PHYSICAL REVIEW D VOLUME 19, NUMBER 10 15 MAY 1979

Dragging effect on the inertial frame and the contribution of matter to the gravitational "constant" in a closed cosmological model of the Brans-Dicke theory*

A. Miyazaki

Department of Physics, Nagoya University, Nagoya, Japan

(Received 7 April 1978; revised manuscript received 28 September 1978)

The perturbation by a spherical rotating shell is investigated in a closed homogeneous and isotropic cosmological model of the Brans-Dicke theory to first order in an angular velocity of the shell. This model has a negative coupling parameter of the scalar field and satisfies the relation $G(t)M/c^2a(t) = \pi$. The inertial frame at the origin is dragged completely with the same angular velocity when the rotating shell covers the whole universe. By a similar perturbation method, the distance dependence of the contribution from matter to the scalar field at the origin is obtained in this model. The contribution from nearby matter is negative because of the negative coupling constant, but the contribution from the whole universe is positive. The gravitational "constant" is almost determined by matter in the distant region.

I. INTRODUCTION

Since general relatively appeared, it has been discussed to what extent Mach's principle is contained in this theory. The interpretation of Mach's principle still remains controversial and contentious, but it is well known that the inertial frame (in which the Coriolis force does not appear) is dragged partially by the rotating body in the framework of general relativity or the modified theory of gravitation.

The investigation of this dragging effect on the inertial frame is mainly classified into the following three types:

(1) Thirring, Bass and Pirani, Okamura *et al.* (Refs. 1–3). They investigated what forces appear in the vicinity of a spherical infinitely thin shell in empty space (Minkowski space) when the shell rotates with a small constant angular velocity ω by means of the weak-field approximation of Einstein's field equations. Thirring[1] calculated up to the $G\omega^2$ term, and indicated that the $G\omega$ term represents the Coriolis force if $GM/Rc^2 = \frac{3}{4}$, where R and M are the radius and the mass of the shell, respectively. Bass and Pirani[2] indicated that the $G\omega^2$ term vanishes and is not the centrifugal force, introducing the elastic stress and the particular distribution of density. Okamura *et al.*[3] calculated up to the $G^2\omega^2$ term and indicated that the $G^2\omega^2$ term represents the centrifugal force if the contradictory condition $GM/Rc^2 = 1260/3737$ still exists. So general relativity does not involve Mach's principle automatically.

These discussions are unsatisfactory because of the weak-field approximation and asymptotic flatness, but they suggest that the degree of dragging of the inertial frame is closely connected with the value of GM/Rc^2.

(2) Brill, Cohen, Lindblom (Refs. 4–7). They discussed this rotating-shell problem in a strong field. They used the Schwarzschild solution as the base metric and calculated the perturbation due to the rotation of the shell up to first order in ω. They indicated that the induced rotation rate of the inertial frame approaches the shell rotation rate, as the radius of the shell equal to its Schwarzschild radius. In this limit the inertial properties of space inside the shell no longer depend on the inertial frames at infinity, but are completely determined by the shell itself (Brill and Cohen[4]).

The Schwarzschild metric has the asymptotically flat region, which is non-Machian. They, however, emphasize that this asymptotically flat region is essential in the definition of frame dragging. They also discussed inertial effects in the gravitational collapse of a rotating shell.

(3) Hönl and Soergel-Fabricius, A. Lausberg (Refs. 8 and 9). Lausberg[9] investigated what perturbation appears in the metric tensor in the first order of an angular velocity when a spherical shell, with the same density as the remaining part of the universe, rotates uniformly in the Einstein universe. He indicated that the dragging coefficient of the inertial frame increases as the rotating shell becomes thicker and reaches unity when the shell covers the whole universe. In his discussion the framework of matter is given first. The Einstein universe is closed and has no asymptotically flat region.

In the present paper we will investigate the dragging effect on the inertial frame in a closed cosmological model[10,11] of the Brans-Dicke theory.[12] This is the extension of discussions (3) in a time-varying case. In this cosmological model, the universe expands forever linearly, but the relation $G(t)M/c^2a(t) = \pi$ is always satisfied. (In the Einstein universe with zero pressure the relation GM/Rc^2

$=\pi/2$ is satisfied.) The dragging coefficient of the inertial frame reaches unity when the rotating shell covers the whole universe.

In Sec. III the distance dependence of the contribution from matter to the gravitational "constant" is investigated in this closed cosmological model by means of a similar perturbation method. According to Brans and Dicke[12] the coupling parameter η of the scalar field must be positive if the contribution to the inertial reaction from nearby matter is to be positive. However, in this model, the coupling parameter is negative ($\eta<-2$). When $3+2\eta<0$, the contribution to the scalar field from nearby matter is negative, but the scalar field is determined by the contribution from the whole universe, and is surely positive in the cosmological solution.

II. COSMOLOGICAL SOLUTION

The field equations of the Brans-Dicke theory[12] are written in our sign convention as

$$G_{\mu\nu}\equiv R_{\mu\nu}-\tfrac{1}{2}Rg_{\mu\nu}$$
$$=\frac{8\pi}{c^4\phi}T_{\mu\nu}-\frac{\eta}{\phi^2}(\phi_{,\mu}\phi_{,\nu}-\tfrac{1}{2}g_{\mu\nu}\phi_{,\lambda}\phi^{,\lambda})$$
$$-\frac{1}{\phi}(\phi_{,\mu;\nu}-g_{\mu\nu}\Box\phi)\,, \tag{1a}$$

$$\Box\phi=\frac{8\pi}{(3+2\eta)c^4}T\,, \tag{1b}$$

where $T_{\mu\nu}$ is the energy-momentum tensor, which has for the perfect fluid the form

$$T_{\mu\nu}=-pg_{\mu\nu}-(\rho+p/c^2)u_\mu u_\nu\,, \tag{2}$$

in which ρ is the density in comoving coordinates, p is the pressure, and u^μ is the four-velocity $dx^\mu/d\tau$ (τ is the proper time). The symbol $\Box$ denotes the generally covariant d'Alembertian $\Box\phi\equiv\phi^{,\mu}{}_{;\mu}$, and the letter η is the coupling parameter between the scalar field ϕ and the contracted energy-momentum tensor T.

For the closed homogeneous and isotropic universe the metric form will be written as

$$ds^2=-dt^2+a^2(t)[d\chi^2+\sin^2\chi(d\theta^2+\sin^2\theta d\varphi^2)]\,. \tag{3}$$

The interaction between galaxies in the universe is negligible, so we can set $p=0$ in the energy-momentum tensor (a dust model). In the present problem the nonvanishing components of the energy-momentum tensor are $T_{00}=-\rho c^2$, and the contracted energy-momentum tensor is $T=\rho c^2$.

So the independent field equations for the metric (3) are

$$2a\ddot{a}+\dot{a}^2+1=-\tfrac{1}{2}\eta a^2\left(\frac{\dot\phi}{\phi}\right)^2-\frac{8\pi}{(3+2\eta)c^2}\frac{a^2\rho}{\phi}\,, \tag{4a}$$

$$\frac{3}{a^2}(\dot{a}^2+1)=\frac{\ddot\phi}{\phi}+\frac{\eta}{3}\left(\frac{\dot\phi}{\phi}\right)^2+\frac{16\pi(1+\eta)}{(3+2\eta)c^2}\frac{\rho}{\phi}\,, \tag{4b}$$

$$\ddot\phi+3\frac{\dot a}{a}\dot\phi=\frac{8\pi}{(3+2\eta)c^2}\rho\,, \tag{4c}$$

where a dot denotes the usual partial derivative with respect to t. We can obtain from the field equations the conservation law of the energy-momentum $T^{\mu\nu}{}_{;\nu}=0$, that is,

$$2\pi^2a^3\rho=M=\text{const}\,, \tag{5}$$

where M is an integral constant and can be regarded as the mass of the whole universe. We can use Eq. (5) as the independent equation instead of Eq. (4a).

According to Mach's ideas the relation $GM/Rc^2\sim 1$ is satisfied, and the degree of dragging of the inertial frame is connected with the value of GM/Rc^2, so we expect the existence of a solution in which the relation $GM/c^2a=\text{const}$ is satisfied.

Let us write

$$a(t)\phi(t)=D=\text{const}. \tag{6}$$

After substitution of Eqs. (5) and (6) into Eq. (4c) and integration we obtain

$$a^2=-\frac{4M}{(3+2\eta)\pi c^2D}(t-t_c)^2+A\,, \tag{7}$$

where A and t_c are integral constants. From Eqs. (4b), (4c), (5), and (6), we obtain

$$\tfrac{1}{6}\eta\dot{a}^2=1-\frac{4M}{3\pi c^2D}\,. \tag{8}$$

Equation (8) indicates that $a(t)$ must be a linear function of t. On account of this and the initial condition $a=0$ at $t=0$, we must set two integral constants t_c, A in Eq. (7) equal to zero. If the conditions

$$(3+2\eta)D<0,\quad \eta\left(1-\frac{4M}{3\pi c^2D}\right)>0,$$
$$\frac{\eta}{3+2\eta}=2-\frac{3\pi c^2D}{2M} \tag{9}$$

are satisfied, Eqs. (6), (7), and (8) are not contradictory to each other and are solutions of the field equations. These conditions are reduced to

$$\eta<-2,\quad G(t)M/c^2a(t)=\pi\,, \tag{10}$$

where $G\equiv(4+2\eta)/(3+2\eta)\phi$. Therefore, a solution satisfying $GM/c^2a=\text{const}$ can exist only when the value of GM/c^2a is π.

Thus, the Brans-Dicke theory has a cosmological solution for the closed homogeneous and isotropic universe:

$$a(t)=[-2/(2+\eta)]^{1/2}t \equiv \alpha t\ ,$$
$$2\pi^2 a^3(t)\rho(t)=M, \quad (11)$$
$$\phi(t)=-[8\pi/(3+2\eta)c^2]\rho(t)t^2\ ,$$

with $\eta<-2$.

III. THE DRAGGING EFFECT ON THE INERTIAL FRAME

In this section we discuss the dragging effect on the inertial frame in this closed cosmological model in the framework of the Brans-Dicke theory. The situation is almost the same as that of Lausberg's arguments[9] applied to the static Einstein universe.

Let us consider in this universe a shell, the volume of which is restricted by the two hypersurfaces $\chi=\chi_0$ and $\chi=\chi_1$, with $0<\chi_0<\chi_1<\pi$. The density of the shell is assumed to be the same as the remaining part of the universe. This shell is now considered to be slowly rotating as a rigid body around the axis $\theta=0$, with an angular velocity $\omega_s = c(d\varphi/dt)_{\text{shell}}$ relative to the remaining part of the universe.

The metric form in the whole universe will be perturbed on analogy of Lausberg's arguments as

$$ds^2=-dt^2+a^2(t)\times\{d\chi^2+\sin^2\chi[d\theta^2+\sin^2\theta(d\varphi-\omega dt/c)^2]\}\ . \quad (12)$$

Owing to the slow rate of rotation we have $c^2 \gg a^2(t)\omega_s^2 > a^2(t)\omega^2$, and hence it is sufficient to calculate up to the first order of an angular velocity ω. So we have the perturbed metric form as

$$ds^2=-dt^2+a^2(t)[d\chi^2+\sin^2\chi(d\theta^2+\sin^2\theta d\varphi^2)] -2\omega(\chi,\theta,t)a^2(t)\sin^2\chi\sin^2\theta d\varphi dt/c\ . \quad (13)$$

From now on the calculations are limited to first order in ω and ω_s.

The nonvanishing components of the Ricci tensor associated with this metric are written as

$$R_{11}=-a\ddot{a}-2\dot{a}^2-2,$$
$$R_{22}=R_{11}\sin^2\chi\ ,$$
$$R_{33}=R_{11}\sin^2\chi\sin^2\theta\ ,$$
$$R_{00}=3\ddot{a}/a\ , \quad (14)$$
$$R_{13}=R_{31}=-(1/2c)a^2\sin^2\chi\sin^2\theta[\partial_\chi\dot{\omega}+3(\dot{a}/a)\partial_\chi\omega]\ ,$$
$$R_{23}=R_{32}=-(1/2c)a^2\sin^2\chi\sin^2\theta[\partial_\theta\dot{\omega}+3(\dot{a}/a)\partial_\theta\omega]\ ,$$
$$R_{30}=R_{03}=(\omega/c)\sin^2\chi\sin^2\theta(a\ddot{a}+3\dot{a}^2) -(1/2c)\sin^2\chi\sin^2\theta(\partial_{\chi\chi}{}^2\omega+4\cot\chi\partial_\chi\omega-4\omega) -(1/2c)\sin^2\theta(\partial_{\theta\theta}{}^2\omega+3\cot\theta\partial_\theta\omega)\ ,$$

where the symbols ∂_χ and ∂_θ denote the usual partial derivatives with respect to χ and θ, respectively. The scalar curvature is

$$R=-(6/a^2)(a\ddot{a}+\dot{a}^2+1)\ . \quad (15)$$

In the present problem the nonvanishing components of the energy-momentum tensor (2) ($p=0$) are

$$T_{00}=-\rho c^2\ ,$$
$$T_{30}=T_{03}=\begin{cases}-\rho c\omega a^2\sin^2\chi\sin^2\theta & \text{(outside of the shell)},\\ -\rho c(\omega-\omega_s)a^2\sin^2\chi\sin^2\theta & \text{(inside volume of the shell)}\ ,\end{cases} \quad (16)$$

and the contracted energy-momentum tensor is $T=\rho c^2$.

In general ϕ is a function of χ, θ, and t, so the generally covariant d'Alembertian is written in the metric form (13) as

$$\Box\phi=-\ddot{\phi}-3\frac{\dot{a}}{a}\dot{\phi}+\frac{1}{a^2}(\partial_{\chi\chi}{}^2\phi+2\cot\chi\partial_\chi\phi) +\frac{1}{a^2\sin^2\chi}(\partial_{\theta\theta}{}^2\phi+\cot\phi\partial_\theta\phi)\ . \quad (17)$$

This perturbed generally covariant d'Alembertian and the contracted energy-momentum tensor is the same as the unperturbed. So ϕ will be a function of only t up to the first order of ω. Therefore the field equations are

$$G_{11}=2a\ddot{a}+\dot{a}^2+1=-\tfrac{1}{2}\eta a^2\left(\frac{\dot{\phi}}{\phi}\right)^2-\frac{8\pi}{(3+2\eta)c^2}\frac{a^2\rho}{\phi}\ ,$$
$$G_{22}=G_{11}\sin^2\chi=[\text{the right-hand side of Eq. (18a)}]\times\sin^2\chi\ , \quad (18a)$$
$$G_{33}=G_{11}\sin^2\chi\sin^2\theta=[\text{the right-hand side of Eq. (18a)}]\times\sin^2\chi\sin^2\theta\ ,$$
$$G_{00}=-\frac{3}{a^2}(\dot{a}^2+1)=-\frac{\ddot{\phi}}{\phi}-\frac{\eta}{2}\left(\frac{\dot{\phi}}{\phi}\right)^2-\frac{16\pi(1+\eta)}{(3+2\eta)c^2}\frac{\rho}{\phi}\ , \quad (18b)$$
$$G_{13}=G_{31}=-(1/2c)a^2\sin^2\chi\sin^2\theta[\partial_\chi\dot{\omega}+3(\dot{a}/a)\partial_\chi\omega]=0\ , \quad (18c)$$
$$G_{23}=G_{32}=-(1/2c)a^2\sin^2\chi\sin^2\theta[\partial_\theta\dot{\omega}+3(\dot{a}/a)\partial_\theta\omega]=0\ , \quad (18d)$$

$$G_{30}=G_{03}=-(\omega/c)\sin^2\chi\sin^2\theta(2a\ddot{a}+3)-(1/2c)\sin^2\chi\sin^2\theta(\partial_{\chi\chi}{}^2\omega+4\cot\chi\partial_\chi\omega-4\omega)$$
$$-(1/2c)\sin^2\theta(\partial_{\theta\theta}{}^2\omega+3\cot\theta\partial_\theta\omega)$$
$$=\left[\frac{8\pi}{c^2}(\omega-\omega_s)\frac{a^2\rho}{\phi}+\frac{\eta\omega}{2c}a^2\left(\frac{\dot{\phi}}{\phi}\right)^2-\frac{\omega}{c}a\mathring{a}\frac{\dot{\phi}}{\phi}+\frac{8\pi\omega}{(3+2\eta)c^3}\frac{a^2\rho}{\phi}\right]\sin^2\chi\sin^2\theta\quad(\omega_s=0\text{ outside the shell}),\tag{18e}$$

$$\ddot{\phi}+3\frac{\mathring{a}}{a}\dot{\phi}=\frac{8\pi}{(3+2\eta)c^2}\rho\,.\tag{18f}$$

Equations (18a), (18b), and (18f) are the same as the unperturbed equations, and determine functions $a(t)$, $\rho(t)$, and $\phi(t)$ completely, that is, cosmological solution (11). The inertial property is determined by Eq. (18e). Using Eqs. (18a), (18b), (18f), and (5) we can reduce Eq. (18e) to

$$\omega\sin^2\chi\left[3-\frac{16M}{3\pi c^2a\phi}-\tfrac{1}{6}\eta a^2\left(\frac{\dot{\phi}}{\phi}\right)^2\right]+\tfrac{1}{2}\sin^2\chi(\partial_{\chi\chi}{}^2\omega+4\cot\chi\partial_\chi\omega-4\omega)+\tfrac{1}{2}(\partial_{\theta\theta}{}^2\omega+3\cot\theta\partial_\theta\omega)$$
$$=-\omega_s\sin^2\chi\frac{4M}{\pi c^2a\phi}\quad(\omega_s=0\text{ outside the shell}).\tag{19}$$

Using Eqs. (6) and (7) ($t_c=A=0$), the inside of the middle brackets of the first term in the left-hand side of Eq. (19) becomes

$$3-\frac{16M}{3\pi c^2a\phi}-\tfrac{1}{6}\eta a^2\left(\frac{\dot{\phi}}{\phi}\right)^2=2-\frac{4M}{\pi c^2D}=\text{const},\tag{20}$$

and the homogeneous equation of Eq. (19) admits a variable separation with respect to χ, θ, and t. A particular solution of the inhomogeneous equation (19) is obviously ω_s.

The time dependence of ω is determined by Eqs. (18c) and (18d), and we can write

$$\omega(\chi,\theta,t)=X(\chi)\Theta(\theta)/a^3(t)\,.\tag{21}$$

Denoting by S the separation constant, two equations arise from the homogeneous equation of Eq. (19):

$$\frac{1}{\Theta}\left(\frac{d^2\Theta}{d\theta^2}+3\cot\theta\frac{d\Theta}{d\theta}\right)=-S\,,\tag{22a}$$

$$2\sin^2\chi\left(2-\frac{4M}{\pi c^2D}\right)+\frac{\sin^2\chi}{X}\left(\frac{d^2X}{d\chi^2}+4\cot\chi\frac{dX}{d\chi}-4X\right)=S\,.\tag{22b}$$

Equation (22a) has a regular solution for $\theta=0$ and π only when $S=(n+2)(n-1)$, where n is an integer, and which is the first derivative of the Legendre polynomials $P_n(\cos\theta)$. As the particular solution does not involve the variable θ, $\Theta(\theta)$ must be constant in order that the solution $\omega(\chi,\theta,t)$ can connect smoothly at $\chi=\chi_0$ and $\chi=\chi_1$. Therefore $n=1$ and $S=0$.

If we use a variable z defined by $2z=\cos\chi+1$, Eq. (22b) reduces to

$$z(1-z)\frac{d^2X}{dz^2}+(\tfrac{5}{2}-5z)\frac{dX}{dz}-\frac{8M}{\pi c^2D}X=0\,.\tag{23}$$

This equation is nothing but the hypergeometric differential equation. The independent solutions in the vicinity of $z=0$ are the hypergeometric functions

$$F(\alpha,\beta,\gamma;z)\,,$$
$$z^{1-\gamma}F(\alpha-\gamma+1,\beta-\gamma+1,2-\gamma;z),\quad|z|<1\tag{24}$$

and in the vicinity of $z=1$

$$F(\alpha,\beta,\gamma+\beta-\gamma+1;1-z)\,,$$
$$(1-z)^{\gamma-\alpha-\beta}F(\alpha-\beta,\gamma-\alpha,\gamma-\alpha-\beta+1;1-z)\,,$$
$$|1-z|<1\tag{25}$$

where

$$\alpha+\beta=4,\quad\alpha\beta=\frac{8M}{\pi c^2D},\quad\gamma=\tfrac{5}{2}\,.\tag{26}$$

The function ω must be regular at $\chi=0$ and $\chi=\pi$. Thus, the complete solution $\omega(\chi,\theta,t)$ is

$$\omega_a(\chi,t)=\frac{A}{a^3(t)}X_a(\chi)\quad(0\le\chi\le\chi_0)\,,$$
$$\omega_b(\chi,t)=\frac{1}{a^3(t)}[B_1X_{b1}(\chi)+B_1X_{b2}(\chi)]+\omega_s\quad(\chi_0\le\chi\le\chi_1)\,,\tag{27}$$
$$\omega_c(\chi,t)=\frac{C}{a^3(t)}X_c(\chi)\quad(\chi_1\le\chi\le\pi)\,,$$

where

$$X_a(\chi)=F(\alpha,\beta,\alpha+\beta-\gamma+1;\ (1-\cos\chi)/2)\,,$$
$$X_{b1}(\chi)=X_c(\chi)=F(\alpha,\beta,\gamma;\ (1+\cos\chi)/2)\,,\tag{28}$$
$$X_{b2}(\chi)=\left(\frac{1+\cos\chi}{2}\right)^{1-\gamma}F(\alpha-\gamma+1,\beta-\gamma+1,2-\gamma;$$
$$(1+\cos\chi)/2)\,,$$

and A, B_1, B_2, and C are arbitrary constants. These constants are determined by means of the conditions that ω_a, ω_b, and ω_c must connect smoothly at $\chi=\chi_0$ and $\chi=\chi_1$, that is,

$$\begin{aligned}
&\omega_a(\chi_0,t)=\omega_b(\chi_0,t)\,,\\
&\omega_b(\chi_1,t)=\omega_c(\chi_1,t)\,,\\
&\partial_\chi\omega_a(\chi_0,t)=\partial_\chi\omega_b(\chi_0,t)\,,\\
&\partial_\chi\omega_b(\chi_1,t)=\partial_\chi\omega_c(\chi_1,t)\,.
\end{aligned} \tag{29}$$

In order that Eq. (29) is satisfied at all times, the particular solution ω_s must have the form $\Omega/a^3(t)$ ($\Omega=\text{const}$).

Now we are interested in the metric form in the vicinity of the origin, so it is enough to determine only the value of A. By solving simultaneously Eqs. (29) we have

$$A=Q(\chi_0,\chi_1)/P(\chi_0,\chi_1)\,, \tag{30}$$

where

$$P(\chi_0,\chi_1)=\begin{vmatrix} X_a(\chi_0) & -X_{b_1}(\chi_0) & -X_{b_2}(\chi_0) & 0\\ 0 & -X_{b_1}(\chi_1) & -X_{b_2}(\chi_1) & X_c(\chi_1)\\ X_a'(\chi_0) & -X_{b_1}'(\chi_0) & -X_{b_2}'(\chi_0) & 0\\ 0 & -X_{b_1}'(\chi_1) & -X_{b_2}'(\chi_1) & X_c'(\chi_1)\end{vmatrix}\,,$$

$$Q(\chi_0,\chi_1)=\begin{vmatrix} \Omega & -X_{b_1}(\chi_0) & -X_{b_2}(\chi_0) & 0\\ \Omega & -X_{b_1}(\chi_1) & -X_{b_2}(\chi_1) & X_c(\chi_1)\\ 0 & -X_{b_1}'(\chi_0) & -X_{b_2}'(\chi_0) & 0\\ 0 & -X_{b_1}'(\chi_1) & -X_{b_2}'(\chi_1) & X_c'(\chi_1)\end{vmatrix}\,, \tag{31}$$

and a prime denotes ∂_χ.

The solution inside the shell is

$$\begin{aligned}\omega_a(\chi_0,\chi_1;\chi,t)=&[Q(\chi_0,\chi_1)/P(\chi_0,\chi_1)a^3(t)]\\ &\times F(\alpha,\beta,\alpha+\beta-\gamma+1;(1-\cos\chi)/2).\end{aligned} \tag{32}$$

At the origin

$$\begin{aligned}\omega_0(\chi_0,\chi_1;t)&=\lim_{\chi\to 0}\omega_a\\ &=Q(\chi_0,\chi_1)/[P(\chi_0,\chi_1)a^3(t)]\,.\end{aligned} \tag{33}$$

This is the induced angular velocity of the inertial frame at the origin by the rotating shell with the angular velocity $\omega_s=\Omega/a^3$ restricted by two hypersurfaces $\chi=\chi_0$ and $\chi=\chi_1$.

In order to find the induced angular velocity ω_0 when the shell covers the whole universe, we need to investigate the asymptotic behavior of the hypergeometric function at $\chi=0$ and $\chi=\pi$. After simple calculations, we find that the value of Q/P converges to Ω when the shell covers the whole universe ($\chi_0\to 0$ and $\chi_1\to\pi$). Therefore, the dragging coefficient of the inertial frame becomes unity:

$$\omega_0/\omega_s=1\,. \tag{34}$$

IV. CONTRIBUTION OF MATTER TO THE GRAVITATIONAL "CONSTANT"

Now let us consider in that universe the same shell, the volume of which is restricted by the two hypersurfaces $\chi=\chi_0$ and $\chi=\chi_1$, with $0<\chi_0<\chi_1<\pi$. How will the scalar field ϕ be perturbed when only the density of this shell changes to $\rho+\Delta\rho$, and the remaining part of the universe is the same? In general the perturbation will depend on χ and t, and will be written as $\phi(t)+\Delta\phi(\chi,t)$.

Therefore the perturbed term $\Delta\phi(\chi,t)$ of the scalar field obeys the partial differential equation if the perturbation is small and the change of the metric is negligible:

$$\begin{aligned}(\ddot{\Delta\phi})+3\frac{\dot a}{a}(\dot{\Delta\phi})-\frac{1}{a^2}\left[\frac{\partial^2}{\partial\chi^2}(\Delta\phi)+2\cot\chi\frac{\partial}{\partial\chi}(\Delta\phi)\right]\\ =\frac{8\pi}{(3+2\eta)c^2}\Delta\rho\quad(\Delta\rho=0\text{ outside the shell}).\end{aligned} \tag{35}$$

The homogeneous equation of Eq. (35) admits the separation with respect to the variables χ and t; let us write

$$\Delta\phi(\chi,t)=X(\chi)T(t)\,. \tag{36}$$

Denoting by S the separation constant, two equations arise:

$$\frac{d^2X}{d\chi^2}+2\cot\chi\frac{dx}{d\chi}-SX=0\,, \tag{37a}$$

$$\frac{d^2T}{dt^2}+3\frac{\dot a}{a}\frac{dT}{dt}-\frac{S}{a^2}T=0\,. \tag{37b}$$

A general solution of Eq. (37a) is

$$X(\chi)=[C_1\cos(1-S)^{1/2}\chi+C_2\sin(1-S)^{1/2}\chi]\sin^{-1}\chi, \tag{38}$$

where C_1 and C_2 are integral constants. On account of Eq. (11), Eq. (37b) reduces to

$$t^2\frac{d^2T}{dt^2}+3t\frac{dT}{dt}-\frac{S}{\alpha^2}T=0\,. \tag{39}$$

A general solution of Eq. (39) is

$$T(t)=\begin{cases}C_1t^{\mu-1}+C_2t^{-\mu-1}, & \text{for }\mu\equiv 2(1+S/\alpha^2)\neq 0\\ t^{-1}(C_1+C_2\ln t), & \text{for }\mu=0\,.\end{cases} \tag{40}$$

Taking the equation $\Delta\rho t^3=\Delta\rho_0 t_0{}^3$ into account, we obtain as a particular solution of the inhomogeneous equation (35)

$$\Delta\phi_p=-\frac{8\pi\Delta\rho_0 t_0{}^3}{(3+2\eta)c^2}\,\frac{1}{t}\,, \tag{41}$$

where an index 0 denotes a present value. As scalar fields in each region must be connected with each other smoothly for all t at the hypersurfaces $\chi=\chi_0$ and $\chi=\chi_1$, the time dependence of the general solution of the homogeneous equation must be the same as that of the particular solution. Therefore $T(t)$ must vary with time as t^{-1}. So the constant μ is equal to zero, and the separation constant S is $-\alpha^2$.

A solution must be regular at $\chi=0$ or $\chi=\pi$, so a complete solution of the perturbed term $\Delta\phi(\chi,t)$ will be written as

$$\Delta\phi_a(\chi,t)=\frac{A}{t}\frac{\sin k\chi}{\sin\chi}\quad(0\le\chi\le\chi_0)\,, \tag{42a}$$

$$\Delta\phi_b(\chi,t)=\frac{1}{t}\left(\frac{B_1\cos k\chi+B_2\sin k\chi}{\sin\chi}\right)-\frac{8\pi\Delta\rho_0 t_0{}^3}{(3+2\eta)c^2}\frac{1}{t}\quad(\chi_0\le\chi\le\chi_1)\,, \tag{42b}$$

$$\Delta\phi_c(\chi,t)=\frac{C}{t}\frac{\sin k(\chi-\pi)}{\sin\chi}\quad(\chi_1\le\chi\le\pi)\,, \tag{42c}$$

where $k\equiv(1+\alpha^2)^{1/2}$, and A, B_1, B_2, and C are integral constants, which are determined by the condition that solutions in each region must be connected with each other smoothly at $\chi=\chi_0$ and $\chi=\chi_1$.

As we are interested in the contribution of matter to the origin, it is enough to determine only a value of A. We obtain as the perturbed scalar field at the origin

$$\Delta\phi_0(t)=\lim_{\chi\to 0}\Delta\phi_a(\chi,t)=-\frac{8\pi}{(3+2\eta)c^2}\frac{\Delta\rho t^2}{\sin k\pi}[k\sin\chi\cos k(\chi-\pi)-\cos\chi\sin k(\chi-\pi)]_{\chi_1}^{\chi_0}\,. \tag{43}$$

This equation determines the contribution to the scalar field at the origin by the density increase $\Delta\rho$ of the shell restricted by the two hypersurfaces $\chi=\chi_0$ and $\chi=\chi_1$ in the background metric (3). If we set $\chi_0=0$ and $\chi_1=\pi$, then we obtain

$$\Delta\phi_0(t)=-\frac{8\pi}{(3+2\eta)c^2}\Delta\rho t^2\,, \tag{44}$$

which we also obtain by replacing ρ with $\rho+\Delta\rho$ in the cosmological solution (11).

In order to evaluate the distance dependence of the contribution, taking the derivative of the expression between the brackets in Eq. (43), it is possible to write Eq. (43) in an integral form

$$\Delta\phi_0(t)=\frac{8\pi\Delta\rho t^2}{(3+2\eta)c^2}\frac{1-k^2}{\sin k\pi}\int_{\chi_0}^{\chi_1}\sin\chi\sin k(\chi-\pi)d\chi\,. \tag{45}$$

The mass of the shell with density $\Delta\rho$ is

$$M_s=4\pi\Delta\rho a^3\int_{\chi_0}^{\chi_1}\sin^2\chi\,d\chi\,. \tag{46}$$

Using a theorem of mean value, when the thickness of the shell is not too large, we obtain

$$\Delta\phi_0(t)=-\frac{2M_s}{(3+2\eta)c^2\alpha t}\frac{\sin k(\chi^*-\pi)}{\sin k\pi\sin\chi^*}\,, \tag{47}$$

where χ^* is some mean value between χ_0 and χ_1. This equation determines the contribution of the mass M_s at the point $\chi=\chi^*$ to the scalar field at the origin.

As $0<\alpha<1/\pi$, the value of $k=(1+\alpha^2)^{1/2}$ is larger than unity, and is very near to unity. On this condition the contribution to $\Delta\phi_0(t)$ evaluated by Eq. (47) from the region inside of $\chi_c=(1-1/k)\pi$ is negative, and the contribution from the region outside of χ_c is positive when $3+2\eta<0$. The contribution from the whole universe is given by integrating Eq. (47), that is, Eq. (44). Therefore, when $3+2\eta<0$, the contribution from the whole universe is positive.

When χ^* is much smaller than π, we obtain

$$\Delta\phi_0(t)=-2\times10^{-23}M_s/r^*\,, \tag{48}$$

where $r^*=a\sin\chi^*$, and we used the value $\eta=-56$ which we evaluated from the perihelion rotation of Mercury.

Let us suppose that the mass of the Galaxy $M_s=3\times10^{44}$ g is concentrated in its center. As the distance from our solar system to it is $r^*=2.5\times10^{22}$ cm, the contribution of the Galaxy to the scalar field is $\Delta\phi_0/\phi=-2\times10^{-8}$, where $\phi=1/G=1.5\times10^7$ dyn^{-1} cm^{-2} g^2. We, however, cannot measure this decrease because the Galaxy really always exists. We can measure only a spatial change of the scalar field by the contribution of the Galaxy. For example, to the change of the position of the Earth by revolution around the Sun, as the diameter of revolution trajectory of the Earth is 3×10^{13} cm, the scalar field changes in $\Delta(\Delta\phi_0)/\phi=2\times10^{-17}$. The change of the gravitational "constant" has the opposite sign.

V. DISCUSSIONS

In this particular cosmological model of the Brans-Dicke theory, as well as in the Einstein universe of general relativity, the dragging coefficient of the inertial frame, that is, the ratio between the angular velocity of the inertial frame at the origin and that of the rotating shell approaches unity when the rotating shell covers the whole universe. This means the complete dragging of the inertial frame. The inertial frame at the origin is completely determined by the rotating shell (matter of the whole universe). First the inertial frame at the origin in this model is at rest, relative to the matter of the universe. As the rotating shell becomes much thicker and more massive, the inertial frame is dragged by the shell more closely. Calculations are limited up to first order in the angular velocity, that is, to the Coriolis force. In this approximation it is indicated that this particular model satisfies Mach's ideas.

The inertial properties of the universe, in this spherical rotating-shell problem, are determined by the (3, 0) component of the field equations. The angular velocity of the inertial frame is connected with a particular solution of this inhomogeneous equation. The particular solution must be the angular velocity of the shell in order that the dragging coefficient becomes unity. The relation $G(t)M/c^2a(t) = \text{const}$ and the linearity of the expansion are essential in this aim and a variable separation of the homogeneous equation in this model.

As the universe expands, the shell restricted by the two hypersurfaces $\chi = \chi_0$ and $\chi = \chi_1$ also expands. For consistency of discussion, the angular velocity ω_s of the shell must vary in time as $a^{-3}(t)$ which corresponds to the conservation of the angular momentum. The angular velocity ω_0 of the inertial frame at the origin also varies in time as $a^{-3}(t)$. So there is no time delay between the angular velocity of the shell and that of the inertial frame.[7] In this model, it is determined whether the particle horizon of the universe does or does not exist, by the value of the coupling parameter η. Even if the horizon exists, when the shell covers the whole universe, the complete dragging of the inertial frame is realized. This means that the Machian inertial interaction propagates beyond the causally related region.

When the rotating shell covers the whole universe, both of the coefficients B_1 and B_2 in Eq. (27) also converge to zero, and only the particular solution ω_s remains. So the metric becomes the same as that obtained by the coordinate transformation.

The present discussions of the dragging effect on the inertial frame are based on the Robertson-Walker metric for the closed homogeneous and isotropic universe. This universe has no asymptotically flat region, which is necessary to define frame dragging, according to Brill *et al.*[5,7] It is surely difficult to define frame dragging in closed models, but we are interested only in a particular observer at the origin of the closed universe. For the observer at the origin it is a fascinating problem how his inertial frame depends on matter. We investigated this problem by calculating the perturbation of the metric due to the spherical rotating shell in comoving coordinates. It is principally possible for the observer at the origin to measure the angular velocity of the inertial frame at the origin ω_0 and of the shell ω_s, relative to the remaining part of the universe.

This cosmological model has the negative coupling parameter of the scalar field contrary to Brans and Dicke. The coupling parameter η must be smaller than -2 in order that the physical quantities are real. In this result, the contribution of matter to the scalar field from nearby matter is negative. However, the contribution from distant matter is positive, and the contribution from the whole universe is also positive. In the flat space, the distance dependence of the contribution to the scalar field is always $-r^{-1}$. However, in the curved (closed) space, this is right only in the neighborhood. The sign of the contribution becomes inverse at a point $[\chi_c = (1 - 1/k)\pi]$ in the closed space, and the summation of the contribution becomes positive. The distant matter is more dominant than nearby matter in determining the scalar field. The contribution to the gravitational "constant" has the opposite sign to that of the scalar field. The nearby mass increases the gravitational constant, and the distant mass decreases it. The gravitational constant is almost determined by matter in the distant region.

ACKNOWLEDGMENTS

The author is grateful to the colleagues in his laboratory for helpful discussions, advice, and reading the manuscript. He would also like to thank Professor T. Takabayasi for constant encouragement.

---

*This is a part of a masters thesis in Nagoya University (1978).

[1] H. Thirring, Phys. Z. 19, 33 (1918).
[2] L. Bass and F. A. E. Pirani, Philos. Mag. 46, 850 (1955).
[3] H. Okamura, T. Ohta, T. Kimura, and K. Hiida, Prog. Theor. Phys. 54, 1872 (1975).
[4] D. R. Brill and J. M. Cohen, Phys. Rev. 143, 1011 (1966).
[5] J. M. Cohen and D. R. Brill, Nuovo Cimento 56B, 209 (1968).
[6] J. M. Cohen, Phys. Rev. 173, 1258 (1968).
[7] L. Lindblom and D. R. Brill, Phys. Rev. D 10, 3151 (1974).
[8] H. Hönl and G. Soergel-Fabricius, Z. Phys. 163, 571 (1961).
[9] A. Lausberg, Astron. Astrophys. 3, 150 (1969).
[10] H. Dehnen and O. Obregon, Astrophys. Space Sci. 14, 454 (1971). After this work was completed, one of the authors indicated that they had already obtained the same solution.
[11] A. Miyazaki, Phys. Rev. Lett. 40, 725 (1978); 40, 1055(E) (1978).
[12] D. Brans and R. H. Dicke, Phys. Rev. 124, 925 (1961).