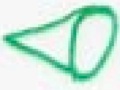
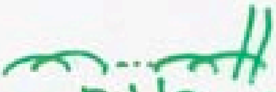
DIHEDRAL GROUPS, G-HILB and $M_\theta(Q,R)$

– Álvaro Nolla de Celis – Japan Dec.'08

$$\begin{tikzcd}
 & \mathbb{C}^2 \circlearrowleft G \subset Gl(2,\mathbb{C}) \text{ small finite} \arrow[d] \\
\text{minimal resolution } Y \arrow[r] & \mathbb{C}^2/G
\end{tikzcd}$$





$E = \bigcup E_i$ excl. divisor

Special McKay Correspondence: Let $\text{Irr}\,G = \{\rho_0, \rho_1, \ldots, \rho_n\}$ irreducible representations of G. Then

$$\{\text{Exceptional Curves } \underset{E_i}{}\} \xleftrightarrow{\text{1-to-1}} \{\text{Irreducible } \underline{\text{Special}} \text{ Representations } \underset{\rho_i}{}\}$$

Example-Notation: $G = \langle \frac{1}{12}(1,7) \rangle := \left\langle \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^7 \end{pmatrix} \,\middle|\, \begin{matrix} \varepsilon^{12}=1 \\ \text{primitive} \end{matrix} \right\rangle$

$\longrightarrow \mathbb{C}^2/G$ toric $\qquad \frac{12}{7} = [2,4,2]$

but G cyclic order 12 ⇒ G has 12 irreducible repr.

Goal: To construct explicitly Y for $G \subset GL(2,\mathbb{C})$ small binary dihedral by giving an affine open cover of it.

$$\boxed{Y = G\text{-Hilb}\,\mathbb{C}^2 = M_\theta(Q,R)}$$

G - Hilb

G-Hilb $\mathbb{C}^2$ = Moduli space param. G-clusters

A G-cluster is a G-invariant 0-dimensional subsqueme $Z \subset \mathbb{C}^2$ such that

$$\mathcal{O}_Z = \mathbb{C}[x,y]/I_Z \underset{\mathbb{C}[G]}{\simeq} \mathbb{C}[G]$$ the regular representation

$$\mathbb{C}[G] = \bigoplus_{\rho_i \in \mathrm{Irr}\, G} \rho_i^{\dim \rho_i}$$

$\Rightarrow$ G-Hilb $\mathbb{C}^2$ parametrizes ideals $I \subset \mathbb{C}[x,y]$ such that $\mathbb{C}[x,y]/I$, as a vector space , has in its basis 1 element in each 1-dim representation

2 " " " 2-dim "

⋮

and so on

Example $\frac{1}{5}(1,2)$; $G = \langle \overset{\alpha}{\begin{pmatrix}\varepsilon & 0\\ 0 & \varepsilon^2\end{pmatrix}} \mid \varepsilon^5=1 \text{ prim.}\rangle \subset GL(2,\mathbb{C})$ $\begin{cases} x \longmapsto \varepsilon x \\ y \longmapsto \varepsilon^2 y\end{cases}$

		α	
	ρ_0	1	$1, x^5, y^5, x^3y, xy^2, \ldots$
→	ρ_1	ε	$x, y^3, x^4y, \ldots$
→	ρ_2	ε^2	$x^2, y, x^5y \ldots$
	ρ_3	ε^3	$x^3, y^4, xy \ldots$
	ρ_4	ε^4	$x^4, y^2, x^2y \ldots$

0
3
1 2
4 0 1
2 3 4 0
0 1 2 3 4 0

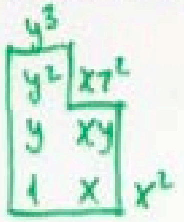
Possible bases of $\mathbb{C}[x,y]/I = \bigoplus \rho_i^{\dim \rho_i}$?

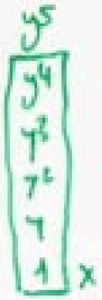
$y \notin$ base $\Rightarrow y^i \notin$ base

$y \in$ base $\Rightarrow x^2 \notin$ base

$x \notin$ base $\Rightarrow x^i \notin$ base

G-graphs:



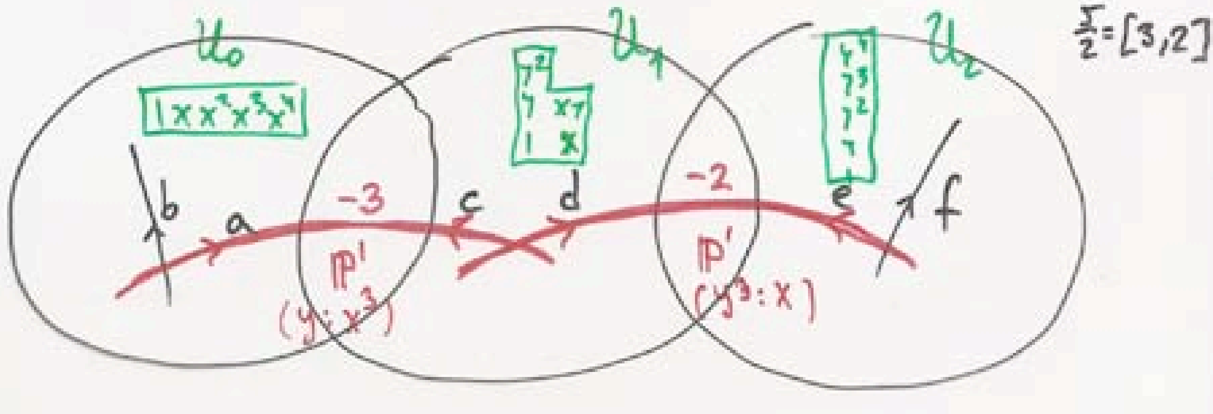


$I_{a,b} = \begin{pmatrix} y = ax^2 \\ x^5 = b\cdot 1\end{pmatrix} \rightsquigarrow \mathbb{C}^2_{a,b}$ $\quad a,b \in \mathbb{C}$

$I_{c,d} = \begin{pmatrix} x^2 = cy \\ y^3 = dx \\ xy^2 = cd\end{pmatrix} \rightsquigarrow \mathbb{C}^2_{c,d}$ $\quad c,d \in \mathbb{C}$

$I_{e,f} = \begin{pmatrix} x = ey^3 \\ y^5 = f\cdot 1\end{pmatrix} \wr \mathbb{C}^2_{e,f}$ $\quad e,f \in \mathbb{C}$

$\Rightarrow G\text{-Hilb}\,\mathbb{C}^2 = \mathbb{C}^2_{a,b} \cup \mathbb{C}^2_{c,d} \cup \mathbb{C}^2_{e,f}$



$\frac{5}{2} = [3,2]$

$\mathcal{M}_\theta(Q,R)$

$\mathcal{M}_\theta(Q,R)$ = Moduli space of θ-stable quiver representations of the McKay quiver (Q,R)

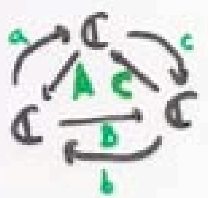
Q = McKay Quiver : V = natural repr. $(G \hookrightarrow GL(2,\mathbb{C}))$
(R = Relations) Form $V \otimes \rho_i = \sum a_{ij}\rho_j \rightsquigarrow \bullet = \rho_i$

$\underset{\rho_i}{\bullet} \rightarrow \underset{\rho_j}{\bullet} \Leftrightarrow a_{ij} \neq 0$

A Representation of Q is $W = (\underbrace{W_i}_{\text{v.e. dim } d_i}, \overset{\text{linear map}}{\varphi_a})_{\underset{\text{vertex}}{i \in Q_0},\ \underset{\text{arrow}}{a \in Q_1}}$

Example : $\frac{1}{3}(1,2)$

Q



W

Dimension vector : $\underline{d} = (\dim W_i)$

Isomorphism classes of quiver representations are orbits under $G = \prod GL(d_i)$ (change of basis)

$$\mathcal{M}_\theta(Q,R) = \mathbb{V}(I_R) /\!\!/_\theta\, G$$

I_R = ideal of relations

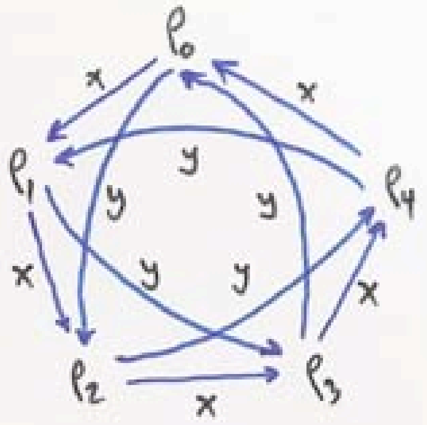
θ generic (semistable $\Rightarrow$ stable)

TAKE $\underline{d} = (\dim \rho_i)$ and $\theta = (-\sum \dim \rho_i, 1, 1, \ldots, 1)$

Then, Stable $\Rightarrow \exists$ $(\dim \rho_i)$ non-zero paths from ρ_0 to any other vertex ρ_i (1-generated)

Ejemplo $\frac{1}{5}(1,2)$

ρ_0	$1, x^5, y^5, x^3y, xy^2, \ldots$
ρ_1	x, y^3, x^4y
ρ_2	$x^2, y, x^5y \ldots$
ρ_3	$x^3, y^4, xy, \ldots$
ρ_4	$x^4, y^2, x^2y, \ldots$

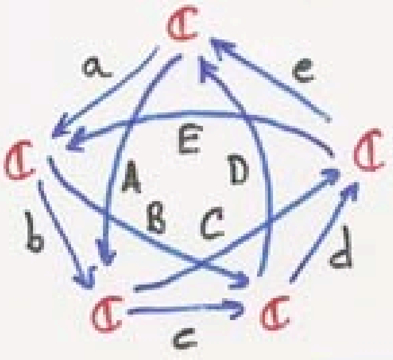


McKay Quiver Q

Representation of Q:

$\underline{d} = (1,1,1,1,1)$

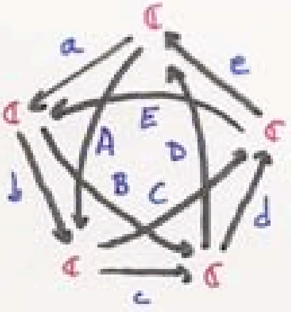
$\theta = (-4,1,1,1,1)$



Relations: "$xy = yx$"

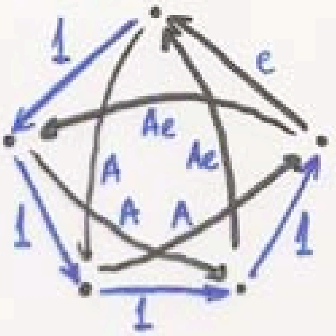
$aB = Ac$
$bC = Bd$
$cD = Ce$
$dE = Da$
$eA = Eb$

$U_0 \subset \mathcal{M}_\theta(Q,R)$ given by
$a\neq 0,\ b\neq 0,\ c\neq 0,\ d\neq 0$



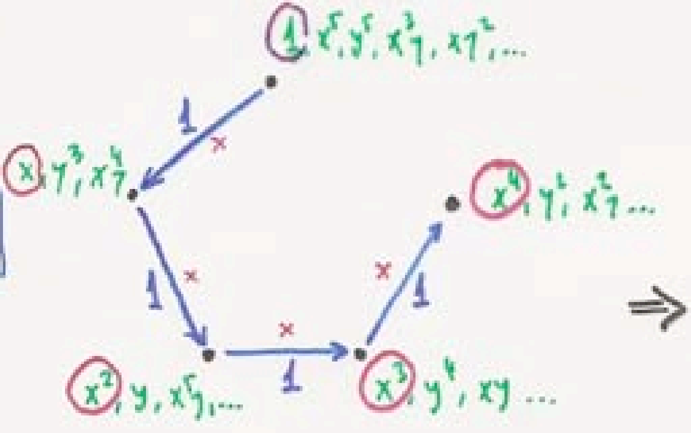
change of basis
+
relations

$\mathcal{M}_\theta(Q,R)$



$\Rightarrow \quad U_0 = \mathbb{C}^2_{A,e}$

G-Hilb

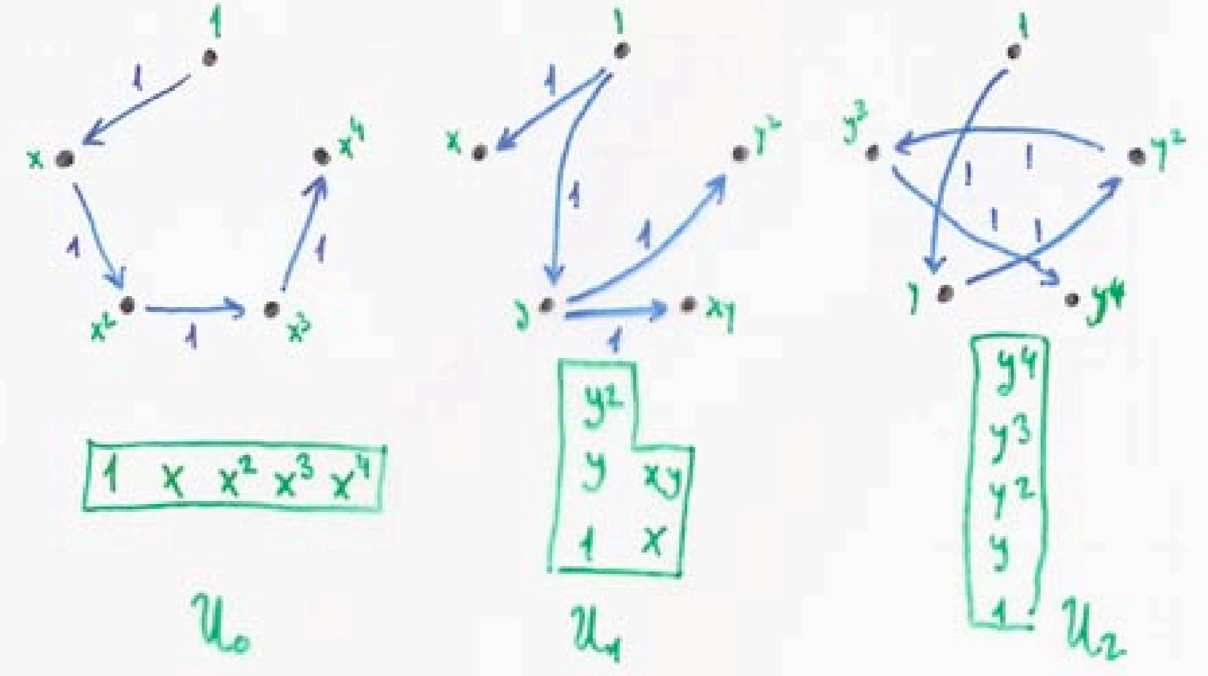


$$\Rightarrow \quad \overset{y}{\boxed{1 \quad x \quad x^2 \quad x^3 \quad x^4}} \ x^5$$

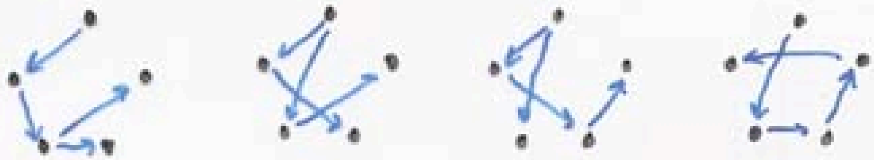
$$I_{A,e} = \begin{pmatrix} x^5 - e \\ y - Ax^2 \end{pmatrix}$$

$$\Rightarrow \quad U_0 = \mathbb{C}^2_{A,e} \subset G\text{-Hilb}\,\mathbb{C}^2$$

⇒ $\mathcal{M}_\theta(Q,R)$ for $\frac{1}{5}(1,2)$ is covered by the following open sets:



Note: There are lot of possibilities !! For example:



... etc

all of them give open sets $U \subset \mathcal{M}_\theta(Q,R)$

⇒ lot of choice in the way of covering $\mathcal{M}_\theta(Q,R)$

⇒ The G-graphs (G-Hilb) determine a particular covering for $\mathcal{M}_\theta(Q,R)$ with the smalles number of open sets.

Binary Dihedral Groups $G \subset GL(2,\mathbb{C})$

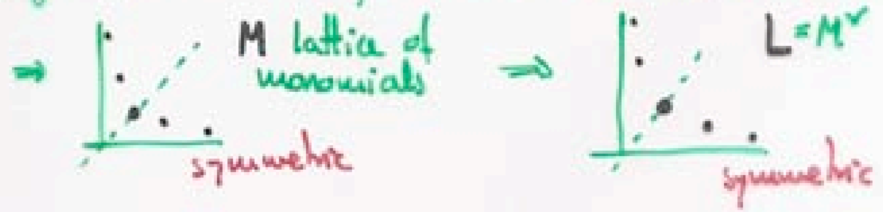
$$G = BD_{2n}(a) := \Big\langle \underset{\overset{\shortparallel}{\alpha}}{\tfrac{1}{2n}(1,a)},\ \underset{\overset{\shortparallel}{\beta}}{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}} \ \Big|\ (2n,a)=1,\ \underbrace{a^2 \equiv 1\ (2n)}_{(*)} \Big\rangle$$

$$\alpha: \begin{cases} x \longmapsto \varepsilon x \\ y \longmapsto \varepsilon^a y \end{cases} \qquad \beta: \begin{cases} x \longmapsto y \\ y \longmapsto -x \end{cases}$$

$|BD_{2n}(a)| = 4n$, $A = \langle \frac{1}{2n}(1,a) \rangle \trianglelefteq G$ index 2 $(\beta^2 \in A)$

- Condition (*) creates lots of symmetry:

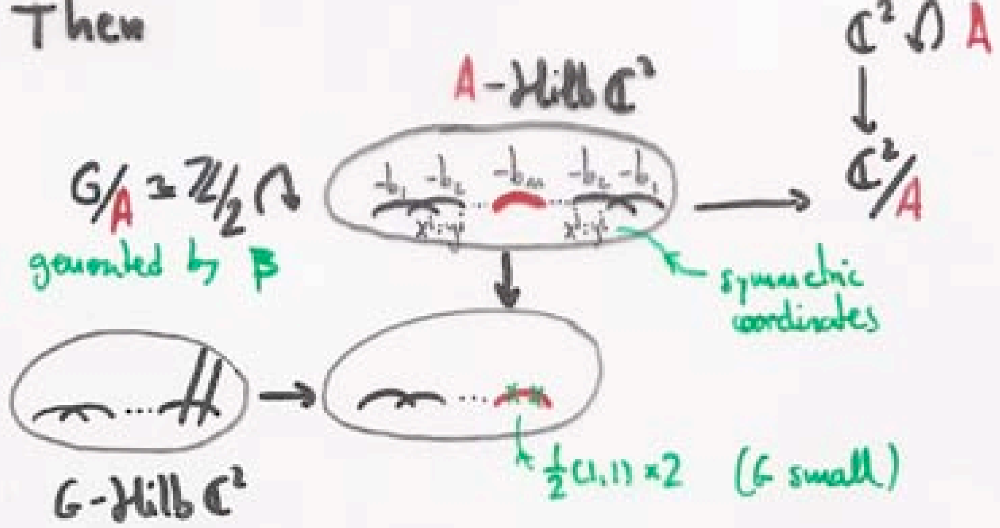
$x^i y^j$ A-inv $\Rightarrow i + aj \equiv 0\ (2n) \Rightarrow ai + j \equiv 0\ (2n) \Rightarrow x^j y^i$ A-inv



$$\Rightarrow \quad \frac{2n}{a} = [b_1, b_2, \dots, b_{m-1}, b_m, b_{m-1}, \dots, b_2, b_1]$$

Symmetric

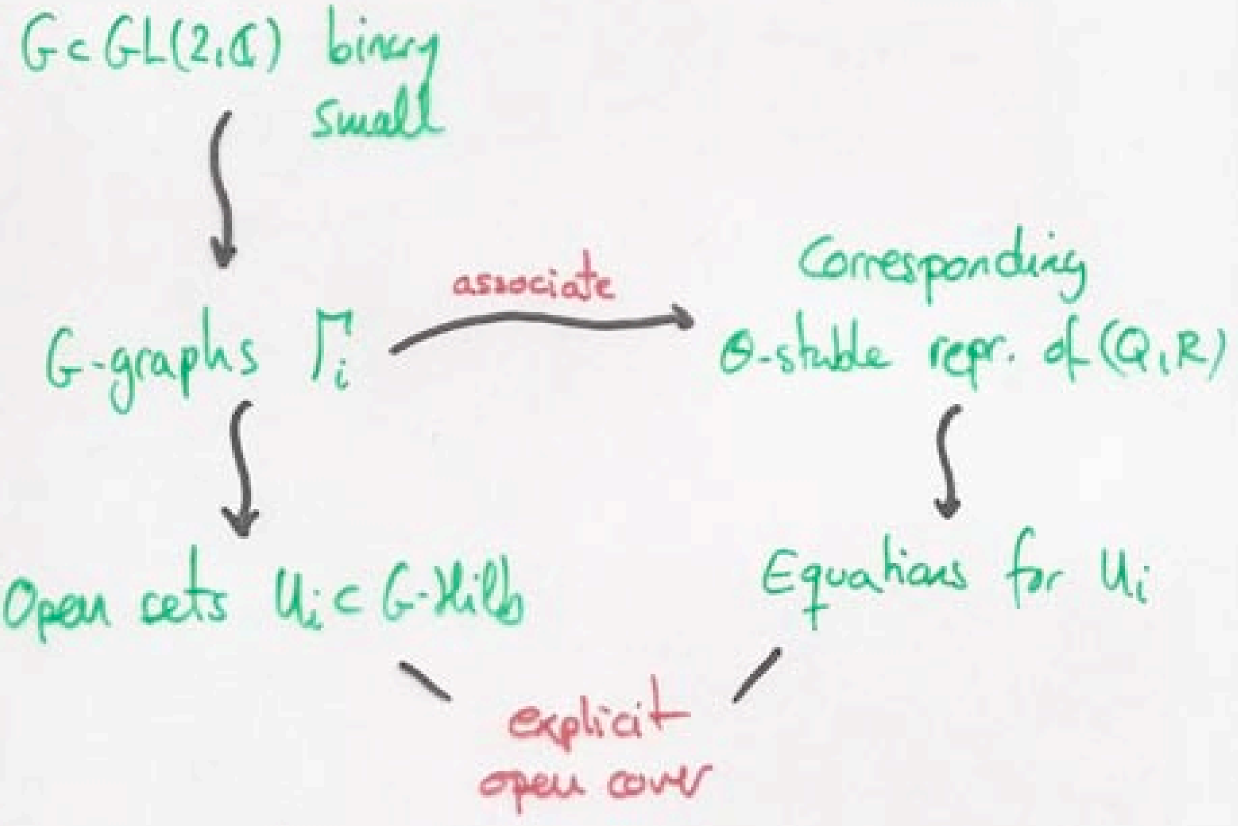
Then



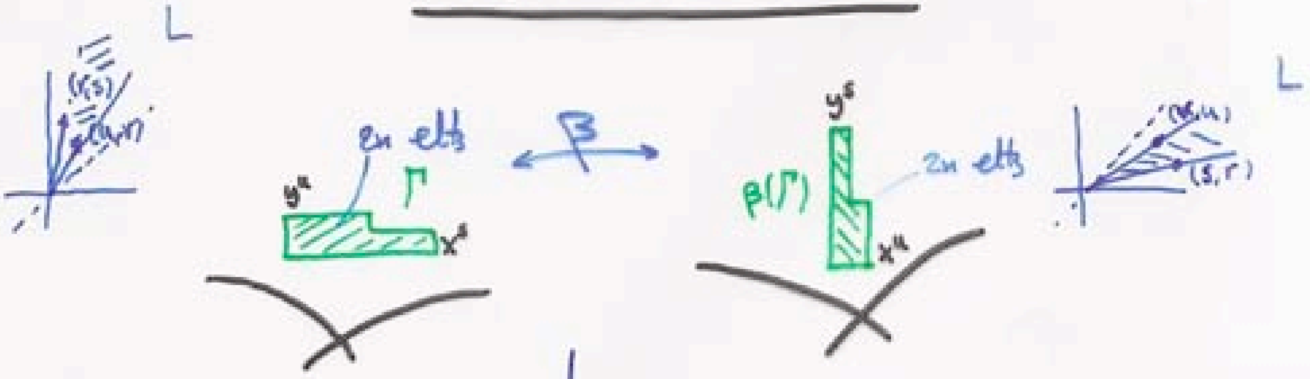
- G-Hilb $\mathbb{C}^2$ = G/A - Hilb (A-Hilb $\mathbb{C}^2$)

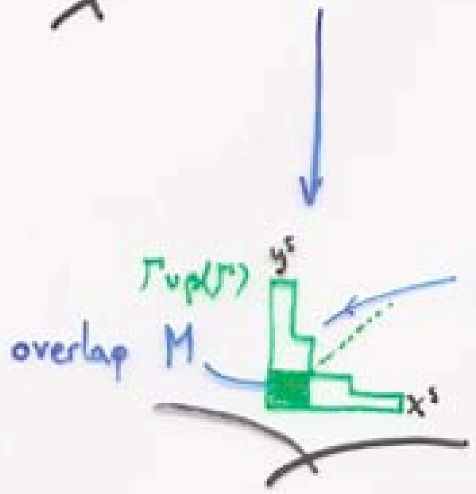
- Thm (Ishii): G-Hilb is the minimal resolution of $\mathbb{C}^2/G$ for $G \subset GL(2,\mathbb{C})$ small finite

Plan of attack :



G-Graphs for $BD_{2n}(a)$

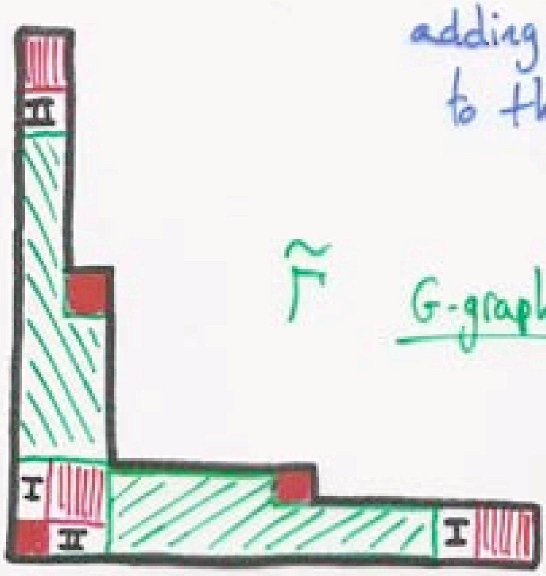




less than 2n elts!

$\Gamma\cup\beta(\Gamma)$ not yet a G-graph

$\exists!$ way of extending $\Gamma\cup\beta(\Gamma)$ to a G-graph $\tilde{\Gamma}$ by adding new elements belonging to the representations in M



$\tilde{\Gamma}$ G-graph

$\tilde{\Gamma}$ is completely determined by (r,s), (u,v)

$\Rightarrow$ G-graph are given by the continued fraction $\frac{2n}{a}$

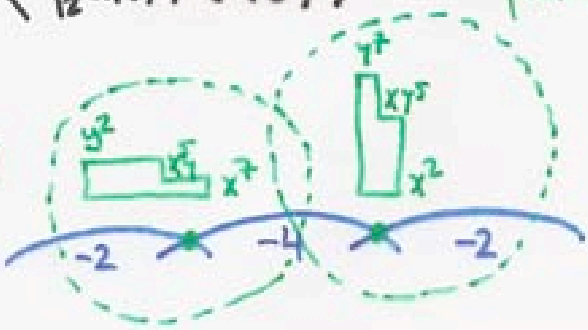
$$BD_{12}(7) = \left\langle \alpha = \frac{1}{12}(1,7),\ \beta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle$$

	α	β	
ρ_0^+	1	1	$1,\ x^{12}+y^{12},\ x^5y-xy^5,\ x^6y^6,\ x^8y^4+x^4y^8$
ρ_0^-	1	-1	$x^{12}-y^{12},\ x^5y+xy^5,\ x^3y^3$
→ ρ_1^+	ε^2	1	$x^2+y^2,\ x^7y-xy^7$
→ ρ_1^-	ε^2	-1	$x^2-y^2,\ x^7y+xy^7$
→ ρ_2^+	ε^4	1	$x^4+y^4,\ x^9y-xy^9,\ x^2y^2$
ρ_2^-	ε^4	-1	$x^4-y^4,\ x^9y+xy^9,\ x^5y^5$
ρ_3^+	-1	1	$x^6+y^6,\ x^{11}y-xy^{11},\ x^4y^2+x^2y^4$
ρ_3^-	-1	-1	$x^6-y^6,\ x^{11}y+xy^{11},\ x^4y^2-x^2y^4$
ρ_4^+	ε^8	1	$x^8+y^8,\ x^6y^2+x^2y^6,\ x^4y^4$
→ ρ_4^-	ε^8	-1	$x^8-y^8,\ xy,\ x^6y^2-x^2y^6$
ρ_5^+	ε^{10}	1	$x^{10}+y^{10},\ x^3y-xy^3$
ρ_5^-	ε^{10}	1	$x^{10}-y^{10},\ x^3y+xy^3$
Q_1	$\begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^7 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	$(x,+y),(y^7,-x^7),(x^6y,-xy^6),(x^2y^5,-x^5y^2)$
Q_2	$\begin{pmatrix} \varepsilon^3 & 0 \\ 0 & \varepsilon^9 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	$(x^3,y^3),(y^9,-x^9),(xy^2,x^2y),(x^8y,-xy^8)$
Q_3	$\begin{pmatrix} \varepsilon^5 & 0 \\ 0 & \varepsilon^{11} \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	$(x^5,y^5),(y^{11},x^{11}),(xy^4,x^4y),(x^{10}y,-xy^{10})$

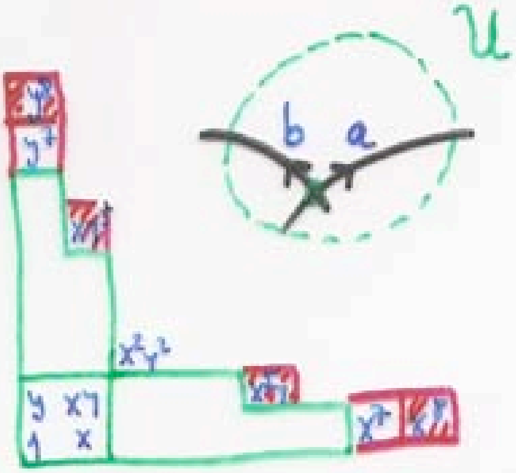
Example of extension

$BD_{12}(7) = \langle \frac{1}{12}(1,7),\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \rangle$ $\rho(I) = (x^2, y^7, xy^5)$

$I = (y^8, x^7, x^5y)$



$\frac{12}{7} = [2,4,2]$



$\rightsquigarrow \tilde{I} = (x^2y^2, x^8-y^8, x^5y-xy^5)$

$x^2y^2, \underset{\text{basis}}{\boxed{x^4+y^4}} \in \rho_4^+ \ ; \ \underset{\text{basis}}{\boxed{xy}}, x^8-y^8 \in \rho_8^- \ ; \ \underset{\text{basis}}{\boxed{1}}, x^5y - xy^5 \in \rho_0^+$

$\Rightarrow \tilde{I} = \left(x^2y^2 - a(x^4+y^4),\ x^8-y^8 - b\,xy \quad x^5y - xy^5 - c \right)$

$\rightarrow$ Equation of $\mathcal{U}$? $\exists\, f(a,b,c)=0$ hypersurface in $\mathbb{C}^3_{b,c,e}$ non-singular

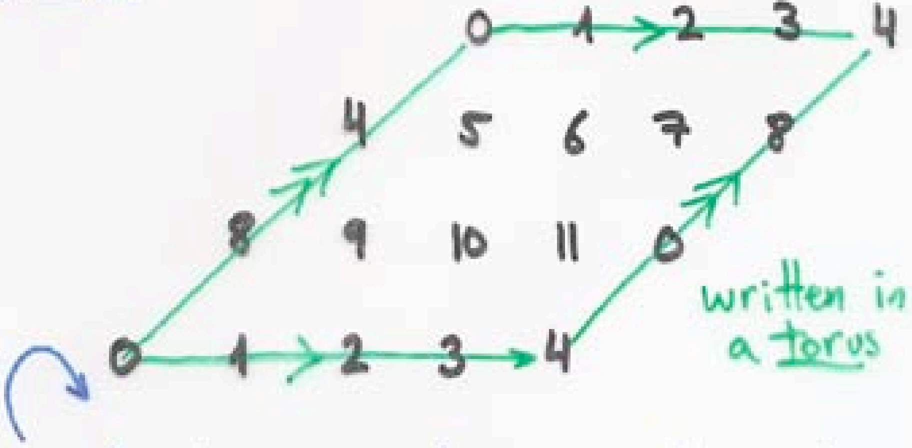
Problems:
- The calculation of f is complicated in general.
- Lots of choices in the selection of the basis elements, e.g. $x^2y^2, x^4+y^4, (x^2+y^2)^2, (x^2-y^2)^2 \in \rho_4^+$ Which one should we choose?

Solu: $\mathcal{M}_\theta(Q,R)$

$$\begin{tikzcd}
\vdots & & \\
y^2 \arrow[u, dashed] & \cdots & \\
y \arrow[u] \arrow[r] & xy & \\
1 \arrow[u] \arrow[r] & x \arrow[u] \arrow[r] & x^2 \arrow[r] & x^3 \arrow[r] & \cdots
\end{tikzcd}$$

$BD_{12}(7) = \langle \frac{1}{12}(1,7), \beta \rangle$

McKay Quiver for $A = \langle \frac{1}{12}(1,7) \rangle$



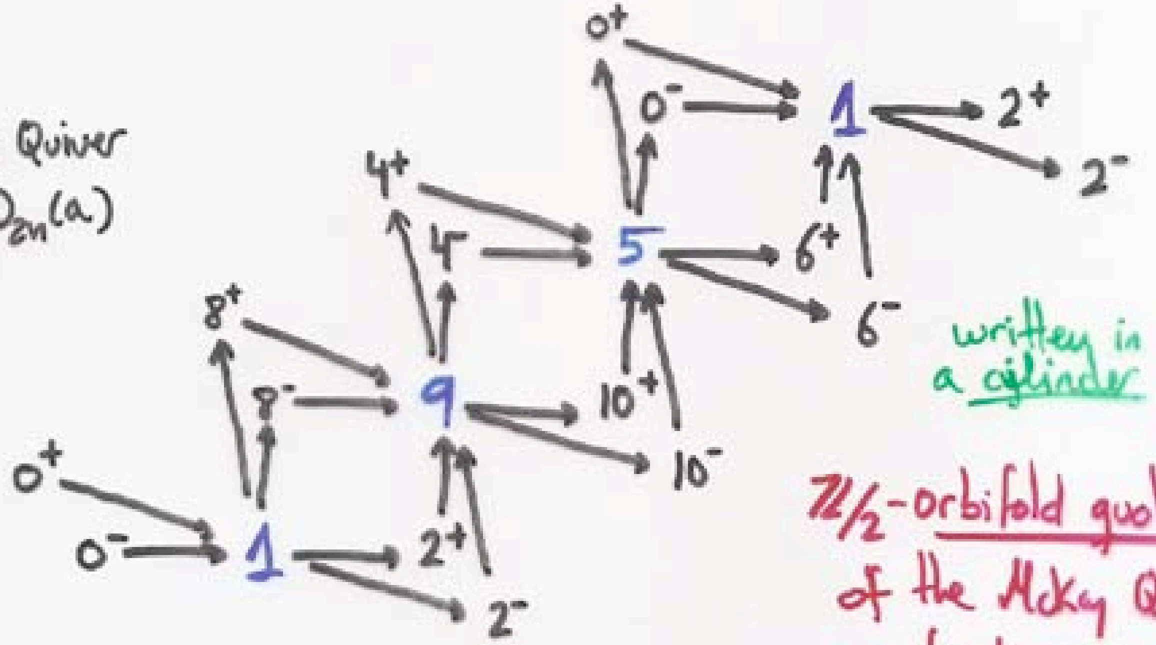
written in a torus

β acts by conjugation on McKay Q(A)

Fixed: $\rho_0 \circlearrowleft$ $\rho_2 \circlearrowleft$ $\rho_4 \circlearrowleft$ $\rho_6 \circlearrowleft$ $\rho_8 \circlearrowleft$ $\rho_{10} \circlearrowleft$

Free orbits: $\rho_1 \rightleftarrows \rho_7$ $\rho_3 \rightleftarrows \rho_9$ $\rho_5 \rightleftarrows \rho_{11}$

McKay Quiver for $BD_{2n}(a)$



written in a cylinder

$\mathbb{Z}/2$-orbifold quotient of the McKay Quiver of $\frac{1}{2n}(1,a)$

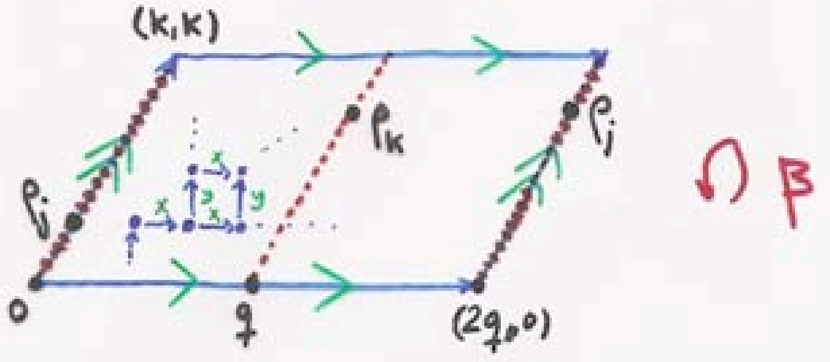
Fixed points ρ_i by β ⇝ two 1-dim repr. ρ_i^+, ρ_i^-

Free orbits $\rho_i \rightleftarrows \rho_j$ ⇝ one 2-dim repr V_i

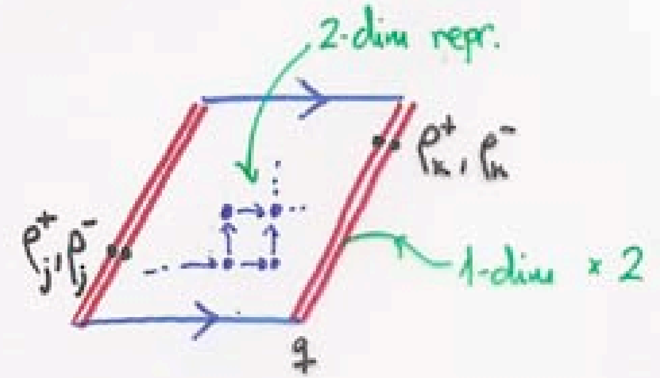
In general,

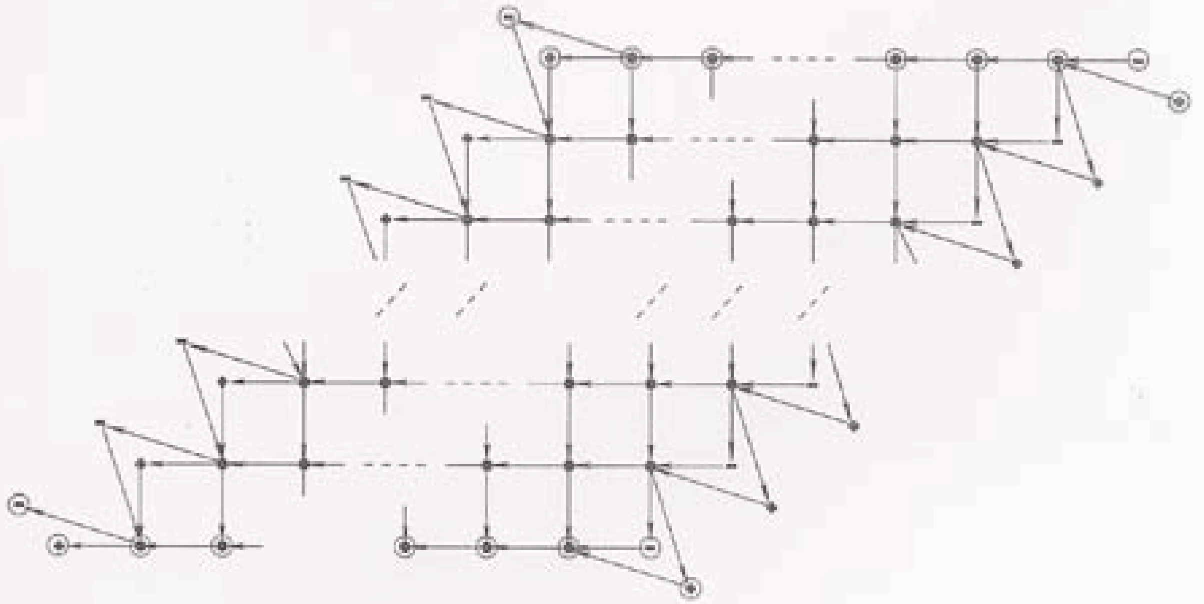
…… = fixed locus

McKay Quiver for $\frac{1}{2n}(1,a)$

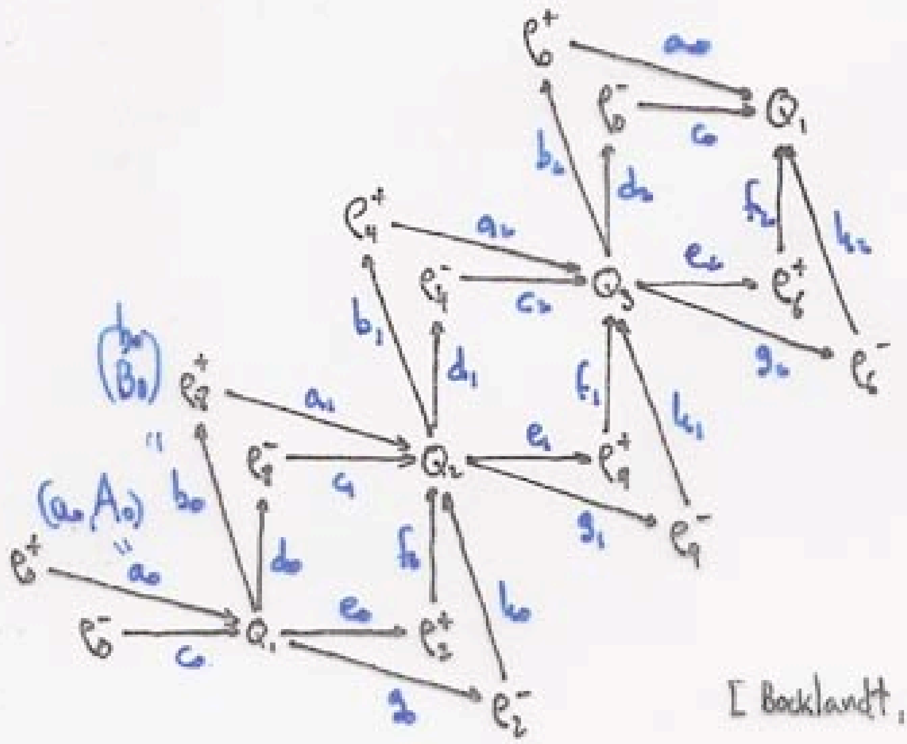


McKay Quiver for $\langle \frac{1}{2n}(1,a), \beta \rangle$





McKay quiver & Relations for $BD_{12}(7)$



[Bocklandt, Schedler, Wemyss '08]

$a_0b_0=0$ $\quad c_0d_0=0$ $\quad f_0e_1=0$ $\quad h_0g_1=0$

$a_1b_1=0$ $\quad c_1d_1=0$ $\quad f_1e_2=0$ $\quad h_1g_2=0$

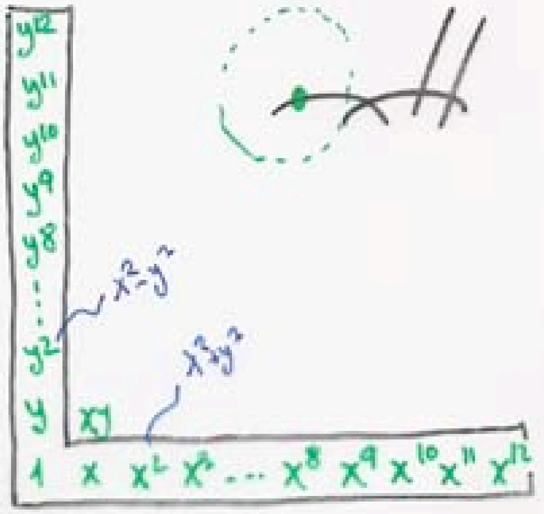
$a_2b_2=0$ $\quad c_2d_2=0$ $\quad f_2e_0=0$ $\quad h_2g_0=0$

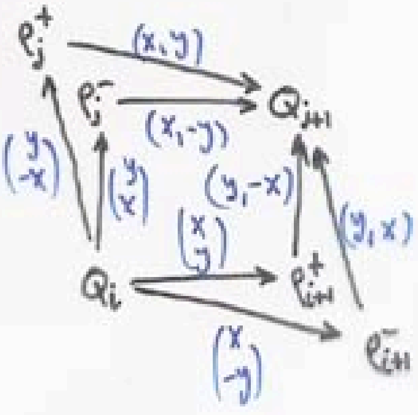
$b_0a_1 + d_0c_1 = e_0f_0 + g_0h_0$

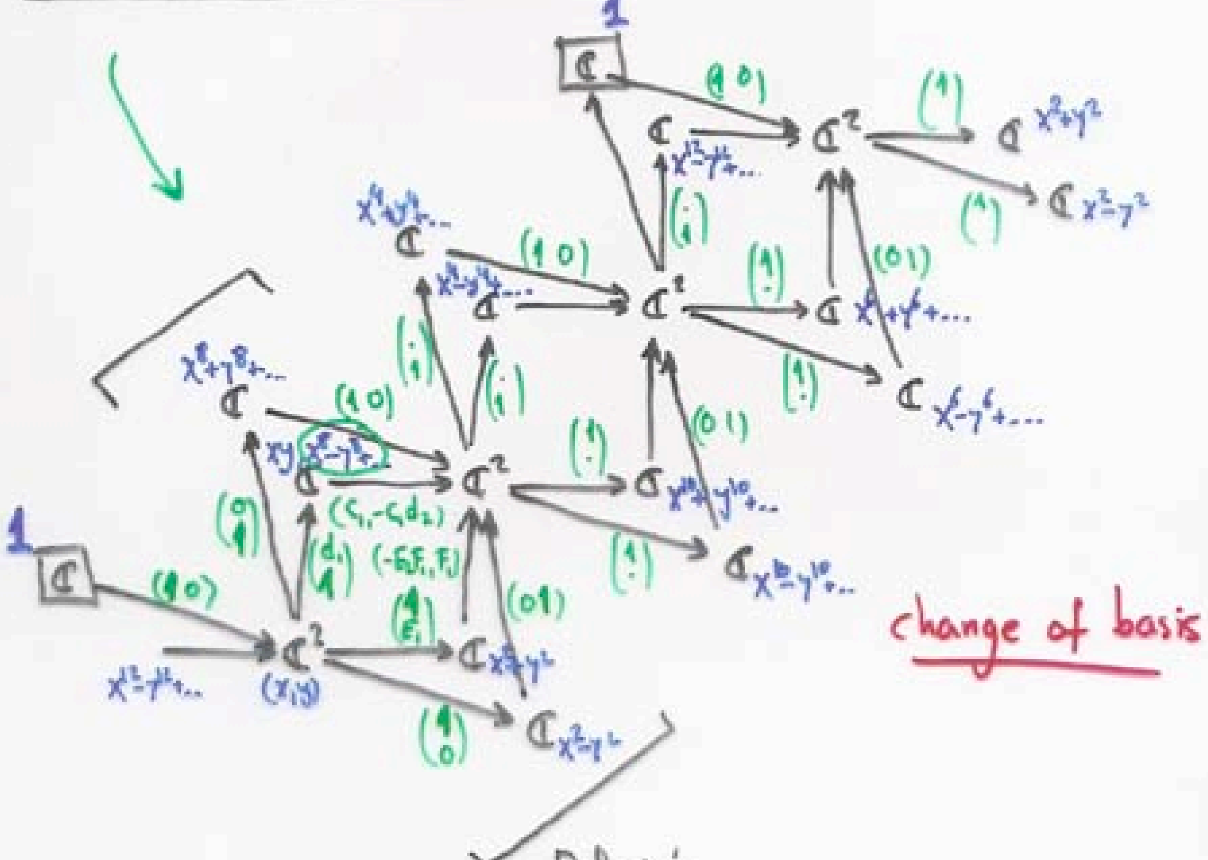
$b_1a_2 + d_1c_2 = e_1f_1 + g_1h_1$

$b_2a_0 + d_2c_0 = e_2f_2 + g_2h_2$

"mesh relations"







Relaciói:

Relations :

$$\begin{pmatrix}0\\1\end{pmatrix}(1\,0)+\begin{pmatrix}d_1\\1\end{pmatrix}(c_1,-c_1d_2)=\begin{pmatrix}1\\E_1\end{pmatrix}(-E_2F_1,F_1)+\begin{pmatrix}1\\0\end{pmatrix}(0\,1)$$

$$\begin{pmatrix}0&0\\1&0\end{pmatrix}+\begin{pmatrix}d_1c_1&-c_1d_1d_2\\c_1&-c_1d_2\end{pmatrix}=\begin{pmatrix}-E_2F_1&F_1\\-E_1E_2F_1&E_1F_1\end{pmatrix}+\begin{pmatrix}0&1\\0&0\end{pmatrix}$$

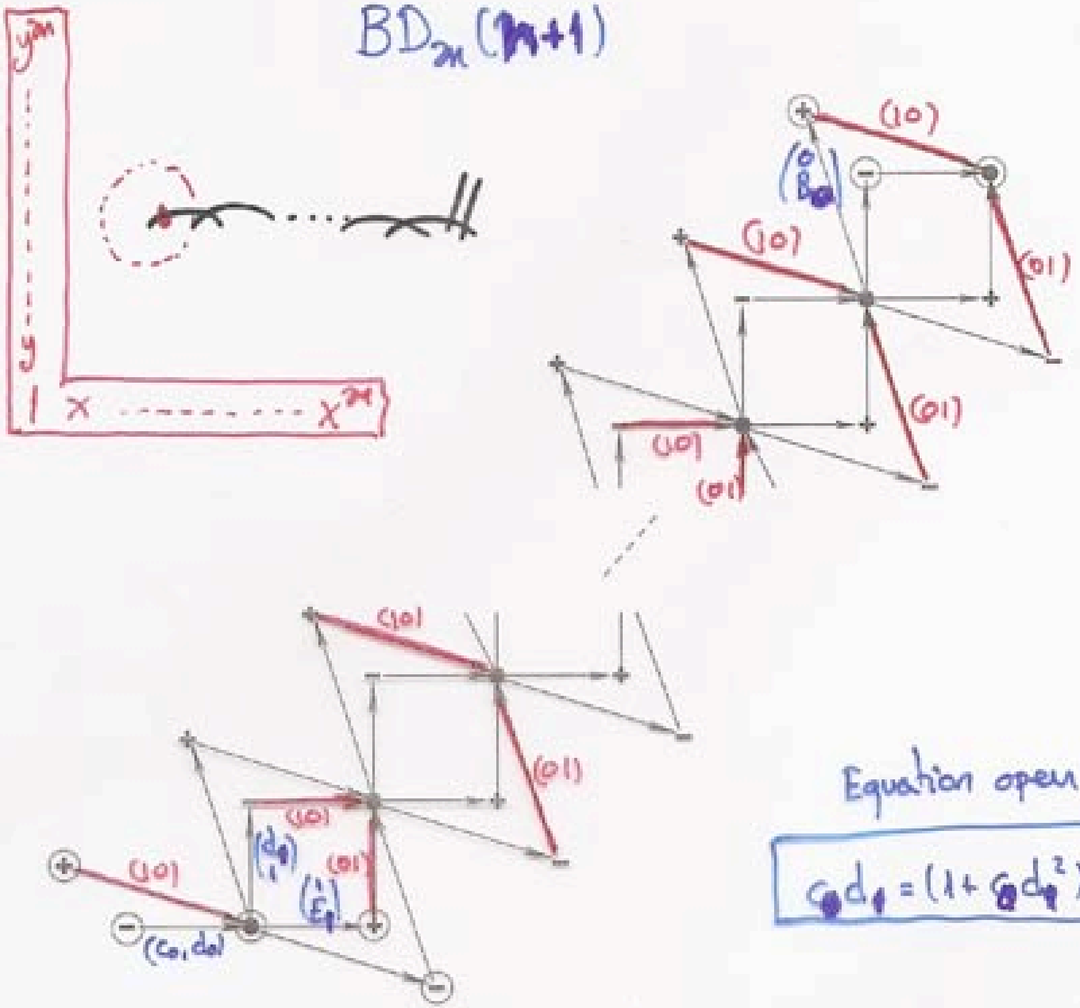
$$\Rightarrow\quad c_1d_1=-E_1F_1 \qquad \underline{F_1=-1-c_1d_1d_2}$$

$$\underline{c_1=-1-E_1E_2F_1} \qquad c_1d_2=-E_1F_1$$

⋮

$$\longrightarrow\quad \boxed{c_0d_1=E_1(1+c_0d_1^2)} \subset \mathcal{M}_\theta(Q,R)$$

$BD_{2n}(n+1)$



Equation open set:

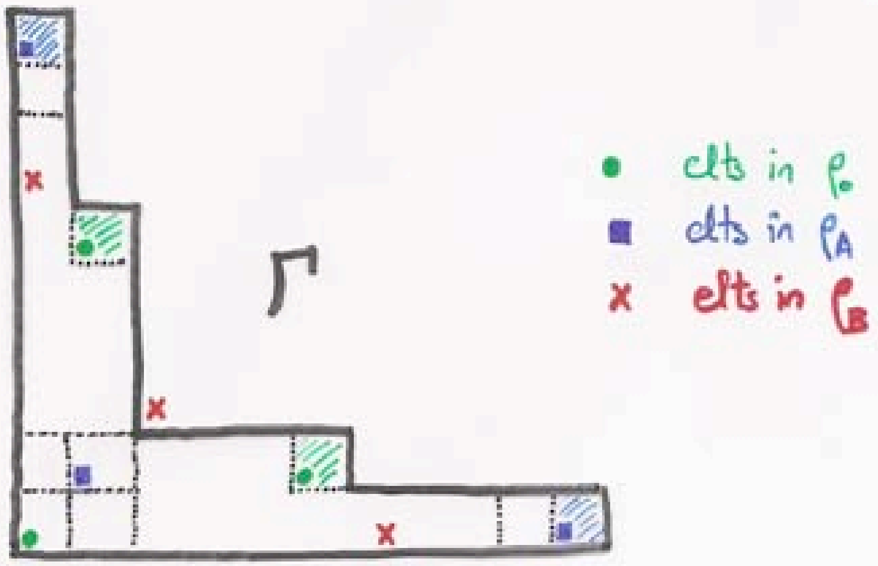
$$\boxed{c_\phi d_\phi = (1 + c_\phi d_\phi^2)\, E_\phi}$$

Ideals:

$$I_{c_0, d_\phi, E_\phi} = \left(\begin{array}{l} 2xy - d_\phi (x^2+y^2)^k (x^2-y^2) \\ 2xy (x^2+y^2)^{k-1} (x^2-y^2) - E_\phi (x^2+y^2) \\ x (x^2+y^2)^{2k-1} (x^2-y^2) - c_0 x - c_0 d_\phi \; y (x^2+y^2)^{k-1} (x^2-y^2) \\ -y (x^2+y^2)^{2k-1} (x^2-y^2) - c_0 y - c_0 d_\phi \, x (x^2+y^2)^{k-1} (x^2-y^2) \\ (x^2+y^2)^{2k-1} (x^2-y^2)^2 - B_\phi \end{array} \right)$$

Special Representations

- In general,

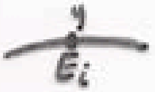


● elts in ρ_0

■ elts in ρ_A

× elts in ρ_B

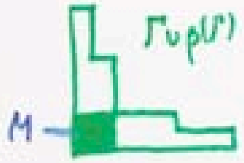
Ishii : $\underset{\substack{E_i \leftrightsquigarrow \rho_i \\ \text{excl} \quad \text{special}}}{y \in E} \Rightarrow I_y / \mathfrak{m} I_y \cong \begin{cases} \rho_i \oplus \rho_0 & \text{if} \\ \rho_i \oplus \rho_j \oplus \rho_0 & \text{if} \end{cases}$



$\Rightarrow$

$\Rightarrow$ Special representations are given by cont. frac. $\frac{2n}{a}$

- **Dimension** of the special representations :



Type A $\rightsquigarrow$ 1-dim special repr.

Type B $\rightsquigarrow$ 2-dim special repr.

$\exists$ only two types of "gluing"