Mutations and noncommutative Donaldson-Thomas theory

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Abstract

Given a quiver with a potential, we can define counting invariants so called noncommutative Donaldson-Thomas invariants. In this note, we study how the generating function of the invariants changes under mutations of the quiver.

1 Quiver with potentials

Let $Q = (I, H)$ be a quiver and ω be a potential which is homogeneous with respect to a degree $H \to \mathbb{Z}_{>0}$. Assume that $A = (Q, \omega)$ is 3-dimensional Calabi-Yau in the sense of Bocklandt [Boc08]. In this section we give some examples of such quivers.

1.1 Conifold

Let $Y_{(-1,-1)} = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ be the resolved conifold and $\pi \colon Y_{(-1,-1)} \to \mathbb{P}^1$ be the projection. The vector bundle $\mathcal{P}_{(-1,-1)} := \mathcal{O}_{Y_{(-1,-1)}} \oplus \pi^*\mathcal{O}_{\mathbb{P}^1}(1)$ is a tilting generator of the derived category $D^b\left(\operatorname{Coh} Y_{(-1,-1)}\right)$ and we have the derived equivalence

$$\mathbb{R}\mathrm{Hom}\left(\mathcal{P}_{(-1,-1)}, -\right) : D^b\left(\operatorname{Coh} Y_{(-1,-1)}\right) \xrightarrow{\sim} D^b\left(A_{(-1,-1)}\text{-mod}\right),$$

where $A_{(-1,-1)} = \mathrm{End}_Y\left(\mathcal{P}_{(-1,-1)}\right)$. Let $Q_{(-1,-1)}$ be the quiver in Figure 1 and $\omega_{(-1,-1)} = a_1b_1a_2b_2 - a_1b_2a_2b_1$. Then we have $A_{(-1,-1)} \simeq (Q_{(-1,-1)}, \omega_{(-1,-1)})$.

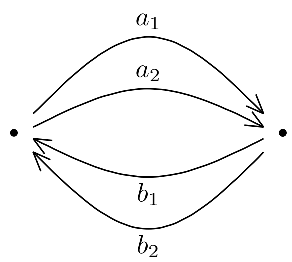

Figure 1: $Q_{(-1,-1)}$

1.2 Local $\mathbb{P}^2$

The next example have been studied carefully by T. Bridgeland ([Bri], [Bri06]). Let $Y_{-3} := \mathcal{O}_{\mathbb{P}^2}(-3)$ be the total space of the canonical bundle on $\mathbb{P}^2$ and $\pi\colon Y_{(-3)} \to \mathbb{P}^2$ be the projection. The vector bundle

$$\mathcal{P}_{(-3)} := \mathcal{O}_{Y_{(-3)}} \oplus \pi^*\mathcal{O}_{\mathbb{P}^2}(1) \oplus \pi^*\mathcal{O}_{\mathbb{P}^2}(2)$$

is a tilting generator of the derived category $D^b\left(\operatorname{Coh} Y_{(-3)}\right)$ and we have the derived equivalence

$$\mathbb{R}\mathrm{Hom}\left(\mathcal{P}_{(-3)}, -\right) : D^b\left(\operatorname{Coh} Y_{(-3)}\right) \xrightarrow{\sim} D^b\left(A_{(-3)}\text{-mod}\right),$$

where $A_{(-3)} = \operatorname{End}_Y\left(\mathcal{P}_{(-3)}\right)$. Let $Q_{(-3)}$ be the quiver in Figure 2 and

$$\omega_{(-3)} = \sum_{\sigma\in\mathfrak{S}_3} \varepsilon(\sigma) c_{\sigma(1)} b_{\sigma(2)} a_{\sigma(3)}.$$

Then we have $A_{(-3)} \simeq \left(Q_{(-3)}, \omega_{(-3)}\right)$.

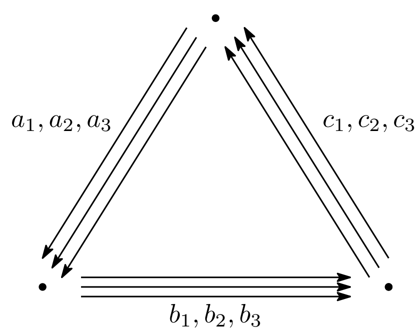

Figure 2: $Q_{(-3)}$

1.3 Geometric engineering

Let Γ be a finite subgroup of $\mathrm{SL}(2,\mathbb{C})$, which acts on the resolved conifold $Y_{(-1,-1)}$ fiber-wisely, and $Y_\Gamma \to X_\Gamma = Y_{(-1,-1)}/\Gamma$ be the crepant resolution (see Figure 3). By the derived McKay correspondence we have

$$D^b\left(\operatorname{Coh} Y_\Gamma\right) \simeq D^b\left(\operatorname{Coh}^\Gamma Y_{(-1,-1)}\right).$$

The Γ-equivariant vector bundle

$$\mathcal{P}_\Gamma := \bigoplus_{\rho\in\operatorname{Irr}(\Gamma)} \mathcal{P}\otimes\rho$$

is a tilting generator of $D^b\left(\operatorname{Coh}^\Gamma Y_{(-1,-1)}\right)$ and the endomorphism algebra $A_\Gamma := \operatorname{End}\left(\mathcal{P}_\Gamma\right)$ can be described as follows: the vertex set of Q_Γ is

$$\{(\epsilon,\rho) \mid \epsilon\in\{0,1\}, \rho\in\operatorname{Irr}(\Gamma)\}.$$

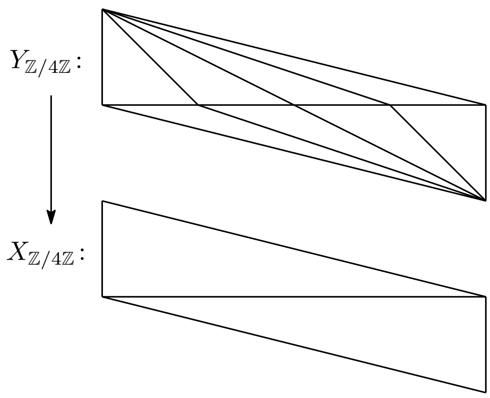

Figure 3: the crepant resolution $Y_{\mathbb{Z}/4\mathbb{Z}} \to X_{\mathbb{Z}/4\mathbb{Z}}$

We have two arrows a^1_ρ and a^2_ρ from $(0,\rho)$ to $(1,\rho)$ for each ρ. If ρ and ρ' are connected by edges in the McKay quiver, we have one arrow from $(1,\rho)$ to $(0,\rho')$ and one arrow from from $(1,\rho')$ to $(0,\rho)$. Let $b_{\rho,\rho'}$ and $b_{\rho',\rho}$ denote these arrows respectively. The potential ω_Γ is the sum of the following elements:

$$a^0_\rho \circ b_{\rho',\rho} \circ a^1_{\rho'} \circ b_{\rho,\rho'}.$$

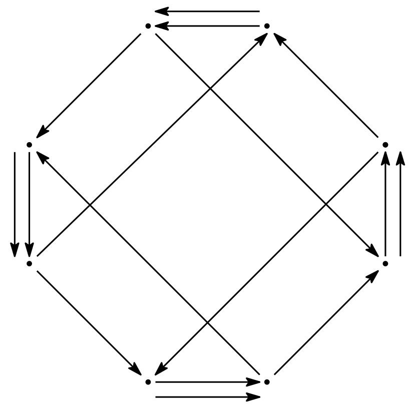

Figure 4: the quiver $Q_{\mathbb{Z}/4\mathbb{Z}}$

1.4 Small crepant resolutions of toric CY 3-folds

Let $X_{a,b}$ be the affine toric Calabi-Yau 3-fold associated with the trapezoid (or possibly triangle) with height 1 and with parallel edges of length a and b. Let σ be a partition of the trapezoid into triangles with areas $1/2$ and $Y^\sigma_{a,b} \to X_{a,b}$ be the associated crepant resolution. The inverse image of $0 \in X_{a,b}$ is the A_{a+b-1} configuration of $(-1,-1)$ or $(0,-2)$-curves. In [Nag], using the result of M. Van den Bergh ([VdB04]), the author constructed a tilting vector bundle with endomorphism algebra $A_\sigma = (Q_\sigma, \omega_\sigma)$. The Q_σ is given by adding some loops to the affine Dynkin quiver of type A_{a+b-1}. Roughly speaking, a vertex with a loop corresponds to a $(0,-2)$-curve. See [Nag] for details.

Example 1.1. *Let $a = 2$, $b = 4$ and σ be the partition in Figure 5. Then the*

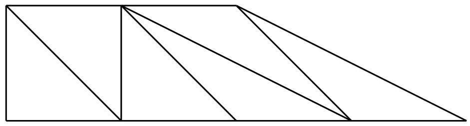

Figure 5: a partition σ

quiver Q_σ is given as in Figure 6.

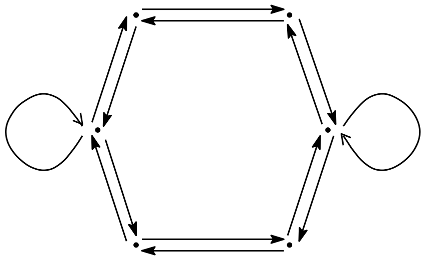

Figure 6: the quiver Q_σ

1.5 Non-toric case: obstructed $(0,-2)$-curve

For $n \geq 2$ we patch two $\mathbb{C}^3$ with coordinates $\{(x,y,z)\}$ and $\{X,Y,Z\}$ respectively by the following transition functions to construct the Calabi-Yau 3-fold $Y^n_{(0,-2)}$:

$$X = x^{-1}, \quad Y = x^2y + xz^n, \quad Z = z.$$

The subvariety $\{y \equiv z \equiv 0\} \cup \{Y \equiv Z \equiv 0\}$ is an obstructed $(0,-2)$-curve. In [AK06] the endomorphism algebra of a tilting vector bundle is computed: the quiver $Q^n_{(0,-2)}$ is given by adding two loops l_0 and l_1 for each vertex to the quiver in Figure 1.

2 Mutations

Let P_k be the projective A-module associated with a vertex $k \in I$ and we set $P := \bigoplus_k P_k (= A)$. We define the new A-module

$$P'_k := \operatorname{coker}\left(P_k \to \bigoplus_{h \in H;\, \mathrm{out}(h)=k} P_{\mathrm{in}(h)}\right).$$

The object $\mu_k(P) = \bigoplus_{l \neq k} P_l \oplus P'_k$ is a tilting generator in $D^b(A\text{-mod})$. Let $\mu_k(A)$ denote the endomorphism algebra $\mathrm{End}\,(\mu_k(P))$.

Example 2.1. *Recall that we take the tilting vector bundle*

$$\mathcal{P}_{(-1,-1)} := \mathcal{O}_{Y_{(-1,-1)}} \oplus \pi^*\mathcal{O}_{\mathbb{P}^1}(1)$$

on the resolved conifold $Y_{(-1,-1)}$ *and get the derived equivalence*

$$D^b(\operatorname{Coh} Y_{(-1,-1)}) \xrightarrow{\sim} D^b(\operatorname{mod} A_{(-1,-1)}).$$

We identify objects in the two categories under the derived equivalence. Let P_0 *and* P_1 *denote the projective* $A_{(-1,-1)}$*-modules* $\mathcal{O}_{Y_{(-1,-1)}}$ *and* $\mathcal{L} := \pi^*\mathcal{O}_{\mathbb{P}^1}(1)$ *respectively. We mutate the quiver* $A_{(-1,-1)}$ *at the vertex* 0*, then we have*

$$P_0' = \operatorname{coker}\left(\mathcal{O}_{Y_{(-1,-1)}} \to \mathcal{L} \oplus \mathcal{L}\right) \simeq \mathcal{L}^2.$$

Hence we have

$$\mu_0(P) = P \otimes \mathcal{L}, \quad \mu_0\left(A_{(-1,-1)}\right)\text{-mod} = A_{(-1,-1)}\text{-mod} \otimes \mathcal{L}.$$

and

$$\mu_0\left(A_{(-1,-1)}\right) \simeq A_{(-1,-1)}.$$

In general, if Q does not have any 1-cycles nor 2-cycles, then $\mu_k(A)$ is given by the mutation of the original quiver with the potential $A = (Q, \omega)$ in the sense of [FZ02] and [DWZ].

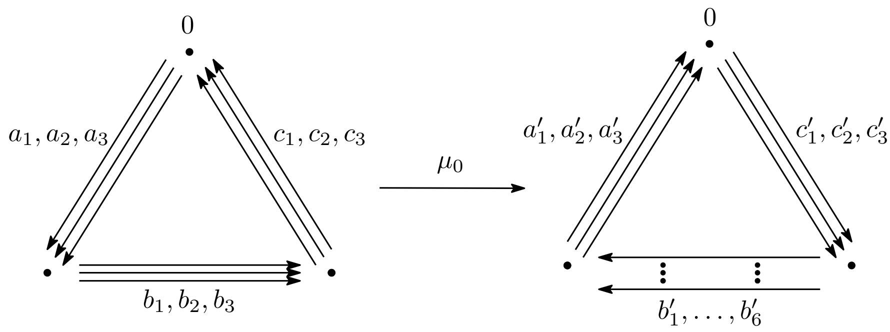

Figure 7: a mutation for $Q_{(-3)}$

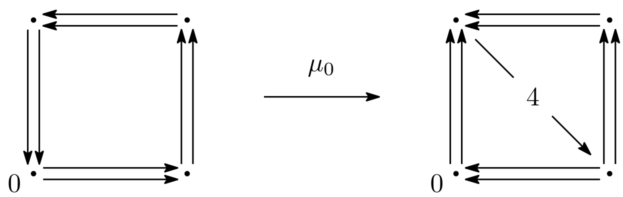

Figure 8: a mutation for $Q_{\mathbb{Z}/2\mathbb{Z}}$

3 Moduli spaces and counting invariants

Let (F, s) be a pair of a finite dimensional A-modules and a map $s \colon P_k \to F$. For a pair of real numbers $\zeta \in \mathbb{R}^I$, a pair (F, s) said to be ζ-(semi)stable if and only if the following two conditions satisfied:

- for any nonzero subobject $0 \neq F' \subseteq F \in A$-mod, we have

$$\underline{\dim}^A V' \cdot \zeta \,(\leq)\, 0,$$

- for any proper subobject $F' \subsetneq F \in A$-mod through which s factors, we have

$$\underline{\dim}^A V' \cdot \zeta \,(\leq)\, \underline{\dim}^A V \cdot \zeta,$$

where $\underline{\dim}^A$ is the composition of the canonical map

$$\mathrm{Obj}(D^b(A\text{-mod})) \to K(A\text{-mod})$$

and the linear map

$$K(A\text{-mod}) \to \mathbb{Z}^I$$

such that $(\underline{\dim}^A V)_i = \dim V_i$ for an A-module V.

By the result of A. King ([Kin94]), for $\mathrm{v} \in \mathbb{Z}^I$ we can construct the moduli space

$$\mathfrak{M}^k_\zeta(\mathrm{v}) := \{(V, s) \mid \underline{\dim}^A V = \mathrm{v},\ \zeta\text{-stable}\}.$$

by geometric invariant theory. We define the counting invariants

$$D^{\mathrm{eu}}_{\zeta,k}(\mathrm{v}) := \chi\left(\mathfrak{M}^k_\zeta(\mathrm{v})\right)$$

as the Euler characteristics of the moduli spaces and their generating function

$$\mathcal{Z}^{\mathrm{eu}}_{\zeta,k}(\mathbf{q}) := \sum_{\mathrm{v}} D^{\mathrm{eu}}_{\zeta,k}(\mathrm{v}) \cdot \mathbf{q}^{\mathrm{v}}.$$

Example 3.1. *In the conifold case*

(1) For ζ_{triv} such that $\zeta^0_{\mathrm{triv}}, \zeta^1_{\mathrm{triv}} > 0$ then

$$\mathfrak{M}^k_{\zeta_{\mathrm{triv}}}(\mathrm{v}) = \begin{cases} \emptyset & \mathrm{v} = 0, \\ \mathrm{pt} & \mathrm{v} \neq 0 \end{cases}$$

and hence $\mathcal{Z}^{\mathrm{eu}}_{\zeta,k} = 1$.

(2) For ζ_{cyclic} such that $\zeta^0_{\mathrm{cyclic}}, \zeta^1_{\mathrm{cyclic}} < 0$ then a pair (F, s) is ζ_{cyclic}-stable if and only if s is surjective. The moduli space have been studied in **non-commutative Donaldson-Thomas theory** *by B. Szendroi ([Sze]). Let $\mathcal{Z}^{\mathrm{eu}}_{\mathrm{NCDT},k}(\mathbf{q})$ denote the generating function of the counting invariants.*

3.1 Remark on virtual counting

Let $\mathcal{M} = \mathcal{M}^k_\zeta(\mathrm{v})$ be the moduli stack of framed representations of the quiver (*without relation*) Q. Taking the trace of the potential, we can define the function on $\mathcal{M}$. Then the moduli space $\mathfrak{M} := \mathfrak{M}^k_{\zeta_{\mathrm{triv}}}(\mathrm{v})$ is the critical locus of this function. We take the Euler characteristic of the Milnor fiber around each critical point to get the constructible function $\nu \colon \mathfrak{M} \to \mathbb{Z}$. The virtual counting of the moduli space is given as the weighted Euler characteristic:

$$D_{\zeta,k}(\mathrm{v}) := \sum_n \chi(\nu^{-1}(n)).$$

The function ν is called *Behrend's constructible function* (or χ*-function.* When the moduli space is compact, the weighted Euler characteristic coincides with the virtual counting ([Beh]), which is defined by integrating the constant function 1 over the virtual fundamental cycle $[\mathfrak{M}]^{\mathrm{vir}}$ ([BF]).

The virtual counting is believe to be the *correct* invariant rather than the Euler characteristic.

One of the reasons is its "deformation invariance". For example, the Donaldson-Thomas invariants of a smooth projective Calabi-Yau 3-fold Y, which are defined as virtual countings of Hilbert schemes of curves, are invariant under the deformation of Y. Though, in our setting deformation invariance is a subtle problem since the moduli is not compact.

Another reason is that the (conjectural) "rationality property" of the generating function (see [PT], [MR]).

Example 3.2. *In the example in §1.5, the generating function of the virtual counting is given by*

$$\begin{aligned}&\mathcal{Z}_{\zeta_{\mathrm{cyclic}},0}(q_0,q_1)\\&=\prod_i\left(1-(-q_0)^iq_1^{i-1}\right)^{ni}\cdot\prod_i\left(1-(-q_0q_1)^i\right)^{-2i}\cdot\prod_i\left(1-(-q_0)^iq_1^{i+1}\right)^{ni},\end{aligned}$$

and the generating function of the Euler characteristics is given by

$$\begin{aligned}\mathcal{Z}^{\mathrm{eu}}_{\zeta_{\mathrm{cyclic}},0}(q_0,q_1)=&\prod_i\left(1+q_0^iq_1^{i-1}+\cdots+q_0^{ni}q_1^{n(i-1)}\right)^i\\&\cdot\prod_i\left(1-q_0q_1^i\right)^{-2i}\cdot\prod_i\left(1+q_0^iq_1^{i+1}+\cdots+q_0^{ni}q_1^{n(i+1)}\right)^i.\end{aligned}$$

When the 3-dimensional Calabi-Yau quiver is derived from a *brane tiling*, then the virtual counting coincides with the Euler characteristic up to sign (see [MR]).

4 Results

For $k\in I$ we define the map $\mu_k\colon\mathbb{Z}^I\to\mathbb{Z}^I$ by

$$(\mu_k(\mathbf{v}))_l=\begin{cases}-\mathbf{v}_k+\sum_{h;\mathrm{out}(h)=k}\mathbf{v}_{\mathrm{in}(h)} & l=k,\\ \mathbf{v}_l & \text{otherwise}\end{cases}$$

for $\mathbf{v}\in\mathbb{Z}^I$. We also define $\mu_k\colon\mathbb{R}^I\to\mathbb{R}^I$ by

$$\mathbf{v}\cdot\zeta=\mu_k(\mathbf{v})\cdot\mu_k(\zeta)$$

for any $\mathbf{v}$ and ζ.

Let $\zeta\in\mathbb{R}^I$ be a generic stability parameter satisfying the following conditions:

- there exists $\eta\in\mathbb{R}^I_{>0}$ such that $\zeta+d\cdot\eta$ on an intersection of two walls for any $d\geq 0$.
- We have the sequence $C_0,\ldots,C_r$ of chambers such that

- $\zeta - d \cdot \eta \in \cup \overline{C_s}$ for any $d \geq 0$,
- for any C_s, there exists some $d \geq 0$ such that $\zeta - d \cdot \eta \in C_s$, and
- suppose $\zeta - d \cdot \eta \in C_s$, $\zeta - d' \cdot \eta \in C_{s'}$ and $s < s'$, then $d > d'$.

- we have the sequence $k_1, \dots, k_r$ of elements in I such that

$$\overline{C_{s-1}} \cap \overline{C_s} \subset W_{\alpha^s} \quad (\alpha^s = \mu_{k_{s-1}} \circ \cdots \circ \mu_{k_1}(\alpha_{k_s})),$$

where α_k denote the simple root vector.

We denote $\mu_s := \mu_{k_s} \circ \cdots \circ \mu_{k_1}$, $\Psi_s := \Psi_{k_s} \circ \cdots \circ \Psi_{k_1}$ and $\mu_\zeta := \mu_r$, $\Psi_\zeta := \Psi_r$.

We set $\mathcal{P} := A$-mod and denote by $\mathcal{P}_\zeta$ the image of the Abelian category $\mu_\zeta(A)$-mod under the equivalence Ψ_ζ^{-1}.

Definition 4.1. *Let (V, s) be a pair of an element $V \in \mathcal{P}_\zeta$ and a map $s \colon P_k \to V$. For $\xi \in \mathbb{R}^{\hat{I}}$, we say (V, s) is $(\xi, \mathcal{P}_\zeta)$-(semi)stable if the following conditions are satisfied:*

(A) for any nonzero subobject $0 \neq S \subseteq V$ in $\mathcal{P}_\zeta$, we have

$$\xi \cdot \underline{\dim}\, S \,(\leq)\, 0,$$

(B) for any proper subobject $T \subsetneq V$ in $\mathcal{P}_\zeta$ which s factors through, we have

$$\xi \cdot \underline{\dim}\, T \,(\leq)\, \xi \cdot \underline{\dim}\, V.$$

From now on, the ζ-(semi)stability for a pair (F, s) with $F \in \mathcal{P} = A$-mod is written as the $(\zeta, \mathcal{P})$-(semi)stability. We set $\xi_{\mathrm{cyclic}} := \mu_\zeta(\zeta)$. Note that $(\xi_{\mathrm{cyclic}})_l < 0$ for any $l \in I$.

Theorem 4.2. (1) [Nag, Lemma 3.5] *Let (F, s) be a $(\zeta, \mathcal{P})$-stable, then $F \in \mathcal{P}_\zeta$.*

(2) [Nag, Proposition 3.6] *Let (F, s) be a $(\zeta, \mathcal{P})$-stable, then (F, s) is $(\xi_{\mathrm{cyclic}}, \mathcal{P}_\zeta)$-stable.*

(3) [Nag, Lemma 3.7] *Let (F, s) be a $(\xi_{\mathrm{cyclic}}, \mathcal{P}_\zeta)$-stable, then $F \in \mathcal{P}$.*

(4) [Nag, Proposition 3.8] *Let (F, s) be a $(\xi_{\mathrm{cyclic}}, \mathcal{P}_\zeta)$-stable, then (F, s) is $(\zeta, \mathcal{P})$-stable.*

This theorem claims that replacing t-structures corresponds to replacing stability conditions. In particular, we can define

$$\mathfrak{M}^k_{\mathcal{P}_\zeta, \zeta_{\mathrm{cyclic}}}(\mathrm{v}) := \{(V, s) \mid V \in \mathcal{P}_\zeta,\ \underline{\dim}^A V = \mathrm{v},\ (\zeta_{\mathrm{cyclic}}, \mathcal{P}_\zeta)\text{-stable}\}$$

which is isomorphic to $\mathfrak{M}^k_{\mathcal{P}, \zeta}(\mathrm{v}) = \mathfrak{M}^k_\zeta(\mathrm{v})$. We can also define the generating function

$$\mathcal{Z}^{\mathrm{eu}}_{\mathcal{P}_\zeta, k}(\mathbf{q}) = \sum_v \chi(\mathfrak{M}^k_{\mathcal{P}_\zeta, \zeta_{\mathrm{cyclic}}}(\mathrm{v})) \cdot \mathbf{q}^{\mathrm{v}}.$$

of the counting invariants.

In [NN] and [Nag], we study how the generating function changes when we replace the stability condition. Now we get the following formula describing how the generating function changes when we mutate the quiver.

Theorem 4.3.

$$\mathcal{Z}^{\mathrm{eu}}_{\mathcal{P}_\varsigma,k}(\mathbf{q}) = \left(1+\mathbf{q}^{\alpha^r}\right)^{(\alpha^r)_k} \mathcal{Z}^{\mathrm{eu}}_{\mathcal{P}_{r-1},k}(\mu_{k_r}(\mathbf{q})),$$

where the left hand side is given by substituting q_k with

$$\left(1+\mathbf{q}^{\alpha^r}\right)^{\langle \alpha_k,\alpha^r\rangle} q_k.$$

Here $\langle -,-\rangle$ is the Euler pairing on $K(D^b(\mathrm{Coh}(Y)))$.

Example 4.4. *In the conifold case, the Euler pairing is trivial and the theorem provides a conceptual interpretation of Young's combinatorial formula ([You]).*

Acknowledgement

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