ALGEBRAS OF FINITE GLOBAL DIMENSION AND SPECIAL COHEN-MACAULAY MODULES

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In my note I will present some results in joint work [IW] with Michael Wemyss on special Cohen-Macaulay modules. We start with explaining briefly the background in noncommutative algebra. After Auslander [A], algebras of finite global dimension are one of the most important subjects in representation theory. For example, famous Auslander-Reiten theory [Y] is based on certain algebras of global dimension two, called *Auslander algebras* (e.g. see [I]). We will explain the connection to special Cohen-Macaulay modules. I recommend anyone who is interested in non-commutative algebra to learn the work of Auslander (especially [A], which is available in [A2]). Additionally, my recent trial [I] to extend this a little bit.

1. Introduction

Λ any ring, M an arbitrary Λ module. We write $\mathrm{pd}M \leq n$ if there exists

$$0 \to P_n \to \dots P_0 \to M \to 0$$

such that each P_i is a projective Λ module. We define $\mathrm{gl.dim}\Lambda := \sup\{\mathrm{pd}M : M \in \mathrm{mod}\Lambda\}$. In this note we study rings with finite global dimension.

Example 1.1. (1) Λ is a commutative complete local k-algebra where k is algebraically closed. Cohen's structure theorem says that $\mathrm{gl.dim}\Lambda$ is finite if and only if Λ is power series ring.

(2) Λ finite dimensional k-algebra. Then by Artin-Wedderburn the global dimension of Λ is 0 if and only if Λ is product of matrix rings over k. The global dimension is ≤ 1 if and only if Λ is morita equivalent to the path algebra of some quiver Q.

There are quite a lot of finite dimensional algebra Λ with global dimension 2, which aren't so nice! For instance

$$\bullet \overset{\cdot\cdot}{\rightrightarrows} \bullet \overset{\cdot\cdot}{\rightrightarrows} \bullet$$

divide by arbitrary relations.

2. The setup

Fix R a commutative ring, complete local so that we get Krull-Schmidt in the category of modules. Denote by d the Krull dimension of R.

Question 2.1. (Auslander [A], 1971): (in modified form) Is there an R-algebra Λ such that

(i) global dimension of Λ is finite,
(ii) the center of Λ is R,
(iii) Λ is a finitely generated R-module.

The last two conditions show the relationship between R and Λ. The idea behind (iii) is that Λ is not much bigger than R.

Question 2.2. (Auslander [A]): Is there $M \in \mathrm{mod}R$ such that the global dimension of $\mathrm{End}_R(R \oplus M)$ is finite.

A positive answer to question 2 implies a positive answer to question 1.

In the rest of this note we will discuss this question in a more explicit setting. Note that question 2 is studied in several contexts recently.

(i) Auslander's *representation dimension* [A], which is defined by

$$\inf\{\text{gl.dimEnd}_R(R \oplus X \oplus \omega) : X \in \text{CM}(R)\} := \text{rep.dim}R$$

for the canonical module ω. The above questions are variants of this.

(ii) Cluster tilting theory as three dimensional Auslander-Reiten theory. See survey [I].

(iii) Noncommutative resolution of singularities [V].

But in (i) and (ii) these both impose more assumptions on the module M in Q2. The idea in (iii) is to construct M using the usual resolution of R and vice versa.

3. Three results

Theorem 3.1 (Auslander [A])**.** *If $d = 0$ (i.e. an Artin ring), then Q2 is true.*

In fact he showed something much stronger, he doesn't assume commutativity. In this case the module M was constructed explicitly as follows: denote by J_R the Jacobson radical of R. Since Artinian there exists m such that $J_R^m = 0$. Have

$$R = R/J_R^m \twoheadrightarrow R/J_R^{m-1} \twoheadrightarrow ... \twoheadrightarrow R/J_R \twoheadrightarrow 0$$

each surjective. Sum them up and take the endomorphism ring.

Theorem 3.2 (König [K], 1991)**.** *If $d = 1$, R is a domain (or reduced, but something similar). Then Q2 is also true.*

In this case let K be the quotient field of R. It is itself infinitely generated, but let $R_0 = R$ and define inductively $R_{i+1} = \{x \in K : xJ_{R_i} \subset J_{R_i}\}$. Get a chain

$$R = R_0 \subset R_1 \subset \ldots \subset R_m$$

which has to stop at the normalization of R, which is R_m (this is where we use the domain bit). Again take the sum of the R_i gives the required module

Want the third case $d = 2$. To do this want to introduce a nice class of modules, the CM modules. Keeping the assumptions on R being commutative complete local,

Definition 3.3. *Let $X \in \text{mod}R$. Define the depth of X to be the maximal length of X-regular sequences. Homolgically, this is equal to*

$$\inf\{i \geq 0 : \text{Ext}_R^i(R/J_R, X) \neq 0\}.$$

We call X a Cohen-Macaulay (CM) R-module if $\text{depth}X = d$

The larger the depth, the nicer the module. We denote CM(R) the category of CM modules. Some general properties:

(i) CM(R) is closed under extensions in mod$R$.

(ii) CM(R) is closed under kernels of epimorphisms.

(iii) Auslander-Buchsbaum formula: if pd$X$ is finite then

$$\text{depth}X + \text{pd}X = \text{depth}R.$$

Consequently if R is regular (i.e. global dimension is finite) then CM(R) is just the projective modules.

This leads to

Question 3.4. Define the *right representation dimension* by

$$\inf\{\text{gl.dimEnd}_R(R \oplus X) : X \in \text{CM}(R)\} := \text{r.rep.dim}R.$$

What can we say about r.rep.dim$R$?

Definition 3.5. *We call R* finite CM type *if there are only finitely many isomorphism classes of indecomposable CM R-modules.*

Example 3.6. (i) So-called simple singularity. A_n is defined as $k[[x_0, x_1, \ldots, x_d]]/f$ where $f = x_0^{n+1} + x_1^2 + x_2^2 + \ldots + x_d^2$. To see this by Knorrer periodicity reduce to either $d = 1$ (easy but a computation) or $d = 2$, which we deal with later. There are also types D_n, E_6, E_7 and E_8.

(ii) 2-dimensional quotient singularities. Take a finite subgroup $G \leq \mathrm{GL}(2, k)$ and consider $R = k[[x, y]]^G$. This has finite CM type. The proof is quite amusing - just show that $\mathrm{CM}(R) = \mathrm{add}k[[x, y]]$.

Now want to go back to Q3: want some nice CM module such that the endomorphism ring has finite global dimension.

Theorem 3.7 (Auslander [A], 1986)**.** *Suppose R is a CM ring of finite CM type. Sum them altogether and take the endomorphism ring. Then this has finite global dimension, and in fact its $\leq max\{2, d\}$.*

4. Krull dimension 2

In the rest of my note, assume the following:

(i) R is still complete local noetherian,
(ii) R is normal domain,
(iii) $d = 2$.

In this setting we have the following:

Remark 4.1. X is CM if and only if it is reflexive i.e. the natural map $X \to X^{**}$ is an isomorphism.

Consequently the functor $*$ is a duality on the category $\mathrm{CM}(R)$.
The key lemma is the following:

Lemma 4.2 (essentially Auslander [A], see also [IW])**.** *Assume $M \in \mathrm{CM}(R)$ is a generator (i.e. has R as a summand). Then the following two conditions are equivalent, for any $n \geq 0$.*

(1) $\mathrm{gl.dimEnd}_R(M) \leq n + 2$.
(2) *for any $X \in \mathrm{CM}(R)$ there is an exact sequence*

$$0 \to M_n \to \ldots \to M_0 \to X \to 0$$

with each $M_i \in \mathrm{add}M$ such that

$$0 \to (M, M_n) \to \ldots \to (M, M_0) \to (M, X) \to 0$$

is exact.

Proof. Denote $\Lambda = \mathrm{End}_R(M)$. Firstly note that we have an equivalence $(M, -) : \mathrm{add}_R M \to \mathrm{add}_\Lambda \Lambda$.

(2)⇒(1) For all $Y \in \mathrm{mod}\Lambda$ take the first two terms

$$P_1 \to P_0 \to Y \to 0$$

in the projective resolution. We can write the projectives as $P_1 = (M, M_1)$ and $P_0 = (M, M_0)$ and the map $P_1 \to P_0$ comes from $M_1 \to M_0$. Taking the kernel of this map (its CM by the depth lemma) and using (2) we get

$$0 \to M_{n+2} \to \ldots \to M_3 \to M_2 \to M_1 \to M_0$$

such that

$$0 \to (M, M_{n+2}) \to \ldots \to (M, M_3) \to (M, M_2) \to (M, M_1) \to (M, M_0)$$

is exact. But this gives projective resolution of X.
(1)⇒(2) is similar, and uses the fact its a generator. □

Now apply this to a special case

Corollary 4.3 (essentially Auslander [A]). *Let* $M \in \mathrm{CM}(R)$ *generator. Then* $\mathrm{add}M = \mathrm{CM}(R)$ *if and only if* $\mathrm{gl.dimEnd}_R(M) = 2$.

Proof. Apply key lemma to $n = 0$ to get $\mathrm{gl.dimEnd}_R(M) \leq 2$. An easy argument shows that it is in fact equality. □

Remark 4.4.
(1) From the above: $\mathrm{r.rep.dim}R \leq 2$ if and only if R has finite CM type.
(2) Most R is not finite CM type! For $d = 2$ only the quotient singularities are finite CM type.

Thus just summing all the CM isn't going to work, need some new idea. This comes from the special CM modules

Definition 4.5 (Wunram [W]). *We call* $M \in \mathrm{CM}(R)$ *special if and only if* $M \otimes_R \omega/T(M \otimes \omega)$ *is CM, where for* $X \in \mathrm{mod}R$ *define*

$$TX := \{x \in X : \exists 0 \neq r \in R, rx = 0\}.$$

Note that by definition the depth of $M \otimes_R \omega/T(M \otimes \omega)$ is always ≥ 1; the specials are those with depth 2. The problem with the definition is that it is hard to handle. Can show the following:

Lemma 4.6 ([IW]). *For* $M \in \mathrm{CM}(R)$ *the following are equivalent.*

(1) M *is special.*
(2) $\mathrm{Ext}^1_R(M, R) = 0$.
(3) $M^* \in \Omega\mathrm{CM}(R)$*, where* $\Omega\mathrm{CM}(R)$ *is the category of first syzygies of CM modules.*

Remark: denote by $\mathrm{SCM}(R)$ the category of special CM modules. The duality $*$ on the level of $\mathrm{CM}(R)$ induces a duality between $\Omega\mathrm{CM}(R)$ and $\mathrm{SCM}(R)$.

Definition 4.7. *Call* R finite SCM type *if there are only finitely many indecomposable special CM modules.*

Could also state the definition on the level of first syzygies.

Theorem 4.8 (Wunram [W]). *All rational normal surface singularities have finite SCM type.*

Converse holds? Remark: Wunram gives a 1-1 correspondence (for rational singularities) between the exceptional curves on the minimal resolution of $\mathrm{Spec}R$ and the indecomposable non-free special CM modules.

Theorem 4.9 ([IW]). *Let* M *be the sum of all the indecomposable modules in* $\Omega\mathrm{CM}(R)$*, then*

$$\mathrm{gl.dimEnd}_R(M) = \begin{cases} 2 & R\text{Gorenstein} \\ 3 & \quad \textit{else} \end{cases}$$

The idea is quite simple: use $n = 1$ case in the previous key lemma. Thus we need to show that for all $X \in \mathrm{CM}(R)$ there exists

$$0 \to M_1 \to M_0 \to X \to 0$$

such that

$$0 \to (M, M_1) \to (M, M_0) \to (M, X) \to 0$$

is exact. The map $M_0 \to X$ is constructed by taking generators of the module (M, X). These give a map $M_0 = M^n \to X$. Applying $(M, -)$ gives a surjective map, thus the map itself has to be a surjection since M is a generator. Since M_0 is a first syzygy of a CM module, the kernel of the map is also a first syzygy of a CM module, so we are done.

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