BROWNIAN MOTION IN A ROTATING FLUID

TOSHIYUKI GOTOH

1985

DEPARTMENT OF APPLIED PHYSICS
NAGOYA UNIVERSITY

BROWNIAN MOTION IN A ROTATING FLUID

BY

TOSHIYUKI GOTOH

1985

DEPARTMENT OF APPLIED PHYSICS

FACULTY OF ENGINEERING

NAGOYA UNIVERSITY

NAGOYA, JAPAN

To my wife Astuko

Brownian Motion in a Rotating Fluid

Contents

Brownian Motion in a Rotating Fluid

1. Intorduction

The first successful theory of Brownian motion was due to Einstein [1]. He obtained the famous formula (the first kind of fluctuation dissipation theorem)

$$D = \frac{k_B T}{\zeta} , \tag{1.1}$$

where D is the diffusion coefficient, k_B the Blotzmann constant, T the absolute temperature and ζ the Stokes friction constant of the particle.

In an attempt to go beyond Einstein's work, Langevin[2] proposed his equation, which in the absence of external macroscopic field reads

$$m \frac{dU_i(t)}{dt} = -\zeta_{ij} U_j(t) + \tilde{F}_i(t), \tag{1.2}$$

$$\langle \tilde{F}_i(t) \rangle = 0 , \tag{1.3a}$$

$$\langle \tilde{F}_i(t) \tilde{F}_j(t') \rangle = 2 k_B T \zeta_{ij} \delta(t-t'), \tag{1.3b}$$

where m is the mass of particle and $U_i(t)$ its velocity, ζ_{ij} the Stokes friction constant, $\tilde{F}_i(t)$ the fluctuating force due to the thermal agitation, $\delta(t)$ Dirac's delta function and $\langle\ \rangle$

denotes the equilibrium ensemble average, and the summation convention is used throughout this paper. Equations (1.2) and (1.3) represent the Gaussian, Markovian stochastic process and equation (1.3b) is also called the fluctuation dissipation theorem.

One may criticize the use of the steady state Stokes friction constant in (1.2). In fact, from the non-equilibrium statistical mechanics, it can be shown that the generalized Langevin equation[3] is of the form

$$m\frac{dU_i(t)}{dt} = -\int_{-\infty}^{t} \gamma_{ij}(t-s)\,U_j(s)\,ds + \tilde{F}_i(t), \qquad (1.4)$$

and

$$\langle \tilde{F}_i(t) \rangle = 0, \qquad (1.5)$$

$$\langle \tilde{F}_i(t)\,\tilde{F}_j(s) \rangle = 2k_BT\,\gamma_{ij}(|t-s|), \qquad (1.6)$$

where $\gamma_{ij}(t)$ is the time dependent friction tensor. This stochastic process is stationary, non-Markovian, Gaussian and the fluctuation-dissipation theorem (1.6) holds. Equations (1.4) to (1.6) are quite general and valid for any system in the thermal equilibrium. However, it is difficult to compute $\gamma_{ij}(t)$ explicitly even for the spherical Brownian prticle. In this circumstances, the fluctuating hydrodyics is very useful and successful. This semi-macroscopic approach was started by Green[4], by Landau and Lifshitz[5] and developed by Fox and Uhlenbeck[6] on the basis of the theory of Onsager and Machlup[7].

In fluctuating hydrodynamics, it is assumed that there are spontaneous local stresses due to thermal agitation in the fluid. Then the basic equations of fluctuating hydrodynamics of the incompressible fluid are assumed to be

$$\rho \frac{\partial \boldsymbol{v}}{\partial t} = -\nabla p + \mu \Delta \boldsymbol{v} + \nabla \cdot \tilde{\boldsymbol{\sigma}} , \qquad (1.7a)$$

$$\nabla \cdot \boldsymbol{v} = 0, \qquad (1.7b)$$

where ρ, μ, p, $\boldsymbol{v}$ and $\tilde{\boldsymbol{\sigma}}(\mathbf{r}, t)$ are the density, viscosity, pressure, velocity vector of the fluid and the random stress tensor due to the thermal agitation, respectively. The random stress tensor $\boldsymbol{\sigma}$ has the following statistical properties

$$\langle \tilde{\sigma}_{ij} (\mathbf{r}, t) \rangle = 0, \qquad (1.8a)$$

$$\langle \tilde{\sigma}_{ij} (\mathbf{r}, t) \tilde{\sigma}_{kl} (\mathbf{r}', t') \rangle = 2 k_B T \mu \gamma_{ijkl} \delta(\mathbf{r}-\mathbf{r}') \delta(t-t'), \qquad (1.8b)$$

$$\gamma_{ijkl} = \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} - \frac{2}{3} \delta_{ij} \delta_{kl} . \qquad (1.8c)$$

It should be noted that the *4*-rank tensor γ_{ijkl} has the properties $\gamma_{ijkl}=\gamma_{jilk}$, $\gamma_{ijkl}=\gamma_{klij}$ and $\gamma_{iijj}=0$. These relations were derived by the use of the Onsager and Machlup theory of the thermal fluctuations[5,6] , in which the assumptions of the stationary Gaussian and Markovian processes, of the linear regression equations of the fluctuations and of the equiliblium distribution are made.

Hereafter we neglect the interaction between the particles. Solving the equations of motion of the fluctuating fluid subject to the appropriate boundary conditions on the particle and eliminating the fluid variables, one can derive the Langevin equation and the fluctuation dissipation theorem for the Brownian particle. This idea was proposed by Zwanzig[8] and by Fox and Uhlenbeck[6] who, however, neglected the inertial effects of fluid. Then they were lead to the classical Langevin equation (1.2) with (1.3). For a spherical Brownian particle the generalized Langevin equation was discussed by Chow and Hermans[9] , while for the particle of arbitrary shape was derived by Hauge and Martin Lof[10] and by Bedeaux and Mazur[11] .

The autocorrelation function of $U_i(t)$ due to the generalized Langevin equation (1.4) to (1.6), which has the long time tail as $\langle U_i(t)U_i(0)\rangle \sim t^{-3/2}$ for large t, was in good agreement with the numerical simulation by Alder and Wainwright[12] and with the experiments(Ohbayashi et al.[13], Paul et al.[14]). Moreover many studies on the Brownian motion in the different physical situations have been made on the basis of the fluctuating hydrodynamics, for example, many particles system of Brownian particles by Mazur[15], Brownian motion of polymer by Jones[16], wall effects on the Brownian motion by Gotoh & Kaneda[17] and so on.

On the other hand, a number of studies have been worked out concerning the fluctuations about steady states far from thermal equilibrium since early 1970's. In these studies, fluctuatig hydrodynamics was also applied to obtain the correlation functions of physical quantities, e.g. velocity-velocity or density-density correlation of the fluid near the Rayleigh-Benard instability by Zaitzev et al.[18], Lesnikov et al.[19], Lekkerkerker et al.[20] and others, then the singularities of the correlations were found. Applying this method to the Brownian motion near the Rayleigh-Bernard instability, Lekkerkerker[21], Garisto and Mazur[22] found that the friction constant and diffusion coefficient of a spherical Brownian particle are proportinal to $\varepsilon^{1/2}$ and $\varepsilon^{-3/2}$ (divergent) as ε tends to zero respectively, where $\varepsilon=(R_c-R)/R_c$, R and R_c are the Rayleigh and critical Rayleigh numbers. However the justification for fluctuating hydrodynamics far from thermal equilibrium was not clear. It was given by Keizer[23] from the viewpoint of the elementary molecular processes(see also Fox[24]).

Very recently many theoretical studies have been made in the fluctuations in fluids when the steady velocity or temperature gradients are applied. Their results[25] predicted the asymmetry in the correlation function of the density-density fluctuations in the frequency space, and they were in good agreement with the experimental data. In these studies, the theories were based on statistical mechanics or fluctuating hydrodynamics. Fluctuating hydrodynamics is convenient and useful to study the nature of fluctuations in the steady states far from equilibrium. In fluctuating hydrodynamics, the basic equations are assumed to be the Navier-Stokes equation with the random stress tensor due to the thermal agitation.

Within the framework of this fluctuating hydrodynamics, Hermans[26] studied the Brownian motion of a spherical particle based on Oseen's approximate equation. However it is well known that Oseen's equation is not a correct approximation near the particle. The full analysis of the Brownian motion based on the Navier-Stokes equation was performed by Kaneda[27]. As is well known the regular pertubation method in a small particle Reynolds number fails in getting a first order correction to Stokes' law(i.e. Whitehead's paradox). Using the matched asymptotic expansions and the correlation functions, which make clearer the boundary conditions at infinity than those of previous studies, he obtained the Langevin equations for the Brownian particle.

In this paper, we consider the Langevin equationss for the Brownian particle immersed in an unbounded fluid which is undergoing rigid rotation with constant angular velocity Ω_f. Since the fluid has the anisotropic nature due to the rotation, it may be expected that the Langevin equations for the Brownian particle are modified and the fluctuation-dissipation relations have the anisotropic character. It is found, in fact, that the rotation gives rise to the anisotropy in the friction constants of the particle and in the correlation functions of the fluctuating forces and torques acting on the particle. It is noted that the expansions of the velocity, pressure and so on can be made not in terms of Ω_f but of $\Omega_f^{1/2}$.

In the following chapters we will analyse this problem by using the matched asymptotic expansions and Kaneda's method of correlation functions. In Chap.2 the basic equations are derived, and in Chaps.3 and 4 we develop the formulation in terms of the correlation functions. In Chaps.5, 6 and 7 the three kinds of fields introduced in Chaps.3 and 4 are solved, and in Chap.8 the correlation functions of the fluctuating forces and torques acting on the particle are computed. In Chap.9 the Langevin equations are derived for the Brownian particle of the arbitrary shape in the rotating fluid, and in Chap.10 the results are discussed.

2. Basic Equations

We consider particles suspending in an infinite region of incompressible fluctuating fluid which is undergoing rigid rotation with constant angular velocity $\boldsymbol{\Omega}'_f$. We assume that the suspension is so dilute that the interaction between the particles can be neglected. A particle of arbitrary shape whose characteristic body dimension a and mass m is assumed to be translating with speed $\boldsymbol{U}'_B$ relative to the systematic unperturbed flow and rotating with angular velocity $\boldsymbol{\Omega}'_B$. The origin O of the Cartesian co-ordinate system is chosen to be fixed to the particle and for the fluid to rotate about the y-axis. In this co-ordinate system, the unperturbed flow is expressed as $\boldsymbol{v}'_0=\boldsymbol{\Omega}'_f \times \boldsymbol{r}'$ (or $=\mathbf{C}'.\boldsymbol{r}'$), here $C'_{ij}=1/2\,|\boldsymbol{\Omega}'_f|\,(\delta_{i1}\delta_{j3}-\delta_{i3}\delta_{j1})$. The motion of the fluid is assumed to be described by the stochastic Landau-Lifshitz equations of motion

$$\rho\left(\frac{\partial \boldsymbol{v}'}{\partial t'}+(\boldsymbol{v}'\cdot\nabla')\boldsymbol{v}'\right)=\nabla'\cdot(\boldsymbol{\tau}'+\tilde{\boldsymbol{\sigma}}')-\rho\frac{d\boldsymbol{U}'_B}{dt'}, \tag{2.1a}$$

$$\nabla'\cdot\boldsymbol{v}'=0, \tag{2.1b}$$

$$\tau'_{ij}=-p'\delta_{ij}+\mu\left(\frac{\partial v'_i}{\partial x'_j}+\frac{\partial v'_j}{\partial x'_i}\right), \tag{2.2}$$

where τ'_{ij} the stress tensor due to the velocity field $\boldsymbol{v}'$ and the pressure p'. It is assumed that the random stress tensor $\tilde{\sigma}'_{ij}$ due to thermal agitation has the stochastic properties (1.8). In (1.8) temperature T is assumed to be constant throughout the fluid and $\langle\ \rangle$ denotes the 'local' equilibrium ensemble average with $\boldsymbol{U}'_B$, $\boldsymbol{\Omega}'_B$ and $\mathbf{C}'$ fixed. The velocity field $\boldsymbol{v}'(\mathbf{r}', t')$ satisfies the following stick boundary condition on the surface S_P of the particle

$$\boldsymbol{v}'(\boldsymbol{r}', t') = \boldsymbol{\Omega}'_B \times \boldsymbol{r}', \qquad \boldsymbol{r}' \text{ on } S_p. \tag{2.3}$$

The motion of the particle is governed by

$$m\frac{d^2\mathbf{X}'_B}{dt'^2} = \mathbf{F}' \equiv \int_{S_p} (\boldsymbol{\tau}' + \tilde{\boldsymbol{\sigma}}') \cdot d\mathbf{S}', \tag{2.4a}$$

$$\mathbf{J}' \cdot \frac{d\boldsymbol{\Omega}'_B}{dt'} = \mathbf{M}' \equiv \int_{S_p} \mathbf{r}' \times (\boldsymbol{\tau}' + \tilde{\boldsymbol{\sigma}}') \cdot d\mathbf{S}', \tag{2.4b}$$

where $\mathbf{X}'_B$ is the position vector of the particle with respect to the laboratory frame, $\boldsymbol{J}'$ the moment of inertia tensor of the particle and $d\boldsymbol{S}'$ the area segment vector taken along the outward normal.

Now let us define the following dimensionless quantities;

$$\boldsymbol{r}' = a\boldsymbol{r}, \qquad t' = t_0 t, \qquad \boldsymbol{v}' = v_0 \boldsymbol{v},$$

$$p' = \frac{\mu U_0}{a} p, \qquad \tilde{\sigma}'_{ij} = \frac{\mu U_0}{a} \sigma_{ij}, \qquad \mathbf{F}' = \mu a U_0 \mathbf{F},$$

$$\mathbf{M}' = \mu a^2 U_0 \mathbf{M}, \qquad \mathbf{J}' = m a^2 \mathbf{J}, \qquad \nu = \frac{\mu}{\rho},$$

$$\lambda t_0 = \frac{a^2}{\nu}, \qquad \Omega_0 = \frac{U_0}{a}, \qquad \lambda = \frac{3}{8\pi} \cdot \frac{\rho}{\rho_B},$$

$$\mathbf{U}'_B = U_0 \mathbf{U}_B, \qquad \boldsymbol{\Omega}'_B = \Omega_0 \boldsymbol{\Omega}_B, \qquad \frac{1}{2} m U_0^2 = \frac{1}{2} k_B T. \tag{2.5}$$

and

$$R = \frac{a U_0}{\nu}, \qquad R_\Omega = \frac{a^2 \Omega_0}{\nu},$$
$$R_\kappa = R \cdot \kappa = \frac{a^2 \Omega'_f}{2\nu}, \qquad \kappa = \frac{\Omega'_f a}{2 U_0}, \tag{2.6}$$

where λ is the Lorentz parameter and κ the dimensionless angular velocity of the systematic unperturbed flow. In terms of these quantities the previous equations of motion and boundary condition become

$$\lambda \frac{\partial \boldsymbol{v}}{\partial t} + R (\boldsymbol{v} \cdot \nabla) \boldsymbol{v} = \nabla \cdot (\boldsymbol{\tau} + \tilde{\boldsymbol{\sigma}}) - \lambda \frac{d \mathbf{U}_B}{dt}, \tag{2.7a}$$

$$\nabla \cdot \boldsymbol{v} = 0, \tag{2.7b}$$

$$\tau_{ij} = -p \delta_{ij} + \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}, \tag{2.7c}$$

$$\langle \tilde{\sigma}_{ij}(\mathbf{r}, t) \rangle = 0, \tag{2.8a}$$

$$\langle \tilde{\sigma}_{ij}(\mathbf{r}_1, t_1)\, \tilde{\sigma}_{kl}(\mathbf{r}_2, t_2) \rangle = \Gamma_{ijkl}\, \delta(\mathbf{r}_1 - \mathbf{r}_2)\, \delta(t_1 - t_2) \tag{2.8b}$$

and

$$\boldsymbol{v} = \boldsymbol{\Omega}_B \times \mathbf{r}, \qquad \text{on} \quad S_p. \tag{2.9}$$

Let us write

$$\boldsymbol{v} = \bar{\boldsymbol{v}} + \tilde{\boldsymbol{v}}, \qquad p = \bar{p} + \tilde{p}, \tag{2.10}$$

where $(\bar{\boldsymbol{v}}, \bar{p})$ is the systematic field satisfying

$$\lambda \frac{\partial \bar{\boldsymbol{v}}}{\partial t} + R\,(\bar{\boldsymbol{v}} \cdot \nabla)\bar{\boldsymbol{v}} = \nabla \cdot \bar{\boldsymbol{\tau}} - \lambda \frac{d\boldsymbol{U}_B}{dt}, \tag{2.11a}$$

$$\nabla \cdot \bar{\boldsymbol{v}} = 0, \tag{2.11b}$$

and the boundary conditions

$$\bar{\boldsymbol{v}} = \boldsymbol{\Omega}_B \times \mathbf{r}, \qquad \text{on} \quad S_p, \tag{2.12a}$$

$$\longrightarrow -\boldsymbol{U}_B + K\, \mathbf{C} \cdot \mathbf{r}, \quad \text{as} \quad |\mathbf{r}| \longrightarrow \infty. \tag{2.12b}$$

Then the fluctuating field $(\tilde{\boldsymbol{v}}, \tilde{p})$ satisfies

$$\lambda \frac{\partial \tilde{\boldsymbol{v}}}{\partial t} + R[(\bar{\boldsymbol{v}}\cdot\nabla)\tilde{\boldsymbol{v}} + (\tilde{\boldsymbol{v}}\cdot\nabla)\bar{\boldsymbol{v}} + (\tilde{\boldsymbol{v}}\cdot\nabla)\tilde{\boldsymbol{v}}] - \nabla\cdot\tilde{\boldsymbol{\tau}} = \nabla\cdot\tilde{\boldsymbol{\sigma}} \tag{2.13a}$$

$$\nabla\cdot\tilde{\boldsymbol{v}} = 0, \tag{2.13b}$$

$$\tilde{\boldsymbol{v}} = 0, \qquad \text{on} \quad S_p, \tag{2.14}$$

where the boundary condition of $\tilde{\boldsymbol{v}}$ at infinity is not specified yet. The equations of motion of the particle are also written as

$$\frac{d^2\boldsymbol{X}_B}{dt^2} = \bar{\mathbf{F}} + \tilde{\mathbf{F}}, \qquad \mathbf{J}\cdot\frac{d\boldsymbol{\Omega}_B}{dt} = \bar{\mathbf{M}} + \tilde{\mathbf{M}}, \tag{2.15a}$$

$$\bar{\mathbf{F}} \equiv \int_{S_p} \bar{\boldsymbol{\tau}}\cdot d\mathbf{S}, \qquad \bar{\mathbf{M}} \equiv \int_{S_p} \mathbf{r}\times\bar{\boldsymbol{\tau}}\cdot d\mathbf{S}, \tag{2.15b}$$

$$\tilde{\mathbf{F}} \equiv \int_{S_p} (\tilde{\boldsymbol{\tau}} + \tilde{\boldsymbol{\sigma}})\cdot d\mathbf{S}, \qquad \tilde{\mathbf{M}} \equiv \int_{S_p} \mathbf{r}\times(\tilde{\boldsymbol{\tau}} + \tilde{\boldsymbol{\sigma}})\cdot d\mathbf{S}, \tag{2.15c}$$

where $(\bar{\boldsymbol{F}},\bar{\boldsymbol{M}})$ and $(\tilde{\boldsymbol{F}},\tilde{\boldsymbol{M}})$ are the systematic force and torque, and the fluctuating force and torque acting on the particle, respectively. It is convenient to introduce the generalized hydrodynamic force;

$$K_i^{\alpha} = \delta^{\alpha 0} F_i + \delta^{\alpha 1} M_i, \qquad \delta^{\alpha\beta} = \delta_{\alpha\beta}, \tag{2.16}$$

where $\alpha=0$ stands for the force and $\alpha=1$ for the torque. Corresponding to (2.16) and noting (2.15 b,c), we can write

$$\begin{aligned} K_i^{\alpha} &= \int_{S_p} \Delta_{ij}^{\alpha} (\tau_{jk} + \tilde{\sigma}_{jk})\, dS_k \\ &= \int_{S_p} \Delta_{ij}^{\alpha} \bar{\tau}_{jk}\, dS_k + \int_{S_p} \Delta_{ij}^{\alpha} (\tilde{\tau}_{jk} + \tilde{\sigma}_{jk})\, dS_k \\ &= \bar{K}_i^{\alpha} + \tilde{K}_i^{\alpha}, \end{aligned} \tag{2.17}$$

where

$$\Delta_{ij}^{\alpha} \equiv \delta^{\alpha 0} \delta_{ij} + \delta^{\alpha 1} \epsilon_{ikj} x_k \tag{2.18}$$

and ε_{ijk} is the alternating tensor. In the present work, we assume that parameters R, R_Ω and R_κ are not only small but more restrictive, namely, they satisfy the conditions;

$$R \ll 1, \qquad R_\Omega \ll 1, \qquad R_\kappa \ll 1, \tag{2.19}$$

$$R \ll R_\kappa^{1/2} \ll 1. \tag{2.20}$$

The second assumption (2.20) shows that the effect of inertia due to the fluid rotation dominates the one due to the translation and rotation of the particle. And if the approximation (2.20) is made, we can neglect the term $R(\tilde{\boldsymbol{v}}\cdot\nabla)\tilde{\boldsymbol{v}}$ in (2.13a) (see Appendix-G).

To simplify notations a formal 4-component vector and tensor notation are introduced. The fields $\tilde{\boldsymbol{v}}$ and $\tilde{p}$ are grouped together to form 4-components vector

$$\tilde{X}_\alpha = \begin{bmatrix} \tilde{X}_1 \\ \tilde{X}_2 \\ \tilde{X}_3 \\ \tilde{X}_4 \end{bmatrix} = \begin{bmatrix} \tilde{v}_1 \\ \tilde{v}_2 \\ \tilde{v}_3 \\ \tilde{p} \end{bmatrix}, \quad \tilde{Y}_\alpha = \begin{bmatrix} \tilde{Y}_1 \\ \tilde{Y}_2 \\ \tilde{Y}_3 \\ \tilde{Y}_4 \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x_m}\tilde{\sigma}_{1m} \\ \frac{\partial}{\partial x_m}\tilde{\sigma}_{2m} \\ \frac{\partial}{\partial x_m}\tilde{\sigma}_{3m} \\ 0 \end{bmatrix}. \quad (2.21)$$

The lower Greek indices take values $1,2,3,4$ where the indices 1,2,3 refer to 3-vector components and the one 4 refers to a scalar field. Roman indices i,j,k etc. will take only the values $1,2,3$, denoting 3-vector components. On the other hand the upper Greek indices take 0 or 1. In this notation a tensor differential operator is defined by

$$L_{\alpha\beta}(x,t) = \begin{pmatrix} L_{11}, & L_{12}, & L_{13}, & L_{14} \\ L_{21}, & L_{22}, & L_{23}, & L_{24} \\ L_{31}, & L_{32}, & L_{33}, & L_{34} \\ L_{41}, & L_{42}, & L_{43}, & L_{44} \end{pmatrix}$$

$$
= \begin{pmatrix}
\lambda\frac{\partial}{\partial t} + R\left(\bar{v}_n \frac{\partial}{\partial x_n} + \frac{\partial \bar{v}_1}{\partial x_1}\right) - \Delta, & R\frac{\partial \bar{v}_1}{\partial x_2}, & R\frac{\partial \bar{v}_1}{\partial x_3}, & -\frac{\partial}{\partial x_1} \\
R\frac{\partial \bar{v}_2}{\partial x_1}, & \lambda\frac{\partial}{\partial t} + R\left(\bar{v}_n \frac{\partial}{\partial x_n} + \frac{\partial \bar{v}_2}{\partial x_2}\right) - \Delta, & R\frac{\partial \bar{v}_2}{\partial x_3}, & -\frac{\partial}{\partial x_2} \\
R\frac{\partial \bar{v}_3}{\partial x_1}, & R\frac{\partial \bar{v}_3}{\partial x_2}, & \lambda\frac{\partial}{\partial t} + R\left(\bar{v}_n \frac{\partial}{\partial x_n} + \frac{\partial \bar{v}_3}{\partial x_3}\right) - \Delta, & -\frac{\partial}{\partial x_3} \\
\frac{\partial}{\partial x_1}, & \frac{\partial}{\partial x_2}, & \frac{\partial}{\partial x_3}, & 0
\end{pmatrix} . \tag{2.22}
$$

Then (2.13) in which the term $R(\tilde{\boldsymbol{v}}.\boldsymbol{\nabla})\tilde{\boldsymbol{v}}$ is dropped can be written simply as

$$
L_{\alpha\beta}(x, t)\tilde{X}_\beta(x, t) = \tilde{Y}_\alpha(x, t), \tag{2.23}
$$

where the relation (2.7c) is used.

3. Formulation based on the correlation functions

Assuming $\lambda \ll 1$, one obtains the equations for the systematic field $(\bar{\boldsymbol{v}}, \bar{p})$ as

$$R(\bar{\boldsymbol{v}}\cdot\nabla)\bar{\boldsymbol{v}} = \nabla\cdot\bar{\boldsymbol{\tau}}, \tag{3.1a}$$

$$\nabla\cdot\bar{\boldsymbol{v}} = 0, \tag{3.1b}$$

$$\bar{\boldsymbol{v}} = \boldsymbol{\Omega}_B \times \boldsymbol{r}, \qquad \text{on} \quad S_p, \tag{3.2a}$$

$$\longrightarrow -\boldsymbol{U}_B + K\,\boldsymbol{C}\cdot\boldsymbol{r}, \qquad \text{as} \quad |\boldsymbol{r}|\to\infty, \tag{3.2b}$$

Since $(\bar{\boldsymbol{v}}, \bar{p})$ is steady and $\tilde{\sigma}_{ij}(\boldsymbol{r}, t)$ is a statistically stationary process, $(\tilde{\boldsymbol{v}}, \tilde{p})$ is also stationary process so that correlation functions $\langle \tilde{v}_i(\boldsymbol{r}, t)\tilde{\sigma}_{kl}(\boldsymbol{x}, t')\rangle$ etc. are functions of $t-t'$. We introduce formal tensor fields defined by

$$\Phi_{\alpha,lm}(\boldsymbol{X}, \boldsymbol{r}; t-t') \equiv \langle \tilde{X}_\alpha(\boldsymbol{X}, t)\tilde{\sigma}_{lm}(\boldsymbol{r}; t')\rangle, \tag{3.3a}$$

$$\Psi_{\alpha,l}^{\ \beta}(\boldsymbol{X}, t-t') \equiv \langle \tilde{X}_\alpha(\boldsymbol{X}, t)\tilde{K}_l^{\beta}(t')\rangle, \tag{3.3b}$$

$$\xi_{\alpha,lm}(\mathbf{x},\mathbf{r},t-t') \equiv \langle Y_\alpha(\mathbf{x},t)\,\tilde{\sigma}_{lm}(\mathbf{r},t')\rangle, \tag{3.4a}$$

$$\eta_{\alpha,l}^{\ \beta}(\mathbf{x},t-t') \equiv \langle Y_\alpha(\mathbf{x},t)\,\tilde{K}_l^{\beta}(t')\rangle \tag{3.4b}$$

and the Fourier time transform

$$\hat{f}(\omega) = \mathcal{F}[f(t)] \equiv \int_{-\infty}^{\infty} f(t)\,e^{i\omega t}\,dt. \tag{3.5}$$

Multiplying (2.13) and (2.14) by $\tilde{\sigma}_{kl}(\mathbf{r},t')$, taking the ensemble average with $\boldsymbol{U}_B$, $\boldsymbol{\Omega}_B$ and $\mathbf{C}$ fixed and after Fourier time transform, we have

$$\hat{L}_{\alpha\beta}(\mathbf{x},\omega)\,\hat{\phi}_{\beta,kl}(\mathbf{x},\mathbf{r},\omega) = \hat{\xi}_{\alpha,kl}(\mathbf{x},\mathbf{r},\omega). \tag{3.6}$$

The boundary conditions are assumed to be

$$\hat{\phi}_{j',kl}(\mathbf{x},\mathbf{r},\omega) = 0, \qquad \mathbf{x} \text{ on } S_p, \tag{3.7a}$$

$$\longrightarrow 0, \qquad \text{as } |\mathbf{x}-\mathbf{r}| \to \infty. \tag{3.7b}$$

In a similar way, one obtains

$$\hat{L}_{\alpha\gamma}(\boldsymbol{x},\omega)\,\hat{\Psi}_{\gamma,\ell}^{\beta}(\boldsymbol{x},\omega)=\eta_{\alpha,\ell}^{\beta}(\boldsymbol{x},\omega). \tag{3.8}$$

with the boundary conditions

$$\hat{\Psi}_{j,\ell}^{\beta}(\boldsymbol{x},\omega)=0, \qquad \boldsymbol{x} \text{ on } S_p \tag{3.9a}$$

$$\longrightarrow 0, \qquad \text{as } |\boldsymbol{x}|\longrightarrow\infty, \tag{3.9b}$$

where

$$\hat{L}_{\alpha\beta}(\boldsymbol{x}.\omega)$$
$$=\begin{bmatrix} -i\omega\lambda+R\left(\bar{v}_n\frac{\partial}{\partial x_n}+\frac{\partial\bar{v}_1}{\partial x_1}\right)-\Delta, & R\frac{\partial\bar{v}_1}{\partial x_2}, & R\frac{\partial\bar{v}_1}{\partial x_3}, & -\frac{\partial}{\partial x_1} \\ R\frac{\partial\bar{v}_2}{\partial x_1}, & -i\omega\lambda+R\left(\bar{v}_n\frac{\partial}{\partial x_n}+\frac{\partial\bar{v}_2}{\partial x_2}\right)-\Delta, & R\frac{\partial\bar{v}_2}{\partial x_3}, & -\frac{\partial}{\partial x_2} \\ R\frac{\partial\bar{v}_3}{\partial x_1}, & R\frac{\partial\bar{v}_3}{\partial x_2}, & -i\omega\lambda+R\left(\bar{v}_n\frac{\partial}{\partial x_n}+\frac{\partial\bar{v}_3}{\partial x_3}\right)-\Delta, & -\frac{\partial}{\partial x_3} \\ \frac{\partial}{\partial x_1}, & \frac{\partial}{\partial x_2}, & \frac{\partial}{\partial x_3}, & 0 \end{bmatrix}. \tag{3.10}$$

Hereafter considering the case $\omega=0$, the equations can be written as

$$L_{\alpha\beta}(\mathbf{x})\,\Phi_{\beta,\ell m}(\mathbf{x},\mathbf{r}) = \xi_{\alpha,\ell m}(\mathbf{x},\mathbf{r}), \qquad (3.11)$$

$$\Phi_{j,\ell m}(\mathbf{x},\mathbf{r}) = 0 \;, \qquad \mathbf{x} \text{ on } S_p, \qquad (3.12a)$$

$$\longrightarrow 0 \;, \qquad \text{as } |\mathbf{x}-\mathbf{r}| \longrightarrow \infty, \qquad (3.12b)$$

$$L_{\alpha\gamma}(\mathbf{x})\,\Psi_{\gamma,\ell}^{\beta}(\mathbf{x}) = \eta_{\alpha,\ell}^{\beta}(\mathbf{x}), \qquad (3.13)$$

$$\Psi_{j,\ell}^{\beta}(\mathbf{x}) = 0 \;, \qquad \mathbf{x} \text{ on } S_p, \qquad (3.14a)$$

$$\longrightarrow 0 \;, \qquad \text{as } |\mathbf{x}| \longrightarrow \infty, \qquad (3.14b)$$

where

$$L_{\alpha\beta}(\mathbf{x}) = L_{\alpha\beta}(\mathbf{x},t)\Big|_{\frac{\partial}{\partial t}=0} = \hat{L}_{\alpha\beta}(\mathbf{x},\omega=0) \qquad (3.15)$$

and $\wedge$ of $\hat{\Phi}$, $\hat{\Psi}$, $\hat{\xi}$ and $\hat{\eta}$ is omitted.

4. Procedure for the calculation of the correlation function of the fluctuating generalized hydrodynamic forces

We consider the correlation function of fluctuating generalized hydrodynamic forces acting on the particle;

$$Y_{ij}^{\alpha\beta} \equiv \int_{-\infty}^{\infty} \langle \widetilde{K}_i^{\alpha}(t)\, \widetilde{K}_j^{\beta}(t') \rangle \, dt = \langle \widetilde{K}_i^{\alpha}\, \widetilde{K}_j^{\beta} \rangle , \quad (4.1)$$

which can be written using (2.17) as

$$= \langle \int_{S_p} \Delta_{il}^{\alpha} \{ \widetilde{\tau}_{lk}(\boldsymbol{x}) + \widetilde{\sigma}_{lk}(\boldsymbol{x}) \} dS_k(\boldsymbol{x}) \cdot \widetilde{K}_j^{\beta} \rangle$$

$$= \int_{S_p} \Delta_{il}^{\alpha} \{ \langle \widetilde{\tau}_{lk}(\boldsymbol{x}) \widetilde{K}_j^{\beta} \rangle + \langle \widetilde{\sigma}_{lk}(\boldsymbol{x}) \widetilde{K}_j^{\beta} \rangle \} dS_k(\boldsymbol{x}) \quad (4.2)$$

Getting the knowledge of $\langle \widetilde{\sigma}_{lk}(x) \widetilde{K}_j^{\beta} \rangle$ and solving the equations (3.13) with (3.14), we can calculate the correlation function (4.1). To obtain the field $\langle \widetilde{\sigma}_{lk}(\boldsymbol{x}) \widetilde{K}_j^{\beta} \rangle$, we define the field $\chi_{\alpha,\gamma}^{\beta}$ and the adjoint field† $\chi_{\alpha,\gamma}^{+\beta}$

† Field $\chi_{\alpha}^{+}(\boldsymbol{r})$ (corresponding to $\chi_{\alpha}^{+} = \chi_{\alpha,\gamma}^{+\beta}$) adjoint to $\chi_{\alpha}(\boldsymbol{r})$ (corresponding to $\chi_{\alpha} = \chi_{\alpha,\gamma}^{\beta}$) is defined by

$$\int \chi_{\alpha}^{+} (L_{\alpha\beta} \chi_{\beta})\, d^3\boldsymbol{r} = \int (L_{\alpha\beta}^{+} \chi_{\beta}^{+}) \chi_{\alpha}\, d^3\boldsymbol{r}.$$

$$\chi_{\alpha,q}^{\beta}(\mathbf{x}) = \begin{bmatrix} \chi_{1q}^{\beta} \\ \chi_{2q}^{\beta} \\ \chi_{3q}^{\beta} \\ \chi_{4q}^{\beta} \end{bmatrix}, \qquad \chi_{\alpha,q}^{+\beta}(\mathbf{x}) = \begin{bmatrix} \chi_{1q}^{+\beta} \\ \chi_{2q}^{+\beta} \\ \chi_{3q}^{+\beta} \\ \chi_{4q}^{+\beta} \end{bmatrix}, \tag{4.3a,b}$$

$$L_{\alpha r}(\mathbf{x})\chi_{r,q}^{\beta}(\mathbf{x}) = 0, \tag{4.4a}$$

$$L_{\alpha r}^{+}(\mathbf{x})\chi_{r,q}^{+\beta}(\mathbf{x}) = 0, \tag{4.4b}$$

$$\chi_{iq}^{\beta}(\mathbf{x}) = \chi_{iq}^{+\beta}(\mathbf{x}) = \Delta_{qi}^{\beta}, \qquad \mathbf{x} \text{ on } S_p, \tag{4.5a}$$

$$\longrightarrow 0, \qquad \text{as } |\mathbf{x}| \to \infty, \tag{4.5b}$$

where

$$L_{\alpha r}^{+}(\mathbf{x})$$

$$= \begin{pmatrix} -R\left(\bar{v}_m\frac{\partial}{\partial x_m} - \frac{\partial \bar{v}_1}{\partial x_1}\right) - \Delta, & R\frac{\partial \bar{v}_2}{\partial x_1}, & R\frac{\partial \bar{v}_3}{\partial x_1}, & -\frac{\partial}{\partial x_1} \\ R\frac{\partial \bar{v}_1}{\partial x_2}, & -R\left(\bar{v}_m\frac{\partial}{\partial x_m} - \frac{\partial \bar{v}_2}{\partial x_2}\right) - \Delta, & R\frac{\partial \bar{v}_3}{\partial x_2}, & -\frac{\partial}{\partial x_2} \\ R\frac{\partial \bar{v}_1}{\partial x_3}, & R\frac{\partial \bar{v}_2}{\partial x_3}, & -R\left(\bar{v}_m\frac{\partial}{\partial x_m} - \frac{\partial \bar{v}_3}{\partial x_3}\right) - \Delta, & -\frac{\partial}{\partial x_3} \\ \frac{\partial}{\partial x_1}, & \frac{\partial}{\partial x_2}, & \frac{\partial}{\partial x_3}, & 0 \end{pmatrix}. \tag{4.6}$$

and define the tensor for the later use

$$\Theta^{+\beta}_{ij\,q} = - \chi^{+\beta}_{4q}\,\delta_{ij} + \frac{\partial}{\partial x_j}\chi^{+\beta}_{iq} + \frac{\partial}{\partial x_i}\chi^{+\beta}_{jq} \tag{4.7}$$

Consider the following identity

$$0 = \chi^{+\beta}_{\alpha,q}(\boldsymbol{x})\cdot\left\{ L_{\alpha\gamma}(\boldsymbol{x})\,\phi_{\gamma,kl}(\boldsymbol{x},\boldsymbol{r}) - \xi_{\gamma,kl}(\boldsymbol{x},\boldsymbol{r})\right\}$$

$$- \phi_{\alpha,kl}(\boldsymbol{x},\boldsymbol{r})\,L^{+}_{\alpha\gamma}(\boldsymbol{x})\,\chi^{+\beta}_{\gamma,q}$$

$$= \chi^{+\beta}_{iq}(\boldsymbol{x})\Big[\Big\{R\Big(\bar{v}_n\frac{\partial}{\partial x_n}\delta_{ij} + \frac{\partial \bar{v}_i}{\partial x_j}\Big) - \Delta\delta_{ij}\Big\}\langle\tilde{v}_j(\boldsymbol{x})\tilde{\sigma}_{kl}(\boldsymbol{r})\rangle$$

$$- \frac{\partial}{\partial x_i}\langle\tilde{p}(\boldsymbol{x})\tilde{\sigma}_{kl}(\boldsymbol{r})\rangle\Big]$$

$$- \langle\tilde{v}_i(\boldsymbol{x})\tilde{\sigma}_{kl}(\boldsymbol{r})\rangle\Big[-\Big\{R\Big(\bar{v}_n\frac{\partial}{\partial x_n}\delta_{ij} - \frac{\partial \bar{v}_j}{\partial x_i}\Big) - \Delta\delta_{ij}\Big\}\chi^{+\beta}_{jq}(\boldsymbol{x})$$

$$- \frac{\partial}{\partial x_i}\chi^{+\beta}_{4q}(\boldsymbol{x})\Big] \tag{4.8}$$

from (3.11) and (4.4b). Then the use of (2.7c), (4.7) and the relation (see Appendix-A)

$$\left\langle \left\{\frac{\partial}{\partial x_j}\chi^{+\beta}_{iq}(\boldsymbol{x})\right\} \tilde{\tau}_{ij}(\boldsymbol{x})\, \tilde{\sigma}_{kl}(\boldsymbol{r})\right\rangle$$

$$= \left\langle \left\{\frac{\partial}{\partial x_j} \tilde{v}_i(\boldsymbol{x})\right\} \cdot \Theta^{+\beta}_{ijq}(\boldsymbol{x})\, \tilde{\sigma}_{kl}(\boldsymbol{r})\right\rangle, \qquad (4.9)$$

yields (4.8) as

$$0 = \frac{\partial}{\partial x_j}\Big[\chi^{+\beta}_{iq}(\boldsymbol{x}) \cdot \left\{ \langle \tilde{\tau}_{ij}(\boldsymbol{x})\, \tilde{\sigma}_{kl}(\boldsymbol{r})\rangle + \langle \tilde{\sigma}_{ij}(\boldsymbol{x})\, \tilde{\sigma}_{kl}(\boldsymbol{r})\rangle\right\}$$

$$- \langle \tilde{v}_i(\boldsymbol{x}) \tilde{\sigma}_{kl}(\boldsymbol{r})\rangle \quad \Theta^{+\beta}_{ijq}(\boldsymbol{x})$$

$$- R\, \chi^{+\beta}_{iq}(\boldsymbol{x}) \langle \tilde{v}_i(\boldsymbol{x})\, \tilde{\sigma}_{kl}(\boldsymbol{r})\rangle \overline{v}_j(\boldsymbol{x})\Big]$$

$$-\left\{\frac{\partial}{\partial x_j} \chi^{+\beta}_{iq}(\boldsymbol{x})\right\} \langle \tilde{\sigma}_{ij}(\boldsymbol{x})\, \tilde{\sigma}_{kl}(\boldsymbol{r})\rangle .$$

Integrating this over the volume Γ bounded externally by the spherical surface S_K with radius K and by the surface S_p of the particle, and using Gauss' theorem and the boundary conditions (3.12) and (4.5a), we have

$$\int_{S_p} \Delta^{\beta}_{qi} \Big\{ \langle \tilde{\tau}_{ij}(\boldsymbol{x})\, \tilde{\sigma}_{kl}(\boldsymbol{r})\rangle$$

$$+ \langle \tilde{\sigma}_{ij}(\boldsymbol{x})\, \tilde{\sigma}_{kl}(\boldsymbol{r})\rangle \Big\}\, dS_j(\boldsymbol{x})$$

$$= \int_{S_k} \Big[\chi_{iq}^{+\beta}(\boldsymbol{x}) \Big\{ < \tilde{\tau}_{ij}(\boldsymbol{x}) \tilde{\sigma}_{kl}(\mathbf{r}) > + < \tilde{\sigma}_{ij}(\boldsymbol{x}) \tilde{\sigma}_{kl}(\mathbf{r}) > \Big\}$$

$$- < \tilde{v}_i(\boldsymbol{x}) \tilde{\sigma}_{kl}(\mathbf{r}) > \quad \Theta_{ijq}^{+\beta}(\boldsymbol{x})$$

$$- R \chi_{iq}^{+\beta}(\boldsymbol{x}) < \tilde{v}_i(\boldsymbol{x}) \tilde{\sigma}_{kl}(\mathbf{r}) > \bar{v}_j(\boldsymbol{x}) \Big] ds_j(\boldsymbol{x})$$

$$- \int_{\Gamma} \Big\{ \frac{\partial}{\partial x_j} \chi_{iq}^{+\alpha}(\boldsymbol{x}) \Big\} < \tilde{\sigma}_{ij}(\boldsymbol{x}) \tilde{\sigma}_{kl}(\mathbf{r}) > d\boldsymbol{x} \ . \tag{4.10}$$

The left hand side of (4.10) can be written by the use of (2.17) and (3.3)

$$< \Big[\int_{S_p} \Delta_{qi}^{\beta} \big\{ \tilde{\tau}_{ij}(\boldsymbol{x}) + \tilde{\sigma}_{ij}(\boldsymbol{x}) \big\} ds_j(\boldsymbol{x}) \Big] \tilde{\sigma}_{kl}(\mathbf{r}) >$$

$$= < \tilde{K}_q^{\beta} \tilde{\sigma}_{kl}(\mathbf{r}) > \ . \tag{4.11}$$

On the other hand, from the assumption (2.8b)

$$< \tilde{\sigma}_{ij}(\boldsymbol{x}) \tilde{\sigma}_{kl}(\mathbf{r}) > = \mathcal{F} \big[< \tilde{\sigma}_{ij}(\boldsymbol{x}, t) \tilde{\sigma}_{kl}(\mathbf{r}, t') > \big]_{\omega=0}$$

$$= \gamma_{ijkl} \, \delta(\boldsymbol{x} - \mathbf{r}) , \tag{4.12}$$

the last term of the right hand side of (4.10) becomes

$$\int_{\Gamma} \left\{ \frac{\partial}{\partial x_j} \chi_{iq}^{+\beta}(\mathbf{x}) \right\} \cdot \langle \tilde{\sigma}_{ij}(\mathbf{x}) \tilde{\sigma}_{kl}(\mathbf{r}) \rangle \, d\mathbf{x}$$

$$= \frac{\partial}{\partial r_k} \chi_{lq}^{+\beta}(\mathbf{r}) + \frac{\partial}{\partial r_l} \chi_{kq}^{+\beta}(\mathbf{r}) . \qquad (4.13)$$

Assuming that if we let $K \to \infty$ the outer surface integral vanishes(see Appendix-E), we obtain

$$\langle \tilde{\sigma}_{kl}(\mathbf{r}) \tilde{K}_q^{\beta} \rangle$$

$$= - \left\{ \frac{\partial}{\partial r_k} \chi_{lq}^{+\beta}(\mathbf{r}) + \frac{\partial}{\partial r_l} \chi_{kq}^{+\beta}(\mathbf{r}) \right\} . \qquad (4.14)$$

It is seen from the above equation that the field $\langle \tilde{\sigma}_{kl}(\mathbf{r}) \tilde{K}_q^{\beta} \rangle$ is expressed by the adjoint field $\chi_{\alpha,q}^{+\beta}(\mathbf{r})$.

5. Calculation of the systematic force and torque

In this chapter, the systematic force and torque acting on the particle are computed. We now introduce the perturbations $\boldsymbol{q}$ and p defined by

$$\bar{\boldsymbol{v}} = k\mathbf{C}\cdot\boldsymbol{r} - \boldsymbol{U}_B + \boldsymbol{q}, \tag{5.1a}$$

$$\bar{P} = P_0 - Rk\,\boldsymbol{r}\cdot\mathbf{C}\cdot\boldsymbol{U}_B + P, \tag{5.1b}$$

where P_0 is the pressure due to the fluid rotation which satisfies

$$P_0 = -\frac{1}{2}Rk^2(x_1^2 + x_3^2). \tag{5.2}$$

Then the equations for the perturbation $(\boldsymbol{q}, p)$ become

$$R(k\mathbf{C}\cdot\boldsymbol{r} - \boldsymbol{U}_B + \boldsymbol{q})\cdot\nabla\boldsymbol{q} + Rk\mathbf{C}\cdot\boldsymbol{r} = -\nabla P + \Delta\boldsymbol{q}, \tag{5.3a}$$

$$\nabla\cdot\boldsymbol{q} = 0, \tag{5.3b}$$

$$\boldsymbol{q} = -k\mathbf{C}\cdot\boldsymbol{r} + \boldsymbol{U}_B + \boldsymbol{\Omega}_B \times \boldsymbol{r}, \quad \text{on } S_p, \tag{5.4a}$$

$$\longrightarrow 0, \quad \text{as } |\boldsymbol{r}| \to \infty. \tag{5.4b}$$

It is well known in low Reynolds number hydrodynamics that one can not obtain the solution of (5.3) subject to (5.4) valid over the whole region by the method of regular perturbation in small R. Here we will consider briefly the reason of this fact. To carry out the regular perturbation method in R, the zeroth order equations are of the form

$$\triangle \mathbf{q}_0 - \nabla p_0 = 0, \qquad \nabla \cdot \mathbf{q}_0 = 0 \qquad (5.5a,b)$$

and the boundary conditions (5.4). The solution of (5.5) is given by the Stokes solution, which has the asymptotic forms;

$$\begin{aligned} \mathbf{q}_0 &= O\left(\frac{1}{r}\right), \\ p_0 &= O\left(\frac{1}{r^2}\right), \end{aligned} \qquad \text{as} \quad |\mathbf{r}| \longrightarrow \infty. \quad (5.6)$$

Now substitute these into (5.3a) again. If the neglected terms in the zeroth order equations are small compared with the retained terms over the whole region, then this approximation is uniformly valid. The results are

omitted terms (left hand side of (5.3a))

$$=O(R\kappa r^{-1})=O(R_\kappa r^{-1}),$$

retained terms (right hande side of (5.3a))

$$=O(r^{-3}),$$

for large $\boldsymbol{r}$ so that for $r>r_*(\equiv R_\kappa^{-1/2} \gg 1)$ the omitted terms become comparable with the retained terms. Thus the zeroth order solution of (5.3) and (5.4) is not uniformly valid, but valid only in the region $r < R_\kappa^{-1/2}$. Therefore, it is neccessary to consider the different expansions corresponding to the inner region (or Stokes region $r < R_\kappa^{-1/2}$) and the outer region (or Oseen region $r > R_\kappa^{-1/2}$), respectively. This method of the expansions is called the method of matched asymptotic expansions, which was developed by Proudman and Pearson[28]. For full discussions see above paper or the textbook of Van Dyke[29]. In this paper, we will use this method to obtain the solution up to $O(R_\kappa^{1/2})$ which is uniformly valid (Brenner and Cox[30]).

5-1. Inner expansions

Under the assumption (2.20), the inner expansions are of the form

$$\mathbf{q} = \mathbf{q}_0(\mathbf{r}) + R_K^{1/2}\,\mathbf{q}_1(\mathbf{r}) + o(R_K^{1/2}), \qquad (5.7a)$$

$$p = p_0(\mathbf{r}) + R_K^{1/2}\,p_1(\mathbf{r}) + o(R_K^{1/2}) \qquad (5.7b)$$

and the systematic force and torque on the particle are also expanded as

$$\bar{\mathbf{F}} = \bar{\mathbf{F}}_0 + R_K^{1/2}\bar{\mathbf{F}}_1 + o(R_K^{1/2}), \qquad (5.8a)$$

$$\bar{\mathbf{M}} = \bar{\mathbf{M}}_0 + R_K^{1/2}\bar{\mathbf{M}}_1 + o(R_K^{1/2}), \qquad (5.8b)$$

where $\bar{\mathbf{F}}_0$ and $\bar{\mathbf{M}}_0$ are the force and torque on the particle due to $(\mathbf{q}_0, p_0)$; $\bar{\mathbf{F}}_1$ and $\bar{\mathbf{M}}_1$ the force and torque due to $(\mathbf{q}_1, p_1)$, respectively. Upon susbstituting (5.7) into (5.3) and (5.4) and equating terms in R_K^0 , one obtains

$$\Delta \mathbf{q}_0 - \nabla p_0 = 0, \qquad \nabla\cdot \mathbf{q}_0 = 0, \qquad (5.9a, b)$$

$$\mathbf{q}_0 = -K\mathbf{C}\cdot\mathbf{r} + \mathbf{U}_B + \boldsymbol{\Omega}_B \times \mathbf{r}, \quad \text{on } S_P, \quad (5.10a)$$

$$\longrightarrow 0, \qquad \text{as } |\mathbf{r}| \longrightarrow \infty. \quad (5.10b)$$

Likewise, equating terms in $R_K^{1/2}$

$$\triangle \boldsymbol{q}_1 - \nabla p_1 = 0, \qquad \nabla\cdot\boldsymbol{q}_1 = 0, \qquad (5.11a,b)$$

$$\boldsymbol{q}_1 = 0, \qquad \text{on } S_p. \qquad (5.12)$$

The boundary condition of the field $(\boldsymbol{q}_1, p_1)$ as $\boldsymbol{r}\to\infty$ is furnished by the matching procedure.

5-2. Outer expansions

Dimensionless outer variable $\tilde{\boldsymbol{r}}$ is defined as follows;

$$\tilde{\boldsymbol{r}} = R_\kappa^{1/2}\, \boldsymbol{r} \qquad (5.13)$$

The outer expansions are

$$\boldsymbol{q} = R_K^{1/2}\, \boldsymbol{Q}_1(\tilde{\boldsymbol{r}}) + o(R_K^{1/2}), \qquad (5.14a)$$

$$p = R_K\, P_1(\tilde{\boldsymbol{r}}) + o(R_K). \qquad (5.14b)$$

If the operator ∇ in (5.3) is rewritten in terms of outer variable and the outer expansions (5.14) are substituted into the resulting equations, it is found that to the lowest order in $R_\kappa^{1/2}$ $(\boldsymbol{Q}_1, P_1)$ satisfies the equations

$$(\mathbf{C}\cdot\tilde{\mathbf{r}})\cdot\tilde{\nabla}\mathbf{Q}_1 + \mathbf{C}\cdot\mathbf{Q}_1 = -\tilde{\nabla}P_1 + \tilde{\Delta}\mathbf{Q}_1, \tag{5.15a}$$

$$\tilde{\nabla}\cdot\mathbf{Q}_1 = 0. \tag{5.15b}$$

The outer boundary condition is

$$\mathbf{Q}_1(\tilde{\mathbf{r}}) \longrightarrow 0, \qquad \text{as} \quad |\tilde{\mathbf{r}}| \longrightarrow \infty \tag{5.16}$$

and there is in addition a matching condition at $\tilde{\mathbf{r}}=0$.

5-3. Zeroth order inner approximation

The solution of (5.9) and (5.10) is clearly the Stokes solution of the problem. In our analysis, we require the knowledge of the Stokes field at great distances from the particle. The asymptotic forms are

$$\mathbf{q}_0(\mathbf{r}) = -\mathbf{S}(\mathbf{r})\cdot\bar{\mathbf{F}}_0 - [\nabla\mathbf{S}(\mathbf{r})]:\bar{\mathbf{B}}_0 + O(r^{-3}), \tag{5.17a}$$

$$P_0(\mathbf{r}) = -\mathbf{t}(\mathbf{r})\cdot\bar{\mathbf{F}}_0 - [\nabla\mathbf{t}(\mathbf{r})]:\bar{\mathbf{B}}_0 + O(r^{-4}), \tag{5.17b}$$

$$\text{as} \quad |\mathbf{r}| \longrightarrow \infty,$$

where

$$(\overline{B}_o)_{ij} - (\overline{B}_o)_{ji} = \epsilon_{ijk}(\overline{M}_o)_k, \tag{5.18a}$$

$$S_{ij}(\mathbf{r}) = \frac{1}{8\pi r}\left(\delta_{ij} + \frac{r_i r_j}{r^2}\right), \tag{5.18b}$$

$$t_j(\mathbf{r}) = \frac{1}{4\pi r^2}\frac{r_j}{r} \tag{5.18c}$$

with $\overline{\boldsymbol{F}}_0$ and $\overline{\boldsymbol{M}}_0$ as the dimensionless Stokes force and torque, and A:B denotes the contraction with respect to two indices.

5-4. First order outer approximation

Expressing the Stokes solution $(\boldsymbol{q}_0, p_0)$ in terms of the outer variable, we obtain

$$\boldsymbol{q}_o = -R_K^{1/2}\,\mathbf{S}(\hat{\mathbf{r}})\cdot\overline{\boldsymbol{F}}_o + O(R_K), \tag{5.19a}$$

$$p_o = -R_K\,\mathbf{t}(\hat{\mathbf{r}})\cdot\overline{\boldsymbol{F}}_o + O(R_K^{3/2}). \tag{5.19b}$$

Hence requirements for $(\boldsymbol{Q}_1, P_1)$ to be properly matched with the inner expansions are

$$\boldsymbol{Q}_1 \sim -\mathbf{S}(\hat{\mathbf{r}})\cdot\overline{\boldsymbol{F}}_o, \tag{5.20a}$$

$$P_1 \sim -\mathbf{t}(\hat{\mathbf{r}})\cdot\overline{\boldsymbol{F}}_o, \qquad \text{as } |\hat{\mathbf{r}}| \longrightarrow 0. \tag{5.20b}$$

The solution of (5.15) subject to (5.16) and (5.20) is given by

$$\mathbf{Q}_1(\tilde{\mathbf{r}}) = -\mathbf{G}(\tilde{\mathbf{r}})\cdot\overline{\mathbf{F}}_0, \tag{5.21a}$$

$$P_1(\tilde{\mathbf{r}}) = -\mathbf{T}(\tilde{\mathbf{r}})\cdot\overline{\mathbf{F}}_0, \tag{5.21b}$$

where the second rank tensor $\mathbf{G}(\tilde{\mathbf{r}})$ and the vector $\mathbf{T}(\tilde{\mathbf{r}})$ satisfy

$$\{(\mathbf{C}\cdot\tilde{\mathbf{r}})\cdot\tilde{\nabla}\}\mathbf{G} + \mathbf{C}\cdot\mathbf{G} = -\tilde{\nabla}\mathbf{T} + \tilde{\Delta}\mathbf{G} + \mathbf{I}\,\delta(\tilde{\mathbf{r}}), \tag{5.22a}$$

$$\tilde{\nabla}\cdot\mathbf{G} = 0, \tag{5.22b}$$

$$\mathbf{G}(\tilde{\mathbf{r}}) \longrightarrow 0, \qquad \text{as } |\tilde{\mathbf{r}}| \longrightarrow 0, \tag{5.23}$$

It is known that the expansions of $(\mathbf{Q}_1, P_1)$ for small $\tilde{\mathbf{r}}$ are of the form (Childress[31])

$$\mathbf{Q}_1(\tilde{\mathbf{r}}) = -\{\mathbf{S}(\tilde{\mathbf{r}}) - \mathbf{H}\}\cdot\overline{\mathbf{F}}_0 + O(\tilde{r}), \tag{5.24a}$$

$$P_1(\tilde{\mathbf{r}}) = -\mathbf{t}(\tilde{\mathbf{r}})\cdot\overline{\mathbf{F}}_0 + O(\tilde{r}^{-1}), \tag{5.24b}$$

where $\mathbf{H}$ is a constant second rank tensor (so called side force tensor).

5-5. First order inner approximation

The boundary conditions of $(\mathbf{q}_0, p_0)$ as $\mathbf{r}\to\infty$ are obtained from (5.24) such as

$$\boldsymbol{q}_1 \sim \boldsymbol{H}\cdot\overline{\boldsymbol{F}}_0, \tag{5.25a}$$

$$p_1 = o(r^{-1}). \tag{5.25b}$$

Therefore the solution of the Stokes problem (5.11), (5.12) and (5.25) is easily found that the asymptotic forms are

$$\boldsymbol{q}_1(\boldsymbol{r}) = \boldsymbol{H}\cdot\overline{\boldsymbol{F}}_0 - \boldsymbol{S}(\boldsymbol{r})\cdot\boldsymbol{b} + O(r^{-2}), \tag{5.26a}$$

$$p_1(\boldsymbol{r}) = -\boldsymbol{t}(\boldsymbol{r})\cdot\boldsymbol{b} + O(r^{-3}), \quad \text{as } |\boldsymbol{r}| \longrightarrow \infty \tag{5.26b}$$

where $\boldsymbol{b}$ is a constant vector.

5-6. Systematic force and torque

It has already been seen that the zeroth order inner approximation (5.17) and the the first order (5.26) are both solutions of the Stokes equations. As is well known in low Reynolds number hydrodynamics (Happel and Brenner[32]), the force and torque acting on the arbitrary shaped particle are give by

$$\bar{\mathbf{F}}_0 = -\mathbf{\Gamma}\cdot\mathbf{U}_B - {}_0\mathbf{\Lambda}\cdot(\mathbf{\Omega}_B - \mathbf{\Omega}_f), \tag{5.27a}$$

$$\bar{\mathbf{M}}_0 = -{}_0\mathbf{\Lambda}^t\cdot\mathbf{U}_B - {}_0\mathbf{D}\cdot(\mathbf{\Omega}_B - \mathbf{\Omega}_f). \tag{5.27b}$$

and

$$\bar{\mathbf{F}}_1 = \mathbf{\Gamma}\cdot\mathbf{H}\cdot\bar{\mathbf{F}}_0 , \tag{5.28a}$$

$$\bar{\mathbf{M}}_1 = {}_0\mathbf{\Lambda}^t\cdot\mathbf{H}\cdot\bar{\mathbf{F}}_0 , \tag{5.28b}$$

where

$$\mathbf{\Omega}_f = -\frac{1}{2}K\,\boldsymbol{\varepsilon}:(\mathbf{C}-\mathbf{C}^t). \tag{5.29}$$

The coefficient $\mathbf{\Gamma}$ is the translation dyadic which depends only upon the shape of the particle. The coefficients ${}_0\mathbf{D}$ and ${}_0\mathbf{\Lambda}$ are the rotation dyadic and the coupling dyadic at the origin which depend upon the particle shape and the location of the origin O. It is well known, moreover, that the following symmetry relations hold for the friction tensors;

$$\Gamma_{ij} = \Gamma_{ji} , \qquad ({}_0D)_{ij} = ({}_0D)_{ji} . \tag{5.30a,b}$$

6. Calculation of the adjoint field

In this chapter we omit indexes such as β, q of (4.3) for the sake of simplicity and recover them when the explicit formulae are needed.

6-1. Inner and outer expansions

In order to seek the asymptotic solution of (4.4b) and (4.5) by the method of matched asymptotic expansions, one can proceed in a way similar to Chap.5. Taking into account the expansions of systematic field $(\bar{\boldsymbol{v}}, \bar{p})$ for small $R_K^{1/2}$, we write the inner expansion as follows

$$\chi^{+\beta}_{\alpha,q} \equiv \chi^{+}_{\alpha} = \chi^{+(0)}_{\alpha}(\boldsymbol{r}) + R_K^{1/2}\, \chi^{+(1)}_{\alpha}(\boldsymbol{r}) + o(R_K^{1/2}). \tag{6.1}$$

A tensor $g^{\alpha\beta}_{iq}$ due to $\chi^{+\beta}_{\gamma,q}$ is defined by

$$g^{\alpha\beta}_{iq} = \int_{S_p} \Delta^{\alpha}_{ik}\, \Theta^{+\beta}_{klq}\, dS_l \tag{6.2}$$

and also be expanded as

$$g_{iq}^{\alpha\beta} = g_{iq}^{\alpha\beta(0)} + R_K^{1/2} g_{iq}^{\alpha\beta(1)} + o(R_K^{1/2}), \tag{6.3}$$

where $g^{(0)}$ and $g^{(1)}$ are tensors due to $\chi^{+(0)}$ and $\chi^{+(1)}$ respectively.

The outer expansions are

$$\chi_j^+ = R_K^{1/2} X_j^+(\tilde{\mathbf{r}}) + o(R_K^{1/2}), \quad (j=1,2,3) \tag{6.4a}$$

$$\chi_4^+ = R_K X_4^+(\tilde{\mathbf{r}}) + o(R_K). \tag{6.4b}$$

Substituting (5.7) and (6.1) into (4.4b) and (4.5), we obtain

$$\Delta \chi_j^{+(0)} - \frac{\partial}{\partial r_j}\chi_4^{+(0)} = 0, \qquad \frac{\partial}{\partial r_j}\chi_j^{+(0)} = 0, \tag{6.5a, b}$$

$$\chi_j^{+(0)} = \Delta_j, \qquad \text{on} \quad S_p, \tag{6.6a}$$

$$\longrightarrow 0, \qquad \text{as} \quad |\mathbf{r}| \longrightarrow \infty. \tag{6.6b}$$

to order R_K^0 and

$$\Delta \chi_j^{+(1)} - \frac{\partial}{\partial r_j}\chi_4^{+(1)} = 0, \qquad \frac{\partial}{\partial r_j}\chi_j^{+(1)} = 0, \tag{6.7a, b}$$

$$\chi_j^{+(1)} = 0, \qquad \text{on} \quad S_p. \tag{6.8}$$

to the first order in $R_K^{1/2}$. The outer boundary condition of

$\chi_j^{+(1)}$ is established by the matching condition. Rewriting equation (4.4b) in terms of the outer variable (5.13) and substituting the outer expansions (6.4) into the resulting euation, we have the equations for $(X_j^{+(1)}, X_4^{+(1)})$

$$-\{(\mathbf{c}\cdot\tilde{\nabla})\cdot\tilde{\nabla}\} X_j^{+(1)} + (C^{+})_{j\ell} X_\ell^{+(1)} = -\frac{\partial}{\partial \tilde{r}_j} X_4^{+(1)} + \tilde{\Delta} X_j^{+(1)}, \tag{6.9a}$$

$$\frac{\partial}{\partial \tilde{r}_j} X_j^{+(1)} = 0. \tag{6.9b}$$

to the lowest order in $R_K^{1/2}$. The outer boundary condition is

$$X_j^{+(1)}(\tilde{\mathbf{r}}) \longrightarrow 0, \qquad as \quad |\tilde{\mathbf{r}}| \longrightarrow \infty \tag{6.10}$$

and in addition $X_j^{+(1)}(\tilde{\mathbf{r}})$ should satisfy the matching condition at $\tilde{r}=0$.

6-2. Zeroth order inner approximation

The solution of (6.5) and (6.6) can be expressed in terms of the Stokes solution. The asymptotic forms of the solution as $\tilde{r} \to \infty$ are

$$\chi_{iq}^{+\alpha(0)}(\mathbf{r}) = -S_{ij}(\mathbf{r})\, g_{jq}^{0\alpha(0)} - \{h_{jq}^{\alpha(0)}\}_l \frac{\partial}{\partial r_l} S_{ij}(\mathbf{r}) + O(r^{-3})$$

$$\chi_{4q}^{+\alpha(0)}(\mathbf{r}) = -t_j(\mathbf{r})\, g_{jq}^{0\alpha(0)} - \{h_{jq}^{\alpha(0)}\}_l \frac{\partial}{\partial r_l} t_j(\mathbf{r}) + O(r^{-4})$$

$$(6.11\,a,\,b)$$

with

$$\{h_{jq}^{\alpha(0)}\}_i - \{h_{iq}^{\alpha(0)}\}_j = \epsilon_{ijl}\, g_{lq}^{1\alpha(0)}, \qquad (6.12)$$

where $g_{ij}^{0\alpha(0)}$ and $g_{ij}^{1\alpha(0)}$ are the i-th component of the dimensionless Stokes force and torque acting on the particle which translates α=0 (rotates α=1) along (about) the j-th axis with the unit velocity (angular velocity), respectively. Thus from (5.27) it follows that

$$g_{ij}^{00(0)} = -\Gamma_{ij}, \qquad g_{ij}^{11(0)} = -({}_0D)_{ij},$$

$$g_{ij}^{01(0)} = g_{ji}^{10(0)} = -({}_0\Lambda)_{ij}. \qquad (6.13)$$

6-3. First order outer approximation

From the asymptotic forms (6.11) the matching conditions of $(\mathrm{X}_j^{+(1)}, \mathrm{X}_4^{+(1)})$ at $\tilde{\boldsymbol{r}}=0$ are obtained as

$$\mathrm{X}_j^{+(1)} \sim - S_{j\ell}(\tilde{\boldsymbol{r}})\, g_\ell^{(0)} \quad , \tag{6.14a}$$

$$\mathrm{X}_4^{+(1)} \sim - t_j(\tilde{\boldsymbol{r}})\, g_j^{(0)} \quad , \tag{6.14b}$$

The solution of (6.9) and (6.10) satisfying (6.14) can be written as

$$\mathrm{X}_j^{+(1)}(\tilde{\boldsymbol{r}}) = - G_{j\ell}^{+}(\tilde{\boldsymbol{r}})\, g_\ell^{(0)} , \tag{6.15a}$$

$$\mathrm{X}_4^{+(1)}(\tilde{\boldsymbol{r}}) = - T_\ell^{+}(\tilde{\boldsymbol{r}}) \cdot g_\ell^{(0)}, \tag{6.15b}$$

where $(\mathbf{G}^{+}, \boldsymbol{T}^{+})$ satisfies

$$-(\mathbf{C}\cdot\tilde{\boldsymbol{r}})\cdot\tilde{\nabla}\,\mathbf{G}^{+} + \mathbf{C}^{t}\cdot\mathbf{G}^{+} = -\tilde{\nabla}\mathbf{T}^{+} + \tilde{\Delta}\mathbf{G}^{+} + \mathbf{I}\,\delta(\tilde{\boldsymbol{r}}), \tag{6.16a}$$

$$\tilde{\nabla}\cdot\mathbf{G}^{+} = 0, \tag{6.16b}$$

$$\mathbf{G}^{+}(\tilde{\boldsymbol{r}}) \longrightarrow 0, \qquad \text{as} \quad |\tilde{\boldsymbol{r}}| \longrightarrow \infty. \tag{6.17}$$

Field $(\mathrm{X}_j^{+(1)}(\tilde{\boldsymbol{r}}), \mathrm{X}_4^{+(1)}(\tilde{\boldsymbol{r}}))$ has the following expansions for small $\tilde{\boldsymbol{r}}$ such as

$$X_j^{+(1)} = -\{ S_{jl}(\hat{\mathbf{r}}) - H^{+}_{jl} \} g_l^{(0)} + O(\tilde{r}), \tag{6.18a}$$

$$X_4^{+(1)} = - t_l(\tilde{\mathbf{r}}) \cdot g_l^{(0)}, \tag{6.18b}$$

here $\mathbf{H}^{+}$ is a constant second rank tensor.

6-4. First order inner approximation

From the expansions (6.18) and the matching principle, the first order inner solution $(\chi_j^{+(1)}(\mathbf{r}), \chi_4^{+(1)}(\mathbf{r}))$ must satisfy the boundary condition;

$$\chi_j^{+(1)} \longrightarrow H^{+}_{jl}\, g_l^{(0)}, \quad \text{as } |\mathbf{r}| \longrightarrow \infty. \tag{6.19}$$

Then the asymptotic forms of Stokes solution of (6.7), (6.8) and (6.19) are

$$\chi_{iq}^{+\alpha(1)}(\mathbf{r}) = H^{+}_{ij}\, g_{jq}^{o\alpha(0)} - S_{ij}(\mathbf{r})\, b_{jq}^{\alpha} + O(r^{-2}), \tag{6.20a}$$

$$\chi_{4q}^{+\alpha(1)}(\mathbf{r}) = - t_j(\mathbf{r})\, b_{jq}^{\alpha} + O(r^{-3}), \quad \text{as } |\mathbf{r}| \to \infty, \tag{6.20b}$$

where b_{jq}^{α} is a constant tensor.

7. Calculation of the field $\psi^{\beta}_{\alpha,l}$

Here for simplicity, notations for $\psi^{\beta}_{\alpha,l}$ in which indeces such as β, l are suppressed, are introduced. Let us write

$$\psi^{\beta}_{j,\ell}(\boldsymbol{r}) = \langle \tilde{v}_j(\boldsymbol{r})\tilde{K}^{\beta}_{\ell}\rangle \equiv \psi_j(\boldsymbol{r})$$

$$\psi^{\beta}_{4,\ell}(\boldsymbol{r}) = \langle \tilde{p}(\boldsymbol{r})\tilde{K}^{\beta}_{\ell}\rangle \equiv \psi_4(\boldsymbol{r})$$

7-1. Inner and outer expansions

Consider the solution of (3.13) and (3.14). The analysis will proceed in a way similar to those of Chaps. 5 and 6. Taking into account the expansions of the systematic field $(\bar{\boldsymbol{v}},\bar{p})$ and the adjoint field χ^{+}_{α}, we have the inner expansion,

$$\psi^{\beta}_{\alpha,\ell} \equiv \psi_{\alpha} = \psi^{(0)}_{\alpha}(\boldsymbol{r}) + R_K^{1/2}\,\psi^{(1)}_{\alpha}(\boldsymbol{r}) + o(R_K^{1/2}) \tag{7.1}$$

and the outer expansions

$$\psi_j = R_K^{1/2}\,\Psi_j(\tilde{\mathbf{r}}) + o(R_K^{1/2}), \quad (j=1,2,3) \qquad (7.2a)$$

$$\psi_4 = R_K\,\Psi_4(\tilde{\mathbf{r}}) + o(R_K) \,. \qquad (7.2b)$$

Using (4.14) and (4.7), and substituting (5.7), (6.1) and (7.1) into (3.13) and (3.14), we obtain

$$\Delta\psi_j^{(0)} - \frac{\partial}{\partial r_j}\psi_4^{(0)} = \Delta\chi_j^{(0)}, \qquad \frac{\partial}{\partial r_j}\psi_j^{(0)} = 0, \qquad (7.3a,b)$$

$$\psi_j^{(0)} = 0 \;, \qquad \text{on} \quad S_p \qquad (7.4a)$$

$$\longrightarrow 0 \;, \qquad \text{as} \quad |\mathbf{r}| \longrightarrow \infty. \qquad (7.4b)$$

to the lowest order in $R_K^{1/2}$ and

$$\Delta\psi_j^{(1)} - \frac{\partial}{\partial r_j}\psi_4^{(1)} = \Delta\chi_j^{(1)}, \qquad \frac{\partial}{\partial r_j}\psi_j^{(1)} = 0, \qquad (7.5a,b)$$

$$\psi_j^{(1)} = 0 \;, \qquad \text{on} \quad S_p \qquad (7.6)$$

to the first order. The outer boundary condition of $\psi_j^{(1)}(\mathbf{r})$ as $r\to\infty$ can also be derived by the matching condition. Likewise the equations for $(\Psi_j^{(1)}(\tilde{\mathbf{r}}), \Psi_4^{(1)}(\tilde{\mathbf{r}}))$ become

$$\{(\mathbf{c}\cdot\tilde{\boldsymbol{r}})\cdot\tilde{\boldsymbol{\nabla}}\}\Psi_j^{(1)} + c_{j\ell}\Psi_\ell^{(1)} = -\frac{\partial}{\partial\tilde{r}_j}\Psi_4^{(1)} + \tilde{\Delta}\Psi_j^{(1)} - \tilde{\Delta}X_j^{+(1)}, \tag{7.7a}$$

$$\frac{\partial}{\partial\tilde{r}_j}\Psi_j^{(1)} = 0, \tag{7.7b}$$

$$\Psi_j^{(1)} \longrightarrow 0, \qquad \text{as} \quad |\tilde{\boldsymbol{r}}| \longrightarrow \infty. \tag{7.8}$$

7-2. Zeroth order inner approximation

The solution of (7.3) and (7.4) is given by

$$\psi_{j,\ell}^{\alpha(0)} = \langle \tilde{v}_j(\boldsymbol{r})\tilde{K}_\ell^\alpha \rangle^{(0)} = 0, \tag{7.9a}$$

$$\begin{aligned}\psi_{4,\ell}^{\alpha(0)} &= \langle \tilde{p}(\boldsymbol{r})\tilde{K}_\ell^\alpha \rangle^{(0)} = -\chi_4^{+\alpha(0)}(\boldsymbol{r}) \\ &= t_j(\boldsymbol{r})\cdot g_{j\ell}^{0\alpha\,(0)} + O(r^{-3}), \\ &\qquad \text{as} \quad |\boldsymbol{r}| \longrightarrow \infty.\end{aligned} \tag{7.9b}$$

7-3. First order outer approximation

The matching conditions for $(\Psi_j^{(1)}(\tilde{\boldsymbol{r}}), \Psi_4^{(1)}(\tilde{\boldsymbol{r}}))$ at $\tilde{\boldsymbol{r}}=0$ are

$$\Psi_j^{(1)} = o(\tilde{r}^{-1}), \quad \Psi_4^{(1)} = t_j(\hat{\mathbf{r}}) g_j . \tag{7.10a,b}$$

The solution of (7.7) and (7.8) which satisfies (7.10) is given by

$$\Psi_j^{(1)}(\tilde{\mathbf{r}}) = -\frac{1}{2}\{ G_{jm}^{+}(\tilde{\mathbf{r}}) - G_{jm}(\tilde{\mathbf{r}})\} \cdot g_m^{(0)} , \tag{7.11a}$$

$$\Psi_4^{(1)}(\tilde{\mathbf{r}}) = -X_4^{+(1)}(\tilde{\mathbf{r}}) - \frac{1}{2}\{ T_m^{+}(\tilde{\mathbf{r}}) - T_m(\tilde{\mathbf{r}})\} g_m^{(0)} . \tag{7.11b}$$

(see Appendix-D). From (5.21a), (5.24a), (6.15a) and (6.18a) the expansion of $\Psi_j^{(1)}(\tilde{\mathbf{r}})$ at $\tilde{\mathbf{r}}=0$ is of the form

$$\begin{aligned}\Psi_j^{(1)}(\tilde{\mathbf{r}}) &= \frac{1}{2}(H_{jm}^{+} - H_{jm}) g_m^{(0)} + O(\tilde{r}) \\ &= a_j + O(\tilde{r}),\end{aligned} \tag{7.12}$$

here

$$\mathbf{a} = \frac{1}{2}(\mathbf{H} - \mathbf{H}^{+}) \cdot \mathbf{g}^{(0)} . \tag{7.15}$$

7-4. First order inner approximation

By the matching principle the field $(\psi_j^{(1)}(\mathbf{r}), \psi_4^{(1)}(\mathbf{r}))$ must satisfy the condition

$$\Psi_j^{(1)}(\mathbf{r}) \longrightarrow a_j , \qquad \text{as} \quad |\mathbf{r}| \longrightarrow \infty . \tag{7.14}$$

Then it is easily found that the asymptotic expressions of the Stokes problem (7.5), (7.6) and (7.14) are

$$\Psi_{j,l}^{\alpha(1)} = \langle \tilde{v}_j(\mathbf{r}) \tilde{K}_l^{\alpha} \rangle^{(1)} = a_{jl}^{\alpha} - S_{jn}(\mathbf{r}) f_{nl}^{\alpha} + O(r^{-2}), \tag{7.14a}$$

$$\Psi_{4,l}^{\alpha(1)} = \langle \tilde{p}(\mathbf{r}) \tilde{K}_l^{\alpha} \rangle^{(1)} = - t_j(\mathbf{r}) f_{jl}^{\alpha} + O(r^{-3}), \tag{7.14b}$$

$$\text{as} \quad |\mathbf{r}| \longrightarrow \infty ,$$

where f_{jl}^{α} is a constant tensor.

8. Computation of the correlation function of the fluctuating generalized hydrodynamic forces

Let us consider the correlation function of the fluctuating generalized hydrodynamic forces (hereafter calling correlation function simply). Corresponding to the expansion of $\psi^{\alpha}_{\gamma,l}$, the correlation function is expanded as

$$Y^{\alpha\beta}_{ij} = Y^{\alpha\beta\,(0)}_{ij} + R_k^{1/2}\, Y^{\alpha\beta\,(1)}_{ij} + o\,(R_k^{1/2}), \tag{8.1}$$

where $Y^{\alpha\beta(0)}_{ij}$ is the zeroth order correlation function due to $\psi^{\alpha(0)}_{\gamma,l}$ and $Y^{\alpha\beta(1)}_{ij}$ the first order one due to $\psi^{\alpha(1)}_{\gamma,l}$. Further by the use of the relations (4.14) and $\frac{\partial}{\partial r_j}\chi^{+(0)}_j = 0$, the equations of motion (7.3) for $\psi^{\alpha(0)}_{\gamma,l}$ can be put into the form

$$\frac{\partial}{\partial r_j}\left\{ <\tilde{\tau}_{ij}(\mathbf{r})\,\tilde{K}^{\alpha}_{\ell}>^{(0)} + <\tilde{\sigma}_{ij}(\mathbf{r})\,\tilde{K}^{\alpha}_{\ell}>^{(0)} \right\} = 0, \tag{8.2a}$$

where

$$<\tilde{\sigma}_{ij}(\mathbf{r})\,\tilde{K}^{\alpha}_{\ell}>^{(0)} = -\left\{ \frac{\partial}{\partial r_j}\chi^{+\alpha(0)}_{i\ell}(\mathbf{r}) + \frac{\partial}{\partial r_i}\chi^{+\alpha(0)}_{j\ell}(\mathbf{r}) \right\}. \tag{8.2b}$$

Similarly for $\psi^{\alpha(1)}_{\gamma,l}$ we have

$$\frac{\partial}{\partial r_j}\left\{ <\tilde{\tau}_{ij}(\mathbf{r})\tilde{K}_\ell^{\alpha}>^{(1)} + <\hat{\sigma}_{ij}(\mathbf{r})\tilde{K}_\ell^{\alpha}>^{(1)} \right\} = 0, \tag{8.3a}$$

here

$$<\hat{\sigma}_{ij}(\mathbf{r})\tilde{K}_\ell^{\alpha}>^{(1)} = -\left\{ \frac{\partial}{\partial r_j}\chi_{i\ell}^{+\alpha(1)}(\mathbf{r}) + \frac{\partial}{\partial r_i}\chi_{j\ell}^{+\alpha(1)}(\mathbf{r}) \right\}. \tag{8.3b}$$

8-1. Zeroth order correlation function

From the definition (4.2) it follows

$$Y_{ij}^{\alpha\beta(0)} = \int_{S_p} \Delta_{ik}^{\alpha} \left\{ <\tilde{\tau}_{k\ell}(\mathbf{r})\tilde{K}_j^{\beta}>^{(0)} + <\tilde{\sigma}_{k\ell}(\mathbf{r})\tilde{K}_j^{\beta}>^{(0)} \right\} dS_\ell(\mathbf{r}). \tag{8.4}$$

Using (2.7c), (7.9) and (8.2b) one can write (8.4) as

$$<\tilde{\tau}_{k\ell}(\mathbf{r})\tilde{K}_j^{\beta}>^{(0)} + <\tilde{\sigma}_{k\ell}(\mathbf{r})\tilde{K}_j^{\beta}>^{(0)} = -\,\Theta_{k\ell j}^{+\beta(0)}(\mathbf{r}), \tag{8.5}$$

where $\Theta_{k\ell j}^{+\beta(0)}(\mathbf{r})$ is the stress tensor due to $\chi_{\gamma,j}^{+\beta(0)}$ (see (4.7)). Thus from (6.2) and (8.4) we obtain

$$Y_{ij}^{\alpha\beta(0)} = -g_{ij}^{\alpha\beta(0)}. \tag{8.6}$$

8-2. First order correlation function

We calculate the first order correlation function (Brenner and Cox[30]). Let $(v^{\beta}_{in}, p^{\beta}_{n})$ be the dimensionless Stokes field which satisfies

$$\frac{\partial}{\partial r_j}\{\tau^{\beta}_{ijn}\}=0, \qquad \frac{\partial}{\partial r_i} v^{\beta}_{in}=0, \qquad (8.7a,b)$$

$$v^{\beta}_{in} = \Delta^{\beta}_{ni}, \qquad \text{on} \quad S_p, \qquad (8.8a)$$

$$\longrightarrow 0, \qquad \text{as} \quad |\mathbf{r}| \longrightarrow \infty, \qquad (8.8b)$$

$$\tau^{\beta}_{ijn} = -p^{\beta}_{n}\delta_{ij} + \frac{\partial}{\partial r_j} v^{\beta}_{in} + \frac{\partial}{\partial r_i} v^{\beta}_{jn}, \qquad (8.9)$$

where τ^{β}_{ijn} is the stress tensor due to $(v^{\beta}_{in}, p^{\beta}_{n})$. This Stokes field is identical with $\chi^{+\alpha(0)}_{\gamma,j}$ defined by (6.5) and (6.6). Therefore, the asymptotic forms of $(v^{\beta}_{in}, p^{\beta}_{n})$ are given by (6.9);

$$v_{im}^{\beta}(\mathbf{r}) = -S_{ij}(\mathbf{r})\, g_{jm}^{0\beta(0)} - \{h_{jm}^{\beta(0)}\} \frac{\partial}{\partial r_l} S_{ij}(\mathbf{r}) + O(r^{-3}), \tag{8.10a}$$

$$P_m^{\beta}(\mathbf{r}) = -t_j(\mathbf{r})\, g_{jm}^{0\beta(0)} - \{h_{jm}^{\beta(0)}\} \frac{\partial}{\partial r_l} t_j(\mathbf{r}) + O(r^{-4}), \tag{8.10b}$$

as $|\mathbf{r}| \longrightarrow \infty$.

Take the scalar product of (8.3a) with v_{in}^{β} , and (8.7a) with $<\hat{v}_i(\mathbf{r})\,\hat{K}_l^{\alpha}>^{(1)} - \chi_{il}^{+\alpha(1)}(\mathbf{r})$ and subtract to obtain

$$0 = v_{im}^{\beta}(\mathbf{r}) \frac{\partial}{\partial r_j} \{ <\hat{\tau}_{ij}(\mathbf{r})\,\hat{K}_l^{\alpha}>^{(1)} + <\hat{\sigma}_{ij}(\mathbf{r})\,\hat{K}_l^{\alpha}>^{(1)} \}$$

$$- \{ <\hat{v}_i(\mathbf{r})\,\hat{K}_l^{\alpha}>^{(1)} - w_{il}^{+\alpha(1)}(\mathbf{r}) \} \frac{\partial}{\partial r_j} \tau_{ijm}^{\beta}(\mathbf{r})$$

$$\tag{8.11}$$

Since $\frac{\partial}{\partial r_j} v_{jm}^{\beta} = \frac{\partial}{\partial r_j} <\hat{v}_j\,\hat{K}_l^{\alpha}>^{(1)} = \frac{\partial}{\partial r_j} \chi_{jl}^{+\alpha(1)} = 0.$ and

$$\left\{\frac{\partial}{\partial r_j} v_{im}^{\beta}(\mathbf{r})\right\} \{ <\hat{\tau}_{ij}(\mathbf{r})\,\hat{K}_l^{\alpha}>^{(1)} + <\hat{\sigma}_{ij}(\mathbf{r})\,\hat{K}_l^{\alpha}>^{(1)} \}$$

$$= \frac{\partial}{\partial r_j} \{ <\hat{v}_i(\mathbf{r})\,\hat{K}_l^{\alpha}>^{(1)} - \chi_{il}^{+\alpha(1)}(\mathbf{r}) \} \{ \tau_{ijm}^{\beta}(\mathbf{r}) \} \tag{8.12}$$

(see Appendix-A), it follows that (8.11) becomes

$$0=\frac{\partial}{\partial r_j}\Big[v_{im}^{\beta}(\mathbf{r})\{<\tilde{\tau}_{ij}(\mathbf{r})\tilde{K}_l^{\alpha}>^{(1)}+<\tilde{\sigma}_{ij}(\mathbf{r})\tilde{K}_l^{\alpha}>^{(1)}\}$$

$$-\{<\hat{v}_i(\mathbf{r})\hat{K}_l^{\alpha}>^{(1)}-\chi_{il}^{+\alpha(1)}(\mathbf{r})\}\,\tau_{ijm}^{\beta}(\mathbf{r})\Big].$$

(8.13)

Integrating this over the volume V_L bounded externally by the surface S_L, of radius L, and by the surface S_P of the particle, yields

$$\Big(\int_{S_L}-\int_{S_P}\Big)\Big[v_{im}^{\beta}(\mathbf{r})\{<\tilde{\tau}_{ij}(\mathbf{r})\tilde{K}_l^{\alpha}>^{(1)}+<\tilde{\sigma}_{ij}(\mathbf{r})\tilde{K}_l^{\alpha}>^{(1)}\}$$

$$-\{<\hat{v}_i(\mathbf{r})\tilde{K}_l^{\alpha}>^{(1)}-\chi_{il}^{+\alpha(1)}(\mathbf{r})\}\,\tau_{ijm}^{\beta}(\mathbf{r})\,dS_j(\mathbf{r})$$

$$=0. \qquad (8.14)$$

Using the boundary condition (8.8a) and (4.2), and letting $L\to\infty$, we can write equation (8.14) as

$$Y_{ml}^{\beta\alpha(1)}=\int_{S_P}\Delta_{im}^{\beta}\{<\tilde{\tau}_{ij}(\mathbf{r})\tilde{K}_l^{\alpha}>^{(1)}+<\tilde{\sigma}_{ij}(\mathbf{r})\tilde{K}_l^{\alpha}>^{(1)}\}dS_j(\mathbf{r})$$

$$=\ (I_{ml}^{\beta\alpha})_1\ +\ (I_{ml}^{\beta\alpha})_2\ +\ (I_{ml}^{\beta\alpha})_3\,,$$

(8.15)

where

$$\left(I_{nl}^{\beta\alpha}\right)_1 = \int_{S_P} \left\{ \langle \tilde{v}_i(\mathbf{r}) \tilde{K}_l^{\alpha} \rangle^{(1)} - \chi_{il}^{+\alpha(1)}(\mathbf{r}) \right\} \tau_{ijm}^{\beta}(\mathbf{r})\, dS_j(\mathbf{r}), \tag{8.16}$$

$$\left(I_{ml}^{\beta\alpha}\right)_2 = \lim_{L\to\infty} \int_{S_L} v_{im}^{\beta}(\mathbf{r}) \left\{ \langle \tilde{\tau}_{ij}(\mathbf{r}) \tilde{K}_l^{\alpha} \rangle^{(1)} + \langle \tilde{\sigma}_{ij}(\mathbf{r}) \tilde{K}_l^{\alpha} \rangle^{(1)} \right\} dS_j(\mathbf{r}), \tag{8.17}$$

$$\left(I_{ml}^{\beta\alpha}\right)_3 = -\lim_{L\to\infty} \int_{S_L} \left\{ \langle \tilde{v}_i(\mathbf{r}) \tilde{K}_l^{\alpha} \rangle^{(1)} - \chi_{il}^{+\alpha(1)}(\mathbf{r}) \right\} \tau_{ijm}^{\beta}(\mathbf{r})\, dS_j(\mathbf{r}). \tag{8.18}$$

Consider $(I_{nl}^{\beta\alpha})_1$. Since $\langle \tilde{v}_i(\mathbf{r}) \tilde{K}_l^{\alpha} \rangle^{(1)} = \chi_{il}^{+\alpha(1)}(\mathbf{r}) = 0$ on S_P, we obtain

$$(I_{nl}^{\beta\alpha})_1 = 0. \tag{8.19}$$

Next consider $(I_{nl}^{\beta\alpha})_2$. The terms in the integrand only contribute to $(I_{nl}^{\beta\alpha})_2$ if the they do not tend to zero faster than r^{-2} as $r \to \infty$. From (8.10) $v_{in}^{\beta} = O(r^{-1})$ and from (2.7c), (6.20), (7.15) and (8.3b)

$$< \tilde{\tau}_{ij}(\mathbf{r})\tilde{K}_{\ell}^{\alpha} >^{(1)} + < \tilde{\sigma}_{ij}(\mathbf{r})\tilde{K}_{\ell}^{\alpha} >^{(1)} = O(r^{-2}),$$
$$\text{as} \quad |\mathbf{r}| \to \infty,$$

therefore. it follows that the integrand is $O(r^{-3})$ as $r \to \infty$. Thus

$$(I_{nl}^{\beta\alpha})_2 = 0. \tag{8.20}$$

Consider now $(I_{nl}^{\beta\alpha})_3$. Since, from (6.20a) and (7.15a),

$$< \vartheta_i(\mathbf{r})\tilde{K}_{\ell}^{\alpha} >^{(1)} - \chi_{i\ell}^{+\alpha(1)}(\mathbf{r}) = O(1), \quad \text{as } |\mathbf{r}| \to \infty,$$

it follows that contribution comes only from the term $v_{in}^{\beta} = O(r^{-1})$ as $r \to \infty$. Hence, from (7.13) and (7.14), (8.18) becomes

$$(I_{m\ell}^{\beta\alpha})_3 = -\left\{\frac{1}{2}(H_{ik}^{+} - H_{ik})g_{k\ell}^{0\alpha(0)} - H_{ik}^{+}g_{k\ell}^{0\alpha(0)}\right\}$$
$$\times \lim_{L\to\infty}\int_{S_L} \tau_{ijn}^{\beta}(\mathbf{r})\,dS_j(\mathbf{r})$$

$$= \frac{1}{2}(H_{ik} + H_{ik}^{+})g_{k\ell}^{0\alpha(0)}$$
$$\times \lim_{L\to\infty}\int_{S_L} \tau_{ijn}^{\beta}(\mathbf{r})\,dS_j(\mathbf{r}). \tag{8.21}$$

On the other hand, integrating (8.7a) over the volume V_L and letting $L \to \infty$, we obtain

$$\int_{S_P} \tau^{\beta}_{ijm}(\mathbf{r})\, dS_j(\mathbf{r}) = \lim_{L\to\infty} \int_{S_L} \tau^{\beta}_{ijm}(\mathbf{r})\, dS_j(\mathbf{r}). \qquad (8.22)$$

From (6.2) and (8.10), (8.22) can be written as

$$g^{o\beta(0)}_{im} = \lim_{L\to\infty} \int_{S_L} \tau^{\beta}_{ijm}(\mathbf{r})\, dS_j(\mathbf{r}). \qquad (8.23)$$

Hence it follows that

$$\left(I^{\beta\alpha}_{m\ell}\right)_3 = \frac{1}{2}\left(H_{ik} + H^{+}_{ik}\right) g^{o\beta(0)}_{im}\, g^{o\alpha(0)}_{k\ell}. \qquad (8.24)$$

Substituting (8.19), (8.20) and (8.24) into equation (8.15), we finally obtain

$$Y^{\beta\alpha(1)}_{m\ell} = \frac{1}{2}\left(H_{ik} + H^{+}_{ik}\right) g^{o\beta(0)}_{im}\, g^{o\alpha(0)}_{k\ell}. \qquad (8.25)$$

9. Langevin equations for the Brownian particle

Here in order to write the Langevin equations of the arbitrary shaped particle immersed in an unbounded rotating fluid, we substitute (5.27) and (5.28) into (2.15a) and change the coordinate system to the laboratory frame, and obtain

$$\frac{d}{dt}\begin{pmatrix} \mathbf{V}_B \\ \mathbf{J}\cdot\boldsymbol{\Omega}_B \end{pmatrix} = -\begin{pmatrix} \mathbf{I}+R_K^{1/2}\boldsymbol{\Gamma}\cdot\mathbf{H}, & \mathbf{I} \\ +R_K^{1/2}\,{}_0\boldsymbol{\Lambda}^t\cdot\mathbf{H}, & \mathbf{I} \end{pmatrix}\begin{pmatrix} \boldsymbol{\Gamma}, & {}_0\boldsymbol{\Lambda} \\ {}_0\boldsymbol{\Lambda}^t, & {}_0\mathbf{D} \end{pmatrix}\begin{pmatrix} \mathbf{V}_B-\mathbf{V}_f \\ \boldsymbol{\Omega}_B-\boldsymbol{\Omega}_f \end{pmatrix} + \begin{pmatrix} \tilde{\mathbf{F}} \\ \tilde{\mathbf{M}} \end{pmatrix}, \qquad (9.1)$$

up to $O(R_K^{1/2})$ in the matrix form; where

$$\mathbf{V}_B = \frac{d\mathbf{X}_B}{dt}, \qquad \mathbf{V}_f = k\,\mathbf{C}\cdot\mathbf{X}_B, \qquad (9.2a,b)$$

here $\mathbf{X}_B$ is the position vector from the origin O which is on the axis of fluid rotation. Furthermore, substitution of (8.6) and (8.25) into (8.1) and the use of (6.13) yields

$$\langle\tilde{F}\rangle = o(R_K^{1/2}), \qquad \langle\tilde{M}\rangle = o(R_K^{1/2}), \qquad (9.3a,b)$$

$$\int_{-\infty}^{\infty} \langle \tilde{F}_i(t) \tilde{F}_j(t') \rangle dt = \Gamma_{ij} + \frac{1}{2} R_K^{1/2} \Gamma_{ik} (H_{kl} + H_{kl}^{+}) \Gamma_{lj} + o(R_K^{1/2}) \tag{9.4}$$

$$\int_{-\infty}^{\infty} \langle \tilde{F}_i(t) \tilde{M}_j(t') \rangle dt = \left(\int_{-\infty}^{\infty} \langle \tilde{M}_i(t) \tilde{F}_j(t') \rangle dt \right)^{t}$$

$$= {}_0\Lambda_{ij} + \frac{1}{2} R_K^{1/2} \Gamma_{ik} (H_{kl} + H_{kl}^{+}) {}_0\Lambda_{lj} + o(R_K^{1/2}), \tag{9.5}$$

$$\int_{-\infty}^{\infty} \langle \tilde{M}_i(t) \tilde{M}_j(t') \rangle dt = {}_0D_{ij} + \frac{1}{2} R_K^{1/2} {}_0\Lambda_{ki} (H_{kl} + H_{kl}^{+}) {}_0\Lambda_{lj} + o(R_K^{1/2}), \tag{9.6}$$

where **H** and **H**$^{+}$ are the side force tensor and the adjoint side force tensor for the rotating fluid(the axis of rotation is X_2), here

$$H_{ij} = H_{ji}^{+} = \begin{pmatrix} h_1 & 0 & h_3 \\ 0 & h_2 & 0 \\ -h_3 & 0 & h_1 \end{pmatrix}, \tag{9.7}$$

$$h_1 = \frac{4}{35}(19 + 9\sqrt{3})\frac{\sqrt{2}\pi^2}{8}\cdot\frac{1}{8\pi^3}, \tag{9.8a}$$

$$h_2 = \frac{16}{21}\pi^2\cdot\frac{1}{8\pi^3}, \tag{9.8b}$$

$$h_3 = \frac{4}{35}(19 - 9\sqrt{3})\frac{\sqrt{2}\pi^2}{8}\cdot\frac{1}{8\pi^3}, \tag{9.8c}$$

$$h_2 > h_1 > h_3 > 0, \tag{9.8d}$$

Thus the fluctuation dissipation relations up to $O(R_\kappa^{1/2})$ are obtained for the Brownian particle of arbitrary shape in a rotating fluid. It should be noted that for the particle of special form with ${}_0\mathbf{\Lambda} = 0$ (e.g. sphere, regullar polyhedra, ellipsoid) the fluid rotation affects the systematic force and the correlation function of fluctuating force to $O(R_\kappa^{1/2})$.

Our formulae (9.1) to (9.8) are quite general, however, it is difficult to get the explicit forms of the friction tensors for the arbitrary shaped particle. Let us consider the special case, i.e. the spherical particle. Since $\Gamma_{ij} = 6\pi\delta_{ij}$, ${}_0\Lambda_{ij} = ({}_0\Lambda)_{ij} = 0$, ${}_0D_{ij} = 8\pi\delta_{ij}$ for a sphere, thus we obtain the Langevin equations in dimensional form

$$m\frac{d\mathbf{V}_B'}{dt'} = -6\pi\mu a(\mathbf{I} + 6\pi R_K^{1/2}\mathbf{H})(\mathbf{V}_B' - \mathbf{C}\cdot\mathbf{X}_B') + \tilde{\mathbf{F}}_1' \quad (9.9a)$$

$$ma^2\,\mathbf{J}\cdot\frac{d\Omega_B'}{dt'} = -8\pi\mu a^3(\Omega_B' - \Omega_f') + \tilde{\mathbf{M}} \quad (9.9b)$$

and the fluctuation dissipation relations

$$\langle \tilde{\mathbf{F}}(t)\rangle = \mu a U_0 \times o(R_K^{1/2}), \quad \langle \tilde{\mathbf{M}}(t)\rangle = \mu a^3 \Omega_0 \times o(R_K^{1/2}), \quad (9.10a, b)$$

$$\int_{-\infty}^{\infty} \langle \tilde{F}_i'(t)\tilde{F}_j'(t')\rangle dt = 2k_B T\cdot 6\pi\mu a \times\{\delta_{ij} + \tfrac{1}{2}R_K^{1/2}\cdot 6\pi(H_{ij} + H_{ij}^{+}) + o(R_K^{1/2})\}, \quad (9.11a)$$

$$\int_{-\infty}^{\infty} \langle \tilde{F}_i'(t)\tilde{M}_j(t')\rangle dt = \int_{-\infty}^{\infty} \langle \tilde{M}_i'(t)\tilde{F}_j'(t')\rangle dt = k_B T\mu a^2 \times o(R_K^{1/2}), \quad (9.11b)$$

$$\int_{-\infty}^{\infty} \langle \tilde{M}_i(t)\tilde{M}_j'(t')\rangle dt = 2k_B T\cdot 8\pi\mu a^3(\delta_{ij} + o(R_K^{1/2})). \quad (9.11c)$$

10. Discussion

In this paper we derived the Langevin equations for the Brownian particle of arbitrary shape in an unbounded rotating fluid from the semi-macroscopic approach based on fluctuating hydrodynamics. In order to treat the nonlinear term of the Navier-Stokes equation, one must take into account the spacial nonuniformity of the validity of the approximation. Using the matched asymptotic expansions for solving the stochastic Navier-Stokes equation at small Reynolds numbers ($R \ll R_\kappa^{1/2} \ll 1$), we obtained the systematic force and torque on the particle and the fluctuation-dissipation relations up to $O(R_\kappa^{1/2})$. It is worth noting that the tensors $\mathbf{H}$ and $\mathbf{H}^+$ are given by the equations;

$$\mathbf{H} = -\{\mathbf{G}(\tilde{\mathbf{r}}) - \mathbf{S}(\tilde{\mathbf{r}})\}_{\tilde{\mathbf{r}}=0} = -\frac{1}{8\pi^3}\int\{\mathbf{G}(\mathbf{k}) - \mathbf{S}(\mathbf{k})\}\,d\mathbf{k}, \tag{10.1a}$$

$$\mathbf{H}^+ = -\{\mathbf{G}^+(\tilde{\mathbf{r}}) - \mathbf{S}(\tilde{\mathbf{r}})\}_{\tilde{\mathbf{r}}=0} = -\frac{1}{8\pi^3}\int\{\mathbf{G}^+(\mathbf{k}) - \mathbf{S}(\mathbf{k})\}\,d\mathbf{k}, \tag{10.1b}$$

from the equations (5.24), (6.18)(see also Appendix B and C). From these equations and the fact that $\mathbf{G}(\tilde{\mathbf{r}})$ (or $\mathbf{G}^+(\tilde{\mathbf{r}})$) is the solution of (5.22) (or (6.16)), it follows that the effects of the fluid rotation to $O(R_\kappa^{1/2})$ are expressed in terms of $\mathbf{H}$ and $\mathbf{H}^+$. Thus if we obtain the Stokes drag and torque on the particle, it is easy to write the Langevin equations and the fluctuation-dissipation relations up to $O(R_\kappa^{1/2})$. It should be noted that the fluctuation dissipation relations consist not only of the side force tensor H but also of the adjoint side force tensor $\mathbf{H}^{+}$ to $O(R_\kappa^{1/2})$. The adjoint side force tensor $\mathbf{H}^{+}$ is derived not from the velocity field but from the adjoint field, that is, we need additional information $\mathbf{H}^{+}$ in order to obtain the fluctuation dissipation relations to $O(R_\kappa^{1/2})$. This differs from the case of Brownian particle in a quiescent fluid, in which the fluctuation dissipation relation can be written only in terms of the friction tensor. In the limit of $R_\kappa^{1/2} \to 0$, our formulae reduce to the well known equations(see Foister[33]).

Now consider the case of spherical particle. It is interesting that to $O(R_\kappa^{1/2})$ the systematic part of side force is independent of the rotation of the particle, and this fact also holds for the correlation functions of fluctuating force and torque. Even if the particle is a sphere, it experiences the side force, while the autocorrelation function of the fluctuating force is diagonal but anisotropic. In particular it should be noted that the fluid rotation affects not only on the components of $\boldsymbol{V}_B'$ perpendicular to the angular velocity vector $\boldsymbol{\Omega}_f'$ of the fluid, but also on the component of

$\boldsymbol{V}_B'$ pararell to it.

Let us consider the diffusion of the spherical Brownian particle under the fluid rotation. In the limitting case of $R_\kappa^{1/2} \to 0$, the fluid rotation can only convect the particle, then the discussions about this situation have already been done (Foister[33], Sancho et al.[34]). Hence we will concern only the effect due to the first order term in $R_\kappa^{1/2}$. Applying the procedure in the derivation of the Fokker-Planck equation to the Langevin equations of a spherical Brownian particle (9.9a), (9.10a), (9.11a), (9.7) and (9.8), we obtain the diffusion equation in a rotating fluid(see Appendix-H)

$$\frac{\partial}{\partial t'}W(\mathbf{X}', t') + \frac{\partial}{\partial \mathbf{X}'}\{\mathbf{U}(\mathbf{X}')W(\mathbf{X}', t')\}$$
$$= \left[D_\perp\left(\frac{\partial^2}{\partial X_1'^2} + \frac{\partial^2}{\partial X_3'^2}\right) + D_{/\!/}\frac{\partial^2}{\partial X_2'^2}\right]W(\mathbf{X}', t') \tag{10.2}$$

where $W(\mathbf{X}', t)$ is the probability distribution function of the Brownian particle and

$$\mathbf{U}(\mathbf{X}') = \mathbf{C}'\cdot\mathbf{X}', \tag{10.3}$$

$$D_\perp = D_0(1 - 6\pi R_\kappa^{1/2} h_1), \qquad D_{/\!/} = D_0(1 - 6\pi R_\kappa^{1/2} h_2), \tag{10.4a,b}$$

$$D_0 = \frac{k_B T}{6\pi\mu a}. \tag{10.5}$$

Since $h_2 > h_1 > 0$, the diffusion coefficients are reduced due to the fluid rotation. Thus the diffusion coefficients will have the weak anisotropy. Solving the equation (10.2) subject to the boundary conditions

$$W(\boldsymbol{X}', t'=0) = \delta(\boldsymbol{X}'), \tag{10.6a}$$

$$W(\boldsymbol{X}', t') \longrightarrow 0, \qquad \text{as} \quad |\boldsymbol{X}'| \longrightarrow \infty, \tag{10.6b}$$

we have

$$W(\boldsymbol{X}', t') = \frac{1}{(4\pi t')^{3/2}(D_\perp^2 D_{/\!/})^{1/2}} \exp\left[-\frac{X_1'^2 + X_3'^2}{4D_\perp t'} - \frac{X_2'^2}{4D_{/\!/} t'}\right]. \tag{10.7}$$

It is easy to see from the above solution that the constant probability surface is a oblate spheroid whose major axis is parpendicular to Ω_f and minor one pararell to Ω_f . It will be hoped that $R_\kappa^{1/2}$ dependence of the diffusion constants (e.g. $D_\perp/D_0 - 1 = 6\pi h_1 R_K^{1/2}$) is examined experimentally though the correction is small; e.g. for the sphere of radius $a = 10^{-3}$ cm, with parameters $k_B = 1.38\times10^{-16}$ erg/deg, $T=300^\circ$ K , $\Omega_f = 10\text{sec}^{-1}$, $\nu = 0.01$ $\text{cm}^2\text{sec}^{-1}$, then $U_0 = 10^{-5/2}$cm/sec, $R = 10^{-7/2}$, $R_\kappa = 10^{-3}$, $R_\kappa^{1/2} = 10^{-3/2}$. The correction will be a few per cent but not be undetectable.

ACKNOWLEDGEMENTS

In presenting this thesis, I would like to express my profound gratitude to Professor Sinzi Kuwabara of Nagoya University, for his constant guidance and encouragement throughout my research period.

I express deeply sincere thanks to Dr. Kaneda for helpful discussions and encouragement. Many thanks also to Dr. M.Ú.Farooq for his cooperation.

Finally, I express my thanks to TOYODA Physical and Chemical Research Institute for financial support.

Toshiyuki Gotoh

Appendix-A The proof of the reciprocal relation

Let $(\boldsymbol{u}, p)$ and $(\hat{\boldsymbol{u}}, \hat{p})$ be any dimensionless fields to satisfy

$$\nabla . \boldsymbol{u} = 0 \ , \quad \nabla . \hat{\boldsymbol{u}} = 0 \tag{A.1a,b}$$

and τ_{ij} , $\hat{\tau}_{ij}$ be corresponding stress tensor defined by

$$\tau_{ij} = -p\,\delta_{ij} + \frac{\partial}{\partial r_j} u_i + \frac{\partial}{\partial r_i} u_j \tag{A.2a}$$

$$\hat{\tau}_{ij} = -\hat{p}\,\delta_{ij} + \frac{\partial}{\partial r_j} \hat{u}_i + \frac{\partial}{\partial r_i} \hat{u}_j \tag{A.2b}$$

Now consider the following quantity

$$\left(\frac{\partial}{\partial r_j} u_i\right) \cdot \hat{\tau}_{ij} = \frac{\partial u_i}{\partial r_j} \left(-\hat{p}\,\delta_{ij} + \frac{\partial \hat{u}_i}{\partial r_j} + \frac{\partial \hat{u}_j}{\partial r_i}\right) . \tag{A.3}$$

Since $\nabla . \boldsymbol{u} = 0$

$$= \frac{\partial u_i}{\partial r_j} \left(\frac{\partial \hat{u}_i}{\partial r_j} + \frac{\partial \hat{u}_j}{\partial r_i} \right) \tag{A.4}$$

and interchanging the indicies of the second term we have

$$= \left(\frac{\partial u_i}{\partial r_j} + \frac{\partial u_j}{\partial r_i}\right) \frac{\partial \hat{u}_i}{\partial r_j} . \tag{A.5}$$

Noting (A.1b) we finally obtain

$$\frac{\partial u_i}{\partial r_j} \hat{\tau}_{ij} = \tau_{ij} \frac{\partial \hat{u}_i}{\partial r_j} . \tag{A.6}$$

Appendix-B Computation of Green's function and side force tensor

In this appendix we compute the side force tensor **H**. A part of this calculation has already been done by Drew[35]. But there are some serious errors in his paper, therefore his results seem to be unacceptable.

Now we define the Fourier transform as

$$f(\mathbf{r}) = \frac{1}{8\pi^3}\int \hat{f}(\mathbf{k})\, e^{i\mathbf{k}\cdot\mathbf{r}}\, d\mathbf{k}, \tag{B.1a}$$

$$\hat{f}(\mathbf{k}) = \int f(\mathbf{r})\, e^{-i\mathbf{k}\cdot\mathbf{r}}\, d\mathbf{r}. \tag{B.1b}$$

Substitution (B.1) in (5.22) leads formally to the system

$$-\left(k_1\frac{\partial}{\partial k_3} - k_3\frac{\partial}{\partial k_1}\right)\hat{G}_{ij} + \hat{G}_{3j}\,\delta_{i1} - \hat{G}_{1j}\,\delta_{i3} \tag{B.2}$$

$$= -ik_i\hat{T}_j - k^2\hat{G}_{ij} + \delta_{ij},$$

$$k_i\cdot\hat{G}_{ij} = 0, \tag{B.3}$$

After eliminating T_j by using (B.3), we have

$$\left(k_1\frac{\partial}{\partial k_3} - k_3\frac{\partial}{\partial k_1}\right)\hat{G}_{ij} - \left(\delta_{i1} - 2\frac{k_i k_1}{k^2}\right)\hat{G}_{3j} + \left(\delta_{i3} - 2\frac{k_i k_3}{k^2}\right)\hat{G}_{1j}$$
$$- k^2\hat{G}_{ij} = -\left(\delta_{ij} - \frac{k_i k_j}{k^2}\right). \qquad (B.4)$$

Following Drew, let us introduce the characteristic co-ordinates

$$k_1 = -\hat{k}\sin\sigma, \qquad k_3 = \hat{k}\cos\sigma,$$

where $k^2 = \hat{k}^2 + k_2^2$ is independent of σ. By cosidering a sequence of chages of the dependent valuables, G_{ij} can be found to be

$$\hat{G}_{11} = \frac{1}{2}\left(k^2\cos^2\sigma + k_2^2\sin^2\sigma\right)\left(\frac{1}{K_+} + \frac{1}{K_-}\right) + \frac{1}{2}k k_2\left(\frac{1}{K_-} - \frac{1}{K_+}\right)$$
$$- \frac{\hat{k}^2}{2k^2}\sin\sigma\cos\sigma\left[(1+\omega)\frac{1}{K_+} + (1-\omega)\frac{1}{K_-}\right], \qquad (B.6)$$

$$\hat{G}_{12} = \frac{k_2\hat{k}}{k^4+\omega^2}\left(\sin\sigma + \frac{2}{k^2}\cos\sigma\right), \qquad (B.7)$$

$$\hat{G}_{13} = \frac{1}{2k^2}\left(k^2\cos^2\sigma + k_2^2\sin^2\sigma\right)\left[(1+\omega)\frac{1}{K_+} + (1-\omega)\frac{1}{K_-}\right]$$
$$+ \frac{k_2}{2k}\left[(1-\omega)\frac{1}{K_-} - (1+\omega)\frac{1}{K_+}\right]$$
$$+ \frac{\hat{k}^2}{2}\left(\frac{1}{K_+} + \frac{1}{K_-}\right)\sin\sigma\cos\sigma, \qquad (B.8)$$

$$\hat{G}_{21} = \frac{-k_2\hat{k}}{k^4+1}(k^2\cos\sigma - \sin\sigma)\left[(1+\frac{k_2}{k})\frac{1}{K_-} + (1-\frac{k_2}{k})\frac{1}{K_+}\right]$$
$$+ \frac{k_2\hat{k}}{k^2}\cdot\frac{1}{k^4+1}(k^2\sin\sigma + \cos\sigma)\left[(1-\omega)(1+\frac{k_2}{k})\frac{1}{K_-}\right.$$
$$\left. + (1+\omega)(1-\frac{k_2}{k})\frac{1}{K_+} + 1\right], \quad (B.9)$$

$$\hat{G}_{22} = \frac{k^2\hat{k}^2}{k^6+4k_2^2}, \quad (B.10)$$

$$\hat{G}_{23} = \frac{-k_2\hat{k}}{k^4+1}(k^2\sin\sigma + \cos\sigma)\left[(1+\frac{k_2}{k})\frac{1}{K_-} + (1-\frac{k_2}{k})\frac{1}{K_+}\right]$$
$$- \frac{k_2\hat{k}}{k^2}\frac{1}{k^4+1}(k^2\cos\sigma - \sin\sigma)\left[(1+\frac{k_2}{k})(1-\omega)\frac{1}{K_-}\right.$$
$$\left. + (1-\frac{k_2}{k})(1+\omega)\frac{1}{K_+} + 1\right], \quad (B.11)$$

$$\hat{G}_{31} = -\frac{1}{2k^2}(k^2\sin^2\sigma + k_2^2\cos^2\sigma)\left[(1+\omega)\frac{1}{K_+} + (1-\omega)\frac{1}{K_-}\right]$$
$$- \frac{k_2}{2k}\left[(1-\omega)\frac{1}{K_-} - (1+\omega)\frac{1}{K_+}\right]$$
$$+ \frac{\hat{k}^2}{2}\left(\frac{1}{K_+} + \frac{1}{K_-}\right)\sin\sigma\cos\sigma, \quad (B.12)$$

$$\hat{G}_{32} = -\frac{k_2\hat{k}}{k^4+\omega^2}\left(\cos\sigma - \frac{2}{k^2}\sin\sigma\right), \quad (B.13)$$

$$\hat{G}_{33} = \frac{1}{2}(k^2\sin^2\sigma + k_2^2\cos^2\sigma)\left(\frac{1}{K_+} + \frac{1}{K_-}\right)$$
$$+ \frac{kk_2}{2}\left(\frac{1}{K_-} - \frac{1}{K_+}\right)$$
$$+ \frac{\hat{k}^2}{2k^2}\left[(1-\omega)\frac{1}{K_-} + (1+\omega)\frac{1}{K_+}\right]\sin\sigma\cos\sigma, \quad (B.14)$$

where

$$\omega = 2\frac{k_2}{k}, \qquad K_\pm = k^4 + (1 \pm \omega)^2. \qquad (B.15, 16)$$

Also the Stokeslet $\hat{S}_{ij}(\boldsymbol{k})$ is of the form

$$\hat{S}_{ij}(\boldsymbol{k}) = \frac{1}{k^2}\left(\delta_{ij} - \frac{k_i k_j}{k^2}\right). \qquad (B.17)$$

From (5.21a) and (5.24a), the side force tensor H_{ij} can be written as

$$H_{ij} = -\left\{ G_{ij}(\tilde{\boldsymbol{r}}) - S_{ij}(\tilde{\boldsymbol{r}}) \right\}_{\tilde{\boldsymbol{r}}=0}$$

$$= -\frac{1}{8\pi^3}\int \left\{ \hat{G}_{ij}(\boldsymbol{k}) - \hat{S}_{ij}(\boldsymbol{k}) \right\} d\boldsymbol{k}\,. \qquad (B.18)$$

This integral can be carried out exactly by changing to spherical co-ordinates (k, δ, σ) where, $\hat{k}^2 = k_1^2 + k_3^2$, $\hat{k} = k \cos\delta$ and $k_2 = k \sin\delta$. The components H_{12}, H_{21}, H_{23} and H_{32} vanish because of the symmetry of G_{ij} and S_{ij} ; e.g $G_{23}(k_2) = -G_{23}(-k_2)$ and $S_{23}(k_2) = -S_{23}(-k_2)$. Results are

$$H_{ij} = \begin{pmatrix} h_1 & 0 & h_3 \\ 0 & h_2 & 0 \\ -h_3 & 0 & h_1 \end{pmatrix}, \qquad (B.19)$$

$$h_1 = \frac{4}{35}(19 + 9\sqrt{3})\frac{\sqrt{2}\pi^2}{8}\cdot\frac{1}{8\pi^3}, \qquad (B.20a)$$

$$h_2 = \frac{16}{21}\pi^2\cdot\frac{1}{8\pi^3}, \qquad (B.20b)$$

$$h_3 = \frac{4}{35}(19 - 9\sqrt{3})\frac{\sqrt{2}\pi^2}{8}\cdot\frac{1}{8\pi^3}. \qquad (B.20c)$$

Appendix-C The proof of the reciprocal relation of of **H** and **H**$^+$

We prove here the reciprocal relation for $G_{ij}(\boldsymbol{x},\boldsymbol{r})$ and $G^+_{ij}(\boldsymbol{x},\boldsymbol{r})$. Let us write equations (5.22) and (6.16) as

$$\frac{\partial}{\partial x_l}\tau_{ilj} - M_{il}G_{lj} = -\delta_{ij}\,\delta(\boldsymbol{x}-\boldsymbol{r}), \tag{C.1a}$$

$$\frac{\partial}{\partial x_i}G_{ij} = 0 \tag{C.1b}$$

and

$$\frac{\partial}{\partial x_l}\tau^+_{ilk} - M^+_{il}G^+_{lk} = -\delta_{ik}\,\delta(\boldsymbol{x}-\boldsymbol{r}'), \tag{C.2a}$$

$$\frac{\partial}{\partial x_i}G^+_{ik} = 0, \tag{C.2b}$$

where

$$\tau_{ilj} = -T_j\delta_{il} + \frac{\partial}{\partial x_l}G_{ij} + \frac{\partial}{\partial x_i}G_{lj}, \tag{C.3a}$$

$$\tau^+_{ilk} = -T^+_k\delta_{il} + \frac{\partial}{\partial x_l}G^+_{ik} + \frac{\partial}{\partial x_i}G^+_{lk}, \tag{C.3b}$$

$$M_{il} = C_{pq} x_q \frac{\partial}{\partial x_p} \delta_{il} + C_{il}, \tag{C.4a}$$

$$M^{+}_{il} = -C_{pq} x_q \frac{\partial}{\partial x_p} \delta_{il} + C_{li}, \tag{C.4b}$$

and $\sim$ of the argument of $\mathbf{G}$ and $\mathbf{G}^{+}$ is omitted for the sake of simplicity. Now take the scalar product of (C.1a) with $G^{+}_{ik}(\boldsymbol{x},\boldsymbol{r}')$, and (C.2a) with $G_{ij}(\boldsymbol{x},\boldsymbol{r})$ and subtract to obtain

$$\begin{aligned} & G^{+}_{ik}(\boldsymbol{x},\boldsymbol{r}')\left\{\frac{\partial}{\partial x_l}\tau_{ilj} - M_{il} G_{lj}\right\} \\ & \quad - G_{ij}(\boldsymbol{x},\boldsymbol{r})\left\{\frac{\partial}{\partial x_l}\tau^{+}_{ilk} - M^{+}_{il} G^{+}_{lk}\right\} \\ & \qquad = - G^{+}_{jk}(\boldsymbol{x},\boldsymbol{r}')\delta(\boldsymbol{x}-\boldsymbol{r}) + G_{kj}(\boldsymbol{x},\boldsymbol{r})\delta(\boldsymbol{x}-\boldsymbol{r}') \end{aligned} \tag{C.5}$$

Substituting (C.3) and (C.4) into (C.5), using the relation (see Appendix-A)

$$\left(\frac{\partial}{\partial x_l} G^{+}_{ik}\right)\tau_{ilj} = \tau^{+}_{ilk}\left(\frac{\partial}{\partial x_l} G_{ij}\right), \tag{C.6}$$

we obtain

$$\begin{aligned} & \frac{\partial}{\partial x_l}\left[G^{+}_{ik}\tau_{ilj} - G_{ij}\tau^{+}_{ilk} - G^{+}_{ik} C_{iq} x_q G_{lj}\right] \\ & \qquad = G_{kj}(\boldsymbol{x},\boldsymbol{r})\delta(\boldsymbol{x}-\boldsymbol{r}') - G^{+}_{jk}(\boldsymbol{x},\boldsymbol{r}')\delta(\boldsymbol{x}-\boldsymbol{r}). \end{aligned} \tag{C.7}$$

Integrating this over the sphere of radius K and using Gauss' theorem yields

$$G_{kj}(\mathbf{r}',\mathbf{r}) - G^{+}_{jk}(\mathbf{r},\mathbf{r}') \tag{C.8}$$
$$= \int_{S_K} \left[G^{+}_{ik} T_{ilj} - G_{ij} T^{+}_{ilk} - G^{+}_{ik} C_{iq} X_q G_{lj} \right] dS_l(\mathbf{x}).$$

Provided that in the limit of $K \to \infty$ the surface integral vanishes, we obtain the reciprocal relation

$$G_{kj}(\mathbf{r}',\mathbf{r}) = G^{+}_{jk}(\mathbf{r},\mathbf{r}'). \tag{C.9}$$

Next consider the relation between $\mathbf{H}$ and $\mathbf{H}^{+}$. From (5.21a), (5.24a), (6.15a) and (6.18a), $\mathbf{H}$ and $\mathbf{H}^{+}$ are defined explicitly as

$$H_{ij} = -\lim_{\mathbf{r}\to 0} \left[G_{ij}(\mathbf{r},0) - S_{ij}(\mathbf{r},0) \right], \tag{C.10a}$$

$$H^{+}_{ij} = -\lim_{\mathbf{r}\to 0} \left[G^{+}_{ij}(\mathbf{r},0) - S_{ij}(\mathbf{r},0) \right], \tag{C.10b}$$

here $\tilde{}$ is also omitted. Because of the well known fact for the Stokeslets

$$S_{ij}(\mathbf{r}_1,\mathbf{r}_2) = S_{ji}(\mathbf{r}_2,\mathbf{r}_1), \tag{C.11}$$

the following identity holds

$$G^{+}_{ij}(\mathbf{r}_1, \mathbf{r}_2) - S_{ij}(\mathbf{r}_1, \mathbf{r}_2) = G^{+}_{ji}(\mathbf{r}_2, \mathbf{r}_1) - S_{ji}(\mathbf{r}_2, \mathbf{r}_1). \tag{C.12}$$

Putting $\boldsymbol{r}_2 = 0$ and letting $\boldsymbol{r}_1 \to 0$ in (C.12), and from (C.10), we finally obtain

$$H^{+}_{ij} = H_{ji} . \tag{C.13}$$

This relation can also be shown by the direct calculation of $\mathbf{H}^{+}$ with the same procedure as $\mathbf{H}$ in appendix-B. The proof by the direct calculation of $\mathbf{H}^{+}$ supports the assumption about the surface integral in eq. (C.8).

Appendix-D Derivation of the first order outer field

In this appendix many indexes such as $\beta, q, (1)$, and $\tilde{}$ of $\mathrm{X}^{+\beta(1)}_{\alpha,q}(\tilde{\boldsymbol{x}})$ and $\Psi^{\beta(1)}_{\alpha,q}(\tilde{\boldsymbol{x}})$ are omitted for the sake of simplicity. Using the notations in Appendix-C, we can write the equations (6.9), (6.10), (6.14), (7.7), (7.8), and (7.10) as

$$\Delta\Psi_j - \frac{\partial}{\partial x_j}\Psi_4 - M_{j\ell}(\boldsymbol{x})\Psi_\ell = \Delta \mathrm{X}^+_j , \qquad (j=1,2,3) \tag{D.1a}$$

$$\frac{\partial}{\partial x_j}\Psi_j = 0, \tag{D.1b}$$

$$\Psi_j = o(x^{-1}), \qquad \text{as } |\boldsymbol{x}| \longrightarrow 0, \tag{D.2a}$$

$$\longrightarrow 0 , \qquad \text{as } |\boldsymbol{x}| \longrightarrow \infty \tag{D.2b}$$

and

$$\Delta \mathrm{X}^+_j - \frac{\partial}{\partial x_j}\mathrm{X}^+_4 - M^+_{j\ell}(\boldsymbol{x})\mathrm{X}^+_\ell = 0 \qquad (j=1,2,3) \tag{D.3a}$$

$$\frac{\partial}{\partial x_j}\mathrm{X}^+_j = 0, \tag{D.3b}$$

$$\mathrm{X}^+_j \longrightarrow -S_{j\ell}(\boldsymbol{x})\cdot g^{(0)}_\ell , \qquad \text{as } |\boldsymbol{x}| \longrightarrow 0, \tag{D.4a}$$

$$\longrightarrow 0 , \qquad \text{as } |\boldsymbol{x}| \longrightarrow \infty. \tag{D.4b}$$

Substitution of (D.3a) into (D.1a) yields

$$\Delta\Psi_j - \frac{\partial}{\partial x_j}(\Psi_4 + X_4^+) - M_{j\ell}(\boldsymbol{x})\Psi_\ell = M_{j\ell}^+(\boldsymbol{x})X_\ell^+ . \qquad (D.5)$$

Let us write the solution of (D.5), (D.1b) and (D.2) as

$$\Psi_j(\boldsymbol{x}) = -\frac{1}{2}\{G_{j\ell}^+(\boldsymbol{x}) - G_{j\ell}(\boldsymbol{x})\}\cdot g_\ell^{(0)} + A_i(\boldsymbol{x}), \qquad (D.6a)$$

$$\Psi_4(\boldsymbol{x}) + X_4^+(\boldsymbol{x}) = -\frac{1}{2}\{T_\ell^+(\boldsymbol{x}) - T_\ell(\boldsymbol{x})\}\cdot g_\ell^{(0)} + B(\boldsymbol{x}). \qquad (D.6b)$$

Substituting (D.6) into (D.5), using (5.22) and (6.16), we obtain the equations for $A_i(\boldsymbol{x})$ and $B(\boldsymbol{x})$;

$$\Delta A_i - \frac{\partial}{\partial x_i}B - M_{i\ell}(\boldsymbol{x})A_\ell = -\frac{1}{2}(M_{i\ell}(\boldsymbol{x}) + M_{i\ell}^+(\boldsymbol{x}))\cdot G_{\ell k}^+\cdot g_k^{(0)}, \qquad (D.7a)$$

$$\frac{\partial}{\partial x_i}A_i = 0, \qquad (D.7b)$$

$$A_i(\boldsymbol{x}) = o(X^{-1}), \qquad \text{as} \quad |\boldsymbol{x}| \longrightarrow 0, \qquad (D.8a)$$

$$\longrightarrow 0, \qquad \text{as} \quad |\boldsymbol{x}| \longrightarrow \infty. \qquad (D.8b)$$

From definitions (C.4) of $\mathbf{M}(\boldsymbol{x})$ and $\mathbf{M}^+(\boldsymbol{x})$ in Appendix-C and the fact $C_{ij} = -C_{ji}$ for pure rotation, it follows that the right hand side of (D.7a) vanishes. Thus

$$A_i(\boldsymbol{x}) = 0, \qquad B(\boldsymbol{x}) = 0. \qquad (D.9a,b)$$

Therefore we finally obtain

$$\Psi_j(\mathbf{x}) = -\frac{1}{2}\left[G^{+}_{j\ell}(\mathbf{x}) - G_{j\ell}(\mathbf{x}) \right] \cdot g^{(0)}_{\ell}, \tag{D.10a}$$

$$\Psi_4(\mathbf{x}) = -X_4(\mathbf{x}) - \frac{1}{2}\left[T^{+}_{\ell}(\mathbf{x}) - T_{\ell}(\mathbf{x}) \right] \cdot g^{(0)}_{\ell}. \tag{D.10b}$$

Appendix-E Estimation of the suface integral in (4.10)

We will now estimate the outer surface integral of (4.10), which can be written as

$$\int_{S_K}\Big[\chi_{i\gamma}^{+\beta}(\boldsymbol{x})\{<\tilde{\tau}_{ij}(\boldsymbol{x})\tilde{\sigma}_{k\ell}(\boldsymbol{r})> + <\tilde{\sigma}_{ij}(\boldsymbol{x})\tilde{\sigma}_{k\ell}(\hat{\boldsymbol{r}})>\}$$
$$-<\tilde{v}_i(\boldsymbol{x})\tilde{\sigma}_{k\ell}(\boldsymbol{r})>\ \Theta_{ij\gamma}^{+\beta}(\boldsymbol{x})$$
$$-R\,\chi_{i\gamma}^{+\beta}(\boldsymbol{x})<\tilde{v}_i(\boldsymbol{x})\tilde{\sigma}_{k\ell}(\boldsymbol{r})>\overline{V}_j(\boldsymbol{x})\Big]\,dS_j(\boldsymbol{x}). \tag{E.1}$$

At large distance of $\boldsymbol{x}$ the major contribution comes from the last term $R\,\chi_{i\gamma}^{+\beta}(\boldsymbol{x})<\tilde{v}_i(\boldsymbol{x})\tilde{\sigma}_{k\ell}(\boldsymbol{r})>\overline{V}_j(\boldsymbol{x})$, therefore it is sufficient to consider only the following quantity

$$J_{k\ell\gamma}^{\beta}(\boldsymbol{r})\equiv\lim_{K\to\infty}R\int_{S_k}\chi_{i\gamma}^{+\beta}(\boldsymbol{x})<\tilde{v}_i(\boldsymbol{x})\tilde{\sigma}_{k\ell}(\boldsymbol{r})>\overline{V}_j(\boldsymbol{x})\,dS_j(\boldsymbol{x}). \tag{E.2}$$

When $\boldsymbol{x}$ lies at large distance in the outer region, from the expansions (5.1a) and (6.4a) we have

$$R\,\bar{\boldsymbol{v}}(\boldsymbol{x}) = R_K^{1/2}\,\mathbf{C}\cdot\tilde{\boldsymbol{x}} + \cdots\cdots, \tag{E.3}$$

$$\begin{aligned}\chi_j^{+}(\boldsymbol{x}) &= R_K^{1/2}\,X_j^{+(1)}(\tilde{\boldsymbol{x}}) + \cdots\cdots \\ &= -R_K^{1/2}\,G_{T_j\cdot\ell}^{+}(\tilde{\boldsymbol{x}})\cdot g_\ell^{(0)} + \cdots\cdots,\end{aligned} \tag{E.4}$$

where (6.15a) is used. While $(\langle\tilde{v}_i\,\tilde{\sigma}_{k\ell}\rangle, \langle\tilde{p}\,\tilde{\sigma}_{k\ell}\rangle)$ satisfies (3.11) or

$$\begin{aligned}L_{\alpha\beta}(\boldsymbol{x})\,\Phi_{\beta,k\ell}(\boldsymbol{x},\boldsymbol{r}) &= \xi_{\alpha,k\ell}(\boldsymbol{x},\boldsymbol{r}) \\ &= \begin{pmatrix} r_{ij\,k\ell}\dfrac{\partial}{\partial x_j}\,\delta(\boldsymbol{x}-\boldsymbol{r}) \\ 0 \end{pmatrix} \\ &= -\begin{pmatrix} r_{ij\,k\ell}\dfrac{\partial}{\partial r_j}\,\delta(\boldsymbol{x}-\boldsymbol{r}) \\ 0 \end{pmatrix}.\end{aligned} \tag{E.6}$$

If we put as follows and use the simplified notation

$$\langle\tilde{v}_i(\boldsymbol{x})\,\tilde{\sigma}_{k\ell}(\boldsymbol{r})\rangle = -\frac{\partial}{\partial r_j}\,D_{ij\,k\ell}(\boldsymbol{x},\boldsymbol{r}) \equiv -\nabla_r\colon\mathbf{D}(\boldsymbol{x},\boldsymbol{r}), \tag{E.7a}$$

$$\langle\tilde{p}(\boldsymbol{x})\,\tilde{\sigma}_{k\ell}(\boldsymbol{r})\rangle = -\frac{\partial}{\partial r_j}\,q_{j\,k\ell}(\boldsymbol{x},\boldsymbol{r}) \equiv -\nabla_r\cdot\boldsymbol{q}(\boldsymbol{x},\boldsymbol{r}), \tag{E.7b}$$

then $(\mathbf{D}(\boldsymbol{x},\boldsymbol{r}),\ \boldsymbol{q}(\boldsymbol{x},\boldsymbol{r}))$ satisfies

$$\mathbf{L}\cdot\begin{pmatrix}\mathbf{D}(\mathbf{x},\mathbf{r})\\ \mathbf{g}(\mathbf{x},\mathbf{r})\end{pmatrix} = \begin{pmatrix}\boldsymbol{\gamma}\,\delta(\mathbf{x}-\mathbf{r})\\ 0\end{pmatrix}. \qquad (E.8)$$

Now introducing the outer variable as

$$\tilde{\mathbf{x}} = R_K^{1/2}\mathbf{x}, \qquad (E.9)$$

expanding $(\mathbf{D}(\mathbf{x},\mathbf{r}),\ \mathbf{g}(\mathbf{x},\mathbf{r}))$ in $R_\kappa^{1/2}$ as

$$\mathbf{D}(\mathbf{x},\mathbf{r}) = R_K^{1/2}\mathbf{D}_1(\tilde{\mathbf{x}},\mathbf{r}) + \cdots\cdots, \qquad (E.10a)$$

$$\mathbf{g}(\mathbf{x},\mathbf{r}) = R_K\,\mathbf{g}_1(\tilde{\mathbf{x}},\mathbf{r}) + \cdots\cdots, \qquad (E.10b)$$

and substituting these into (E.6), we obtain

$$\tilde{\Delta}\mathbf{D}_1 - \tilde{\nabla}\mathbf{g}_1 - \mathbf{M}(\tilde{\mathbf{x}})\cdot\mathbf{D}_1 = -\boldsymbol{\gamma}\delta(\tilde{\mathbf{x}}-\tilde{\mathbf{r}}) \qquad (E.11a)$$

and

$$\tilde{\nabla}\cdot\mathbf{D}_1 = 0 \qquad (E.11b)$$

to the lowest order in $R_\kappa^{1/2}$, Equations (E.11) are similar to equations (5.22) for $\mathbf{G}(\tilde{\mathbf{x}})$, then we may expect that $\mathbf{D}_1(\tilde{\mathbf{x}},\tilde{\mathbf{r}})$ has the same structure as $\mathbf{G}(\tilde{\mathbf{x}})$, in particular the same asymptotic behavior. Substituting (E.3), (E.4) and (E.10a) into (E.2), we obtain

$$J_{k\ell q}^{\beta}(\tilde{\mathbf{r}}) = -R_K \lim_{K\to\infty} \int_{S_K} G^{+}_{ij}\, g^{o\beta(o)}_{jq} \cdot \left\{\frac{\partial}{\partial \tilde{r}_m} D_{1,imk\ell}(\tilde{\mathbf{x}},\tilde{\mathbf{r}})\right\} \cdot C_{np}\tilde{x}_p\, dS_n(\tilde{\mathbf{x}}). \tag{E.12}$$

It is known that the far field structure of $\mathbf{G}(\tilde{\mathbf{x}})$ and $\mathbf{G}^{+}(\tilde{\mathbf{x}})$ has the cubical cone (see Appendix-F), i.e.

$$G_{ij}(\tilde{\mathbf{x}}),\quad G^{+}_{ij}(\tilde{\mathbf{x}}) \sim \frac{1}{|\tilde{x}_2|}\ . \quad \text{for} \quad \frac{(\tilde{x}_1^2+\tilde{x}_3^2)^{1/2}}{|\tilde{x}_2|^{1/3}} \le O(1).$$

$$\text{as} \quad |\tilde{x}_2| \longrightarrow \infty\ . \tag{E.13}$$

Therefore from (E.12) and (E.13) the major contribution of the surface integral of (E.12) for fixed $\tilde{\mathbf{r}}$ and for large $\tilde{\mathbf{x}}$ is of the form

$$\lim_{K\to\infty} \int_{S_K} G^{+}_{ij}(\tilde{\mathbf{x}}) \cdot \left\{\frac{\partial}{\partial \tilde{r}_m} D_{1,imk\ell}(\tilde{\mathbf{x}},\tilde{\mathbf{r}})\right\} C_{np}\tilde{x}_p\, dS_n(\tilde{\mathbf{x}}) \cdot g^{o\beta(o)}_{jq}$$

$$\sim \frac{1}{|\tilde{x}_2|} \cdot \frac{1}{|\tilde{x}_2 - \tilde{r}_2|^2} |\tilde{x}_2|\,|\tilde{x}_2|^{2/3} \sim |\tilde{x}_2|^{-\frac{4}{3}}. \tag{E.14}$$

Thus letting $K \to \infty$, we obtain the following estimation for any $\mathbf{r}$;

$$J_{k\ell q}^{\beta}(\mathbf{r}) = o\,(R_K). \tag{E.15}$$

Appendix-F The structure of Green's function

The structure of Green's function $\mathbf{G}(\tilde{x})$ is far more complicated. However, if we restrict attention to the flow at large distance, i.e. $|\tilde{x}| \gg 1$, it is not difficult to investigate the main effect of the rotation[36]. In order to see the wake structure, it is easy to consider the component $G_{22}(\tilde{x})$, which is given by

$$G_{22}(\tilde{\boldsymbol{x}}) = \frac{1}{8\pi^3}\int \frac{(k_1^2+k_3^2)k^2}{k^6+4k_2^2} e^{i\boldsymbol{k}\cdot\tilde{\boldsymbol{x}}} d\boldsymbol{k}. \tag{F.1}$$

Now we introduce cylindrical co-ordinates as

$$k_1 = \hat{k}\cos\theta, \qquad k_3 = \hat{k}\sin\theta, \tag{F.2a}$$

and

$$\tilde{\mathbf{b}} = (\tilde{x}_1, \tilde{x}_3), \tag{F.2b}$$

where θ denotes the angle between $\tilde{\boldsymbol{b}}$ and $\hat{\boldsymbol{k}}$, then (F.1) becomes

$$G_{22}(\tilde{\boldsymbol{x}}) = \frac{1}{8\pi^3}\int_0^\infty \hat{k}\,d\hat{k}\int_0^{2\pi} d\theta \int_{-\infty}^{\infty} dk_2 \frac{\hat{k}^2(k_2^2+\hat{k}^2)\,e^{i\hat{k}\tilde{b}\cos\theta + ik_2\tilde{x}_2}}{(k_2^2+\hat{k}^2)^3+4k_2^2}. \tag{F.3}$$

Poles of the integrand occur when

$$(\hat{k}^2 + k_2^2)^3 + 4k_2^2 = 0. \tag{F.4}$$

If $G_{22}(\tilde{\boldsymbol{x}})$ is expanded for large $|\tilde{x}_2|$, the dominant contribution to the integral with respect to k_2 occurs when the imaginary part of the root of (F.4) is smallest, and this implies that a neighbourhood of the origin $\boldsymbol{k} = 0$ is to be considered. The minimum imaginary part of k_2 occurs where $\hat{k}$ is small

$$k_2 = \frac{i}{2}\hat{k}^3 + O(\hat{k}^7). \tag{F.5}$$

By this approximation the integral of (F.3) can be written as

$$G_{22}(\tilde{\boldsymbol{x}}) \sim \frac{1}{8\pi^3}\int_0^\infty \hat{k}\,d\hat{k}\int_0^{2\pi} d\theta \int_{-\infty}^{\infty} dk_2 \,\frac{\hat{k}^4 e^{i\hat{k}\tilde{b}\cos\theta + ik_2\tilde{x}_2}}{\hat{k}^6 + 4k_2^2} + \cdots\cdots, \tag{F.6}$$

This integral is carried out, thus we obtain

$$G_{22}(\tilde{\boldsymbol{x}}) \sim \frac{1}{4\pi|\tilde{x}_2|}\int_0^\infty s^2 e^{-s^3} J_0(\xi s)\,ds + O(|\tilde{x}_2|^{-1}), \tag{F.7}$$

where

$$\xi = (\tilde{x}_1^2 + \tilde{x}_3^2)^{1/2}\left(\frac{2}{|\tilde{x}_2|}\right)^{1/3}, \tag{F.8}$$

and J_0 is the zeroth order Bessel function. It is easily seen that the width of viscous wake grows as $(\tilde{x}_2)^{1/3}$ and it is expected that the other components of $\mathbf{G}(\tilde{\boldsymbol{x}})$ and also of $\mathbf{G}^+(\tilde{\boldsymbol{x}})$ have the same wake structure.

Appendix-G The effects of $R(\tilde{\boldsymbol{v}}.\boldsymbol{\nabla})\tilde{\boldsymbol{v}}$ term

Now we estimate the magnitude of the $R(\tilde{\boldsymbol{v}}.\boldsymbol{\nabla})\tilde{\boldsymbol{v}}$ term. If $R(\tilde{\boldsymbol{v}}.\boldsymbol{\nabla})\tilde{\boldsymbol{v}}$ term is retained, to (3.11) and (3.13) are to be added respectively terms,

$$R < \tilde{v}_j(\boldsymbol{x},t)\frac{\partial}{\partial x_j}\tilde{v}_i(\boldsymbol{x},t)\,\tilde{\sigma}_{k\ell}(\boldsymbol{r},t')> \qquad (G.1)$$

and

$$R < \tilde{v}_j(\boldsymbol{x},t)\frac{\partial}{\partial x_j}\tilde{v}_i(\boldsymbol{x},t)\,\tilde{K}^{\beta}_{\ell}(\boldsymbol{r},t')>. \qquad (G.2)$$

Here we assume that the probability distribution of $\tilde{\sigma}_{ij}$ is nearly Gaussian. Then the triple moment $<\tilde{\sigma}\tilde{\sigma}\tilde{\sigma}>$ vanishes and the fourth order moment $<\tilde{\sigma}\tilde{\sigma}\tilde{\sigma}\tilde{\sigma}>$ may be expressed in terms of the second order moment $<\tilde{\sigma}\tilde{\sigma}>$. Similarly the fourth order moments $<\tilde{v}\tilde{v}\tilde{v}\tilde{K}>$, $<\tilde{v}\tilde{v}\tilde{v}\tilde{\sigma}>$, $<\tilde{v}\tilde{v}\tilde{v}\tilde{v}>$ etc. may be expressed in the lowest order in $R_K^{1/2}$ in terms of the second order moments as $<\tilde{v}\tilde{v}>$, $<\tilde{v}\tilde{K}>^{(0)}$, $<\tilde{v}\tilde{\sigma}>^{(0)}$ etc., which are solutions of (2.13a) with all R-terms dropped.

Consider the following triple moments

$$\hat{\mathbf{u}}(\mathbf{r},t\,|\,\boldsymbol{x},\boldsymbol{x}',t') \equiv \langle \tilde{\boldsymbol{v}}(\mathbf{r},t)\tilde{\boldsymbol{\sigma}}(\boldsymbol{x},t)\tilde{\boldsymbol{\sigma}}(\boldsymbol{x}',t')\rangle, \qquad (G.3)$$

$$\hat{\mathbf{p}}(\mathbf{r},t\,|\,\boldsymbol{x},\boldsymbol{x}',t') \equiv \langle \tilde{p}(\mathbf{r},t)\tilde{\boldsymbol{\sigma}}(\boldsymbol{x},t)\tilde{\boldsymbol{\sigma}}(\boldsymbol{x}',t')\rangle, \qquad (G.4)$$

$$\hat{\boldsymbol{\tau}}(\mathbf{r},t\,|\,\boldsymbol{x},\boldsymbol{x}',t') \equiv \langle \tilde{\boldsymbol{\tau}}(\mathbf{r},t)\tilde{\boldsymbol{\sigma}}(\boldsymbol{x},t)\tilde{\boldsymbol{\sigma}}(\boldsymbol{x}',t')\rangle, \qquad (G.5)$$

which satisfy, from (2.13a),

$$\lambda\frac{\partial\hat{\mathbf{U}}}{\partial t} + R[(\bar{\boldsymbol{v}}\cdot\nabla_r)\hat{\mathbf{U}} + (\hat{\mathbf{U}}\cdot\nabla_r)\bar{\boldsymbol{v}}] - \nabla_r\cdot(\hat{\boldsymbol{\tau}} + \langle\tilde{\boldsymbol{\sigma}}\tilde{\boldsymbol{\sigma}}\tilde{\boldsymbol{\sigma}}\rangle)$$

$$= -R\langle(\tilde{\boldsymbol{v}}\cdot\nabla_r)\tilde{\boldsymbol{v}}\,\tilde{\boldsymbol{\sigma}}\tilde{\boldsymbol{\sigma}}\rangle. \qquad (G.6)$$

From the above discussions, the fourth order moments in (G.6) can be expressed in terms of the second order moments to the lowest order in $R_\kappa^{1/2}$, so that the right hand side of (G.6) is $O(R)$. Then this implies that $(\hat{\mathbf{U}},\hat{\mathbf{P}})$ is $O(R)$. Similar arguments are available also to the triple moments $\langle\tilde{v}_i(\boldsymbol{x},t)\tilde{v}_j(\boldsymbol{x},t)\tilde{\sigma}_{k\ell}(\mathbf{r},t')\rangle$ and $\langle\tilde{v}_i(\boldsymbol{x},t)\tilde{v}_j(\boldsymbol{x},t)\,{}_\alpha\tilde{K}_\ell(t')\rangle$, that is, it is natural to assume that (G.1) and (G.2) are $O(R^2)$.

Next consider the contribution of $R(\tilde{\boldsymbol{v}}.\nabla)\tilde{\boldsymbol{v}}$ term to $\langle\tilde{\boldsymbol{F}}\rangle$ and $\langle\tilde{\boldsymbol{M}}\rangle$. Taking the conditional ensemble average of (2.13a), we obtain

$$\lambda\frac{\partial}{\partial t}\langle\tilde{v}\rangle + R[(\bar{v}\cdot\nabla)\langle\tilde{v}\rangle + (\langle\tilde{v}\rangle\cdot\nabla)\bar{v}]$$
$$- \nabla\cdot(\langle\tilde{\tau}\rangle + \langle\tilde{\sigma}\rangle) = -R\langle(\tilde{v}\cdot\nabla)\tilde{v}\rangle, \qquad (G.7)$$

From previous discussions, it is found that the right hand side of (G.7) is $O(R)$. Therefore within the approximation neglecting $o(R_\kappa^{1/2})$, the $R(\tilde{v}.\nabla)\tilde{v}$ term can be neglected.

≡

Appendix-H Derivation of the diffusion equation

Rewriting the Langevin equation (9.9a) in the form of simultaneous equations

$$\dot{\mathbf{V}} = -\mathbf{B}\{\mathbf{V} - \mathbf{U}(\mathbf{X})\} + \frac{1}{m}\tilde{\mathbf{F}}, \tag{H.1}$$

$$\dot{\mathbf{X}} = \mathbf{V}, \tag{H.2}$$

$$\mathbf{B} = \zeta(\mathbf{I} + 6\pi R_K^{1/2}\mathbf{H}), \qquad \zeta = \frac{6\pi\mu a}{m}, \tag{H.3}$$

we obtain for the average values occuring in the Fokker-Planck equation

$$\lim_{\Delta t\to 0} \frac{\langle \Delta \mathbf{V}\rangle}{\Delta t} = -\mathbf{B}\{\mathbf{V} - \mathbf{U}(\mathbf{X})\}, \quad \lim_{\Delta t\to 0} \frac{\langle \Delta \mathbf{X}\rangle}{\Delta t} = \mathbf{V}, \tag{H.4,5}$$

$$\lim_{\Delta t\to 0} \frac{\langle \Delta V_i \Delta V_j\rangle}{\Delta t} = \frac{2k_B T\zeta}{m} Z_{ij} \tag{H.6}$$

$$\lim_{\Delta t\to 0} \frac{\langle \Delta X_i \Delta X_j\rangle}{\Delta t} = 0, \tag{H.7}$$

$$Z_{ij} = \begin{pmatrix} 1+6\pi R_K^{1/2} h_1 & 0 & 0 \\ 0 & 1+6\pi R_K^{1/2} h_2 & 0 \\ 0 & 0 & 1+6\pi R_K^{1/2} h_1 \end{pmatrix},$$

where the correlation time of the random forces is assumed to be small compared to Δt and (9.10a), (9.11a) are used. Using these quantities, we get

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial \mathbf{V}}\cdot\{-\mathbf{B}\cdot[\mathbf{V}-\mathbf{U}(\boldsymbol{X})]P\} + \frac{k_B T \zeta}{m}\frac{\partial}{\partial V_i}\left(Z_{ij}\frac{\partial}{\partial V_j}P\right), \tag{H.8}$$

$$\frac{\partial W}{\partial t} = -\frac{\partial}{\partial \boldsymbol{X}}(\mathbf{V}W) - \frac{\partial}{\partial \mathbf{V}}\{-\mathbf{B}(\mathbf{V}-\mathbf{U}(\boldsymbol{X}))W\} + \frac{k_B T \zeta}{m}\frac{\partial}{\partial V_i}\left(Z_{ij}\frac{\partial}{\partial V_j}W\right), \tag{H.9}$$

where $P(\boldsymbol{V},t)$ and $W(\boldsymbol{X},\boldsymbol{V},t)$ are the probability density functions of the particle in $\boldsymbol{V}$ and $(\boldsymbol{X},\boldsymbol{V})$ space, respectively. First consider (H.8). It is easily shown that this has the Maxwell distribution

$$P = \exp\left[-\frac{m(\mathbf{V}-\mathbf{U}(\boldsymbol{X}))^2}{2k_B T}\right]. \tag{H.10}$$

Next consier (H.9). After some manipulation we can write as

$$\begin{aligned}\frac{\partial W}{\partial t} = \left(\frac{\partial}{\partial \mathbf{V}} - \mathbf{B}^{-1}\cdot\frac{\partial}{\partial \boldsymbol{X}}\right)\cdot\Big\{ & \mathbf{B}\cdot(\mathbf{V}-\mathbf{U}(\boldsymbol{X}))W \\ & + \mathbf{Z}\cdot\left[\left(\frac{\partial}{\partial \mathbf{V}} - \mathbf{B}^{-1}\cdot\frac{\partial}{\partial \boldsymbol{X}}\right) + 2\mathbf{B}^{-1}\cdot\frac{\partial}{\partial \boldsymbol{X}}\right]W\Big\} \\ & - \frac{\partial}{\partial \boldsymbol{X}}(\mathbf{U}(\boldsymbol{X})W) + \left(\mathbf{B}^{-1}\cdot\frac{\partial}{\partial \boldsymbol{X}}\right)\cdot\mathbf{Z}\cdot\left(\mathbf{B}^{-1}\cdot\frac{\partial}{\partial \boldsymbol{X}}\right)W.\end{aligned} \tag{H.11}$$

Now integrating this equation along the straight line

$$\mathbf{X} - \mathbf{B}^{-1}\cdot\mathbf{V} = \mathbf{r} = \text{const.} \tag{H.12}$$

from $\boldsymbol{V}=-\infty$ *to* $+\infty$, one gets

$$\frac{\partial}{\partial t}\int_{\mathbf{X}-\mathbf{B}^{-1}\cdot\mathbf{V}=\mathbf{r}} W\, d\mathbf{V} = \int_{\mathbf{X}-\mathbf{B}^{-1}\cdot\mathbf{V}=\mathbf{r}} \left[-\frac{\partial}{\partial \mathbf{X}}(\mathbf{U}(\mathbf{X})W) + (\mathbf{B}^{-1}\cdot\frac{\partial}{\partial \mathbf{X}})\cdot\mathbf{Z}\cdot(\mathbf{B}^{-1}\cdot\frac{\partial}{\partial \mathbf{X}})W\right] d\mathbf{V}. \tag{H.13}$$

If we may neglect the changes in $\boldsymbol{U}(\boldsymbol{X})$ under the $\boldsymbol{V}$ integration, we can write

$$\frac{\partial}{\partial t}W = -\frac{\partial}{\partial \mathbf{X}}(\mathbf{U}(\mathbf{X})W) + (\mathbf{B}^{-1}\cdot\frac{\partial}{\partial \mathbf{X}})\cdot\mathbf{Z}\cdot(\mathbf{B}^{-1}\cdot\frac{\partial}{\partial \mathbf{X}})W. \tag{H.14}$$

After a little manipulation we obtain

$$\frac{\partial}{\partial t}W = -\frac{\partial}{\partial \mathbf{X}}\cdot(\mathbf{U}(\mathbf{X})W) + \left[D_{\perp}\left(\frac{\partial^2}{\partial X_1^2} + \frac{\partial^2}{\partial X_3^2}\right) + D_{/\!/}\frac{\partial^2}{\partial X_3^2}\right]W. \tag{H.15}$$

References

1. Einstein A. Investigation on the theory of the Brownian movement Dover, New York (1956)
2. Langevin P. Comptes.Rendus. 146 530 (1908) See also "Selected papers on Noise and Stochastic Processes". N.Wax (ed.) Dover, New York (1954)
3. Mori H. Prog.Theor.Phys. 33A 127 (1967)
4. Green M.S. J.Chem.Phys. 22 398 (1954)
5. Landau L.D. and Lifshitz E.M. Fluid Mechanics, Pergamon Press, London (1959)
6. Fox.R.F. and Uhlenbeck G.E. Phys.Fluids 13 1893 (1970)
7. Onsager L. and Machlup S. Phys.Rev. 91 1505 (1953)
8. Zwanzig R. J.Res.Natl.Bur.Std. (U.S.) 68B 143 (1964)
9. Chow T.S. and Hermans J.J. J.Chem.Phys. 56 3150 (1972)
10. Hauge E.H. and Martin-Lof A. J.Stat.Phys. 7 259 (1973)
11. Bedeaux D. and Mazur P. Physica 76 247 (1974)
12. Alder B.J. and Wainwright T.E. Phys.Rev. A2 2005(1970)
13. Ohbayashi K., Kohno T. and Utiyama H. Phys.Rev.A27 2632 (1983)
14. Paul G.L. and Pussy P.N. J.Phys. A14 3301 (1981)
15. Mazur P. Physica 110A 128 (1982)
16. Jones R.B. Physica 100A 417 (1980)

17. Gotoh T. and Kaneda Y. J.Chem.Phys. 76 3193 (1982)

18. Zaitsev V.M. and Shliomis M.I. Sov.Phys. JETP 32 866 (1971)

19. Lesnikov V.P. and Fisher I.Z. Sov.Phys. JETP 40 667 (1975)

20. Lekkerkerker H.N.W. and Boon J.P. Phys.Rev. A10 1355 (1974)

21. Lekkerkerker H.N.W. Physica 80A 415 (1975)

22. Garisto F. and Mazur P. Physica 99A 365 (1978)

23. Keizer J. Phys.Fluids 21 198 (1978)

24. Fox R.F. J.Math.Phys. 19 1993 (1978)

25. Trembley A.M.S., Arai M. and Siggia E.D. Phys.Rev. A23 1451 (1981) See also references cited there.

26. Hermans J.J. Physica 109A 295 (1981)

27. Kaneda Y. Unpublished note

28. Proudman I. and Pearson J.R.A. J.Fluid Mech. 2 237 (1957)

29. Van Dyke M. Perturbation methodsin fluid mechanics Academic Press, New York and London (1964)

30. Brenner H. and Cox R.G. J.Fluid Mech. 17 561 (1963)

31. Childress S. J.Fluid Mech. 20 305 (1964)

32. Happel J. and Brenner H. Low Reynolds number

hydrodynamics Printice-Hall (1965)

33. Foister R.T. and Van De Ven T.G.M. J.Fluid Mech. 96 105 (1980)

34. San Miguel M. and Sancho J.M. Physica 99A 357 (1979)

35. Drew D.A. J.Fluid Mech. 88 393 (1978)

36. Herron I.H., Davis S.H. and Bretherton F.P. J.Fluid Mech. 68 209 (1975)