NONCOMMUTATIVE RESOLUTION VIA FROBENIUS MORPHISMS AND D-MODULES

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This is a note of my talk at Nagoya University in November, 2008. The aim of the talk was to outline my joint work with Yukinobu Toda [6].

1. Motivation

Throughout the note, we work over an algebraically closed field k of characteristic $p > 0$.

Consider a finite group $G \subset GL_d(k)$ of order prime to p. Then the associated G-Hilbert scheme $\mathrm{Hilb}^G(\mathbb{A}^d)$ is a blowup of the quotient variety $X := \mathbb{A}^d/G$. Namely we have a natural projective and birational morphism

$$\mathrm{Hilb}^G(\mathbb{A}^d) \to X.$$

This is a restriction of the Hilbert-Chow morphism.

On the other hand, for each $e \in \mathbb{Z}_{\geq 0}$, the e-th F-blowup of X, $\mathrm{FB}_e(X)$, is defined as the universal flattening of the e-th k-linear Frobenius

$$F^e : X_e \to X.$$

It was found [9] that for sufficiently large e, $\mathrm{Hilb}^G(\mathbb{A}^d)$ and $\mathrm{FB}_e(X)$ are isomorphic. The isomorphism was constructed as follows: Indeed there is a canonical morphism

$$\mathrm{Hilb}^G(\mathbb{A}^d) \to \mathrm{FB}_e(X)$$

for any e. A point of $\mathrm{Hilb}^G(\mathbb{A}^d)$ is identified with a 0-dimensional subscheme of $\mathbb{A}^d$ of length $l := \sharp G$, write $Z \subset \mathbb{A}^d$. Then pull it back by the Frobenius $\mathbb{A}^d_e \to \mathbb{A}^d$ and obtain a subscheme $\tilde{Z} \subset \mathbb{A}^d_e$ of length lp^{de}. Then take the quotient scheme $\tilde{Z}/G$, which is a subscheme of X and corresponds to a point of $\mathrm{FB}_e(X)$. It defines the map $\mathrm{Hilb}^G(\mathbb{A}^d) \to \mathrm{FB}_e(X)$, which coincides with the morphism $\mathrm{Hilb}^G(\mathbb{A}^d) \to X$ if $e = 0$.

One of the motivations of our work is to understand the isomorphism from the viewpoint of noncommutative geometry.

Bridgeland-King-Reid [2] proved that if $G \subset SL_d(k)$, $d = 2, 3$, then the derived category of coherent sheaves on $\mathrm{Hilb}^G(\mathbb{A}^d)$ is equivalent to that of G-equivariant ones on $\mathbb{A}^d$.

$$D(\mathrm{Hilb}^G(\mathbb{A}^d)) \cong D^G(\mathbb{A}^d)$$

The equivalence is obtained as the Fourier-Mukai transform associated to the universal family in the following diagram,

$$\begin{array}{ccc} \text{Univ. fam.} & \longrightarrow & \mathbb{A}^d \\ \downarrow & & \downarrow \\ \mathrm{Hilb}^G(\mathbb{A}^d) & \longrightarrow & X \end{array}$$

But we have a similar diagram associated to the F-blowup,

$$\begin{array}{ccc} \text{Univ. fam.} & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathrm{FB}_e(X) & \longrightarrow & X \end{array}$$

The other motivation of our work is to obtain a similar result as the Bridgeland-King-Reid's one for the F-blowup.

We look at the problems from the viewpoint of the noncommutative resolution after Van den Bergh [7].

2. Morita equivalence of the skew group ring and a ring of differential operators.

Let $G \subset GL_d(k)$ be as before, $S := k[x_1, \dots, x_d]$ with the natural G-action and $R := S^G$ the ring of invariants. Thus we have $\mathbb{A}^d = \operatorname{Spec} S$ and $X = \operatorname{Spec} R$. Moreover we assume that G is small, that is, there is no reflection. Then the skew group ring $S * G$ is canonically isomorphic to the endomorphism ring $\mathrm{End}_R(S)$ of S as an R-module, and then we write $A := S * G = \mathrm{End}_R(S)$. This ring is a noncommutative resolution of R in the sense that they have finite global dimension. Using the noncommutative ring A, Van den Bergh translated the Bridgeland-King-Reid equivalence into his framework of the noncommutative resolution [7] as follows: A G-equivariant coherent sheaf on $\mathbb{A}^d$ is nothing but a (left) A-module. On the other hand, $\mathrm{Hilb}^G(\mathbb{A}^d)$ is (an irreducible component of) the moduli space of stable A-modules. Thus the equivalence is interpreted as a derived equivalence between the abelian category of A-modules and that of coherent sheaves on the moduli space of A-modules.

$$D(\text{the moduli space of } A\text{-modules}) \cong D(A).$$

Now the translated statement is applicable to a broader range of issues.

Pursuing the analogy between the G-Hilbert scheme and the F-blowup, we shall consinder the ring $D_e := \mathrm{End}_R(R^{1/q})$, $q = p^e$, in place of $A = \mathrm{End}_R(S)$.

$$\begin{array}{c} S \\ \text{Galois} \uparrow \rightsquigarrow A = \mathrm{End}_R(S) = S * G \\ R \end{array}$$

$$\begin{array}{c} R^{1/q} \\ \text{purely inseparable} \uparrow \rightsquigarrow D_e = \mathrm{End}_R(R^{1/q}) \\ R \end{array}$$

Proposition 2.1. *For sufficiently large e, A and D_e are Morita equivalent.*

Outline of the proof. The proposition follows from the fact that as R-modules, S is a direct summand of $(R^{1/q})^{\oplus l}$ for some l, and vice versa. Indeed we saw [6] that S and $R^{1/q}$ are full modules of covariants, that is, they contain as a a summand every indecomposable module of covariant. □

Since Morita equivalent rings have the same global dimension, we obtain:

Corollary 2.2. *D_e is a noncommutative resolution of R in the sense that it has a finite global dimension.*

We can see that the F-blowup is an irreducible component of the moduli space of stable D_e-modules. But for $e \gg 0$, since the categories of A-modules and D_e-modules are equivalent, we obtain isomorphic moduli spaces. Thus the isomorphism $\mathrm{Hilb}^G(\mathbb{A}^d) \cong \mathrm{FB}_e(\mathbb{A}^d/G)$ is now a direct consequence of the Morita equivalence. Moreover applying Van den Bergh's interpretation of the Bridgeland-King-Reid derived equivalence to the F-blowup and the ring D_e, we see that if $G \subset SL_d(k)$, $d = 2, 3$, then for $e \gg 0$, $\mathrm{FB}_e(X)$ and D_e are derived equivalent.

$$D(\mathrm{FB}_e(X)) \cong D(D_e)$$

Remark 2.3. The stability of modules, in fact, depends on a parameter called the stability condition. For a general stability condition, the stable $S * G$-modules are called G-constellations [3].

Remark 2.4. If G is abelian and if $q > \sharp G$, then the assertion of the proposition holds. But in the non-abelian case, we have not obtained such an effective estimate on how large e is enough.

Remark 2.5. Each element of D_e is a differential operator on $R^{1/q}$. Moreover the ring

$$\bigcup_{e \geq 0} \mathrm{End}_{R^{p^e}}(R)$$

is the ring of all differential operators on R.

In the Galois theory for purely inseparable extensions, derivations play a role of automorphisms in the Galois theory of normal extensions (see [4]). Hence it seems natural that differential operators appear instead of the group G of automorphisms.

3. D_e for some other singularities

Let now R be a Noetherian complete local domain over k. The ring $D_e := \mathrm{End}_R(R^{1/q})$ is well-defined for such R, not only in the case of quotient singularities. Therefore it is natural to ask

Problem 3.1. When is D_e a noncommutative resolution?

We have proved that the answer is affirmative in the following cases:

(1) the 1-dimensional case
(2) the singularity of type A_1 (in odd characteristic), that is, R is of the form

$$k[[x_0, x_1, \dots, x_d]]/(x_0^2 + x_1^2 + \cdots + x_d^2).$$

In the first case, for sufficiently large q, we have that $\bar{R} \subset R^{1/q}$, where $\bar{R}$ is the normalization. For such q, indeed D_e is Morita equivalent to $\bar{R}$.

In the second case, we can see that for $e > 0$, $R^{1/q}$ is a representation generator, that is, contains as a summand every indecomposable maximal Cohen-Macaulay module. Then from a theorem of Auslander [1] (see also [5]), D_e has finite global dimension.

Problem 3.2. Suppose that R has finite representation type, that is, there are only finitely many indecomposable maximal Cohen-Macaulay modules up to isomorphisms. Then for sufficiently large q, is $R^{1/q}$ a representation generator? In particular, what about the case of simple singularities?

4. F-blowup as the moduli space

We saw that in the case of quotient singularities, the F-blowup is the moduli space of D_e-modules. We can also show that this holds for F-pure singularities. A k-algebra R is F-pure if the inclusion map $R \hookrightarrow R^{1/p}$ splits as an R-module map. Then for any e, we can write $R^{1/q} \cong R \oplus M$ for some R-module M. From this decomposition and some parameters, we obtain the stability condition of D_e-modules [7].

Proposition 4.1. *Let $X := \operatorname{Spec} R$. Then $\mathrm{FB}_e(X)$ is canonically identified with the irreducible component dominating X of stable D_e-module with respect to the mentioned stability condition.*

Remark 4.2. When X is F-pure, the sequence of F-blowups is monotone, that is, $\mathrm{FB}_{e+1}(X)$ dominates $\mathrm{FB}_e(X)$ for every e (see [8]).

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