_{30}a_{03} \qquad c_{0+}c_{+1}c_{12}c_{23}k_6 = a_{03}c_{30}$$

where the bizarre labeling will soon become clear. Pick dimension vector $(1,1,1,1,1)$ and stability condition $(-4,1,1,1,1)$, where the -4 is in the position of $\star$. We are used to this now - it means that a module M of dimension vector $(1,1,1,1,1)$ is θ-stable if and only if for every vertex in the quiver there is a non-zero path from $\star$ to that vertex. Consequently we have the following five open sets:



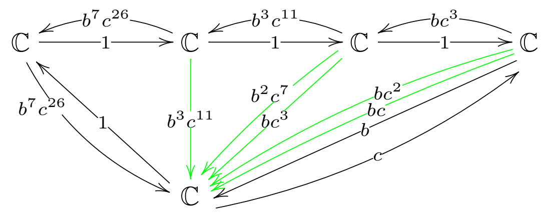


This is why we choose the stability $(-4,1,1,1,1)$, since it is 'blind' to the extra green arrows. Notice no matter how many extra green arrows we add to the above quiver, picking the dimension and stability as above the moduli is still covered by five open sets. Call the open set in the top left U_0. I shall now show that $U_0 \cong \mathbb{C}^2$, i.e. the open set U_0 is parameterized by two variables b and c subject to no relation.

Place b in the position of c_{30} and c in the position of a_{03}. I claim that every other arrow is determined by these. Well

$$\begin{aligned} k_6 c_{0+}c_{+1}c_{12}c_{23} = c_{30}a_{03} \quad &\Rightarrow \quad k_6 = bc \\ k_5 c_{0+}c_{+1}c_{12}c_{23} = k_6 a_{03} \quad &\Rightarrow \quad k_5 = k_6 c = bc^2 \\ a_{32}c_{23} = k_5 a_{03} \quad &\Rightarrow \quad a_{32} = k_5 c = bc^3 \\ &\vdots \end{aligned}$$

Continuing in this fashion (it is best done visually; I will explain this in the lecture), we get



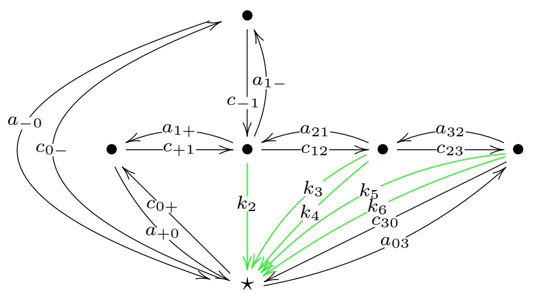
and so indeed this open set is just $\mathbb{C}^2$. Now the next open set U_1 is also just $\mathbb{C}^2$, and is parameterized by the variables in the c_{23} and a_{32} positions. By changing basis at the vertex 3 it immediate from the above picture (multiply all arrows out of vertex 3 by c, divide all arrows into vertex 3 by c) that the glue is

$$U_0 \ni (b,c) \leftrightarrow (b^{-1}, bc^4) \in U_1$$

Example 2.2. We are now going to explicitly resolve the singularity $\mathbb{C}^2/\mathbb{D}_{56,15}$, where

$$\mathbb{D}_{56,15} := \left\langle \begin{pmatrix} \varepsilon_{30} & 0 \\ 0 & \varepsilon_{30}^{-1} \end{pmatrix}, \begin{pmatrix} 0 & \varepsilon_4 \\ \varepsilon_4 & 0 \end{pmatrix}, \begin{pmatrix} \varepsilon_{82} & 0 \\ 0 & \varepsilon_{82} \end{pmatrix} \right\rangle$$

is a non-abelian group of order 2460. I claim this is really easy, once you know the reconstruction algebra. In this case it is

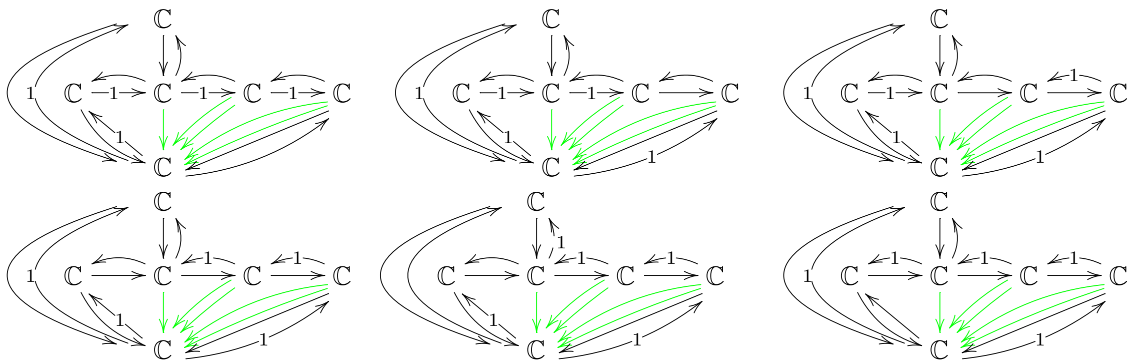


$$\begin{gathered} c_{0+}c_{+1} - c_{0-}c_{-1} = 4A_{01} \\ c_{0+}a_{+0} = c_{0-}a_{-0} \\ a_{-0}c_{0-} = c_{-1}a_{1-} \\ a_{1+}c_{+1} = a_{1-}c_{-1} \\ \hline a_{+0}c_{0+} = c_{+1}a_{1+} \\ a_{1+}c_{+1} = k_2 a_{03}a_{32}a_{21} \qquad a_{03}a_{32}a_{21}k_2 = c_{0+}a_{+0} \\ k_2 c_{0+}c_{+1} = c_{12}a_{21} \\ a_{21}c_{12} = k_3 a_{03}a_{32} \qquad a_{03}a_{32}k_3 = c_{0+}c_{+1}k_2 \\ k_3 c_{0+}c_{+1}c_{12} = k_4 a_{03}a_{32} \qquad c_{0+}c_{+1}c_{12}k_3 = a_{03}a_{32}k_4 \\ k_4 c_{0+}c_{+1}c_{12} = c_{23}a_{32} \\ a_{32}c_{23} = k_5 a_{03} \qquad a_{03}k_5 = c_{0+}c_{+1}c_{12}k_4 \\ k_5 c_{0+}c_{+1}c_{12}c_{23} = k_6 a_{03} \qquad c_{0+}c_{+1}c_{12}c_{23}k_5 = a_{03}k_6 \\ k_6 c_{0+}c_{+1}c_{12}c_{23} = c_{30}a_{03} \qquad c_{0+}c_{+1}c_{12}c_{23}k_6 = a_{03}c_{30} \end{gathered}$$

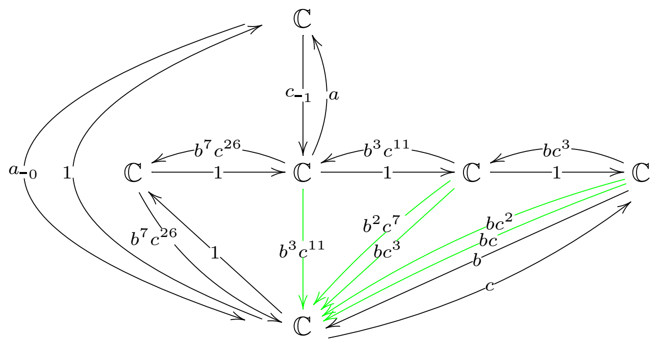
Note that the relations below the horizontal line are exactly the same as for the toric example we did earlier. Choose dimension vector $(1,1,1,1,1,1)$ and stability $(-5,1,1,1,1,1)$ where the -5 corresponds to the position $\star$. Its not too hard to see[1] that the moduli is covered by

[1]If you get stuck, use the only non-monomial relation. If you're still stuck, try using the non-monomial relation again.

the following six open sets



Denote these by U_0, U_1, U_2, U_3, U_+ and U_- respectively. Lets look at U_0. Setting $a = a_{1-}$, $b = c_{30}$ and $c = a_{03}$ then using *exactly* the same calculation as the toric example earlier, U_0 looks like



now subject to the 4 new relations above the horizontal line. But these give that $a_{-0} = b^7c^{26}$, $c_{-1} = 1 - 4b^4c^{15}$ and $a(1 - 4b^4c^{15}) = b^7c^{26}$. Thus our open set is $\mathbb{C}^3_{a,b,c}$ subject to the one equation $a(1 - 4b^4c^{15}) = b^7c^{26}$. Note that basically everything in this calculation is the same as the toric case, except the one non-monomial relation ends up giving us a hypersurface in $\mathbb{C}^3$.

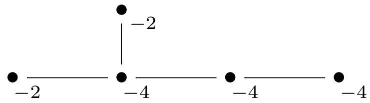
The other open sets are done similarly, and all follow very quickly from the toric case. We find that we can read off the co-ordinates in the following positions (I've also stated their abstract equations):

$$\begin{array}{ccc}
U_0 & (a_{1-}, c_{30}, a_{03}) & a(1 - 4b^4c^{15}) = b^7c^{26} \\
U_1 & (a_{1-}, c_{23}, a_{32}) & a(1 - 4bc^4) = b^2c^7 \\
U_2 & (a_{1-}, c_{12}, a_{21}) & a(1 - 4c) = bc^2 \\
U_3 & (a_{1-}, a_{1+}, c_{+1}) & a(c - 4) = bc \\
U_+ & (c_{0-}, a_{1+}, a_{-0}) & b(a^2c + 4) = ac \\
U_- & (c_{0+}, a_{1-}, a_{+0}) & b(a^2c - 4) = ac
\end{array}$$

Actually there is a choice of coordinate in U_3 above since we can pick the position c_{-1} instead of c_{+1}; denoting d as this new third coordinate changes the abstract equation to a, b, d subject to $ad = b(4 - d)$. With respect to the above ordering, the gluing of these open sets is:

$$\begin{array}{ccc}
U_0 \ni (a, b, c) & \leftrightarrow & (a, c^{-1}, c^4b) \in U_1 \\
U_1 \ni (a, b, c) & \leftrightarrow & (a, c^{-1}, c^4b) \in U_2 \\
U_2 \ni (a, b, c) & \leftrightarrow & (ca, c^3b, c^{-1}) \in U_3 \\
U_3 \ni (a, b, d) & \leftrightarrow & (a^{-1}, b, a^2d) \in U_+ \\
U_3 \ni (a, b, c) & \leftrightarrow & (b^{-1}, a, b^2c) \in U_-
\end{array}$$

The dual graph in this example is



3. The $SL(2)$ McKay Correspondence: Preliminaries

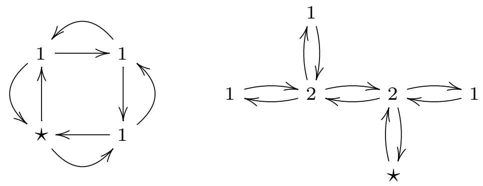
The last section was very geometrical; I'll now come back and motivate the algebraic side. If we take a finite subgroup G of $\mathrm{GL}(2,\mathbb{C})$ we would like to use the representation theory of G to resolve the singularity $\mathbb{C}^2/G$. In this section I'll introduce the notions needed to explain the classical McKay correspondence (i.e. when $G \leq \mathrm{SL}(2,\mathbb{C})$), but I'll define everything when $G \leq GL(2,\mathbb{C})$.

The geometry of $\mathbb{C}^2/G$ is a function of two variables, the group G and the natural representation $V = \mathbb{C}^2$. Changing either may change the geometry. Consequently the representation theory by itself will tell us nothing about the geometry (since it is only a function of one variable, namely the group G), so we have to enrich the representations with the action of G on V. We will do this in three ways: the first is as follows

Definition 3.1. *For given finite G acting on $\mathbb{C}^2 = V$, the McKay quiver is defined to be the quiver with vertices corresponding to the isomorphism classes of indecomposable representations, and the number of arrows from ρ_1 to ρ_2 is defined to be*

$$\dim_{\mathbb{C}}\mathrm{Hom}_{\mathbb{C}G}(\rho_1, \rho_2 \otimes V)$$

Example 3.2. For the groups $\frac{1}{4}(1,3)$ and $BD_{4\cdot 3}$ inside $\mathrm{SL}(2,\mathbb{C})$ the McKay quivers are



respectively, where the number on a vertex is the dimension of the representation at that vertex.

Beware that sometimes the McKay quiver is defined with the arrows reversed, i.e. the number of arrows from ρ_1 to ρ_2 is $\dim_{\mathbb{C}}\mathrm{Hom}_{\mathbb{C}G}(\rho_2, \rho_1 \otimes V)$. This is just a convention, so it doesn't really matter.

The second way we are going to encode the geometry into the representation theory is to treat every representation as a semi-invariant, and take the corresponding endomorphism ring:

Definition 3.3. *For a representation ρ, form $\rho \otimes_{\mathbb{C}} \mathbb{C}[x,y]$. Now G acts on both sides of the tensor, so we can form $(\rho \otimes \mathbb{C}[x,y])^G$, which is a CM module over the invariant ring $\mathbb{C}[x,y]^G = (\rho_0 \otimes \mathbb{C}[x,y])^G$ where ρ_0 is the trivial representation. We denote $S_\rho := (\rho \otimes \mathbb{C}[x,y])^G$ and call it the CM module associated to ρ. Denote $A := \mathrm{End}_{\mathbb{C}[x,y]^G}\left(\oplus_{\rho \in \mathrm{Irr} G}(\rho \otimes \mathbb{C}[x,y])^G\right)$*

In fact the above gives a 1-1 correspondence between the representations and the CM modules. You should perhaps view the CM modules as being 'better' than the representations, since they generalize to the non-quotient singularity case.

The third way to encode the geometry into the representation theory is done as follows:

Definition 3.4. *Define the skew group ring $\mathbb{C}[x,y]\#G$ to be the vector space $\mathbb{C}[x,y] \otimes_{\mathbb{C}} \mathbb{C}G$ with multiplication given by*

$$(f_1 \otimes g_1) \cdot (f_2 \otimes g_2) := (f_1(g_1 \cdot f_2)) \otimes g_1 g_2$$

You should view this as the algebra put together from $\mathbb{C}G$ and $\mathbb{C}[x,y]$ in a natural way, and it is the twist in the multiplication which is encoding the action of G on V. Note that a $\mathbb{C}[x,y]\#G$ module is exactly the same thing as $\mathbb{C}[x,y]$ module M (=coherent sheaf on $\mathbb{C}^2$) with a compatible G-action; i.e. a G-action such that

$$g(f \cdot m) = g(f) \cdot g(m) \quad \text{for all } f \in \mathbb{C}[x,y], g \in G, m \in M$$

Thus we can think of modules for $\mathbb{C}[x,y]\#G$ a little more geometrically as G-equivariant sheaves on $\mathbb{C}^2$.

The following theorem due to Auslander tells us that our two naturally defined algebras give us the same answer:

Theorem 3.5. *If $G \leq GL(2,\mathbb{C})$ is small (i.e. contains no pseudoreflections) then*

$$\mathbb{C}[x,y]\#G \cong \mathrm{End}_{\mathbb{C}[x,y]^G}\left(\oplus_{\rho\in\mathrm{Irr}G} S_\rho^{\oplus \dim\rho}\right).$$

Consequently (killing multiplicity) $A = \mathrm{End}_{\mathbb{C}[x,y]^G}(\oplus_{\rho\in\mathrm{Irr}G} S_\rho)$ is Morita equivalent to the skew group ring $\mathbb{C}[x,y]\#G$.

Actually the three ways of encoding the geometry onto the representation theory give us the same answer:

Lemma 3.6. *The underlying quiver of $\mathbb{C}[x,y]\#G$ (and thus $A = \mathrm{End}_{\mathbb{C}[x,y]^G}(\oplus_{\rho\in\mathrm{Irr}G} S_\rho)$ when the group is small) is the McKay quiver.*

The relations on the McKay quiver that give the Morita equivalence with the skew group ring (at least in the case when G is small) are known as the mesh relations from AR theory. Perhaps more will be said about this later.

4. The $SL(2)$ McKay Correspondence

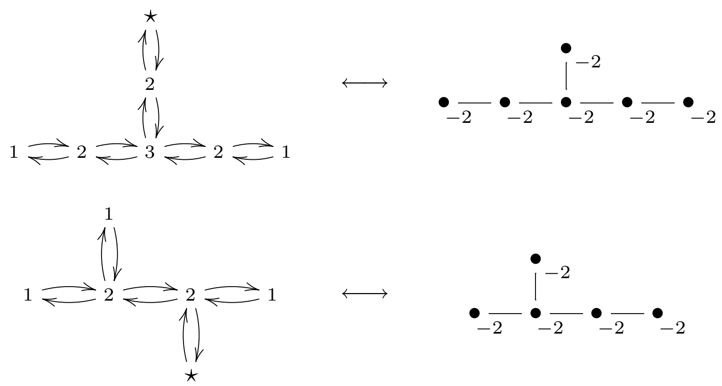
The last section introduced the algebra and notation, in this section we get to the point. Let $G \leq \mathrm{SL}(2,\mathbb{C})$ and denote by $\widetilde{X} \to \mathbb{C}^2/G$ the minimal resolution. Firstly, there is a 1-1 correspondence

$$\{\text{exceptional curves}\} \leftrightarrow \{\text{non-trivial irreducible representations}\}$$

where recall that the right hand side is in 1-1 correspondence with the non-free CM modules. I emphasize that so far this is a numerical correspondence, we want more structure. McKay observed that

$$\{\text{dual graph}\} \rightleftarrows \text{McKay quiver}$$

where we go from one side to the other by deleting (or adding) the vertex corresponding to the trivial representation. For example



If we consider an algebra instead of just a quiver (by adding relations, which are the pre-projective relations if you know about these things) we can say more. Firstly the above correspondence becomes

$$\{\text{dual graph}\} \quad \rightleftarrows \quad \text{quiver of } \mathrm{End}_{\mathbb{C}[x,y]^G}(\oplus_{\rho\in\mathrm{Irr}G} S_\rho)$$

In fact there are also statements about the derived category and quiver GIT. I will now summarize all this into one theorem. By Theorem 3.5 we can replace $\mathbb{C}[x,y]\#G$ by $\mathrm{End}_R(\oplus_{\rho\in\mathrm{Irr}G} S_\rho)$ throughout.

Theorem 4.1 (SL$(2,\mathbb{C})$ McKay Correspondence)**.** *Let G be a finite subgroup of* SL$(2,\mathbb{C})$*, denote $R=\mathbb{C}[x,y]^G$, $X=\mathrm{Spec}R$ and $\widetilde{X}\xrightarrow{\pi}X$ the minimal resolution. Then*

(i) *There is a 1-1 correspondence*

$$\{\text{exceptional curves}\} \leftrightarrow \{\text{indecomposable non-free CM modules}\}$$

(ii) *The McKay quiver gives the dual graph $\widetilde{X}$ after we delete the trivial vertex. The only possibilities are the Dynkin diagrams of type ADE.*

(iii) *The co-efficients in Z_f correspond to the dimensions of the representations associated to the vertices.*

(iv) $D^b(\mathfrak{mod}\mathbb{C}[x,y]\#G)\approx D^b(\mathfrak{coh}\widetilde{X})$.

(v) *Considering $\mathbb{C}[x,y]\#G$, take the dimension vector α given by the co-efficients in Z_f. Then for* any *generic stability condition θ,*

$$\mathfrak{M}^s_\theta \xrightarrow{\pi} X$$

is the minimal resolution.

I should add some remarks. View (i) as a numerical correspondence, to which (ii) and (iii) adds more structure. To improve this we have to add in relations (i.e. we need to consider an algebra, not just a quiver) and as soon as we do this we can start talking about modules, and so consider (iv) and (v). The derived equivalence in (iv) can be seen using either Fourier-Mukai transforms or tilting. Perhaps (v) is the strongest statement.

I should also say that via Artin-Verdier we can view this correspondence geometrically on the minimal resolution in terms of full sheaves and their Chern classes. This is important when considering Wunram's generalization later.

The above theorem fails for GL$(2,\mathbb{C})$ but I shall explain how to modify the above so that properties (i)-(iv) hold. Property (v) the way it is stated will turn out to be false even after the modification, however there will be one particular stability condition which will work.