Asymptotic Stability for Some Systems of Semilinear Volterra Diffusion Equations

半線型ボルテラ拡散方程式系の漸近安定性

山田 義雄

主　論　文

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Semilinear Volterra Diffusion Equations

by

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Asymptotic Stability for Some Systems of Semilinear Volterra Diffusion Equations

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1. INTRODUCTION

Let Ω be a bounded domain in R^n with smooth boundary $\partial\Omega$. We consider the following problem for $u_i(x,t)$, $i = 1,2,\cdots,N$, with $x = (x_1,x_2,\cdots,x_n) \in \Omega$ and $t \in R^1$:

$$\frac{\partial u_i}{\partial t} = \mu_i \Delta u_i + \sum_{j=1}^{n} \left(B_{ij}u_j + \int_{-\infty}^{t} C_{ij}(t-s)u_j(s)ds\right) + f_i(u_{1,t},u_{2,t},\cdots,u_{N,t}), \qquad (i=1,2,\cdots,N), \quad \text{in} \quad \Omega \times (0,+\infty), \tag{1.1}$$

$$\frac{\partial u_i}{\partial \nu} = 0, \qquad (i=1,2,\cdots,N), \quad \text{on} \quad \partial\Omega \times (0,+\infty), \tag{1,2}$$

$$u_i = \phi_i, \qquad (i=1,2,\cdots,N), \quad \text{in} \quad \Omega \times (-\infty,0], \tag{1.3}$$

where each μ_i is a positive constant, Δ is the Laplace operator $(\sum_{j=1}^{n} \partial^2/\partial x_j^2)$, for any $t \geqq 0$ each $u_{i,t}$ represents a function on $\Omega \times (-\infty,0]$ defined by $u_{i,t}(x,\theta) = u_i(x,t+\theta)$ with $x \in \Omega$ and $\theta \in (-\infty,0]$, $\partial/\partial\nu$ denotes the outward normal derivative to $\partial\Omega$ and each ϕ_i is a given function on $\Omega \times (-\infty,0]$. In (1.1), B_{ij} and $C_{ij}(t)$ are bounded linear operators on some function spaces, $C_{ij}(t)$ are integrable over $[0,+\infty)$ in the operator norm and $f_i(\psi_1,\psi_2,\cdots,\psi_N)$ are, in a sense, higher-order terms with respect to $(\psi_1,\psi_2,\cdots,\psi_N)$, which satisfy $f_i(0,0,\cdots,0) = 0$ and a certain type of smallness condition to be made precise later.

Existence, uniqueness and stability of solutions for similar Volterra integrodifferential equations have been studied by many authors (see e.g, [1, 2,5,13,14,27,29] in the finite dimensional case and [15,17,18,20-25,28,30] in the infinite dimensional case). Among others, Schiaffino and Tesei([22,23,25]) have treated equations of the form (1.1) with (1.2) (or zero Dirichlet boundary condition) to get some sufficient conditions for the local asymptotic stability of the trivial solution within the framework of $C(\bar{\Omega})$-theory (or $C_0(\bar{\Omega})$-theory) in the case $1 \leqq n \leqq 3$.

The main purpose of the present paper is to investigate stability properties of solutions for (1.1)-(1.3) in the framework of L^p-theory for general $n \geqq 1$. By setting $u(t) = {}^t(u_1(t), u_2(t), \cdots, u_N(t)) \in L^p(\Omega; R^N)$, the problem (1.1)-(1.3) may be written in the abstract form

$$\frac{du}{dt}(t) = -A_p u(t) + Bu(t) + \int_{-\infty}^{t} C(t-s)u(s)ds + f(u_t), \quad 0 < t < +\infty, \tag{1.4}$$

$$u(t) = \phi(t) \equiv {}^t(\phi_1(t), \phi_2(t), \cdots, \phi_N(t)), \qquad -\infty < t \leqq 0, \tag{1.5}$$

where $-A_p$ is a suitable closed linear operator generating an analytic semi-group $\{\exp(-tA_p)\}$ on $L^p(\Omega; R^N)$, $B = (B_{ij})$ and $C(t) = (C_{ij}(t))$ are bounded linear operators on $L^p(\Omega; R^N)$ and $f = {}^t(f_1, f_2, \cdots, f_N)$ is a nonlinear, in general unbounded operator in $L^p(\Omega; R^N)$. Local existence and uniqueness of (strong) solutions for (1.4) and (1.5) will be obtained with use of fractional power spaces of A_p. Global existence and asymptotic behavior of solutions for (1.4) will be studied by regarding (1.4) as a nonlinear perturbation equation of

$$\frac{du}{dt} = -A_p u + Bu + \int_{-\infty}^{t} c(t-s)u(s)ds, \quad 0 < t < +\infty, \tag{1.6}$$

so that it is important to establish stability theory for (1.6).

It seems that there are two approaches to the stability analysis of (1.6). One approach is to study solutions from a point of view of semi-group theory (cf. [1,14,15]). This approach is very effective in the finite delay case (see

Hale [6] or Travis and Webb [26] for example). The other is to study asymptotic stability properties of a unique fundamental solution associated with (1.6)(cf. [5,13,22,23,25]). By a fundamental solution, we mean a map $R(t): L^p(\Omega; R^N) \to L^p(\Omega; R^N)$ with $0 \leqq t < \infty$ such that $u(t) = R(t)a$ for $a \in L^p(\Omega; R^N)$ is a solution of

$$u(t) = \exp(-t(A_p-B))a + \int_0^t \exp(-(t-s)(A_p-B)) \int_0^s C(s-r)u(r)drds, \qquad (1.7)$$

which is a mild form of (1.6) and (1.5) with initial function $\phi(t) = a$ for $t = 0$ and $\phi(t) = 0$ for $-\infty < t < 0$. The fundamental solution is very useful to give representation of mild solutions of nonhomogeneous equations

$$\frac{du}{dt} = -A_p u + Bu + \int_{-\infty}^t C(t-s)u(s)ds + g, \quad 0 < t < +\infty. \qquad (1.8)$$

Actually the mild solution of (1.8) with (1.5) is expressed, in terms of R(t), as

$$u(t) = R(t)\phi(0) + \int_0^t R(t-s)(g(s) + h(s))ds, \quad 0 \leqq t < +\infty, \qquad (1.9)$$

where $h(t) = \int_{-\infty}^0 C(t-s)\phi(s)ds$.

In this paper, stability theory for (1.6) will be developed by means of the fundamental solution because a semi-group approach brings about some disadvantages in our situation. First, when we construct a semi-group on a suitable function space in such a way as Barbu and Grossman [1] or Miller [14], it is difficult to derive decaying estimates of the semi-group directly from the characteristic equation. (Decaying properties of the semi-group constructed by Miller are obtained with the aid of asymptotic stability properties of the fundamental solution.) As is shown in Levin and Nohel [12] and Plant [17], rate of decay of solutions for functional differential equations with infinite time delay is not, in general, expoential; estimates of exponential type require a fairly strong spectral condition. Second, the semi-group is not an analytic semi-group; so that it will become complicate to treat (1.4) (which contains a nonlinear unbounded operator f) via the variation of constants formula.

We make use of the theory of Fourier-Laplace transform to get some asymptotic stability properties of the fundamental solution $R(t)$ associated with (1.6). The Laplace transform of $R(t)$ is $(\lambda + A_p - B - \hat{C}(\lambda))^{-1}$, where $\hat{C}(\lambda)$ denotes the Laplace transform of $C(t)$. However, the theory of Fourier-Laplace transform is not available for general Banach space, so that we consider (1.6) in the Hilbert space $L^2(\Omega; R^N)$ $(p = 2)$. Under a spectral condition $(\lambda + A_2 - B - \hat{C}(\lambda))u \neq 0$ for all $\operatorname{Re} \lambda \geqq 0$ and $u\ (\neq 0) \in L^2(\Omega; R^N)$ and some additional conditions, it will be shown that, for any $a \in L^2(\Omega; R^N)$, $R(t)a$ is integrable over $[0,+\infty)$ in $L^2(\Omega; R^N)$-norm and decays to zero like t^{-1} as $t \to \infty$. Moreover, by virtue of the above refered representation formula (1.9), the solution of (1.4) and (1.5), as an $L^2(\Omega; R^N)$-valued function, can be represented in terms of $R(t)$. These results will help us to derive asymptotic stability properties for (1.4) and (1.5).

Here we shall state some differences between the results of Schiaffino and Tesei and ours. In [22], [23] and [25], a fundamental solution of a certain class of partial Volterra integrodifferential equations is studied in the Banach space $C(\bar{\Omega})$ (or $C_0(\bar{\Omega})$). They have intended to establish the integrability over $[0,+\infty)$ of the fundamental solution in the operator norm, which makes it possible to carry out stability analysis for nonlinear Volterra integrodifferential equations along the same line as Grossman and Miller [5]. For this purpose, the original equation is reduced to a family of suitable approximate equations on finite dimensional spaces where the theory of Fourier-Laplace transform is available. However, the requirement $1 \leqq n \leqq 3$ is needed to ensure the validity of the approximation procedure.

The plan of this paper is as follows. In section 2 we shall give some notation and local existence results of solutions to a system of partial functional differential equations. Section 3 is devoted to the study of some asymptotic properties of the fundamental solution associated with (1.6) in $L^2(\Omega; R^N)$. In Section 4 we shall make use of the results in Section 3 to obtain the global existence and stability of solutions for (1.4) and (1.5) in the framework of L^p-theory. In Section 5 our theory will be applied to some nonlinear Volterra diffusion systems arising in mathematical biology.

In this paper, our discussion is restricted to the case of homogeneous Neumann boundary conditions, but the case of homogeneous Dirichlet boundary conditions will be treated with similarly except global stability in Section 5.

2. SOME PRELIMINARIES

2.1. Notation

Let I be a metric space and let X be a Banach space with norm $\|\cdot\|_X$. We denote by $C(I; X)$ the space of bounded and uniformly continuous functions from I to X with norm

$$\|u\|_{C(I;\,X)} = \sup_{t\in I} \|u(t)\|_X .$$

For each $\alpha > 0$, $C^\alpha(I; X)$ is the space of functions $u\colon I \to X$ such that u and its derivatives up to order $[\alpha]$ belong to $C(I; X)$, and its $[\alpha]$-derivative is uniformly Hölder continuous with exponent $\alpha - [\alpha]$ if α is not an integer. The norm of $C^\alpha(I; X)$ is defined in the standard way (see [7]). When an X-valued function is defined on I, we sometimes write $u \in C^\alpha(I'; X)$ with $I' \subset I$ to mean that the restriction of u onto I' belongs to $C^\alpha(I'; X)$.

Let Ω be a measure space. For each $1 \leq p < \infty$, $L^p(\Omega; X)$ denotes the space of measurable functions from Ω to X with finite norm

$$\|u\|_{L^p(\Omega;\, X)} = \{\int_\Omega \|u(t)\|_X^p \, dt\}^{1/p}.$$

For each $1 \leq p < \infty$ and integer $k \geq 1$, $W^{k,p}(\Omega; X)$ denotes the usual Sobolev space of measurable functions u on Ω such that u and its distributional derivatives up to order k belong to $L^p(\Omega; X)$.

For a measurable function $u\colon [0,\infty) \to X$, its Laplace transform $\hat{u}$ is defined by

$$\hat{u}(\lambda) = \int_0^\infty e^{-\lambda t}\, u(t)dt,$$

whenever this integral exists. If $u \in L^1(0,\infty; X)$, it is easily seen that $\hat{u}(\lambda)$ is analytic in $\operatorname{Re}\lambda > 0$ and continuous in $\operatorname{Re}\lambda \geq 0$.

Finally we introduce the Hardy-Lebesgue class $H^p(\beta; X)$: the class of functions $f(\lambda)$ from $\operatorname{Re}\lambda > \beta$ into X, which are analytic in $\operatorname{Re}\lambda > \beta$ and satisfy the following conditions.

(i) $\sup_{\xi>\beta} \{\int_{-\infty}^{\infty} \|f(\xi+i\eta)\|_X^p \, d\eta\}^{1/p} < \infty.$

(ii) $f(\beta+i\eta) = \lim_{\xi\to\beta} f(\xi+i\eta)$ exists almost everywhere and belongs to $L^p(-\infty,\infty; X)$.

For details, see Hille and Phillips [8, pp. 227-229].

2.2. Local Existence Results

In this subsection we shall give some preliminary results on the existence of local solutions for semilinear diffusion systems with infinite time delay.

In what follows, Ω denotes a bounded domain in R^n with smooth boundary $\partial\Omega$. Let $\alpha > 0$ and let w be a bounded and uniformly continous function from $(-\infty,\alpha]$ to $C(\bar{\Omega}; R^1)$. For any $0 \leq t \leq \alpha$, we define $w_t \in C((-\infty,0]; C(\bar{\Omega};R^1))$ by $w_t(\theta) = w(t+\theta)$ with $-\infty < \theta \leq 0$. Consider the following problem for

$u_i = u_i(x,t)$, $i = 1,2,\cdots,N$,

$$\begin{aligned} &\partial u_i/\partial t = \mu_i \Delta u_i + f_i(u_{1,t}, u_{2,t}, \cdots, u_{N,t}), && \text{in} \quad \Omega \times (0,+\infty), \\ &\partial u_i/\partial \nu = 0, && \text{on} \quad \partial\Omega \times (0,+\infty), \\ &u_i = \phi_i, && \text{in} \quad \Omega \times (-\infty,0], \end{aligned} \tag{2.1}$$

with positive constants μ_i $(i = 1,2,\cdots,N)$. Here $f = (f_1, f_2, \cdots, f_N)$ is assumed to satisfy

(F.1) $f\colon C((-\infty,0]; X_\infty) \to X_\infty$ is Lipschitz continuous on every bounded subset of $C((-\infty,0]; X_\infty)$, where $X_\infty = C(\overline{\Omega}; R^N)$ with norm

$$\|v\|_\infty = \sum_{j=1}^{N} \|v_j\|_{C(\overline{\Omega};\, R^1)} \quad \text{for} \quad v = (v_1, v_2, \cdots, v_N) \in X_\infty.$$

Given initial functions are assumed to satisfy $\phi = (\phi_1, \phi_2, \cdots, \phi_N) \in C^\sigma((-\infty,0], X_\infty)$ with $0 < \sigma < 1$: that is

$$\sup_{t \leqq 0} \|\phi(t)\|_\infty + \sup_{\substack{t,s \leqq 0 \\ t \neq s}} \frac{\|\phi(t) - \phi(s)\|_\infty}{|t-s|^\sigma} < \infty. \tag{2.2}$$

Throughout this paper, let p be a fixed positive number satisfying $p > n$ and $p \geqq 2$. We shall treat (2.1) in $X_p = L^p(\Omega; R^N)$ with norm

$$\|v\|_p = \{\sum_{j=1}^{N} \|v_j\|^p_{L^p(\Omega;\, R^1)}\}^{1/p} \quad \text{for} \quad v = (v_1, v_2, \cdots, v_N) \in X_p.$$

Define a closed linear operator A_p in X_p with domain $D(A_p)$ by

$$A_p = \begin{pmatrix} -\mu_1\Delta & 0 & \cdots & 0 \\ 0 & -\mu_2\Delta & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & -\mu_N\Delta \end{pmatrix} \tag{2.3}$$

$$D(A_p) = \{u = {}^t(u_1, u_2, \cdots, u_N) \in W^{2,p}(\Omega; R^N);\ \frac{\partial u}{\partial \nu} = 0 \ \text{ on } \ \partial\Omega\}.$$

It is well known that $-A_p$ generates an analytic semi-group of bounded linear operators $\{\exp(-tA_p)\}_{t\geqq 0}$ on X_p, which satisfies

$$\|A_p^\alpha \exp(-tA_p)u\|_p \leqq M_\alpha t^{-\alpha} e^{\gamma t}\|u\|_p \quad \text{for } \alpha \geqq 0,\ t>0 \text{ and } u\in X_p, \tag{2.4}$$

with some positive numbers M_α and γ (see Henry [7] or Krein [11]). For each $0 < \alpha < 1$, introduce the fractional power space $D(A_p^\alpha)$ equipped with the graph norm of A_p^α

$$\|u\|_{p,\alpha} = \|u\|_p + \|A_p^\alpha u\|_p \qquad \text{for } u \in D(A_p^\alpha).$$

Then, by Glushko-Krein theorem [4]([7, Theorem 1.6.1]), the following inclusion relation holds;

$$D(A_p^\alpha) \hookrightarrow C^\mu(\overline{\Omega};\ R^N) \qquad \text{if } 0 \leqq \mu < 2\alpha - \frac{n}{p}, \tag{2.5}$$

where "$\hookrightarrow$" means that the inclusion is continuous. Since $p > n$, this relation, in particular, implies

$$D(A_p^\alpha) \hookrightarrow \{u \in C^1(\overline{\Omega};\ R^N);\ \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega\},$$

provided $(p+n)/2p < \alpha < 1$. Hence, for $(p+n)/2p < \alpha < 1$, there exists a positive number c_α satisfying

$$\|u\|_\infty + \sum_{i=1}^n \|\frac{\partial u}{\partial x_i}\|_\infty \leqq c_\alpha \|u\|_{p,\alpha} \qquad \text{for all } u \in D(A_p^\alpha). \tag{2.6}$$

We rewrite (2.1) in the abstract form

$$\begin{aligned} &du(t)/dt + A_p u(t) = f(u_t), && 0 < t < +\infty, \\ &u(t) = \phi(t), && -\infty < t \leqq 0, \end{aligned} \tag{2.7}$$

where $u = {}^t(u_1, u_2, \cdots, u_N)$. We seek a *strong solution* u of (2.7) on $(-\infty, T]$ (with some $T > 0$), which satisfies (i) $u \in C((-\infty,T];\ X_p) \cap C^1((0,T];\ X_p)$, (ii) $u(t) = \phi(t)$ for $t \leqq 0$, (iii) $u(t) \in D(A_p)$ for $0 < t \leqq T$ and $A_p u \in C((0,T];\ X_p)$ and (iv) $du(t)/dt + A_p u = f(u_t)$ for $0 < t \leqq T$.

Local existence result for (2.7) can be given under the following condition on f, which is slightly weaker than (F.1).

(F.1)' As an operator from $C((-\infty,0]; X_\infty)$ to X_p, f is Lipschitz continuous on every bounded subset of $C((-\infty,0]; X_\infty)$.

THEOREM 2.1. Assume (F.1)' and $(p+n)/2p < \alpha < 1$. Then, for each ϕ satisfying (2.2) and $\phi(0) \in D(A_p^\alpha)$, there exists a positive number T such that (2.7) has a unique strong solution u on $(-\infty,T]$ satisfying $u \in C([0,T]; D(A_p^\alpha))$.

Remark 2.1. By virtue of (2.5), the solution u in Theorem 2.1 is smooth in the sense $u \in C([0,T]; C^1(\bar{\Omega}; R^N))$. Moreover, it is possible to show $u \in C([0,T]; D(A_p)) \cap C^1([0,T]; X_p)$ if $\phi(0) \in D(A_p)$.

Proof of Theorem 2.1. It is very convenient to reduce (2.7) to the integral equation

$$u(t) = \exp(-tA_p)\phi(0) + \int_0^t \exp(-(t-s)A_p)f(u_s)ds, \quad \text{with} \quad u_0 = \phi, \tag{2.8}$$

for $t \geqq 0$. One can show the existence and uniqueness of a solution $u \in C((-\infty,T]; X_\infty) \cap C([0,T]; D(A_p^\alpha))$ (with some $T > 0$) of (2.8) in the standard manner based on the contraction mapping principle (see Henry [7, Theorem 3.3.3] for example). For the sake of completeness, we shall give the proof.

Take a sufficiently large number m satisfying $m > \|\phi(0)\|_{p,\alpha}$ and define a complete metric space K by

$$K = \{u \in C([0,T]; D(A_p^\alpha)); \ u(0) = \phi(0) \ \text{ and } \sup_{0 \leqq t \leqq T} \|u(t)\|_{p,\alpha} \leqq m\},$$

where T is a positive number to be determined later. For each $u \in K$, denote the right-hand side of (2.8) by $(Su)(t)$ for $0 \leqq t \leqq T$. Since $\phi(0) \in D(A_p^\alpha)$, it is easy to see $Su \in C([0,T]; D(A_p^\alpha))$.

We now observe that, by virtue of (F.1)' and (2.6), the following inequalites hold for a suitable constant $C(m) > 0$;

$$\|f(u_t)\|_p \leqq C(m) \qquad \text{for} \quad u \in K \quad \text{and} \quad 0 \leqq t \leqq T \tag{2.9}$$

and

$$\|f(u_t) - f(v_t)\|_p \leqq C(m) \sup_{0 \leqq s \leqq t} \|u(s) - v(s)\|_{p,\alpha} \tag{2.10}$$

$$\text{for} \quad u,v \in K \quad \text{and} \quad 0 \leqq t \leqq T.$$

Therefore, it follows from (2.4) and (2.9) that

$$\sup_{0 \leqq t \leqq T} \|Su(t)\|_{p,\alpha} \leqq \|\phi(0)\|_{p,\alpha} + C(m)e^{\gamma T}\{M_0 T + M_\alpha T^{1-\alpha}/(1-\alpha)\} \tag{2.11}$$

for $u \in K$. Similarly, making use of (2.10) one can obtain

$$\begin{aligned} &\sup_{0 \leqq t \leqq T} \|Su(t) - Sv(t)\|_{p,\alpha} \\ &\leqq C(m)e^{\gamma T}\{M_0 T + M_\alpha T^{1-\alpha}/(1-\alpha)\} \sup_{0 \leqq t \leqq T} \|u(t) - v(t)\|_{p,\alpha} \end{aligned} \tag{2.12}$$

for $u,v \in K$. Since $m > \|\phi(0)\|_{p,\alpha}$, it follows from (2.11) and (2.12) that S is a strictly contraction mapping from K into itself if $T > 0$ is sufficiently small. Hence, the fixed point theorem is applied to show that (2.8) has a unique solution $u \in C((-\infty,T];\, X_\infty) \cap C([0,T];\, D(A_p^\alpha))$.

We next prove that this solution u actually satisfies (2.7). It is well known (see [7] or [11]) that, if $f(u_t)\colon (0,T] \to X_p$ is Hölder continuous, the function u given by (2.8) is a strong solution of (2.7). Therefore, in view of (F.1)' and (2.2), it suffices to show the Hölder continuity of $u\colon [0,T] \to X_\infty$. For this purpose, we employ the method used by Pazy [16, pp. 30-31].

Let $t, t+h \in [0,T]$ with $h > 0$. From (2.8) we have

$$\begin{aligned} u(t+h) - u(t) &= \{\exp(-hA_p) - I\}\exp(-tA_p)\phi(0) + \int_t^{t+h} \exp(-(t+h-s)A_p)f(u_s)ds \\ &\quad + \int_0^t \{\exp(-hA_p) - I\}\exp(-(t-s)A_p)f(u_s)ds \\ &\equiv I_1 + I_2 + I_3. \end{aligned}$$

For any $0 \leqq \beta < \alpha$, each $A_p^\beta I_i$ will be estimated separately. Since

$$A_p^\beta I_1 = \int_t^{t+h} A_p^\beta \frac{d}{ds} \exp(-sA_p)\phi(0)ds = -\int_t^{t+h} A_p^{1+\beta-\alpha}\exp(-sA_p)A_p^\alpha\phi(0)ds,$$

it follows from (2.4) that

$$\begin{aligned}\|A_p^\beta I_1\|_p &\leqq M_{1+\beta-\alpha}e^{\gamma T}\{(t+h)^{\alpha-\beta} - t^{\alpha-\beta}\}\ \|A_p^\alpha\phi(0)\|_p/(\alpha-\beta) \\ &\leqq M_{1+\beta-\alpha}e^{\gamma T}\ \|A_p^\alpha\phi(0)\|_p\ h^{\alpha-\beta}/(\alpha-\beta).\end{aligned} \tag{2.13}$$

By (2.4) and (2.9),

$$\|A_p^\beta I_2\|_p \leqq C(m)M_\beta e^{\gamma T}h^{1-\beta}/(1-\beta). \tag{2.14}$$

Use the following inequality to get a bound for $\|A_p^\beta I_3\|_p$: for any t, $t+h \in (0,T]$ (with $h > 0$), $0 \leqq \delta \leqq 1$ and $u \in X_p$,

$$\|A_p^\beta\{\exp(-hA_p) - I\}\exp(-tA_p)u\|_p \leqq M h^\delta t^{-\beta-\delta}\ \|u\|_p$$

with some $M > 0$ independent of t, h and u (see [16, Lemma 5.1] or [7, Theorem 1.4.3]). Hence,

$$\|A_p^\beta I_3\|_p \leqq MC(m)h^\delta\int_0^t (t-s)^{-\beta-\delta}ds \leqq MC(m)h^\delta t^{1-\beta-\delta}/(1-\beta-\delta) \tag{2.15}$$

if $0 < \delta < 1-\beta$. These estimates (2.13), (2.14) and (2.15) yield the Hölder continuity of $A_p^\beta u\colon [0,T] \to X_p$ with exponent $\alpha-\beta$ for any $0 \leqq \beta < \alpha$. This fact together with (2.5) implies $u \in C^{\alpha-\beta}([0,T];\ X_\infty)$ for $n/2p < \beta < \alpha$. Thus the proof is complete. Q.E.D.

Finally we state a result which will be useful in the study of stability properties.

THEOREM 2.2. Under the assumptions of Theorem 2.1, let u be the strong solution of (2.7) on $(-\infty,T]$ $(T > 0)$, which satisfies

$$\|u(t)\|_p \leqq m_1 \quad \text{and} \quad \|f(u_t)\|_p \leqq m_2, \qquad 0 \leqq t \leqq T,$$

with some $m_1, m_2 > 0$. Then the following statements hold true.

(i) If $T \geqq 1$, then for any $0 \leqq \beta < 1$,

$$\|A_p^{\beta}u(t)\|_p \leqq M_{\beta}e^{\gamma}\{m_1 + m_2/(1-\beta)\}, \qquad 1 \leqq t \leqq T.$$

(ii) If $0 < T < 1$, then, for any $0 \leqq \beta \leqq \alpha$,

$$\|A_p^{\beta}u(t)\|_p \leqq e^{\gamma}\{M_0 \|A_p^{\beta}\phi(0)\|_p + m_2M_{\beta}/(1-\beta)\}, \quad 0 \leqq t \leqq T.$$

Proof. In the case $T \geqq 1$, we have

$$u(t) = \exp(-A_p)u(t-1) + \int_{t-1}^{t} \exp(-(t-s)A_p)f(u_s)ds \tag{2.16}$$

for $1 \leqq t \leqq T$. Applying A_p^{β} $(0 \leqq \beta < 1)$ to the both sides of (2.16) and making use of (2.4) we get the assertion (i). To show (ii), it suffices to use (2.8) in place of (2.16); the conclusion is obtained similarly. Q.E.D.

Remark 2.2. Under some circumstances, maximum principle arguments are used to get an a priori estimates for $\|u(t)\|_{\infty}$. As is seen from the proofs of Theorems 2.1 and 2.2, it is possible to extend local solutions u over any inerval on which an a priori bound for $\|u(t)\|_{\infty}$ is derived.

3. STABILITY FOR LINEAR VOLTERRA DIFFUSION SYSTEMS

In this section, stability theory for linear Volterra diffusion systems with infinite time delays will be developed in the framework of L^2-theory. Let us consider the following problem for $u(x,t) = {}^t(u_1(x,t), u_2(x,t), \cdots, u_N(x,t))$,

$$\begin{aligned} &\partial u/\partial t = D\Delta u + Bu + \int_{-\infty}^{t} C(t-s)u(s)ds + g, && \text{in} \quad \Omega \times (0,+\infty), \\ &\partial u/\partial \nu = 0, && \text{on} \quad \partial\Omega \times (0,+\infty), \\ &u = \phi, && \text{in} \quad \Omega \times (-\infty,0], \end{aligned} \tag{3.1}$$

where D is an n by n diagonal matrix whose (i,i)-component is $\mu_i > 0$, and $g = {}^t(g_1, g_2, \cdots, g_N)$ and $\phi = {}^t(\phi_1, \phi_2, \cdots, \phi_N)$ are given functions defined on $\Omega \times [0,+\infty)$ and $\Omega \times (-\infty,0]$, respectively. In what follows, $B = (B_{ij})$ and $C(t) = (C_{ij}(t))$ appearing in (3.1) are assumed to satisfy the following conditions.

(B) B is a bounded linear operator from X_q $(= L^q(\Omega; R^N))$ to X_q with $q = 2$ and $q = p$.

(C)(i) For each $0 \leqq t < \infty$, $C(t)$ is a bounded linear operator from X_q to X_q with $q = 2$ and $q = p$.

(ii) $C(\cdot) \in L^1(0,+\infty; L(X_q))$ for $q = 2$ and $q = p$, where $L(X_q)$ denotes the Banach space of bounded linear operators from X_q to X_q equipped with the operator norm $|||\cdot|||_q$.

(C)' In addition to (C), $tC(\cdot) \in L^1(0,+\infty; L(X_2))$.

Remark 3.1. In this section, we have only to assume (B) and (C) for $q = 2$ to discuss the stability of (3.1) in X_2. The conditions for $q = p$ will be needed in the next section to study nonlinear perturbation problems of (3.1) in X_p.

We consider (3.1) in the Hilbert space X_2. Define a closed linear operator $\tilde{A}$ with dense domain $D(\tilde{A})$ in X_2 by

$$\tilde{A}u = -D\Delta u - Bu \quad \text{for} \quad u \in D(\tilde{A}) = \{u \in W^{2,2}(\Omega; R^N);\ \frac{\partial u}{\partial \nu} = 0 \ \text{ on } \ \partial\Omega\}.$$

Then, $-\tilde{A}$ is a generator of an analytic semi-group $\{\exp(-t\tilde{A})\}$, $t \geqq 0$, on X_2, which satisfies

$$|||\exp(-t\tilde{A})|||_2 \leqq e^{\beta t}, \quad t \geqq 0, \tag{3.2}$$

where $\beta = |||B|||_2$. The original problem (3.1) can be written in the abstract form

$$du(t)/dt + \tilde{A}u = \int_0^t C(t-s)u(s)ds + h(t), \quad t > 0,$$
$$u(0) = a, \tag{3.3}$$

with $a = \phi(0)$ and $h(t) = g(t) + \int_{-\infty}^0 C(t-s)\phi(s)ds$.

We deal with the integral equation related to (3.3):

$$u(t) = \exp(-t\tilde{A})a + \int_0^t \exp(-(t-s)\tilde{A})\{\int_0^s C(s-r)u(r)dr + h(s)\}ds, \quad t \geqq 0. \tag{3.4}$$

It is easy to see that, for each $a \in X_2$ and $h \in C([0,T]; X_2)$ with any $T > 0$, there exists a unique continuous function $u: [0,T] \to X_2$ satisfying (3.4). Such a function u is called a *mild solution* of (3.3). We define a linear operator $R(t): X_2 \to X_2$, $0 \leqq t < \infty$, by

$$R(t)a = u(t) \qquad \text{for} \quad a \in X_2, \tag{3.5}$$

where u is the mild solution of (3.3) with $h \equiv 0$. Clearly,

$$R(t) = \exp(-t\tilde{A}) + \int_0^t \exp(-(t-s)\tilde{A})\{\int_0^s C(s-r)R(r)dr\}ds \tag{3.6}$$

for $t \geqq 0$. Therefore, by virtue of (3.2),

$$\begin{aligned} |||R(t)|||_2 &\leqq e^{\beta t} + \int_0^t e^{\beta(t-s)}\{\int_0^s |||C(s-r)|||_2 |||R(r)|||_2 dr\}ds \\ &= e^{\beta t}[1 + \int_0^t e^{-\beta r}|||R(r)|||_2\{\int_0^{t-r} e^{-\beta s}|||C(s)|||_2 ds\}dr]. \end{aligned} \tag{3.7}$$

Since $C(\cdot) \in L^1(0,+\infty; L(X_2))$, an application of Gronwall's inequality to (3.7) yields

$$|||R(t)|||_2 \leqq e^{\omega t}, \quad t \geqq 0, \quad \text{with} \quad \omega = \beta + \int_0^\infty e^{-\beta t}|||C(t)|||_2 dt. \tag{3.8}$$

Summarizing these results we have

THEOREM 3.1. The operator $R(t)$ defined by (3.5) has the following properties.

(i) For each $a \in X_2$, $R(t)a$ is continuous in $t \in [0,+\infty)$.

(ii) There exists a positive number ω satisfying (3.8).

(iii) For each $a \in X_2$ and $h \in C([0,T]; X_2)$ with any $T > 0$, the mild solution u of (3.3) is given by

$$u(t) = R(t)a + \int_0^t R(t-s)h(s)ds, \quad t \geqq 0. \tag{3.9}$$

Proof. Since (i) and (ii) are evident from the preceding consideration, it suffices to show (iii).

We remark here that the Laplace transform $\hat{R}(\lambda)$ of $R(t)$ is given by

$$\hat{R}(\lambda) = (\lambda I + \tilde{A} - \hat{C}(\lambda))^{-1} \quad \text{for } \operatorname{Re} \lambda > \omega. \tag{3.10}$$

To see this, we take the Laplace transform of (3.6) and use the convolution theorem. The laplace transform of $\exp(-t\tilde{A})$ is $(\lambda I + \tilde{A})^{-1}$ for $\operatorname{Re} \lambda > \beta$; so that

$$\hat{R}(\lambda) = (\lambda I + \tilde{A})^{-1}(I + \hat{C}(\lambda)\hat{R}(\lambda)) \quad \text{for } \operatorname{Re} \lambda > \omega,$$

from which (3.10) follows.

We are going to prove (3.9) by assuming $h \in L^1(0,\infty; X_2)$ in addition to the t-continuity. Since the mild solution u satsifies (3.4), it is possible to show, in the same way as (3.8),

$$\|u(t)\|_2 \leqq (\|a\|_2 + \int_0^\infty \|h(s)\|_2 ds)e^{\omega t}, \quad t \geqq 0,$$

which enables us to define $\hat{u}(\lambda)$ for $\operatorname{Re} \lambda > \omega$. Taking the Laplace transform of (3.4) leads to

$$\hat{u}(\lambda) = (\lambda I + \tilde{A})^{-1}(a + \hat{C}(\lambda)\hat{u}(\lambda) + \hat{h}(\lambda)) \quad \text{for } \operatorname{Re} \lambda > \omega,$$

from which it follows that

$$\hat{u}(\lambda) = (\lambda I + \tilde{A} - \hat{C}(\lambda))^{-1}(a + \hat{h}(\lambda)) = \hat{R}(\lambda)(a + \hat{h}(\lambda)), \quad \text{for } \operatorname{Re} \lambda > \omega,$$

where (3.10) has been used.

On the other hand, the Laplace transform of the right-hand side of (3.9) is equal to $\hat{R}(\lambda)(a + \hat{h}(\lambda))$. Therefore, by the uniqueness theorem for Laplace transforms (see e.g., Hille and Phillips [8, Theorem 6.3.2], we obtain the validity of (3.9).

Finally we shall show (3.9) without assumig $h \in L^1(0,\infty; X_2)$. For this purpose, it is sufficient to prove (3.9) for $t \in [0,T]$ with any $T > 0$. Modify $h(t)$ for $t > T$ so that the modified function $\tilde{h}(t)$ belongs to $C([0,\infty); X_2) \cap L^1(0,\infty; X_2)$. The corresponding mild solution $\tilde{u}$ of (3.3) satisfies (3.9) for all $t \geqq 0$ (replace h by $\tilde{h}$). Since $\tilde{u}$ coincides with u on $[0,T]$, (3.5) is valid for $0 \leqq t \leqq T$. Thus the proof is complete. Q.E.D.

On account of Theorem 3.1 (iii), $R(t)$ is called a *fundamental solution* of

$$du(t)/dt + \tilde{A}u(t) = \int_0^t C(t-s)u(s)ds. \tag{3.11}$$

(In [5], Grossman and Miller call $R(t)$ a differential resolvent.)

Remark 3.2. Suppose that $|||C(t)|||_2$ is bounded on any compact subset of $[0,+\infty)$. Then, for each $0 < \alpha \leqq 1$, we can show that $R(t)$ maps $D(A_2^\alpha)$ to $D(\tilde{A}) = D(A_2)$ for $t > 0$, where A_2 is defined by (2.3) with $p = 2$, and that $u(t) = R(t)a$ for $a \in D(A_2^\alpha)$ is a strong solution of (3.3) with $h \equiv 0$ (cf. the proof of Theorem 2.1). Furthermore, if the Hölder continuity of $t \to C(t)$ in $L(X_2)$-norm is assumed, the above result holds true for every $0 \leqq \alpha \leqq 1$.

Before proceding to further investigation of asymptotic prperties of $R(t)$, it is convenient to state the following result.

LEMMA 3.2. For every $\mathrm{Re}\,\lambda \geqq 0$, one of the following statements is true.

(i) For each $v \in X_2$, there exists a unique element $u \in D(\tilde{A})$ such that $(\lambda I + \tilde{A} - \hat{C}(\lambda))u = v$.

(ii) There exists a non-trivial element $u \in D(\tilde{A})$ such that $(\lambda I + \tilde{A} - \hat{C}(\lambda))u = 0$.

This lemma is shown by the method used for the proof of F. Riesz-Schauder theorem (see e.g., Yosida [31]).

We say that

$$L(\lambda)u \equiv (\lambda I + \tilde{A} - \hat{C}(\lambda))u = 0 \tag{3.12}$$

is the *characteristic problem* associated with (3.11). If $L(\lambda)u = 0$ for some $u \not\equiv 0$, such λ is called a *characteristic value* of (3.12).

Suppose that

(A) the characteristic problem (3.12) has no characteristic values λ such that $\mathrm{Re}\,\lambda \geqq 0$.

Lemma 3.2 assures that, for every $\mathrm{Re}\,\lambda \geqq 0$, $(\lambda I + \tilde{A} - \hat{C}(\lambda))^{-1}$ exists as a bounded linear operator in X_2. It is analytic for $\mathrm{Re}\,\lambda > 0$ and continuous for $\mathrm{Re}\,\lambda \geqq 0$. Moreover, we can show

LEMMA 3.3. If (A), (B) and (C) are fulfilled, there exists a positive number K such that

$$|||(\lambda I + \tilde{A} - \hat{C}(\lambda))^{-1}|||_2 \leqq \frac{K}{1+|\lambda|} \qquad \text{for all } \mathrm{Re}\,\lambda \geqq 0. \tag{3.13}$$

Proof. It suffices to prove (3.13) for large $|\lambda|$ (with $\mathrm{Re}\,\lambda \geqq 0$). Recall $\tilde{A} = -D\Delta - B$. Since $-D\Delta$ is a non-negative self-adjoint operator in X_2,

$$|||(\lambda I - D\Delta)^{-1}|||_2 \leqq \frac{1}{|\lambda|}$$

is valid for $\mathrm{Re}\,\lambda \geqq 0$ $(\lambda \not\equiv 0)$. Hence, in view of $C \in L^1(0,\infty; L(X_2))$, there exist constants $0 < k < 1$ and $c > 0$ such that

$$|||(B + \hat{C}(\lambda))(\lambda I - D\Delta)^{-1}|||_2 \leqq \{|||B|||_2 + |||\hat{C}(\lambda)|||_2\}/|\lambda| \leqq k \tag{3.14}$$

for all $|\lambda| \geqq c$ with $\mathrm{Re}\,\lambda \geqq 0$. Consequently,

$$(\lambda I + \tilde{A} - \hat{C}(\lambda))^{-1} = (\lambda I - D\Delta)^{-1} \sum_{m=0}^{\infty} \{(B + \hat{C}(\lambda))(\lambda I - D\Delta)^{-1}\}^m$$

and, by (3.14),

$$|||(\lambda I + \tilde{A} - \hat{C}(\lambda))^{-1}|||_2 \leqq (1-k)^{-1}|\lambda|^{-1},$$

which completes the proof. Q.E.D.

We are now ready to establish asymptotic stability properties of $R(t)$.

THEOREM 3.4. Let (A), (B) and (C) be satisfied.

(i) For each $2 \leqq q < \infty$, there exists a positive number N_q such that

$$\int_0^{\infty} \|R(t)a\|_2^q\,dt \leqq N_q \|a\|_2^q \quad \text{for all} \quad a \in X_2. \tag{3.15}$$

(ii) Assume (C)' in place of (C). Then for each $1 \leqq q < \infty$ there exists a positive number N_q satisfying (3.15). Moreover,

$$(1+t)\|R(t)a\|_2 \leqq M\|a\|_2 \quad \text{for all} \quad t \geqq 0 \quad \text{and} \quad a \in X_2, \tag{3.16}$$

with some $M > 0$.

Proof. (i) Set $u(t) = R(t)a$ for $a \in X_2$. Then, $\hat{u}(\lambda) = (\lambda I + \tilde{A} - \hat{C}(\lambda))^{-1}a$ for $\mathrm{Re}\,\lambda \geqq 0$, by (3.10). Since $\hat{u}(\lambda)$ is analytic for $\mathrm{Re}\,\lambda > 0$ and continuous for $\mathrm{Re}\,\lambda \geqq 0$, it follows from Lemma 3.3 that $\hat{u}(\lambda)$ belongs to the Hardy-Lebesgue class $H^r(0; X_2)$ for any $r > 1$ (see the definition in Section 2). In particular, for $r = 2$, it is possible to follow the arguments of K. Yosida [31, pp. 163-165] to show that

$$u(t) = \frac{1}{2\pi} \lim_{m\to\infty} \int_{-m}^{m} e^{it\eta}\hat{u}(i\eta)\,d\eta \tag{3.17}$$

holds for $t \geqq 0$ in the sense of $L^2(-\infty,+\infty; X_2)$ and that the right-hand side of (3.17) vanishes for $t < 0$. Hence, Parseval's equality, together with Lemma 3.3, gives

$$\int_0^\infty \|u(t)\|_2^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \|\hat{u}(i\eta)\|_2^2 d\eta$$
$$\leqq \frac{1}{2\pi}\int_{-\infty}^{\infty} K^2(1+|\eta|)^{-2}d\eta\ \|a\|_2^2 \equiv N_2\ \|a\|_2^2 . \tag{3.18}$$

Moreover, making use of the interpolation theorem of M. Riesz-Thorin (see e.g., [19, Theorems 9.8 and 9.17]), we can derive the following estimate: if $2 \leqq q < \infty$ and $1/q + 1/r = 1$, then

$$\{\int_0^\infty \|u(t)\|_2^q dt\}^{1/q} \leqq c\{\int_{-\infty}^{\infty} \|\hat{u}(i\eta)\|_2^r d\eta\}^{1/r} \leqq cK\{\int_{-\infty}^{\infty} (1+|\eta|)^{-r}d\eta\}^{1/r}\ \|a\|_2 ,$$

with some $c > 0$. Thus (3.15) has been proved.

(ii) We shall employ the technics used by Friedman and Shinbrot [3]. The Laplace transform of $-tu(t)$ is $d\hat{u}(\lambda)/d\lambda$, which is given by

$$\frac{d}{d\lambda}\ \hat{u}(\lambda) = \frac{d}{d\lambda}(\lambda I + \tilde{A} - \hat{C}(\lambda))^{-1}a$$
$$= -(\lambda I + \tilde{A} - \hat{C}(\lambda))^{-1}(I - \frac{d}{d\lambda}\hat{C}(\lambda))(\lambda I + \tilde{A} - \hat{C}(\lambda))^{-1}a.$$

Since $d\hat{C}(\lambda)/d\lambda$ is the Laplace transform of $-tC(t) \in L^1(0,\infty;\ X_2)$ (by (C)'), we can find, with use of Lemma 3.3, a positive number K_1 such that

$$\|\frac{d}{d\lambda}\hat{u}(\lambda)\|_2 \leqq K_1(1+|\lambda|)^{-2}\ \|a\|_2 \qquad \text{for } a \in X_2 \text{ and } \operatorname{Re}\lambda \geqq 0, \tag{3.19}$$

which implies $d\hat{u}(\lambda)/d\lambda \in H^r(0;\ X_2)$ for any $r \geqq 1$. The identity

$$-tu(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{it\eta}(\frac{d\hat{u}}{d\lambda})(i\eta)d\eta$$

being valid with $u(t) \equiv 0$ for $t < 0$, (3.19) enables us to derive (3.16).

Now observe that Parseval's equality together with (3.19) yields

$$\int_0^\infty t^2\ \|u(t)\|_2^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \|(\frac{d\hat{u}}{d\lambda})(i\eta)\|_2^2 d\eta \leqq K_1'\ \|a\|_2^2 \tag{3.20}$$

with some $K_1' > 0$. Consequently, by (3.18) and (3.20),

$$\int_0^\infty \|u(t)\|_2 dt = \int_0^1 \|u(t)\|_2 dt + \int_1^\infty \|u(t)\|_2 dt$$
$$\leqq \{\int_0^1 \|u(t)\|_2^2 dt\}^{1/2} + \{\int_1^\infty t^{-2}dt\}^{1/2}\{\int_1^\infty t^2\ \|u(t)\|_2^2 dt\}^{1/2}$$

$$\leqq (\sqrt{N_2} + \sqrt{K_1'}\,) \|a\|_2 .$$

Thus we have shown (3.15) for $q = 1$ and $2 \leqq q < \infty$. The results for $1 < q < 2$ follow from those for $q = 1$ and $q = 2$ by Hölder's inequality. Q.E.D.

COROLLARY 3.5. In addition to (A), (B) and (C), assume $t^j C(t) \in L^1(0,+\infty; L(X_2))$ for $j = 1,2,\cdots,m$. Then

$$\sup_{t \geqq 0} \{(1+t)^m \|R(t)a\|_2\} < \infty \quad \text{for all } a \in X_2.$$

Proof. Put $u(t) = R(t)a$. Since $d^j\hat{u}(\lambda)/d\lambda^j$ is the Laplace transform of $(-t)^j u(t)$ for any $j = 1,2,\cdots,m$, we see from our assumptions that

$$\|d^j\hat{u}(\lambda)/d\lambda^j\|_2 = \|(d^j/d\lambda^j)(\lambda I + \tilde{A} - \hat{C}(\lambda))^{-1}a\|_2 \leqq K_j(1+|\lambda|)^{-2} \|a\|_2$$

with some $K_j > 0$. Hence, the conclusion easily follows by using

$$(-t)^j u(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{it\eta}(d^j\hat{u}/d\lambda^j)(i\eta)\, d\eta. \qquad \text{Q.E.D.}$$

Theorem 3.4 and Corollary 3.5 give conditions under which the fundamental solution behaves asymptotically like an inverse power of t. To derive asymptotic stability of exponential type, we need fairly stronger assumptions (cf. Barbu-Grossman [1]).

COROLLARY 3.6. In addition to (A), (B) and (C), let $e^{\delta t}C(\cdot) \in L^1(0,\infty; L(X_2))$ be fulfilled for some $\delta > 0$. If (3.12) has no characteristic values for $\mathrm{Re}\,\lambda \geqq -\rho$ with some $0 < \rho < \delta$, there exists a positive number M' such that

$$e^{\rho t}\|R(t)a\|_2 \leqq M' \|a\|_2 \quad \text{for all } t \geqq 0 \text{ and } a \in X_2.$$

Proof. Note that $v(t) = e^{\rho t}R(t)a$ is a mild solution of (3.11) with $\tilde{A}$ and $C(t)$ replaced by $\tilde{A}-\rho$ and $e^{\rho t}C(t) \in L^1(0,\infty; L(X_2))$. Therefore, the related chacteristic problem is

$$L(\lambda-\rho)u = \{(\lambda-\rho)I + \tilde{A} - \hat{C}(\lambda-\rho)\}u = 0,$$

which has no characteristic values λ such that $\mathrm{Re}\,\lambda \geqq 0$. By virtue of Theorem 3.4, we get the conclusion. Q.E.D.

Remark 3.3. When $\tilde{A}$ and $C(t)$ in (3.11) represent N by N matrices such that $C(\cdot) \in L^1(0,\infty)$, Grossman and Miller [5] established the equivalence of (i) the asymptotic stability of the trivial solution, (ii) the integrability of $R(t)$ and (iii) $\det(\lambda I + \tilde{A} - \hat{C}(\lambda)) \neq 0$ for $\mathrm{Re}\,\lambda \geqq 0$. For the infinite dimensional case, Schiaffino and Tesei [22] have treated (3.11) in a Banach space X by assuming some spectral condition (which is essentially the same as (A)) and the existence of suitable projection operators onto finite dimensional invariant subspaces. In [22] approximate problems on finite dimensional spaces are considered to derive the integrability of $\|R(t)\|_{L(X)}$ itself, because the theory of Fourier transform is not available for infinite dimensional Banach spaces. The same idea is used in [23] and [25].

In contrast to the result of [22], we are contented with the integrability of $\|R(t)a\|_2$ for every $a \in X_2$. Hence the theory of Fourier-Laplace transform can be applied to our stability analysis without reduction onto finite dimensional spaces.

Finally we shall show asymptotic behavior of mild solutions for inhomogeneous equations (3.3).

THEOREM 3.7. Assume (A), (B) and (C)'. For every $a \in X_2$ and $h \in L^1(0,\infty; X_2) \cap C([0,T]; X_2)$ with any $T > 0$, the mild solution u of (3.3) satisfies

$$\int_0^\infty \|u(t)\|_2 dt < \infty \quad \text{and} \quad \lim_{t\to\infty} \|u(t)\|_2 = 0. \tag{3.21}$$

Proof. Use representation (3.9) in Theorem 3.1. By virtue of Theorem 3.4, one obtains

$$\|u(t)\|_2 \leqq M\{(t+1)^{-1}\|a\|_2 + \int_0^t (t-s+1)^{-1}\,\|h(s)\|_2 ds\}.$$

The integral term in the bracket is the convolution of an L^1-function with a bounded function tending to zero as $t \to \infty$; so that it converges to zero as $t \to \infty$. Hence $\lim_{t\to\infty} \|u(t)\|_2 = 0$. We next make use of (3.15); then

$$\int_0^\infty \|u(t)\|_2 dt \leqq \int_0^\infty \|R(t)a\|_2\, dt + \int_0^\infty \{\int_0^t \|R(t-s)h(s)\|_2 ds\} dt$$

$$= \int_0^\infty \|R(t)a\|_2\, dt + \int_0^\infty \{\int_s^\infty \|R(t-s)h(s)\|_2 dt\} ds$$

$$\leqq N_1\{\|a\|_2 + \int_0^\infty \|h(s)\|_2 ds\} < \infty.$$

Q.E.D.

Remark 3.4. Let (B) and (C)' be fulfilled and let $a \in X_2$ and $h \in L^1(0,\infty; X_2) \cap C([0,T]; X_2)$ for any $T > 0$. Then the condition (A) is necessary and sufficient for every mild solution of (3.3) to have properties (3.21). Indeed, the sufficiency part is Theorem 3.7 itself and the necessity part can be proved essentially in the same way as Miller [13, Corollary 1].

4. STABILITY FOR SEMILINEAR VOLTERRA DIFFUSION SYSTEMS

The preceding results will be used to investigate asymptotic stability properties for the initial boundary value problem (1.1)-(1.3); this problem is regarded as a nonlinear perturbation problem of (3.1). We assume that the linear operators $B = (B_{ij})$ and $C(t) = (C_{ij}(t))$ fulfill conditions (B) and (C)' in Section 3. Moreover, the nonlinear operator $f = {}^t(f_1, f_2, \cdots, f_N)$ satisfies, besides (F.1)',

(F.2) For each $q = 2$, p, there exists a continuous increasing function Φ_q with $\Phi_q(0) = 0$ such that

$$\|f(\psi)\|_q \leqq \Phi_q(\sup_{-\infty<\theta\leqq 0} \|\psi(\theta)\|_\infty) \|\psi(0)\|_q, \tag{4.1}$$

or

$$\|f(\psi)\|_q \leqq \Phi_q(\sup_{-\infty<\theta\leqq 0} \|\psi(\theta)\|_\infty) \int_{-\infty}^0 k(-\theta)\|\psi(\theta)\|_q d\theta, \tag{4.2}$$

for all $\psi \in C((-\infty,0]; X_\infty)$. In (4.2), k is a non-negative function such that k, $tk \in L^1(0,\infty; R^1)$.

Typical examples of such f are given by N-vectors composed of functions of the form

$$\psi_i(x,0)\int_{-\infty}^0\int_\Omega G(x,y,\theta)\psi_j(y,\theta)dyd\theta, \quad \text{or} \quad \int_{-\infty}^0\int_\Omega G(x,y,\theta)\psi_i(y,\theta)\psi_j(y,\theta)dyd\theta,$$

$$i,j = 1,2,\cdots,N,$$

with an appropriate function G.

We now treat with (1.1)-(1.3) in X_p: for $u = {}^t(u_1,u_2,\cdots,u_N)$,

$$\begin{aligned} &du(t)/dt + A_p u(t) = Bu(t) + \int_{-\infty}^t C(t-s)u(s)ds + f(u_t), && 0 < t < +\infty, \\ &u(t) = \phi(t), && -\infty < t \leqq 0, \end{aligned} \tag{4.3}$$

where A_p is defined by (2.3) and a given initial function $\phi = {}^t(\phi_1,\phi_2,\cdots,\phi_N)$ is assumed to satisfy

$$\begin{aligned} &\phi \in C^\sigma((-\infty,0]; X_\infty) \text{ with } 0 < \sigma < 1 \text{ and} \\ &\phi(0) \in D(A_p^\alpha) \text{ with } (p+n)/2p < \alpha < 1. \end{aligned} \tag{4.4}$$

Theorem 2.1 assures the existence of some interval $(-\infty,T_0]$ (with $T_0 > 0$) on which (4.3) has a unique strong solution u satisfying $u \in C([0,T_0]; D(A_p^\alpha))$. In order to extend this local solution over $(-\infty,T_1]$ with $T_1 > T_0$, it is sufficient to get an a priori estimate for $\|u(t)\|_{p,\alpha}$ on $[0,T_1]$.

Our main theorem reads as follows.

THEOREM 4.1. Let (A) be satisfied with $\tilde{A} = -D\Delta - B$ and $D(\tilde{A}) = \{u \in W^{2,2}(\Omega; R^N); \partial u/\partial\nu = 0 \text{ on } \partial\Omega\}$. Then, for any $0 < \varepsilon \leqq \varepsilon_0$ with some ε_0, there exists a positive number $\delta(\varepsilon)$ such that, if

$$\max\{\|\phi(0)\|_{p,\alpha},\ \sup_{\theta\leqq 0}\|\phi(\theta)\|_\infty\} \leqq \delta(\varepsilon),$$

then (4.3) has a unique strong solution u on $(-\infty,+\infty)$ satisfying

$$\|u(t)\|_{p,\alpha} \leqq \varepsilon \qquad \text{for all} \quad 0 \leqq t < +\infty$$

and

$$\lim_{t\to\infty} \|u(t)\|_{p,\beta} = 0 \qquad \text{for any} \quad 0 \leqq \beta < 1.$$

As a simple consequence of Theorem 4.1, we can show (use (2.6))

COROLLARY 4.2. The solution u in Theorem 4.1 satisfies

$$\lim_{t\to\infty} u(x,t) = 0 \quad \text{uniformly for} \quad x \in \overline{\Omega}.$$

Remark 4.1. Consider semilinear parabolic differential equations (without time delay) of the form

$$du/dt + Au = f(u),$$

where f is a nonlinear higher-order term satisfying $f(0) = 0$. It is well known (see Kielhöfer [9],[10] for example) that the trivial solution is asymptotically stable in a suitable sense if every point of the spectrum of $-A$ has a negative real part. This spectral condition corresponds to (A) in our time delay case.

Proof Theorem 4.1. We prove this theorem by assuming (4.1) in (F.2); the proof in the case where (4.2) is fulfilled will be carried out similarly.

Set

$$\delta = \max\{\|\phi(0)\|_{p,\alpha},\ \sup_{\theta\leqq 0}\|\phi(\theta)\|_\infty\}.$$

By (F.2), there exists a positive number ε_0 such that

$$N_1\Phi_2(c_\alpha\varepsilon_0) < 1, \tag{4.4}$$

where N_1 and c_α are positive constants in (3.15) and (2.6), respectively. For any $0 < \varepsilon \leqq \varepsilon_0$, we take any initial function ϕ such that

$$\delta \leqq c_\alpha \varepsilon \quad \text{and} \quad \delta < \varepsilon. \tag{4.5}$$

Suppose for the time being that, foy any fixed $T > 0$, (4.3) has a strong solution u on $(-\infty,T]$ such that $u \in C([0,T]; D(A_p^\alpha))$ and

$$\|u(t)\|_{p,\alpha} \leqq \varepsilon \qquad \text{for} \quad 0 \leqq t \leqq T. \tag{4.6}$$

This estimate, together with (2.6) and (4.5), yields

$$\|u(t)\|_\infty \leqq c_\alpha \varepsilon \qquad \text{for} \quad -\infty < t \leqq T. \tag{4.7}$$

We shall derive various estimates of u, which will depend on ε and δ, by assuming (4.6) (and, therefore, (4.7)). It is very convenient to represent the solution, as a function from $[0,T]$ to X_2, by means of the fundamental solution $R(t)$ constructed in Section 3 (use Theorem 3.1):

$$u(t) = R(t)\phi(0) + \int_0^t R(t-s)\{f(u_s)+h(s)\}ds, \quad \text{with} \quad u_0 = \phi. \tag{4.8}$$

Here $h(t) = \int_{-\infty}^0 C(t-s)\phi(s)ds$, which satisfies

$$\|h(t)\|_p \leqq \sup_{\theta\leqq 0} \|\phi(\theta)\|_p \int_0^\infty |||C(s)|||_p ds \quad \text{for all} \quad t \geqq 0, \tag{4.9}$$

and

$$\int_0^\infty \|h(s)\|_2 ds \leqq \sup_{\theta\leqq 0} \|\phi(\theta)\|_2 \int_0^\infty s|||C(s)|||_2 ds. \tag{4.10}$$

By (4.1) and (4.7), $\|f(u_s)\|_2 \leqq \Phi_2(c_\alpha\varepsilon) \|u(s)\|_2$ for $0 \leqq s \leqq T$; so that application of Theorem 3.4 to (4.8) gives

$$\|u(t)\|_2 \leqq M\{\|\phi(0)\|_2 + \int_0^\infty \|h(s)\|_2 ds + \Phi_2(c_\alpha\varepsilon)\int_0^t \|u(s)\|_2 ds\}, \tag{4.11}$$

for $0 \leqq t \leqq T$. We use Theorem 3.4 again to derive

$$\int_0^t \|u(s)\|_2 ds \leqq \int_0^t \|R(s)\phi(0)\|_2 ds + \int_0^t [\int_0^s \|R(s-r)\{f(u_r) + h(r)\}\|_2 dr]ds$$

$$= \int_0^t \|R(s)\phi(0)\|_2 ds + \int_0^t [\int_r^t \|R(s-r)\{f(u_r) + h(r)\}\|_2 ds] dr$$

$$\leqq N_1[\ \|\phi(0)\|_2 + \int_0^t \{\ \|f(u_r)\|_2 + \ \|h(r)\|_2\} dr].$$

Since $\int_0^t \|f(u_r)\|_2 dr$ is bounded by $\Phi_2(c_\alpha\varepsilon)\int_0^t \|u(r)\|_2 dr$ and $N_1\Phi_2(c_\alpha\varepsilon) \leqq N_1\Phi_2(c_\alpha\varepsilon_0) < 1$ by (4.4), we can show after some rearrangements

$$\int_0^t \|u(s)\|_2 ds \leqq N_1\{\|\phi(0)\|_2 + \int_0^\infty \|h(s)\|_2 ds\}/(1-N_1\Phi_2(c_\alpha\varepsilon)), \quad 0 \leqq t \leqq T.$$

In view of (4.10), the above estimate implies

$$\int_0^t \|u(s)\|_2 ds \leqq K_1(\varepsilon;\delta), \qquad 0 \leqq t \leqq T, \tag{4.12}$$

for some constant $K_1(\varepsilon;\delta)$ (depending on ε and δ, but not on T), which is a continuous increasing function of $\delta \geqq 0$ for each $\varepsilon \geqq 0$ and satisfies $K_1(\varepsilon;0) = 0$. We shall denote such positive constants by $K_i(\varepsilon;\delta)$ $(i = 1,2,3,\cdots)$. The substitution of (4.12) into (4.11) yields

$$\|u(t)\|_2 \leqq K_2(\varepsilon;\delta), \qquad 0 \leqq t \leqq T, \tag{4.13}$$

which, together with (4.7), gives by virtue of Hölder's inequality

$$\|u(t)\|_p \leqq K_3(\varepsilon;\delta), \qquad 0 \leqq t \leqq T, \tag{4.14}$$

with some $K_3(\varepsilon;\delta)$.

We now invoke Theorem 2.2 to evaluate $\|A_p^\alpha u(t)\|_p$. By (4.1), (4,7) and (4.9),

$$\begin{aligned} &\|Bu(t) + \int_{-\infty}^t C(t-s)u(s)ds + f(u_t)\|_p \\ &\leqq K_3(\varepsilon;\delta)\{\,|||B|||_p + \Phi_p(c_\alpha\varepsilon) + \int_0^t |||C(s)|||_p ds\} + \|h(t)\|_p \leqq K_4(\varepsilon;\delta), \end{aligned} \tag{4.15}$$

with some $K_4(\varepsilon;\delta) > 0$. Therefore, (4.14) and (4.15) enables us to apply Theorem 2.2 to derive

$$\|A_p^{\alpha}u(t)\|_p \leqq K_5(\varepsilon;\delta), \qquad 0 \leqq t \leqq T. \tag{4.16}$$

It follows from (4.14) and (4.16) that

$$\|u(t)\|_{p,\alpha} \leqq K_3(\varepsilon;\delta) + K_5(\varepsilon;\delta) \equiv K_6(\varepsilon;\delta) \quad \text{for} \quad 0 \leqq t \leqq T.$$

Choose a small number $\delta(\varepsilon) > 0$ such that $K_6(\varepsilon;\delta(\varepsilon)) < \varepsilon$; this is possible by the property of $K_i(\varepsilon;\delta)$. When the initial function ϕ satisfies $\delta \leqq \delta(\varepsilon)$, the preceding consideration shows that, if $\|u(t)\|_{p,\alpha} \leqq \varepsilon$ on $[0,T]$, then u satisfies $\|u(t)\|_{p,\alpha} \leqq K_6(\varepsilon;\delta(\varepsilon)) < \varepsilon$ on $[0,T]$. From this fact we can conclude that, for any $0 < \delta \leqq \delta(\varepsilon)$, estimate (4.6) actually holds true on $[0,T]$. Since $T > 0$ is arbitrary, (4.3) has a (unique) strong solution u on $(-\infty,+\infty)$ satisfying

$$\|u(t)\|_{p,\alpha} \leqq \varepsilon \qquad \text{for} \quad 0 \leqq t \leqq +\infty. \tag{4.17}$$

Thus the stability part of Theorem 4.1 has been established.

Before proving

$$\lim_{t\to\infty} \|u(t)\|_{p,\beta} = 0 \quad \text{for any} \quad 0 \leqq \beta < 1, \tag{4.18}$$

we observe here that Theorem 2.2 (i) combined with (4.14) and (4.15) (which hold on $[0,+\infty)$) gives

$$\sup_{t\geqq 1} \|A_p^{\gamma}u(t)\|_p < \infty \qquad \text{for any} \quad 0 \leqq \gamma < 1. \tag{4.19}$$

We shall employ the usual energy method to show (4.18). Multiplying (1.1) by u_i and integrating over Ω we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|u(t)\|_2^2 + \sum_{i=1}^{n} \mu_i \|\operatorname{grad} u_i(t)\|_2^2 \\ &= (Bu(t) + \int_{-\infty}^{t} C(t-s)u(s)ds + f(u_t),u(t))_2, \end{aligned} \tag{4.20}$$

where $|\operatorname{grad} v|^2 = \sum_{i=1}^{n} |\partial v/\partial x_i|^2$ and $(\cdot,\cdot)_2$ denotes the inner product of X_2.

The right-hand side of (4.20) is majorized by

$$\{|||B|||_2 + \Phi_2(c_\alpha\varepsilon)\} \|u(t)\|_2^2 + \{\sup_{-\infty<s\leqq t} \|u(s)\|_2 \int_0^\infty |||C(t)|||_2 dt\} \|u(t)\|_2,$$

which is integrable on $[0,+\infty)$ because (4.12) and (4.13) remain valid for $0 \leqq t < +\infty$. Therefore, integrating (4.20) with respect to t, we see $\|\mathrm{grad}\, u(\cdot)\|_2 \in L^2(0,+\infty;\, R^1)$, which assures $(d/dt)\|u(t)\|_2^2 \in L^1(0,+\infty;\, R^1)$ (use (4.20) again). Consequently, since $\|u(\cdot)\|_2^2 \in L^1(0,+\infty;\, R^1)$, we have

$$\lim_{t\to\infty} \|u(t)\|_2 = 0. \tag{4.21}$$

Moreover, $\|u(t)\|_\infty$ being bounded for all $t \geqq 0$ by (4.17) and (2.6), it follows from (4.21), with the aid of Hölder's inequality, that

$$\lim_{t\to\infty} \|u(t)\|_p = 0. \tag{4.22}$$

We invoke moment inequality for fractional powers of A_p (see Krein [11, Chapter 1]); for $0 < \beta < \gamma$ and $u \in D(A_p^\gamma)$

$$\|A_p^\beta u\|_p \leqq c(\beta,\gamma)\, \|u\|_p^{(\gamma-\beta)/\gamma}\, \|A_p^\gamma u\|_p^{\beta/\gamma}, \tag{4.23}$$

which, together with (4.19) and (4.22), proves (4.18). Q.E.D.

Remark 4.2. In [23] and [25], Schiaffino and Tesei have treated equations of the form (4.1) in the framework of $X = C(\overline{\Omega})$ (or $C_0(\overline{\Omega})$). After showing that the related fundamental solution $R(t)$ belongs to $L^1(0,+\infty;\, L(X))$ (see Remark 3.3), they have established the X-asymptotic stability of the trivial solution along the same line as Grossman and Miller [5]. However, some regularity results are needed to ensure $R \in L^1(0,+\infty;\, L(X))$, which restricts the space dimension $(n \leqq 3)$.

We have obtained the local asymptotic stability of the trivial solution of (4.3) in the sense of Theorem 4.1. However, it seems difficult to know the decaying rate of the bounded global solution in Theorem 4.1. The decaying estimate is given by assuming the stronger spectral condition.

THEOREM 4.3. Assume that $e^{\delta t}C(\cdot) \in L^1(0,+\infty; L(X_2))$ for some $\delta > 0$ and that the characteristic problem (3.12) has no characteristic values for $\mathrm{Re}\,\lambda \geqq -\rho$ with some $0 < \rho < \delta$. (The function k in (4.2) should satisfy $e^{\rho t} k \in L^1(0,+\infty; R^1)$.) Then there exists a positive number ε_1 with the following property: for each $0 \leqq \beta < 1$ and $0 < \varepsilon \leqq \varepsilon_1$, any strong solution u of (4.3) satisfying $\sup_{-\infty<t<+\infty} \|u(t)\|_\infty \leqq \varepsilon$ decays like

$$\|A_p^\beta u(t)\|_p = O(e^{-\zeta t}) \qquad \text{as} \quad t \to \infty, \tag{4.24}$$

with some $\zeta = \zeta(p,\beta,\varepsilon) > 0$.

Proof. We prove this theorem in the case where (4.1) is fulfilled.

Let u be the strong solution of (4.3) such that $\sup_{-\infty<t<+\infty} \|u(t)\|_\infty \leqq \varepsilon$. The solution is given by (4.8) and, by Corollary 3.6, the fundamental solution satisfies $|||R(t)|||_2 \leqq M'e^{-\rho t}$, $0 \leqq t < +\infty$, with some $M' > 0$. Therefore,

$$\begin{aligned} e^{\rho t} \|u(t)\|_2 &\leqq M'[\, \|\phi(0)\|_2 + \int_0^t e^{\rho s}\{ \|f(u_s)\|_2 + \|h(s)\|_2\}ds] \\ &\leqq M'[\, \|\phi(0)\|_2 + \sup_{\theta\leqq 0} \|\phi(\theta)\|_2 \int_0^t e^{\rho s}\{\int_s^\infty |||C(r)|||_2 dr\}ds \\ &\qquad + \Phi_2(\varepsilon)\int_0^t e^{\rho s} \|u(s)\|_2 ds] \\ &\leqq M'' + M'\Phi_2(\varepsilon)\int_0^t e^{\rho s} \|u(s)\|_2 ds, \end{aligned} \tag{4.25}$$

with a positive constant M''. An application of Gronwall's inequality to (4.25) leads to

$$e^{\rho t} \|u(t)\|_2 \leqq M''\exp\{M'\Phi_2(\varepsilon)t\} \qquad \text{for} \quad 0 \leqq t < +\infty.$$

If we take $\varepsilon_1 > 0$ such that $M'\Phi_2(\varepsilon_1) = \rho$, then, for any $0 < \varepsilon < \varepsilon_1$,

$$\|u(t)\|_2 = O(\exp(-\zeta_1 t)) \quad \text{with} \quad \zeta_1 = \rho - M'\Phi_2(\varepsilon) \quad \text{as} \quad t \to \infty.$$

Hence, by the uniform boundedness of $\|u(t)\|_\infty$ and Hölder's inequality, we get

$$\|u(t)\|_p = O(\exp(-\zeta_2 t) \quad \text{with} \quad \zeta_2 = 2\zeta_1/p \quad \text{as} \quad t \to \infty. \tag{4.26}$$

Since Theorem 2.2 assures $\sup_{t \geq 1} \|A_p^\gamma u(t)\|_p < \infty$ for any $0 \leq \gamma < 1$, (4.24) easily follws from (4.23) and (4.26). Q.E.D.

5. STABILITY OF PREY-PREDATOR SYSTEMS WITH DIFFUSION AND MEMORY EFFECTS

Example 5.1. We consider the situation in which a prey species and a predator species live in Ω; the predator species feeds only on the prey species, while the prey species feeds on the resources in Ω. Moreover, each species is assumed to move from regions of high concentration to regions of low one, but it is not allowed to leave Ω. Let $u_1(x,t)$ and $u_2(x,t)$ denote the population density of the prey species and the predator species at position $x \in \Omega$ and time $t \in R^1$. In consideration of memory effects of the interaction between two species, we assume that their relatioship is described by

$$\begin{aligned} \partial u_1(x,t)/\partial t &= \mu_1 \Delta u_1(x,t) + u_1(x,t)\{a_1 - b_1 u_1(x,t) - c_1 g_1(x,u_{2,t})\} \\ \partial u_2(x,t)/\partial t &= \mu_2 \Delta u_2(x,t) + u_2(x,t)\{-a_2 - b_2 u_2(x,t) + c_2 g_2(x,u_{1,t})\} \end{aligned} \tag{5.1}$$

$$\text{for} \quad x \in \Omega,\ t > 0,$$

$$\partial u_i(x,t)/\partial \nu = 0, \qquad (i = 1,2), \qquad \text{for} \quad x \in \partial\Omega,\ t > 0, \tag{5.2}$$

$$u_i(x,t) = \phi_i(x,t), \qquad (i = 1,2), \qquad \text{for} \quad x \in \Omega,\ t \leq 0, \tag{5.3}$$

where positive constants μ_i, a_i, b_i and c_i (i=1,2) denote diffusion constants, growth rate of u_1 for i=1 (death rate of u_2 for i=2), self-inhibitory crowding effects on the growth of u_i, and interactions between two species. The notation $g_i(x,v)$, for $i = 1,2$ and $v \in C((-\infty;0];\ C(\overline{\Omega};\ R^1))$, means

$$g_i(x,v) = \int_{-\infty}^{0} \int_{\Omega} G_i(x,y,\theta)v(y,\theta)dyd\theta,$$

where $G_i(x,y,\theta)$ is a non-negative function which is continuous in $(x,y) \in \overline{\Omega} \times \overline{\Omega}$ for each $\theta \in (-\infty,0]$ and measurable in $\theta \in (-\infty,0]$ for each $(x,y) \in \overline{\Omega} \times \overline{\Omega}$. Moreover, G_i is assumed to satisfy

$$\int_{\Omega} G_i(x,y,\theta)dx = \int_{\Omega} G_i(x,y,\theta)dy = h_i(\theta), \quad \theta \leqq 0,$$

with

$$\int_{-\infty}^{0} h_i(\theta)d\theta = 1 \quad \text{and} \quad \theta h_i(\theta) \in L^1(-\infty,0;\ R^1).$$

For the derivation of 'historical' or 'hereditary' terms, see the book of Volterra [27].

Since we are interested in asymptotic behavior of non-negative solutions, we assume that a pair of given initial functions (ϕ_1,ϕ_2) is non-negative (but, not identically zero) and, for the sake of simplicity, sufficiently smooth with $\partial\phi_i/\partial\nu(x,0) = 0$ at $x \in \partial\Omega$ for $i = 1,2$. The local existence of solutions (u_1,u_2) to (5.1)-(5.3) follows from Theorem 2.1. Furthermore, the comparison theorem for parabolic differential equations assures the global existence of the solutions such that

$$\begin{aligned} 0 &\leqq u_1(x,t) \leqq m_1 \equiv \max\{a_1/b_1,\ \sup_{\theta\leqq 0} \|\phi_1(\theta)\|_\infty\}, \\ 0 &\leqq u_2(x,t) \leqq m_2 \equiv \max\{(-a_2+c_2m_1)/b_2,\ \sup_{\theta\leqq 0} \|\phi_2(\theta)\|_\infty\}, \end{aligned} \tag{5.4}$$

for $x \in \overline{\Omega}$ and $t \in R^1$, where $\|\cdot\|_\infty$ denotes the $C(\overline{\Omega};\ R^1)$-norm. In (5.4) the strict positivity of $u_i(x,t)$ $(i=1,2)$ holds for $x \in \overline{\Omega}$ and $t > 0$.

We observe here that, in addition to the trivial equilibrium point (0,0), (5.1) has spatially homogeneous and non-negative equilibrium points

$$(a_1/b_1, 0) \quad \text{and, if} \quad a_1/b_1 \geqq a_2/c_2,$$

$$(u_1^*, u_2^*) \equiv \left(\frac{a_1b_2+a_2c_1}{b_1b_2+c_1c_2},\ \frac{a_1c_2-a_2b_1}{b_1b_2+c_1c_2}\right).$$

The asymptotic behavior of the solutions for (5.1)-(5.3) has been discussed by Pozio [18], where the global asymptotic stability of the above equilibrium points is studied with use of attractivity of a suitable family of invariant convex sets. (See also the results of Schiaffino [21], where global asymptotic stability for some specific competition models is treated by an elegant application of the maximum principle as in [20].)

We shall give another proof of stability results due to Pozio.

THEOREM 5.1 (cf. [18, Theorem 3]). A pair of the solutions (u_1, u_2) of (5.1)-(5.3) has the following properties.

(i) If $a_1/b_1 < a_2/c_2$, then

$$\lim_{t\to\infty} (u_1(x,t), u_2(x,t)) = (a_1/b_1, 0) \quad \text{uniformly for} \quad x \in \overline{\Omega}. \tag{5.5}$$

(ii) If $a_1/b_1 \geqq a_2/c_2$ and $b_1 b_2 > c_1 c_2$, then

$$\lim_{t\to\infty} (u_1(x,t), u_2(x,t)) = (u_1^*, u_2^*) \quad \text{uniformly for} \quad x \in \overline{\Omega}. \tag{5.6}$$

Proof. (i) The comparison theorem for parabolic differential equations plays a crucial role. Since u_1 and u_2 are non-negative, we see

$$u_1(x,t) \leqq \overline{w}(t) \qquad \text{for} \quad x \in \overline{\Omega} \quad \text{and} \quad t \geqq 0, \tag{5.7}$$

where $\overline{w}$ is the solution of $\overline{w}' = \overline{w}(a_1 - b_1\overline{w})$, $t \geqq 0$, with $\overline{w}(0) = \|\phi_1(0)\|_\infty$. Since $\lim_{t\to\infty} \overline{w}(t) = a_1/b_1$, some positive constants δ and T_1 satisfy

$$-a_2 - b_2 u_2 + c_2 g_2(x, u_{1,t}) \leqq -\delta \qquad \text{for} \quad x \in \overline{\Omega} \quad \text{and} \quad t \geqq T_1,$$

because $a_2/c_2 > a_1/b_1$. Therefore, the comparison theorem for u_2 helps us to show

$$\lim_{t\to\infty} u_2(x,t) = 0 \qquad \text{uniformly for} \quad x \in \overline{\Omega}. \tag{5.8}$$

Since (5.8) implies that for any $\varepsilon > 0$

$$a_1 - b_1 u_1 - c_1 g_1(x, u_{2,t}) \geqq (a_1 - \varepsilon) - b_1 u_1, \qquad x \in \overline{\Omega},\ t \geqq T_2,$$

with a suitable $T_2 = T_2(\varepsilon)$, another application of the comparison theorem to u_1 yields

$$u_1(x,t) \geqq \underline{w}(t) \quad \text{for } x \in \overline{\Omega} \text{ and } t \geqq T_2, \tag{5.9}$$

where $\underline{w}$ is the solution of $\underline{w}' = \underline{w}(a_1 - \varepsilon - b_1\underline{w})$, $t \geqq T_2$, with $\underline{w}(T_2) = \min_{x\in\overline{\Omega}} u_1(x,T_2) > 0$. Since $\lim_{t\to\infty} \underline{w}(t) = (a_1-\varepsilon)/b_1$ and $\varepsilon > 0$ is arbitrary, we see from (5.7) and (5.9)

$$\lim_{t\to\infty} u_1(x,t) = a_1/b_1 \quad \text{uniformly for } x \in \overline{\Omega}.$$

(ii) We shall show (5.6) essentially in the same way as [30]. Define the following non-negative functional

$$\begin{aligned} E(u,u^*) &= \int_\Omega \{u(x) - u^* - u^* \log u(x)/u^*\}dx \quad &\text{if } u^* > 0, \\ &= \int_\Omega u(x)dx &\text{if } u^* = 0. \end{aligned}$$

We consider only the case $u_2^* > 0$. Let (u_1,u_2) be the solutions of (5.1)-(5.3) and let k be a positive number to be determined later. We use the identity:

$$\begin{aligned} &\frac{d}{dt}\{kE(u_1(t),u_1^*) + E(u_2(t),u_2^*)\} + k\mu_1\int_\Omega |\text{grad } u_1(x,t)|^2/u_1(x,t)^2dx \\ &+ \mu_2\int_\Omega |\text{grad } u_2(x,t)|^2/u_2(x,t)^2dx + kb_1 \|\tilde{u}_1(t)\|_2^2 + b_2 \|\tilde{u}_2(t)\|_2^2 \\ &= -kc_1(g_1(\cdot,\tilde{u}_{2,t}),\tilde{u}_1(t))_2 + c_2(g_2(\cdot,\tilde{u}_{1,t}),\tilde{u}_2(t))_2, \end{aligned} \tag{5.10}$$

where $(\cdot,\cdot)_2$ is the innner product of $L^2(\Omega; R^1)$, $\|v\|_2^2 = (v,v)_2$ and $\tilde{u}_i(t) = u_i(t) - u_i^*$ $(i=1,2)$. By Hausdorff-Young's inequality,

$$\begin{aligned} \|g_i(\cdot,v_t)\|_2 &\leqq \int_{-\infty}^0 h_i(\theta)\ \|v_t(\theta)\|_2 d\theta \\ &\leqq \sup_{\theta\leqq 0} \|v(\theta)\|_2 \int_{-\infty}^{-t} h_i(\theta)d\theta + \int_0^t h_i(s-t)\ \|v(s)\|_2 ds, \end{aligned}$$

for $i = 1,2$ and $v \in C((-\infty,t]; C(\overline{\Omega}; R^1))$ with $t > 0$. Therefore, using

Hausdorff-Young's inequality again, we get

$$\left|\int_0^T (g_i(\cdot,v_t),w(t))_2 dt\right| \leq \sup_{\theta\leq 0}\|v(\theta)\|_2 \sup_{0\leq t\leq T}\|w(t)\|_2 \int_0^\infty \theta h_i(-\theta)d\theta$$

$$+ |||v|||_{2,T}\, |||w|||_{2,T} \qquad \text{for any } T > 0,$$

where the $L^2(0,T;\, L^2(\Omega;\, R^1))$-norm is simply denoted by $|||\cdot|||_{2,T}$. Consequently, integrating (5.10) over $[0,T]$ with any $T > 0$ and recalling (5.4), one can obtain the following estimates:

$$|||\mathrm{grad}\ u_i|||_{2,T} \leq C \quad \text{for} \quad i = 1,2 \quad \text{and}$$

$$kb_1|||\tilde{u}_1|||_{2,T}^2 + b_2|||\tilde{u}_2|||_{2,T}^2 - (kc_1+c_2)|||\tilde{u}_1|||_{2,T}|||\tilde{u}_2|||_{2,T} \leq C,$$

where C is a positive constant independent of T. Since we can take $k > 0$ such that $(kc_1+c_2)^2 < 4kb_1b_2$ (use $b_1b_2 > c_1c_2$), the above estimates imply $\tilde{u}_i = u_i - u_i^* \in L^2(0,+\infty;\, W^{1,2}(\Omega;\, R^1))$ for $i = 1,2$. To accomplish the proof, it suffices to repeat the arguments used in the proof of (4.18) with a slight modification. Q.E.D.

Remark 5.1. As is seen from the above proof, Theorem 5.1 (i) remains true even if $b_2 = 0$.

Remark 5.2. If (5.2) is replaced by homogeneous Dirichlet boundary conditions, the study of asymptotic behavior of solutions becomes considerably difficult. For a single species model, see the paper of Schiaffino and Tesei [24], where global stability is discussed by means of the monotone method based on the comparison theorem.

Example 5.2. Theorem 4.1 enables us to proceed to further investigation of the asymptotic stability of (u_1^*,u_2^*) with $u_2^* > 0$ when explicit forms of G_i are given; say,

$G_i(x,y,\theta) = \delta(x-y)h_i(\theta)$, where δ is the Dirac function, or

$$G_i(x,y,\theta) = U_i(x,y,-\theta)h_i(\theta),$$

where $U_i(x,y,t)$ is the fundamental solution of $\partial u_i/\partial t = \mu_i \Delta u_i$ with $\partial u_i/\partial \nu = 0$ on $\partial\Omega$. In order to simplify the computation, we consider the following special forms of g_i:

$$g_1(x,v) = v(x,0) \quad \text{and} \quad g_2(x,v) = \omega\int_{-\infty}^{0} e^{\omega\theta}v(x,\theta)d\theta \quad \text{with} \quad \omega > 0,$$

for $v \in C((-\infty,0]; C(\overline{\Omega}; R^1))$. Initial conditions (5.3) may be replaced by

$$u_1(x,t) = \phi_1(x,t) \quad \text{and} \quad u_2(x,0) = \phi_2(x,0), \quad x \in \overline{\Omega},\ t \leqq 0.$$

Such an example has been discussed in [2] and [29] without diffusion. The following theorem gives generalization of the result of Wörz-Busekros [29, Theorem 9].

THEOREM 5.2. Set $\alpha = b_1u_1^* + b_2u_2^*$ and $\beta = b_1b_2u_1^*u_2^*$.

(i) Assume $c_1c_2/b_1b_2 < \alpha(\omega^2 + \alpha\omega + \beta)/\beta\omega$. If $\sup_{t\leq 0} \|\phi_1(t) - u_1^*\|_\infty$ and $\sum_{i=1}^{2} (\|\phi_i(0) - u_i^*\|_\infty + \|\Delta\phi_i(0)\|_\infty)$ are sufficiently small, then (5.6) holds true.

(ii) Assume $c_1c_2/b_1b_2 < 8$. Then every solution of (5.1)-(5.3) satisfies (5.6).

Remark 5.3. Note that $\min \{\alpha(\omega^2 + \alpha\omega + \beta)/\beta\omega;\ \omega > 0\} = \alpha(\alpha+2\sqrt{\beta})/\beta \geqq 8$, because $\alpha \geqq 2\sqrt{\beta}$. Hence, (ii) is a special case of (i) in Theorem 5.2.

Proof of Theorem 5.2. (i) Put $\tilde{u} = {}^t(\tilde{u}_1,\tilde{u}_2) \equiv {}^t(u_1-u_1^*,u_2-u_2^*)$ to apply Theorem 4.1 to (5.1)-(5.3). Then $\tilde{u}$ satisfies (4.3) with $Bu = {}^t(-b_1u_1^*u_1-c_1u_1^*u_2, -b_2u_2^*u_2)$, $C(t)u = {}^t(0,c_2u_2^*\omega e^{-\omega t}u_1)$ for $u = {}^t(u_1,u_2)$ and $f(\psi) = {}^t(-\psi_1(0)\{b_1\psi_1(0)+c_1\psi_2(0)\},-\psi_2(0)\{b_2\psi_2(0)-c_2\omega\int_{-\infty}^{0} e^{\omega\theta}\psi_1(\theta)d\theta\})$ for $\psi = {}^t(\psi_1,\psi_2) \in C((-\infty,0];C(\overline{\Omega}; R^2))$. It is easy to verify (B), (C)', (F.1)' and (F.2). The spectral condition (A) is equivalent to

$$\det\begin{pmatrix} \lambda+\mu_1\zeta_j+b_1u_1^* & c_1u_1^* \\ -c_2u_2^*\omega/(\lambda+\omega) & \lambda+\mu_2\zeta_j+b_2u_2^* \end{pmatrix} \neq 0 \quad \text{for } \operatorname{Re}\lambda \geq 0 \text{ and } j = 0,1,2,\cdots, \tag{5.11}$$

where $0=\zeta_0<\zeta_1\leqq\zeta_2\leqq\cdots$ are eigenvalues of the problem $-\Delta u = \zeta u$ with zero Neumann condition. After some tedious calculations, it is shown with the aid of Hurwitz' criterion that (5.11) is fulfilled if and only if $c_1c_2/b_1b_2 < \alpha(\omega^2+\alpha\omega+\beta)/\beta\omega$. Hence Theorem 4.1 yields the conclusion.

(ii) The functional method developed in the proof of Theorem 5.1 (ii) is valid if we carry out more delicate calculations. The integral over $[0,T]$ of the right-hand side of (5.10) is expressed as

$$\begin{aligned}
&\{-kc_1\int_0^T(\tilde u_2(t),\tilde u_1(t))_2dt + \omega c_2\int_0^T(\int_0^t e^{-\omega(t-s)}\tilde u_1(s)ds,\tilde u_2(t))_2dt\} \\
&\qquad + \{\omega c_2\int_0^T(\int_{-\infty}^0 e^{-\omega(t-s)}(\phi_1(s)-u_1^*)ds,\tilde u_2(t))_2dt\} \\
&\equiv I_1 + I_2.
\end{aligned}$$

In view of (5.4), the last term is bounded by a positive constant independent of T, while, by Parseval's theorem for the Fourier transform,

$$I_1 = (2\pi)^{-1}\operatorname{Re}\int_{-\infty}^{+\infty}\{-kc_1+\omega c_2/(\omega+i\eta)\}(\hat u_1(i\eta;T),\hat u_2(i\eta;T))_2d\eta,$$

where $\hat u_j(x,i\eta;T) = \int_{-\infty}^{+\infty}e^{-i\eta t}\,\tilde u_j(x,t;T)dt$ $(j=1,2)$ with $\tilde u_j(x,t;T) = \tilde u_j(x,t)$ for $t\in[0,T]$ and $\tilde u_j(x,t;T) = 0$ for $t\in(-\infty,+\infty)\setminus[0,T]$. Therefore, by virtue of Plancherel's theorem

$$\begin{aligned}
|I_1| &\leqq \frac{1}{2\pi}\sup_{\eta\in R^1}\left|-kc_1+\frac{\omega c_2}{\omega+i\eta}\right| \{\int_{-\infty}^{+\infty}\|\hat u_1(i\eta;T)\|_2^2d\eta\}^{1/2}\{\int_{-\infty}^{+\infty}\|\hat u_2(i\eta;T)\|_2^2d\eta\}^{1/2} \\
&= \sup_{\eta\in R^1}\left|-kc_1+\frac{\omega c_2}{\omega+i\eta}\right| \;|||\tilde u_1|||_{2,T}|||\tilde u_2|||_{2,T}\,.
\end{aligned}$$

In order to complete the proof, it is sufficient to verify that there exists a positive number k such that

$$\sup_{\eta \in R^1} \left| -kc_1 + \frac{\omega c_2}{\omega + i\eta} \right|^2 < 4b_1 b_2 k,$$

which is possible if $8b_1b_2 > c_1c_2$. Q.E.D.

Remark 5.4. Consider the case $b_2 = 0$ in this example. Although the first estimate of (5.4) is valid, the uniform boundedness of u_2 is not derived directly from the comparison theorem. Therefore, the functional method developed before is not applicable in this case.

However, Theorem 4.1 is available to study the local asymptotic stability of equilibrium points in the sense of Theorem 5.2 (i). In fact, (5.11) (with $b_2 = 0$) is equivalent to

$$c_1c_2u_2^* < b_1(\omega + b_1u_1^*),$$

which assures that (u_1^*, u_2^*) is locally asymptotically stable.

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