Minimal Cubature Formulae for Spherically Symmetric Integrals and Tight Euclidean Designs

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Contents

Chapter 1

Introduction

Let Ω be a domain of $\mathbb{R}^n$ and μ be a measure on Ω. Under the assumption that polynomials up to a sufficient large degree are integrable with respect to μ, we consider the integral over Ω

$$\mathcal{I}[f] = \int_{\Omega} f(\boldsymbol{x}) \mathrm{d}\mu(\boldsymbol{x}). \tag{1.1}$$

The cubature formula, which is the main theme of this thesis, serves a method of study in various fields, for example, numerical analysis, mathematical finance and algebraic combinatorics.

Let X be a finite subset in Ω and w be a positive weight function on w. We say that a finite weighted set (X, w) forms a *cubature formula of degree t for the integral $\mathcal{I}$* if the identity

$$\mathcal{I}[f] = \sum_{\xi \in X} w(\xi) f(\xi) \tag{1.2}$$

holds for every $f \in \mathcal{P}_t(\mathbb{R}^n)$, the space of polynomials of degree at most t in n variables. When $n = 1$, the name quadrature formulae may be used in place of cubature formulae.

If continuous functions can be approximated by polynomials, the integral $\mathcal{I}$ for a continuous function f may be approximated numerically by the cubature formula ([56]). However, the role of cubature formulae is not limited to the simple numerical computation of integrals. For example, Lyons and Victoir [51] introduced the concept of cubature formula on Wiener space which is motivated by the application to mathematical finance and so on, and Teichmann [60] applied it to compute the Greeks which are important as hedging parameters in mathematical finance.

It is a matter of course that cubature formulae with smaller number of points are better in application, and we are interested in the existence of such formulae. Tchakaloff's theorem [59] is a fundamental result on the number of points of a cubature formula and says that a cubature formula of degree t using at most $\dim(\mathcal{P}_t(\mathbb{R}^n))(= \binom{n+t}{t})$ points exists. As a natural problem, we look for cubature formulae with as small number of points as possible.

There are several studies on the lower bound for the number of points X in a cubature formula and it is known (see, e.g., [56]) that

$$|X| \geq \dim(\mathcal{P}_{[t/2]}(\Omega)), \tag{1.3}$$

where $\mathcal{P}_{[t/2]}(\Omega)$ is the space of all polynomials of degree at most t in n variables restricted to Ω and $[t/2]$ means the greatest integer less than or equal to $t/2$. For the case when $t = 2e+1$ and the integral is *centrally symmetric*, that is, $\mathcal{I}[f] = 0$ for any odd polynomial f, this bound was improved by Möller [41, 42] and Mysovskikh [46] as follows:

$$|X| \geq \begin{cases} 2\dim \mathcal{P}_e^*(\Omega) - 1 & \text{if } e \equiv 0 \pmod 2 \text{ and } \mathbf{0} \in X, \\ 2\dim \mathcal{P}_e^*(\Omega) & \text{otherwise,} \end{cases} \tag{1.4}$$

where $\mathcal{P}_e^*(\Omega)$ denotes the subspace of $\mathcal{P}_e(\Omega)$ consisting of all polynomials of degree at most e with the same parity as e (see Section 2.1 for details). A cubature formula of degree t is said to be *minimal* if it attains the equality in (1.3) or (1.4), according to whether t is an even or odd integer and $\mathbf{0} \in X$ or not.

The purpose of this thesis is to study the existence and nonexistence of minimal cubature formulae for spherically symmetric integrals. Let $\langle \boldsymbol{x}, \boldsymbol{y} \rangle$ be the usual inner product on $\mathbb{R}^n$ and $\|\boldsymbol{x}\| = \sqrt{\langle \boldsymbol{x}, \boldsymbol{x} \rangle}$. We say that the integral (1.1) is *spherically symmetric* if Ω is of the form $\{\boldsymbol{x} \in \mathbb{R}^n \mid q_1 \leq \|\boldsymbol{x}\| < q_2\}$, $0 \leq q_1 < q_2 \leq \infty$, and the measure μ has a positive weight function $W(\boldsymbol{x})$ which is a function of $\|\boldsymbol{x}\|$. We use the same symbol $W(\boldsymbol{x})$ for $W(u)$ when $\|\boldsymbol{x}\| = u$. Namely, the spherically symmetric integral $\mathcal{I}$ is of the following form:

$$\mathcal{I}[f] = \frac{1}{V} \int_\Omega f(\boldsymbol{x}) W(\|\boldsymbol{x}\|) \mathrm{d}\boldsymbol{x} = \frac{1}{V} \int_{q_1}^{q_2} \left(\int_{S^{n-1}} f(u\boldsymbol{y}) \mathrm{d}\sigma(\boldsymbol{y}) \right) u^{n-1} W(u) \mathrm{d}u, \tag{1.5}$$

where $V = \int_\Omega W(\|\boldsymbol{x}\|)\mathrm{d}\boldsymbol{x}$, σ denotes the unnormalized surface measure on the unit sphere S^{n-1} and $|S^{n-1}| = \int_{S^{n-1}} \mathrm{d}\sigma(\boldsymbol{x})$.

We should note here that the connection between minimal cubature formulae and orthogonal polynomials was found about one century ago (see, e.g., [1, 21]) and has been studied extensively (see, e.g., [23, 38, 47, 55, 56]). In the one dimensional case, the points of a minimal quadrature formula of degree $2e-1$ for a univariate Gaussian integral are exactly the roots of the Laguerre polynomial of degree e, as noticed by Gauss (see, e.g., [38, 56]). In the multidimensional case, Radon [52] constructed a two dimensional minimal cubature formula of degree 5 by taking the roots of multivariate orthogonal polynomials into account and Mysovskikh [44] characterized the minimal cubature formulae by using reproducing kernels (see Section 4.1). We also use several results related to orthogonal polynomials to study the existence and nonexistence of minimal cubature formulae for spherically symmetric integrals.

On the other hand, in algebraic combinatorics, Delsarte, Goethals and Seidel [26] introduced the concept of spherical design in order to study geometrical structures of finite subsets on S^{n-1}. We say that a finite subset X on S^{n-1} is a *spherical t-design*, if the identity

$$\frac{1}{|S^{n-1}|}\int_{S^{n-1}} f(\boldsymbol{x})\mathrm{d}\sigma(\boldsymbol{x}) = \frac{1}{|X|}\sum_{\xi\in X} f(\xi)$$

holds for every $f \in \mathcal{P}_t(\mathbb{R}^n)$. A spherical design is a cubature formula on the unit sphere all of whose weights are equal, and said to be *tight* if it gives rise to the minimal cubature formula for $\Omega = S^{n-1}$. Hence we may say that a tight spherical t-design is good in numerical computation. Moreover, for example, a tight spherical $2e$-design is an e-distance set, that is, the number of values of the inner products between the distinct vectors in the design is equal to e, and this design attains the upper bound of the cardinality of an e-distance set on S^{n-1} (see Theorem 2.10 and 2.11 for details). This advantage in the sense of coding theory is the second merit of the tight spherical design. So many authors in algebraic combinatorics have been studying spherical designs with the relation to other fields. For example, the existence problem of tight spherical design has a deep connection with Lehmer's conjecture in number theory (see, e.g., [11]).

Moreover, Neumaier and Seidel [48] proposed to study Euclidean design, which is a cubature formula for an integral on the multiple concentric spheres. The results of the theory of Euclidean designs are useful in our study and we recall it here. Let X be a finite set in $\mathbb{R}^n$. We set $\{r_1, r_2, \ldots, r_p\} = \{\|\xi\| \mid \xi \in X\}$, $S_i = \{\boldsymbol{x} \in \mathbb{R}^n \mid \|\boldsymbol{x}\| = r_i\}$ and $S = \cup_{i=1}^p S_i$ and $X_i = X \cap S_i$ for $i = 1, \ldots, p$. Letting w be a positive function on X and setting, $W_i = \sum_{\xi\in X_i} w(\xi)$ for $i = 1, \ldots, p$, we say that a finite weighted set (X, w) is a Euclidean t-design supported by a union S of p concentric spheres if the identity

$$\sum_{i=1}^{p} \frac{W_i}{|S_i|}\int_{S_i} f(\boldsymbol{x})\mathrm{d}\sigma_i(\boldsymbol{x}) = \sum_{\xi\in X} w(\xi)f(\xi)$$

holds for every $f \in \mathcal{P}_t(\mathbb{R}^n)$, where σ_i is the unnormalized surface measure on S_i and $|S_i| = \int_{S_i} \mathrm{d}\sigma_i(\boldsymbol{x})$. We note that the definition given above coincide with that of spherical design when $p = 1, r_1 = 1$ and $w \equiv 1$.

A Euclidean t-design supported by a union S of p concentric spheres is said to be *tight* if it gives rise to the minimal cubature formula for $\Omega = S$. Recently, many authors have been studying the existence and nonexistence of tight Euclidean designs (see, e.g., [2, 3, 5, 6, 8, 9, 10, 12, 14, 32]). Among these works, Bannai and Bannai [8] showed the classification of tight Gaussian 4-designs, namely, minimal cubature formulae of degree 4 for a multivariate Gaussian integral (see Definition 2.19). Gaussian integrals are important in many fields of science. What is important in this thesis is the following equivalence between the Euclidean and the Gaussian designs. A Euclidean t-design is regarded as a cubature formula of degree t for an integral on a union S of p concentric spheres. On the other hand, as is shown in in Proposition 2.20, a Gaussian design is also a Euclidean design. Hence, in order to study cubature formulae, we may use the results of Euclidean design.

This thesis is organized as follows. After preparing basic notions concerning spherical design and Euclidean design in the next chapter, we study in Chapter 3 the existence and nonexistence of minimal cubature formulae for a bivariate Gaussian integral in the same framework of [13, 35]. We characterize the radii of concentric spheres by which the points of minimal cubature formula are supported in terms of Laguerre polynomials. We also specify the two dimensional tight Euclidean t-designs on p concentric spheres with $p \geq [t/4] + 1$. Combining these results, we derive necessary conditions for the existence of minimal cubature formulae of degree $4k + 1$ and $4k + 3$ for the bivariate Gaussian integral whose points are supported by $k + 1$ concentric spheres without the origin, and show that there exists no minimal cubature formula of degree $2e + 1$ whose points are supported by $[e/2] + 1$ concentric spheres without the origin only when $e = 1$.

In Chapter 4, following [34], we study the existence and nonexistence of minimal cubature formulae for the multivariate spherically symmetric integral $\mathcal{I}$. Combining the theories on reproducing kernels and that on Euclidean designs, we show that, if there exists a minimal cubature formula of degree $4k + 1$, then the points in the formula are supported by $k + 1$ concentric spheres including the origin, and the weights in the formula are constant on each concentric sphere. We show that equivalence of minimal cubature formula of degree 5 and 4 which includes the origin and a spherical tight 5- and 4-design, We also show that there exists no minimal cubature formula of degree 9 for some classical spherically symmetric integrals.

The results in Chapter 5 are based on [36]. We construct a four dimensional tight Euclidean 5-design on 3 concentric spheres without the origin. We also show that this design does not have an algebraic structure called the coherent configuration, which all of known examples of tight Euclidean designs have. Bannai and Bannai [12] showed that, if (X, w) is a tight Euclidean design supported by p concentric spheres with $p = 1, 2$ and $\mathbf{0} \notin X$, then X has the structure of a coherent configuration. However, the case when $p \geq 3$, it does not seem to be clear whether the tight Euclidean design has the structure of a coherent configuration or not. Our design is important in this sense.

In Chapter 6, we give several remarks on minimal cubature formulae for spherically symmetric integrals, tight Euclidean designs and related topics.

In the last part of this thesis, Appendix A, we first give a list of all known tight and almost tight Gaussian t-design supported by $[t/4]+1$ concentric spheres. Secondary we give the explicit form of the 4-th modified reproducing kernel for the spherically symmetric integral $\mathcal{I}$, which we use to derive necessary conditions for the existence of minimal cubature formulae of degree 9 in Chapter 4.

Chapter 2

Preliminaries

2.1 The linear spaces of polynomials

We first give the notation for the linear spaces of polynomials which we frequently use in this thesis. Let $\mathcal{P}(\mathbb{R}^n) = \mathbb{R}[x_1, \ldots, x_n]$ be the vector space of polynomials in n variables $x_1, \ldots, x_n$. Let $\mathrm{Hom}_l(\mathbb{R}^n)$ be the subspace of $\mathcal{P}(\mathbb{R}^n)$ spanned by homogeneous polynomials of degree l. Let $\mathcal{P}_l(\mathbb{R}^n) = \oplus_{i=0}^{l}\mathrm{Hom}_l(\mathbb{R}^n)$ and $\mathcal{P}_l^*(\mathbb{R}^n) = \oplus_{i=0}^{[l/2]}\mathrm{Hom}_{l-2i}(\mathbb{R}^n)$. Let $\mathrm{Harm}(\mathbb{R}^n)$ be the subspace of $\mathcal{P}(\mathbb{R}^n)$ which consists of all the harmonic polynomials. Let $\mathrm{Harm}_l(\mathbb{R}^n) = \mathrm{Harm}(\mathbb{R}^n) \cap \mathrm{Hom}_l(\mathbb{R}^n)$. For a subset Ω in $\mathbb{R}^n$, let $\mathcal{P}(\Omega)$, $\mathcal{P}_l(\Omega)$, $\mathcal{P}_l^*(\Omega)$, $\mathrm{Hom}_l(\Omega)$, $\mathrm{Harm}(\Omega)$, $\mathrm{Harm}_l(\Omega)$ be the sets of corresponding polynomials restricted to Ω. For example, $\mathcal{P}_l^*(\Omega) = \{f|_\Omega \mid f \in \mathcal{P}_l^*(\mathbb{R}^n)\}$.

Let S be a union of p concentric spheres centered at the origin. We define $\epsilon_S \in \{0, 1\}$ by

$$\epsilon_S = 1 \quad \text{if } \mathbf{0} \in S, \quad \epsilon_S = 0 \quad \text{if } \mathbf{0} \notin S.$$

The following lemma is useful to represent lower bounds for the number of points in a cubature formula of degree t (see [48] and also [5] for a useful list).

Lemma 2.1. *We have the following equalities:*

(i)

$$\dim(\mathcal{P}_e(\mathbb{R}^n)) = \binom{n+e}{n}.$$

(ii)

$$\dim(\mathcal{P}_e^*(\mathbb{R}^n)) = \sum_{i=0}^{[e/2]} \binom{n+e-2i-1}{e-2i}.$$

(iii) *When* $p \le [(e+\epsilon_S)/2]$,

$$\dim(\mathcal{P}_e(S)) = \epsilon_S + \sum_{i=0}^{2(p-\epsilon_S)-1} \binom{n+e-i-1}{e-i} < \dim(\mathcal{P}_e(\mathbb{R}^n)).$$

(iv) *When* $p \ge [(e+\epsilon_S)/2]+1$,

$$\dim(\mathcal{P}_e(S)) = \sum_{i=0}^{e} \binom{n+e-i-1}{e-i} = \binom{n+e}{e}.$$

(v) *When* $p \ge [e/2]+1$,

$$\dim(\mathcal{P}_e^*(S)) = \sum_{i=0}^{[e/2]} \binom{n+e-2i-1}{e-2i}.$$

(vi) *When* $p \le [e/2]$ *and* e *is an odd integer or is an even integer and* $\mathbf{0} \notin S$,

$$\dim(\mathcal{P}_e^*(S)) = \sum_{i=0}^{p-1} \binom{n+e-2i-1}{e-2i} < \dim(\mathcal{P}_e^*(\mathbb{R}^n)).$$

(vii) *When* $p \le [e/2]$, e *is an even integer and* $\mathbf{0} \in S$,

$$\dim(\mathcal{P}_e^*(S)) = 1 + \sum_{l=0}^{p-2} \binom{n+e-2i-1}{e-2i} < \dim(\mathcal{P}_e^*(\mathbb{R}^n)).$$

We note that, if the number of concentric spheres is small, the dimensions of $\mathcal{P}_e(S)$ or $\mathcal{P}_e^*(S)$ is less than the dimensions of $\mathcal{P}_e(\mathbb{R}^n)$ or $\mathcal{P}_e^*(\mathbb{R}^n)$, respectively. Moreover, if the interior of Ω is not empty, then the dimensions of $\mathcal{P}_e(\Omega)$ or $\mathcal{P}_e^*(\Omega)$ is equal to the dimensions of $\mathcal{P}_e(\mathbb{R}^n)$ or $\mathcal{P}_e^*(\mathbb{R}^n)$, respectively.

2.2 Minimal cubature formulae

We recall the best results on lower bounds for the number of points of a cubature formula of degree t and the definition of a minimal cubature formula. Let Ω be a subset of $\mathbb{R}^n$ and μ be a measure on Ω. We assume all polynomials of up to sufficient large degrees are integrable. We consider the integral

$$\mathcal{I}[f] = \int_\Omega f(\boldsymbol{x}) \mathrm{d}\mu(\boldsymbol{x}).$$

The lower bound for the number of points in a cubature formula X of degree $2e$ for $\mathcal{I}$ is well known (see, e.g., [56]).

Theorem 2.2. *Let (X, w) be a cubature formula of degree $2e$ for the integral $\mathcal{I}$. Then*

$$|X| \geq \dim(\mathcal{P}_e(\Omega)). \tag{2.1}$$

We say that the integral $\mathcal{I}$ is centrally symmetric if the identity

$$\int_\Omega f(\boldsymbol{x}) \mathrm{d}\mu(\boldsymbol{x}) = 0$$

holds for every $f \in \mathrm{Hom}_{2l+1}(\mathbb{R}^n)$ with any $l \geq 0$. For a cubature formula of degree $2e+1$, the best lower bound for the number of points in the formula is given by Möller [41, 42] and Mysovskikh [46].

Theorem 2.3. *Assume that the integral $\mathcal{I}$ is centrally symmetric. For a cubature formula (X, w) of degree $2e+1$ for $\mathcal{I}$, it holds that*

$$|X| \geq \begin{cases} 2\dim(\mathcal{P}_e^*(\Omega)) - 1 & \text{if } e \equiv 0 \pmod 2 \text{ and } \mathbf{0} \in X, \\ 2\dim(\mathcal{P}_e^*(\Omega)) & \text{otherwise.} \end{cases} \tag{2.2}$$

Definition 2.4. *Let (X, w) be a cubature formula of degree $2e$ for the integral $\mathcal{I}$. Then we say that (X, w) forms a minimal cubature formula of degree $2e$ for $\mathcal{I}$ if the equality holds in the inequality (2.1).*

Definition 2.5. *Assume that the integral $\mathcal{I}$ is centrally symmetric. A cubature formula (X, w) of degree $2e+1$ for $\mathcal{I}$ is called a minimal cubature formula of degree $2e+1$ for $\mathcal{I}$ if one of the following condition holds*[1]*:*

(i) *e is an even integer, $\mathbf{0} \in X$ and $|X| = 2\dim(\mathcal{P}_e^*(\Omega)) - 1$.*

(ii) *e is an even integer and $\mathbf{0} \notin X$ or e is an odd integer, and $|X| = 2\dim(\mathcal{P}_e^*(\Omega))$.*

Möller [41] also showed that the minimal cubature formulae of degree $2e+1$, which excludes the case when e is an even integer and $\mathbf{0} \notin X$, have the symmetry with respect to the origin.

Theorem 2.6. *Assume that the integral $\mathcal{I}$ is centrally symmetric. For a cubature formula (X, w) of degree $2e+1$ for $\mathcal{I}$, we assume that one of the following conditions:*

[1] In [35], we defined a minimal cubature formula (X, w) of degree $4k+1$ only when X satisfies $|X| = 2\dim(\mathcal{P}_{2k}^*(\Omega)) - 1$ with $\mathbf{0} \in X$. Since we would like to consider the existence of a cubature formula of degree $4k+1$ whose number of points attains $|X| = 2\dim(\mathcal{P}_e^*(\Omega))$ with $\mathbf{0} \notin X$, we change the definition of minimal.

(i) *e is an even integer, $\mathbf{0} \in X$ and $|X| = 2\dim(\mathcal{P}_e^*(\Omega)) - 1$.*

(ii) *e is an odd integer and $|X| = 2\dim(\mathcal{P}_e^*(\Omega))$.*

Then X is antipodal and the weight function w is centrally symmetric, that is, $-\xi \in X$ and $w(-\xi) = w(\xi)$ for any $\xi \in X$.

2.3 Spherical designs

In this section, we give a brief review on spherical designs and tight spherical designs. We recall the definition of a spherical t-design, which was introduced by Delsarte, Goethals and Seidel [26].

Definition 2.7. *A finite set X on the unit sphere S^{n-1} is called a spherical t-design, if the identity*

$$\frac{1}{|S^{n-1}|}\int_{\boldsymbol{x}\in S^{n-1}} f(\boldsymbol{x})\mathrm{d}\sigma(\boldsymbol{x}) = \frac{1}{|X|}\sum_{\xi\in X} f(\xi)$$

holds for every $f \in \mathcal{P}_t(\mathbb{R}^n)$.

By the definition of spherical designs, the formula given by a spherical t-design is a cubature formula of degree t for the integral over S^{n-1}. Since the integral given in the definition of a spherical t-design is centrally symmetric, we can apply (2.1) and (2.2) for spherical t-designs. On the other hand, Delsarte, Goethals and Seidel [26] independently found the same lower bounds.

Theorem 2.8. *Let X be a spherical t-design on S^{n-1}.*

(i) *If $t = 2e$, then the following holds:*

$$|X| \geq \dim(\mathcal{P}_e(S^{n-1})).$$

(ii) *If $t = 2e+1$, then the following holds:*

$$|X| \geq 2\dim(\mathcal{P}_e^*(S^{n-1})).$$

Definition 2.9. *Let X be a spherical t-design. Then we say that X is a tight spherical t-design, if the equality holds in the inequality of Theorem 2.8.*

We recall the following three theorems due to Delsarte, Goethals and Seidel [26], which are useful to study the shapes of spherical designs. We say that X is called an s-distance set, if $|\{\langle \xi, \eta \rangle \mid \xi, \eta \in X,\ \xi \neq \eta\}| = s$. Upper bounds for the cardinalities of s-distance sets X on S^{n-1} are also given.

Theorem 2.10. *For an s-distance set X in S^{n-1}, it holds that*

$$|X| \leq \dim(\mathcal{P}_s(S^{n-1})). \tag{2.3}$$

Moreover if X is antipodal then it holds that

$$|X| \leq 2\dim(\mathcal{P}^*_{s-1}(S^{n-1})). \tag{2.4}$$

The following theorem shows a close relationship between tight spherical designs and distance sets whose cardinalities attain the equality in (2.3) or (2.4).

Theorem 2.11. *Let X be a spherical t-design on S^{n-1}.*

(i) *X is a tight spherical $2e$-design if and only if X is an e-distance set.*

(ii) *X is a tight spherical $(2e+1)$-design if and only if X is an antipodal $(e+1)$-distance set.*

Let us denote by $C_l^{(n-2)/2}(u)$ the univariate Gegenbauer polynomial of degree l with parameter $(n-2)/2$; see [30, pp.174–178]. This polynomial is given by the generating function

$$\frac{1}{(1-2ux+x^2)^{(n-2)/2}} = \sum_{k=0}^{\infty} C_k^{(n-2)/2}(u)x^k.$$

For future use in Chapter 4, we write down the first few of them;

$$\begin{aligned} &C_0^{(n-2)/2}(u) \equiv 1, \quad C_1^{(n-2)/2}(u) = (n-2)u, \\ &C_2^{(n-2)/2}(u) = \frac{(n-2)(nu^2-1)}{2}, \quad C_3^{(n-2)/2}(u) = \frac{n(n-2)u((n+2)u^2-3)}{6}, \\ &C_4^{(n-2)/2}(u) = \frac{(n-2)n((n+2)(n+4)u^4 - 6(n+2)u^2+3)}{24}. \end{aligned} \tag{2.5}$$

For $\xi, \eta \in X$ and $\alpha, \beta \in \{\langle \xi, \eta \rangle \mid \xi, \eta \in X,\ \xi \neq \eta\} \cup \{1\}$, put

$$p_{\alpha,\beta}(\xi, \eta) = |\{\zeta \in X \mid \langle \xi, \zeta \rangle = \alpha, \langle \zeta, \eta \rangle = \beta\}|.$$

We consider the Gegenbauer expansion of the l-th monomial u^l

$$u^l = \sum_{k=0}^{l} f_{l,k} C_k^{(n-2)/2}(u).$$

In particular, $f_{l,0}$'s are given as

$$f_{l,0} = \begin{cases} \dfrac{l!(n-2)!!}{l!!(n+l-2)!!} & \text{if } l \equiv 0 \pmod 2, \\ 0 & \text{if } l \equiv 1 \pmod 2. \end{cases} \tag{2.6}$$

The following theorem gives information about the inner products of points in spherical designs. It will play an important role when we derive necessary conditions of the existence of a minimal cubature formula in Chapter 4.

Theorem 2.12. *Let X be a spherical t-design on S^{n-1}. Then, for nonnegative integers i, j with $i + j \leq t$ and for $\xi, \eta \in X$ with $\langle \xi, \eta \rangle = \gamma$, it holds that*

$$\begin{aligned} &\sum_{\alpha,\beta \in \{\langle \xi,\eta \rangle \mid \xi,\eta \in X,\ \xi \neq \eta\}} \alpha^i \beta^j p_{\alpha,\beta}(\xi, \eta) \\ &\qquad = |X| \sum_{k=0}^{\min\{i,j\}} \frac{f_{i,k} f_{j,k} (d-2)}{d+2k-2} C_k^{(n-2)/2}(\gamma) - \gamma^i - \gamma^j + \delta_{1,\gamma}, \end{aligned}$$

where $\delta_{1,\gamma}$ is the Kronecker delta.

By virtue of the following well known theorem (see, e.g., [7, 10]), the shape of tight spherical designs X on S^1 are completely described.

Proposition 2.13. *X is a tight spherical t-design of S^1 if and only if X is a regular $(t+1)$-gon.*

As is seen from the next result by Bannai and Bannai [16, 17], tight spherical t-designs are very special and hardly exist for $n \geq 3$ or $t \geq 4$.

Theorem 2.14. *Assume $n \geq 3$. If a tight spherical t-design exists on S^{n-1}, then t is in $\{1, 2, 3, 4, 5, 7, 11\}$. Moreover, if $t = 11$, then $n = 24$.*

The last case actually corresponds to the existence of a tight spherical 11-design on S^{23}, and its uniqueness is proved by Bannai and Sloane [20].

Theorem 2.15. *There exists a unique (up to orthogonal transformations) tight spherical 11-design on S^{23}, namely, the 196560 minimal vectors of the Leech lattice.*

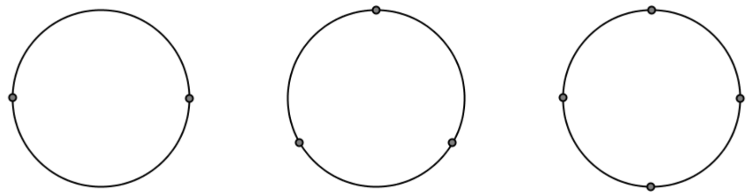
Following is known (see [11]): classification of tight spherical t-designs for each $t \in \{1, 2, 3\}$.

- 1-designs: $\{\xi, -\xi\}$ is a tight spherical 1-design for any $\xi \in S^{n-1}$.

- 2-designs: A finite subset X of S^{n-1} is a tight spherical 2-design if and only if X is a regular simplex.

- 3-designs: A finite subset X of S^{n-1} is a tight spherical 3-design if and only if X is isometric to $\{\pm \boldsymbol{e}_i \mid 1 \leq i \leq n\}$, where $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\}$ is the canonical basis of $\mathbb{R}^n$.

Figure 2.1: Tight spherical $1, 2, 3$-designs on S^1



The classifications of the tight spherical $4, 5, 7$-designs are still open problem ([19]). It is shown that the existence of a tight spherical 4-design on S^{n-2} implies the existence of a tight spherical 5-design on S^{n-1} and vice versa (see [19, 43]). Followings are the only known examples so far (see [11]):

- 4-designs: 27 point set on S^5 related to the E_6 root system, 275 point set on S^{21}.

- 5-designs: set of the 12 vertices of the icosahedron on S^2, set of 126 vectors of the E_7 root system on S^6, 552 point set on S^{22}.

- 7-designs: set of 240 vectors of the E_8 root system on S^7, 4600 point set on S^{23}, which is a section of the Leech lattice.

2.4 Euclidean designs

We give a review on theory of Euclidean designs in order to study cubature formulae for spherically symmetric integrals. Let X be a finite set in $\mathbb{R}^n$ and put $\{r_1, r_2, \ldots, r_p\} = \{\|\xi\| \mid \xi \in X\}$ with $r_1 > r_2 > \cdots > r_p \geq 0$. We set $S_i = \{\boldsymbol{x} \in \mathbb{R}^n \mid \|\boldsymbol{x}\| = r_i\}$ and $X_i = X \cap S_i$ for $i = 1, \ldots, p$. Letting w be a positive function on X, set $W_i = \sum_{\xi \in X_i} w(\xi)$ for $i = 1, \ldots, p$. On each S_i we consider the surface measure σ_i. For $S_i \neq \{\boldsymbol{0}\}$, we put $|S_i| = \int_{S_i} \mathrm{d}\sigma_i(\boldsymbol{x})$. Letting $S = \cup_{i=1}^{p} S_i$, we say X is supported by S.

Definition 2.16. *Let X be a finite set in $\mathbb{R}^n$ supported by a union S of p concentric spheres and w be a positive weight function on X. Then we say that (X, w) is a Euclidean t-design supported by S if*

$$\sum_{i=1}^{p} \frac{W_i}{|S_i|} \int_{S_i} f(\boldsymbol{x}) \mathrm{d}\sigma_i(\boldsymbol{x}) = \sum_{\xi \in X} w(\xi) f(\xi) \tag{2.7}$$

holds for every $f \in \mathcal{P}_t(\mathbb{R}^n)$.

If $S_p = \{\boldsymbol{0}\}$, then we define $|S_p|^{-1} \int_{S_p} f(\boldsymbol{x}) \mathrm{d}\sigma_p(\boldsymbol{x}) = f(\boldsymbol{0})$. We will write a Euclidean design (X, w) simply X when no confusion arise. We note that a Euclidean design is a spherical design if X is a finite subset of S^{n-1} and $w \equiv 1$.

The following theorem, which gives an equivalent condition for Euclidean t-designs, was proved by Neumaier and Seidel [48].

Theorem 2.17. *Let X be a finite set in $\mathbb{R}^n$ and w be a positive function defined on X. Then the following conditions are equivalent:*

(i) *(X, w) is a Euclidean t-design.*

(ii) $\sum_{\xi \in X} w(\xi) f(\xi) = 0$ *for every $f \in \|\boldsymbol{x}\|^{2j} \mathrm{Harm}_l(\mathbb{R}^n)$ with $1 \le l \le t$, $0 \le j \le [(t-l)/2]$.*

By using the above theorem, we show the important connection between cubature formulae for a spherically symmetric integral and Euclidean designs. Letting $\Omega = \{\boldsymbol{x} \in \mathbb{R}^n \mid q_1 \le \|\boldsymbol{x}\| < q_2\}$, $0 \le q_1 < q_2 \le \infty$, and $W(\boldsymbol{x})$ be a radial weight function of $\|\boldsymbol{x}\|$, we consider the spherically symmetric integral $\mathcal{I}$ given by (1.5).

Theorem 2.18. *If (X, w) forms a cubature formula of degree t for the spherically symmetric integral $\mathcal{I}$ given by (1.5), then (X, w) is a Euclidean t-design.*

Proof. It follows that for $l \ge 1$ and $\phi \in \mathrm{Harm}_l(\mathbb{R}^n)$,

$$\begin{aligned} \sum_{\xi \in X} w(\xi) \|\xi\|^{2j} \phi(\xi) &= \frac{1}{V} \int_{\Omega} \|\boldsymbol{x}\|^{2j} W(\|\boldsymbol{x}\|) \mathrm{d}\boldsymbol{x} \\ &= \frac{1}{V} \int_{q_1}^{q_2} u^{l+2j+n-1} \left(\int_{S^{n-1}} \phi(\boldsymbol{y}) \mathrm{d}\sigma(\boldsymbol{y}) \right) W(u) \mathrm{d}u = 0. \end{aligned}$$

Since $\int_{S^{n-1}} \phi(\boldsymbol{y}) \mathrm{d}\sigma(\boldsymbol{y}) = 0$ for $\phi \in \mathrm{Harm}_l(\mathbb{R}^n)$, we obtain the desired result by Theorem 2.17. □

In Chapters 3 and 4, we study the existence and nonexistence of a minimal cubature formula of degree $2e + 1$ for a multivariate Gaussian integral. From the point of view of algebraic combinatorics, this formula have been investigated as a Gaussian design which is a special class of a Euclidean design (see [8, 32]).

Definition 2.19. *Let α be a positive real number. Let X be a finite set in $\mathbb{R}^n$ and w be a positive function on X. Then we say that (X, w) is a Gaussian t-design, if the identity*

$$\frac{1}{V}\int_{\mathbb{R}^n} f(\boldsymbol{x})e^{-\alpha\|\boldsymbol{x}\|^2}\mathrm{d}\boldsymbol{x} = \sum_{\xi\in X} w(\xi)f(\xi) \tag{2.8}$$

holds for every f in $\mathcal{P}_t(\mathbb{R}^n)$, where $V = \int_{\mathbb{R}^n} e^{-\alpha\|\boldsymbol{x}\|^2}\mathrm{d}\boldsymbol{x}$.

We will write a Gaussian design (X, w) simply X when no confusion can arise. Since multivariate Gaussian integrals are spherically symmetric, we obtain the following proposition by using Theorem 2.18 (see [8]).

Proposition 2.20. *A Gaussian t-design is a Euclidean t-design.*

On the other hand, by the definition of Euclidean designs, the formula given by a Euclidean t-design is a cubature formula of degree t for an integral on a union S of p concentric spheres. Since the integral given in the definition of a Euclidean t-design is centrally symmetric, we can also apply (2.1) or (2.2) for Euclidean t-designs. Hence we get the following well known theorem.

Theorem 2.21. *Assume that X is a Euclidean $2e$-design supported by a union S of p concentric spheres. Then the following holds.*

$$|X| \geq \dim(\mathcal{P}_e(S)).$$

The following theorem was shown in [41]. When $r_p > 0$, Delsarte and Seidel [27] gave the same lower bound under the assumption that X is antipodal. Moreover, when $r_p \geq 0$, Bannai [5] also gave the same lower bound under the assumption that X is antipodal.

Theorem 2.22. *Let X be a Euclidean $(2e+1)$-design supported by a union S of p concentric spheres. Then the followings hold:*

$$|X| \geq \begin{cases} 2\dim(\mathcal{P}_e^*(S)) - 1 & \textit{if } e \equiv 0 \pmod 2 \textit{ and } \mathbf{0} \in X \\ 2\dim(\mathcal{P}_e^*(S)) & \textit{otherwise.} \end{cases} \tag{2.9}$$

It is shown in [41] that any Euclidean $(2e + 1)$-design whose number of points attains the equality in (2.9) has antipodality.

Theorem 2.23. *Let (X, w) be a Euclidean $(2e + 1)$-design supported by a union S of p concentric spheres. Assume that the following* (i) *or* (ii) *holds:*

(i) *e is an even integer, $\mathbf{0} \in X$ and $|X| = 2\dim(\mathcal{P}_e^*(S)) - 1$.*

(ii) *e is an odd integer and* $|X| = 2\dim(\mathcal{P}_e^*(S))$.

Then X is antipodal and the weight w is centrally symmetric.

In [41], Möller has not mentioned the antipodality of a Euclidean $(2e+1)$-design whose number of points attains the equality in (2.9) when e is an even integer and $\mathbf{0} \notin X$ since it was out of his scope. We show it under some additional conditions.

Theorem 2.24. *Let (X, w) be a Euclidean $(2e+1)$-design supported by a union S of p concentric spheres. Assume that e is an even integer, $\mathbf{0} \notin X$ and that the following are satisfied:*

(i) $|X| = 2\dim(\mathcal{P}_e^*(S))$.

(ii) *For any line l passing through the origin, let $Y \subset X \cap l$. If $\xi \neq -\eta$ for any $\xi, \eta \in Y$, then $|Y| \leq e/2 + 1$.*

Then X is antipodal and the weight w is centrally symmetric.

We will give the proof of Theorem 2.24, which is similar to that in [41], in the next section.

Definition 2.25. *Let X be a Euclidean t-design supported by a union S of p concentric spheres. Then we say that X is a tight Euclidean t-design on p concentric spheres if one of the following conditions holds:*

(i) $t = 2e$ *and* $|X| = \dim(\mathcal{P}_e(S))$.

(ii) $t = 2e+1$, *e is an even integer,* $\mathbf{0} \in X$ *and* $|X| = 2\dim(\mathcal{P}_e^*(S)) - 1$.

(iii) $t = 2e+1$, *e is an even integer and $\mathbf{0} \notin X$ or e is an odd integer, and* $|X| = 2\dim(\mathcal{P}_e^*(S))$.

Definition 2.26. *Let X be a tight Euclidean t-design on p concentric spheres. We say that X is a tight Euclidean t-design of $\mathbb{R}^n$ if one of the following conditions holds:*

(i) $t = 2e$ *and* $\dim(\mathcal{P}_e(S)) = \dim(\mathcal{P}_e(\mathbb{R}^n))$.

(ii) $t = 2e+1$ *and* $\dim(\mathcal{P}_e^*(S)) = \dim(\mathcal{P}_e^*(\mathbb{R}^n))$.

Definition 2.27. *If X is a tight Euclidean t-design of $\mathbb{R}^n$ with $\mathbf{0} \notin X$, then we call $X \cup \{\mathbf{0}\}$ an almost tight Euclidean t-design of $\mathbb{R}^n$.*

Definition 2.28. *We say that X is a tight or an almost tight Gaussian t-design if X is a tight or almost tight Euclidean t-design of $\mathbb{R}^2$, respectively.*

If X is an almost tight Gaussian t-design of $\mathbb{R}^n$, then $X \setminus \{\mathbf{0}\}$ is not a Gaussian t-design of $\mathbb{R}^n$. However, $X \setminus \{\mathbf{0}\}$ is a tight Euclidean t-design. In this viewpoint, the definition of almost tight is important as a classification of tight Euclidean design.

In the latter chapters, we study the existence problem of minimal cubature formulae for spherically symmetric integrals. The following properties are useful in the study on the number of concentric spheres by which a minimal cubature formula are supported.

Proposition 2.29. (i) *A tight Euclidean $2e$-design X of $\mathbb{R}^n$ is supported by at least $[(e+\epsilon_S)/2]+1$ concentric spheres. Moreover, if $\mathbf{0} \in X$, then e is an even integer and X is supported by $e/2+1$ concentric spheres.*
(ii) *A tight Euclidean $(2e+1)$-design X of $\mathbb{R}^n$ is supported by at least $[e/2]+1$ concentric spheres including the origin. Moreover, if $\mathbf{0} \in X$, then e is an even integer and X is supported by $e/2+1$ concentric spheres including the origin.*

Proposition 2.30. *Let X be a tight Gaussian $(2e+1)$-design, that is, a minimal cubature formula of degree $(2e{+}1)$ for a multivariate variable Gaussian integral. Then X is supported by at least $[e/2]+1$ concentric spheres. Moreover, if $\mathbf{0} \in X$, then e is an even integer and X is supported by $e/2+1$ concentric spheres including the origin.*

The following proposition also implies that the shape of a tight Euclidean $(2e{+}1)$-design on p concentric spheres.

Proposition 2.31. *Let (X, w) be a tight Euclidean $(2e{+}1)$-design on p concentric spheres. Then the followings hold:*

(i) *If e is an odd integer, then X is antipodal, $\mathbf{0} \notin X$, and the weight w is centrally symmetric.*

(ii) *If e is an even integer and $\mathbf{0} \in X$, then X is antipodal and the weight w is centrally symmetric.*

(iii) *If e is an even integer, $\mathbf{0} \notin X$, and $p \le e/2+1$ (exactly speaking if Theorem 2.3.9 (2) holds), then X is antipodal and the weight w is centrally symmetric.*

For a tight Euclidean design (X, w) on p concentric spheres, the following proposition implies the upper bound of the number of inner products among the points in X_i.

Proposition 2.32. (i) *Let X be a tight Euclidean $2e$-design on p concentric spheres. Then each X_i is an at most e-distance set.*
(ii) *Let X be a tight Euclidean $(2e+1)$-design on p concentric spheres. Then each X_i is an at most $(e+1)$-distance set.*

For a Euclidean design (X, w) supported by p concentric spheres, we set $\tilde{X}_i = \{\xi/\|\xi\| \mid \xi \in X_i\}$ for $i = 1, \ldots, p$. Then the following theorem in [5] implies that each $\tilde{X}_i$ is a spherical design.

Theorem 2.33. (i) *Let X be a tight Euclidean $2e$-design on p concentric spheres. If $e - p + \epsilon_S \geq 0$, then each $\tilde{X}_i$ is a spherical $(2e - 2p + 2\epsilon_S + 2)$-design.*
(ii) *Let X be a tight Euclidean $(2e+1)$-design on p concentric spheres. If $e - p + \epsilon_S \geq 0$, then each X_i is an antipodal spherical $(2e - 2p + 2\epsilon_S + 3)$-design.*

We should note here that the original statement of Theorem 2.33 by Bannai [5] also deals with the relation between tight Euclidean design and distance sets.

2.5 Proof of Theorem 2.24

In the study of the shapes of tight Euclidean $(2e+1)$-designs which does not include the origin, Theorem 2.24 in the previous section is important. Hence we give a proof of this section.

Firstly, we introduce the positive inner product on $\mathcal{P}_{e+1}(S)$;

$$\langle g, f \rangle_{\mathcal{I}_{\mathrm{E}}} = \sum_{i=1}^{p} \frac{W_i}{|S_i|} \int_{S_i} g(\boldsymbol{x}) f(\boldsymbol{x}) \mathrm{d}\sigma_i(\boldsymbol{x}),$$

and we write it simply $\langle g, f \rangle$ in this section. Since $\mathcal{P}_{e+1}(S) = \mathcal{P}_e^*(S) + \mathcal{P}_{e+1}^*(S)$ and $\langle g, f \rangle = 0$ for any $g \in \mathcal{P}_e^*(S)$ and $f \in \mathcal{P}_{e+1}^*(S)$, $\mathcal{P}_{e+1}(S) = \mathcal{P}_e^*(S) + \mathcal{P}_{e+1}^*(S)$ is an orthogonal decomposition.

Let $X = \{\xi_1, \ldots, \xi_N\}$. For each $\xi_i \in X$, we denote by L_i an element of the dual space $\hat{\mathcal{P}}_{e+1}(S)$ of $\mathcal{P}_{e+1}(S)$ defined by $L_i(f) = f(\xi_i)$. Consider the subspace $\langle L_i|_{\mathcal{P}_e^*(S)} \mid 1 \leq i \leq N \rangle$ of the dual $\hat{\mathcal{P}}_e^*(S)$ of $\mathcal{P}_e^*(S)$. For each $\psi \in \hat{\mathcal{P}}_e^*(S)$, by the definition of a Euclidean design, there exists a polynomial $g \in \mathcal{P}_e^*(S)$ satisfying $\psi(f) = \langle g, f \rangle = \sum_{i=1}^{N} w(\xi_i) g(\xi_i) f(\xi_i)$ for any $f \in \mathcal{P}_e^*(S)$. This implies

$$\psi = \sum_{i=1}^{N} w(\xi_i) g(\xi_i) L_i|_{\mathcal{P}_e^*(S)} \in \langle L_i|_{\mathcal{P}_e^*(S)} \mid 1 \leq i \leq N \rangle$$

and $\langle L_i|_{\mathcal{P}_e^*(S)} \mid 1 \le i \le N\rangle = \hat{\mathcal{P}}_e^*(S)$. Hence we obtain

$$\dim(\langle L_i|_{\mathcal{P}_e^*(S)} \mid 1 \le i \le N\rangle) = \dim(\hat{\mathcal{P}}_e^*(S)) = \dim(\mathcal{P}_e^*(S)).$$

Now, let $b = \dim(\mathcal{P}_e^*(S))$ and $L_1|_{\mathcal{P}_e^*(S)},\ L_2|_{\mathcal{P}_e^*(S)}, \ldots,\ L_b|_{\mathcal{P}_e^*(S)}$ be a basis of $\langle L_i|_{\mathcal{P}_e^*(S)} \mid 1 \le i \le N\rangle$. Since X does not contain $\mathbf{0}$, we can take a polynomial $h_1 \in \mathrm{Hom}_1(\mathbb{R}^n)$ satisfying $h_1(\xi_i) \ne 0$ for $i = 1, \ldots, N$. Then $h_1\mathcal{P}_e^*(S)$ is a subspace of $\mathcal{P}_{e+1}^*(S)$. Then we can easily check that $L_1|_{h_1\mathcal{P}_e^*(S)},\ L_2|_{h_1\mathcal{P}_e^*(S)}, \ldots,\ L_b|_{h_1\mathcal{P}_e^*(S)}$ are linearly independent. Hence $L_1|_{\mathcal{P}_{e+1}^*(S)},\ L_2|_{\mathcal{P}_{e+1}^*(S)}, \ldots,\ L_b|_{\mathcal{P}_{e+1}^*(S)}$ are linearly independent. This implies

$$\dim(\langle L_i|_{\mathcal{P}_e^*(S)} \mid 1 \le i \le N\rangle) \le \dim(\langle L_i|_{\mathcal{P}_{e+1}^*(S)} \mid 1 \le i \le N\rangle).$$

Hence we obtain

$$\begin{aligned} |X| &\ge \dim(\langle L_i \mid 1 \le i \le N\rangle) \\ &= \dim(\langle L_i|_{\mathcal{P}_{e+1}^*(S)} \mid 1 \le i \le N\rangle) + \dim(\langle L_i|_{\mathcal{P}_e^*(S)} \mid 1 \le i \le N\rangle) \\ &\ge 2\dim(\mathcal{P}_e^*(S)). \end{aligned} \tag{2.10}$$

By assumption e is an even integer and $|X| = 2\dim(\mathcal{P}_e^*(S)) = 2b\ (= N)$. Therefore (2.10) implies that $L_1,\ L_2,\ \ldots,\ L_N$ are linearly independent in $\hat{\mathcal{P}}_{e+1}(S)$ and

$$\dim(\langle L_i|_{\mathcal{P}_{e+1}^*(S)} \mid 1 \le i \le 2b\rangle) = \dim(\langle L_i|_{\mathcal{P}_e^*(S)} \mid 1 \le i \le 2b\rangle) = b.$$

Hence $L_1|_{\mathcal{P}_{e+1}^*(S)}, \ldots, L_b|_{\mathcal{P}_{e+1}^*(S)}$ is a basis of $\langle L_i|_{\mathcal{P}_{e+1}^*(S)} \mid 1 \le i \le 2b\rangle$. Fix an $i \in \{b+1, \ldots, 2b\}$ arbitrarily. Then we can express

$$L_i|_{\mathcal{P}_{e+1}^*(S)} = \sum_{j=1}^{b} c_{i,j} L_j|_{\mathcal{P}_{e+1}^*(S)}.$$

Choose $h \in \mathrm{Hom}_1(S)$ satisfying $h(\xi_i) = 0$. Since $h\mathcal{P}_e^*(S) \subset \mathcal{P}_{e+1}^*(S)$, we have

$$0 = L_i(hf) = \sum_{j=1}^{b} c_{i,j} L_j(hf) = \sum_{j=1}^{b} c_{i,j} h(\xi_j) f(\xi_j)$$

for any $f \in \mathcal{P}_e^*(S)$. Thus

$$\sum_{j=1}^{b} c_{i,j} h(\xi_j) L_j|_{\mathcal{P}_e^*(S)} = 0.$$

Hence we obtain $c_{i,j}h(\xi_j) = 0$ for $j = 1, \ldots, b$. Let $\{j \mid c_{i,j} \ne 0,\ 1 \le j \le b\} = \{1, 2, \ldots, r\}$. Then $h(\xi_j) = 0$ for $j = 1, \ldots, r$. Since h was chosen arbitrarily, $\xi_1, \ldots, \xi_r$ lie on any hyperplane passing through 0 and ξ_i. Hence $\xi_i, \xi_1, \ldots, \xi_r$ lie on a line passing through 0. Hence there exist distinct real numbers $\lambda_{i,j} \neq 0,\ 1$, satisfying $\xi_j = \lambda_{i,j}\xi_i$ for $j = 1, \ldots, r$. Let us consider the following matrix of size $(r+1) \times (e/2+1)$.

$$\begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ 1 & \lambda_{i,1}^2 & \cdots & \lambda_{i,1}^{e-2} & \lambda_{i,1}^{e} \\ 1 & \lambda_{i,2}^2 & \cdots & \lambda_{i,2}^{e-2} & \lambda_{i,2}^{e} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \lambda_{i,r}^2 & \cdots & \lambda_{i,r}^{e-2} & \lambda_{i,r}^{e} \end{bmatrix}.$$

Since there exists a polynomial $f \in \mathrm{Hom}_{2l+1}(S) \subset \mathcal{P}_{e+1}^*(S)$ satisfying $f(\xi_i) \neq 0$ for any l with $0 \leq l \leq e/2$, we have

$$f(\xi_i) = L_i(f) = \sum_{j=1}^{r} c_{i,j} L_j(f) = \sum_{j=1}^{r} c_{i,j} f(\xi_j) = f(\xi_i) \sum_{j=1}^{r} c_{i,j} \lambda_{i,j}^{2l+1}. \tag{2.11}$$

This implies

$$\sum_{j=1}^{r} c_{i,j} \lambda_{i,j} \lambda_{i,j}^{2l} = 1, \text{ for } l = 0, 1, \ldots, \frac{e}{2}.$$

Since $\lambda_{i,j} \neq 0$, the $r+1$ row vectors in the matrix given above are linearly dependent. On the other hand, if $\lambda_{i,j_1} = -\lambda_{i,j_2}$ holds for some $j_1,\ j_2$ satisfying $1 \leq j_1 \neq j_2 \leq r \leq b$, then $\xi_{j_1} = -\xi_{j_2}$ holds. Since e is even, this implies that $L_{j_1}(f) = f(\xi_{j_1}) = f(-\xi_{j_2}) = f(\xi_{j_2}) = L_{j_2}(f)$ holds for any $f \in \mathcal{P}_e^*(S)$. This contradicts the assumption that $L_1|_{\mathcal{P}_e^*(S)}, \ldots, L_b|_{\mathcal{P}_e^*(S)}$ are linearly independent. Hence $\lambda_{i,1}^2,\ \lambda_{i,2}^2, \ldots,\ \lambda_{i,r}^2$ are distinct to each other and none of the pair of two vectors in $\xi_1, \ldots \xi_r$ is antipodal to each other. If $\lambda_{i,j}^2 \neq 1$ for any $j = 1, 2, \ldots, r$, then none of the pair of two vectors in $r+1$ point set $\xi_i, \xi_1, \ldots, \xi_r$ is antipodal to each other. Hence by assumption we obtain $r + 1 \leq e/2 + 1$. Since $1,\ \lambda_{i,1}^2, \ldots, \lambda_{i,r}^2$ are distinct to each other the $r+1$ row vectors of the matrix given above cannot be linearly dependent. This is a contradiction. Therefore there exists a $j \in \{1, \ldots, r\}$ satisfying $\lambda_{i,j} = -1$ and $\xi_i = -\xi_j$. Thus we have seen that for any $i \in \{b+1, \ldots, 2b\}$, there exists $j \in \{1, \ldots, b\}$ satisfying $\xi_i = -\xi_j$. Hence X is an antipodal set. Next, we will prove that the weight w is centrally symmetric. Since e is an even integer, we have

$$\sum_{i=1}^{N} w(\xi_i) f(\xi_i) = \sum_{i=1}^{N} w(\xi_i) f(-\xi_i) \text{ for any } f \in \mathcal{P}_e^*(S), \tag{2.12}$$

$$0 = \sum_{i=1}^{N} w(\xi_i) f(\xi_i) = -\sum_{i=1}^{N} w(\xi_i) f(-\xi_i) \text{ for any } f \in \mathcal{P}_{e+1}^*(S). \tag{2.13}$$

This implies

$$\sum_{i=1}^{N} w(\xi_i)f(\xi_i) = \sum_{i=1}^{N} w(\xi_i)f(-\xi_i) = \sum_{i=1}^{N} w(-\xi_i)f(\xi_i) \text{ for any } f \in \mathcal{P}_{e+1}(S).$$

Hence

$$\sum_{i=1}^{N} w(\xi_i)L_i = \sum_{i=1}^{N} w(-\xi_i)L_i.$$

Since $L_1, \ldots, L_N$ are linearly independent in $\hat{\mathcal{P}}_{e+1}(S)$, we must have $w(\xi_i) = w(-\xi_i)$ for $i = 1, \ldots, N$. This completes the proof of Theorem 2.24.

2.6 Coherent configurations

In this section, we introduce coherent configuration in order to study the algebraic structure of Euclidean designs. Higman [33] gave the definition of it by abstracting combinatorial structure from permutation groups which are not necessarily transitive. A coherent configuration is a system of a finite set X and their ordered pairs $\mathcal{R}_1, \cdots, \mathcal{R}_q$ which satisfies

(i) $X \times X = \bigcup_{i=1}^{q} \mathcal{R}_i$ is a partition;

(ii) There exists some $1 \le s \le q$ such that $\{(\boldsymbol{x}, \boldsymbol{x}) \mid \boldsymbol{x} \in X\} = \bigcup_{i=1}^{s} \mathcal{R}_i$;

(iii) There exists some $1 \le i' \le q$ such that ${}^t\mathcal{R}_i = \mathcal{R}_{i'},\ 1 \le i \le q$, where ${}^t\mathcal{R}_i = \{(\boldsymbol{x}, \boldsymbol{y}) \mid (\boldsymbol{y}, \boldsymbol{x}) \in \mathcal{R}_i\}$;

(iv) For each $i, j, k \in \{1, \ldots, q\}$, $|\{\boldsymbol{z} \in X \mid (\boldsymbol{x}, \boldsymbol{z}) \in \mathcal{R}_i,\ (\boldsymbol{z}, \boldsymbol{y}) \in \mathcal{R}_j\}|$ is independent of $(\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{R}_k$.

In particular, the coherent configuration with $s = 1$ in (ii) is called an association scheme ([18, 25]).

Let (X, w) be a Euclidean t-design on p concentric spheres $S_1, \cdots, S_p$. For given λ, μ with $1 \le \lambda, \mu \le p$, let

$$A_{\lambda,\mu} = \begin{cases} \left\{ \frac{\langle \boldsymbol{x}, \boldsymbol{y} \rangle}{r_\lambda r_\mu} \middle| \ \boldsymbol{x} \in X_\lambda, \boldsymbol{y} \in X_\mu, \boldsymbol{x} \neq \boldsymbol{y} \right\} = \{\alpha_{\lambda,\mu}^{(u)} \mid u = 1, \ldots, s_{\lambda,\mu}\} & \text{if } \lambda \neq \mu, \\ \left\{ \frac{\langle \boldsymbol{x}, \boldsymbol{y} \rangle}{r_\lambda^2} \middle| \ \boldsymbol{x}, \boldsymbol{y} \in X_\lambda \right\} = \{\alpha_{\lambda,\lambda}^{(u)} \mid u = 0, 1, \ldots, s_{\lambda,\lambda}\} & \text{if } \lambda = \mu, \end{cases}$$

where $\alpha_{\lambda,\lambda}^{(0)} = 1$. For each $\alpha_{\lambda,\mu}^{(u)} \in A_{\lambda,\mu}$, $u = 1 - \delta_{\lambda,\mu}, \ldots, s_{\lambda,\mu}$, let

$$\mathcal{R}_{\lambda,\mu,u} = \left\{ (\boldsymbol{x}, \boldsymbol{y}) \in X_\lambda \times X_\mu \middle| \ \frac{\langle \boldsymbol{x}, \boldsymbol{y} \rangle}{r_\lambda r_\mu} = \alpha_{\lambda,\mu}^{(u)} \right\},$$

where $\delta_{\lambda,\mu}$ means the Kronecker delta. For each $(\boldsymbol{x},\boldsymbol{y}) \in \mathcal{R}_{\lambda,\mu,u}$, we let

$$p(\alpha^{(v)}_{\lambda,\nu}, \alpha^{(\omega)}_{\nu,\mu}, \boldsymbol{x}, \boldsymbol{y}) = \left|\left\{ \boldsymbol{z} \in X_\nu \mid (\boldsymbol{x},\boldsymbol{z}) \in \mathcal{R}_{\lambda,\nu,v},\ (\boldsymbol{z},\boldsymbol{y}) \in \mathcal{R}_{\nu,\mu,\omega} \right\}\right|.$$

Definition 2.34. *Let (X, w) be a Euclidean on p concentric spheres. Then we say that X has the structure of a coherent configuration if it satisfies the following condition corresponding to (iv) above: For each $1 \le \lambda, \nu, \mu \le p$ and $1-\delta_{\lambda,\mu} \le u \le s_{\lambda,\mu}$, $1-\delta_{\lambda,\nu} \le v \le s_{\lambda,\nu}$, $1-\delta_{\nu,\mu} \le \omega \le s_{\nu,\mu}$, $p(\alpha^{(v)}_{\lambda,\nu}, \alpha^{(\omega)}_{\nu,\mu}, \boldsymbol{x}, \boldsymbol{y})$ is independent of $(\boldsymbol{x},\boldsymbol{y}) \in \mathcal{R}_{\lambda,\mu,u}$.*

In [26], Delsarte, Goethals and Seidel showed that, if $t \ge 2s_{1,1} - 2$ and X is a spherical t-design, then $(X, \{R_{1,1,u}\}_{0 \le u \le s_{1,1}})$ has the structure of an association scheme. Recently, Bannai and Bannai [12] extended their result to Euclidean designs with $p \ge 2$, as the following theorem shows:

Theorem 2.35. *Let (X, w) be a Euclidean t-design on p concentric spheres with $\mathbf{0} \notin X$. Assume that $w(\xi) \equiv w_j$ for every $\xi \in X_j$, $1 \le j \le p$. Moreover we assume that for any $1 \le \lambda, \mu, \nu \le p$, one of the following conditions is satisfied:*

(i) *$s_{\lambda,\nu} + s_{\nu,\mu} \le t - 2(p-2)$.*

(ii) *$-\xi \in X$ for each $\xi \in X$, and $s_{\lambda,\nu} + s_{\nu,\mu} - \delta_{\lambda,\nu} - \delta_{\nu,\mu} \le t - 2(p-2)$.*

Then X has the structure of a coherent configuration.

We note that for given λ, μ, ν, the inequality of (i) or (ii) implies that $p(\alpha^{(v)}_{\lambda,\nu}, \alpha^{(\omega)}_{\nu,\mu}, \boldsymbol{x}, \boldsymbol{y})$ is independent of $(\boldsymbol{x},\boldsymbol{y}) \in R_{\lambda,\mu,u}$; see [12] for the details. We also note here that the original statement of Theorem 2.35 by [12] does not assume $\mathbf{0} \notin X$.

If X is a tight Euclidean $2e$-design or $(2e+1)$-design on p concentric sphere, then each X_i is an at most e- or $(e+1)$-distance set by Proposition 2.32, respectively. Thus Bannai and Bannai [12] showed the following result.

Corollary 2.36. *If X is a tight Euclidean t-design (X, w) on p concentric sphere with $p = 1, 2$ and $\mathbf{0} \in X$, then X has the structure of a coherent configuration.*

It does not seem to be clear whether a tight Euclidean design X on p concentric spheres with $\mathbf{0} \notin X$ has the structure of a coherent configuration for $p \ge 3$. In Chapter 6, we give a tight Euclidean 5-design X on 3 concentric spheres with $\mathbf{0} \notin X$. We note that this design does not have the structure of a coherent configuration, which all of known examples of tight Euclidean design have.

2.7 Some related results on Euclidean designs

In this section, for the convenience of future work, we give a brief review on the results about tight Euclidean designs.

- In the two dimensional case, Bajnok constructed examples of tight Euclidean t-designs on p concentric spheres for any t ([2]). It is proved in Chapter 3 that if p is at most $[t/4]+1$, then it is similar to one of the examples given by Bajnok (see also [13]).

- X is a tight Euclidean 2-design of $\mathbb{R}^n$ if and only if X is an $(n+1)$-points 1-inner product set with a negative inner product (see [15]).

- An $(n+1)$ point set X in $\mathbb{R}^n$ is the set of points in a tight Euclidean 2-design of $\mathbb{R}^n$ if and only if there exists a negative real number α and $\langle \xi, \eta \rangle = \alpha$ holds for any distinct points $\xi, \eta \in X$, that is, X is a 1-inner product set; see, e.g., [28] for s-inner product sets. A necessary and sufficient conditions for the $(n+1)$ vectors to be a tight Euclidean 2-design of $\mathbb{R}^2$ in terms of their lengths is given ([15]).

- The set of points in a tight Euclidean 3-design of $\mathbb{R}^n$ is similar to the $2n$ point set $\{\pm r_i \boldsymbol{e}_i \mid 1 \leq i \leq n\}$ with any positive real numbers $r_1, \ldots, r_n$ and weight $w(r_i \boldsymbol{e}_i) = 1/nr_i^2$ (see [5]).

- Let X be a tight Euclidean 5-design of $\mathbb{R}^n$ supported by 2 concentric spheres. Then $\mathbf{0} \in X$ and $X \setminus \{\mathbf{0}\}$ is similar to a tight spherical 5-design or one of the four tight Euclidean 5-designs of $\mathbb{R}^n$, with $n = 2, 3, 5, 6$ (see [5] for the explicit structures). Moreover, we construct a tight Euclidean 5-design supported by 3 concentric spheres which has no coherent configurations. It is shown in Chapter 5.

- Let X be a tight Euclidean 7-design with $p = 2$. Then $\mathbf{0} \notin X$ and $n = 2, 4, 7$ ([10]).

- Examples of Euclidean t-designs are constructed using orbits of hyperoctahedral groups in $\mathbb{R}^n$. For example, tight Euclidean 5-design of $\mathbb{R}^3$ with $p = 2$, tight Euclidean 7-design of $\mathbb{R}^3$ with $p = 3$ and tight Euclidean 7-design of $\mathbb{R}^4$ with $p = 2$ are given in the manner ([3]).

- Let $X = X_1 \cup X_2$ with $|X_1| \leq |X_2|$ be a tight Euclidean 4-design of $\mathbb{R}^n$. If $X_1 = \{\mathbf{0}\}$ then X_2 is similar to a tight spherical 4-design. If $|X_1| = n+1$, then $n = 2, 4, 5,$ 6 and 22. In particular, the example constructed in $\mathbb{R}^{22}$ has the structure of combinatorial tight 4-$(23, 7, 1)$ design which is known to be unique (up to taking complimentary design) combinatorial tight 4-design, i.e., tight 4-design in Johnson association schemes. If $|X_1| = n + 2$, then $n = 4$. An example of tight Euclidean 4-design of $\mathbb{R}^{22}$ with $|X_1| = 33$, and $|X_2| = 243$ was constructed. This example has the structure of tight Euclidean 4-design of the Hamming scheme $H(11, 3)$. For the explicit structures of the designs see [6]. It is conjectured in [6] that for $|X_1| \geq n+3$, then $n = (2k - 1)^2 - 3$ with some integer $k \geq 1$.

- Rigidity of tight Euclidean t-designs of $\mathbb{R}^n$ and tight Euclidean t-designs on p concentric spheres are studied in [15] (see also [4, 37]). A Euclidean t-design $X = \{\xi_1, \xi_2, \ldots, \xi_N\}$ with weight w is strongly non-rigid if for any positive real number ϵ, there exists a Euclidean t-design $X' = \{\xi'_1, \xi'_2, \ldots, \xi'_N\}$ with weight w' satisfying $\|\xi_i - \xi'_i\| < \epsilon$, $|w(\xi_i) - w'(\xi'_i)| < \epsilon$, and there exist distinct j_1, j_2 satisfying $\|\xi_{j_1}\| = \|\xi_{j_2}\|$ and $\|\xi'_{j_1}\| \neq \|\xi'_{j_2}\|$. A tight Euclidean 4-design of $\mathbb{R}^2$ with $p = 2$ is proved to be strongly non-rigid and existence of tight Euclidean 4-designs of $\mathbb{R}^2$ with $p = 3$, and 4 were proved. The tight Euclidean 5-design of $\mathbb{R}^2$ (which is antipodal) is proved to be strongly non-rigid and existence of tight Euclidean 4-designs of $\mathbb{R}^2$ with $p = 3$, and 4 was proved (which are antipodal). Also tight spherical 3-design on S^1 is strongly non-rigid and existence of antipodal tight Euclidean 3-design of $\mathbb{R}^2$ with $p = 2$ is proved.

Chapter 3

Minimal cubature formulae for a bivariate Gaussian integral

The purpose of this chapter is to study the existence and nonexistence of minimal cubature formulae of degree $2e+1$ for a bivariate Gaussian integral. In numerical analysis, a minimal cubature formula has long been studied in connection with the theory of orthogonal polynomials (see, e.g., [24, 45, 61]). In this chapter, Laguerre polynomials play an important role.

Proposition 2.30 implies that a minimal cubature formula of degree $2e+1$ supported by at least $[e/2]+1$ concentric spheres. When $e = 2k$ and $p = [e/2]+1$, there is a famous result: Verlinden and Cools [61] and Cools and Schmid [24] showed that there exists a minimal cubature formula of degree $4k+1$ for a bivariate Gaussian integral which includes the origin only when $k = 1$. On the other hand, little is known on the existence of minimal cubature formulae of degree $2e+1$ whose points are supported by $[e/2]+1$ concentric spheres without the origin.

The main result in this chapter is that there exists a minimal cubature formula of degree $2e+1$ for a bivariate Gaussian integral whose points are supported by $[e/2]+1$ concentric spheres without the origin only when $e = 1$. In Section 3.1, we characterize the points and weights of a cubature formula for a bivariate Gaussian integral in terms of roots of Laguerre polynomials. In Section 3.2, we give the structure of the points and the weights in a tight Euclidean design on p concentric spheres and of $\mathbb{R}^2$ ([13]). In Section 3.3, by using results obtained in Section 3.1 and 3.2, we derive necessary conditions for the existence of a minimal cubature formula of degree $4k+1$ and $4k+3$ whose points are supported by $k+1$ concentric spheres without the origin in terms of Laguerre polynomials. Moreover, we see that there exists no tight minimal cubature formula of degree $2e+1$ whose points are supported by $[e/2]+1$ concentric spheres without the origin for any integer $e \geq 2$.

3.1 Cubature formulae and Laguerre polynomial

In this section we show relationships between cubature formulae for a bivariate Gaussian integral and Laguerre polynomials, and review some elementary facts on Laguerre polynomials.

Let Y be a finite set in $[0,\infty)$, λ be a positive weight function on Y and q be a nonnegative integer. We say (Y,λ) forms a quadrature formula of degree t with respect to $u^q e^{-u}$ if the identity

$$\int_0^\infty f(u)\cdot u^q e^{-u}\mathrm{d}u = \sum_{\nu\in Y}\lambda(\nu)f(\nu)$$

holds for every $f \in \mathcal{P}_t(\mathbb{R})$. When the elements of Y and the values of λ are explicitly written, say $Y=\{\nu_1,\cdots,\nu_n\}$ and $\lambda(\nu_i)=\lambda_i$, we write $(\{\nu_1,\cdots,\nu_n\},\{\lambda_1,\cdots,\lambda_n\})$ for the quadrature formula (Y,λ).

Throughout this chapter, let $\mathcal{I}$ be a bivariate Gaussian integral, that is,

$$\mathcal{I}[f]=\frac{1}{\pi}\int_{\mathbb{R}^2} f(x,y)e^{-(x^2+y^2)}\mathrm{d}x\mathrm{d}y.$$

A cubature formula for the bivariate Gaussian integral $\mathcal{I}$ can be reduced to a quadrature formula, as the following shows.

Lemma 3.1. *Assume that (X,w) forms a cubature formula for $\mathcal{I}$ and that X are supported by a union of p concentric spheres S_i with radius r_i. Let $R_i=r_i^2$ for $i=1,\dots,p$. Then the following hold:*

(i) *$(\{R_1,\cdots,R_p\},\{W_1,\cdots,W_p\})$ forms a quadrature formula of degree e with respect to e^{-u}.*

(ii) *If $r_p=0$, then $(\{R_1,\cdots,R_{p-1}\},\{W_1R_1,\cdots,W_{p-1}R_{p-1}\})$ forms a quadrature formula of degree $e-1$ with respect to ue^{-u}.*

Proof. (i) It follows that for each $l=0,\dots,e$,

$$\int_0^\infty u^l e^{-u}\mathrm{d}u = \mathcal{I}[(x^2+y^2)^l] = \sum_{(\xi,\eta)\in X} w(\xi,\eta)(\xi^2+\eta^2)^l = \sum_{i=1}^p W_iR_i^l. \tag{3.1}$$

(ii) The result follows by an argument similar to that used in the proof of (i). □

Let l, α, β be nonnegative integers with $\alpha \geq \beta$. We denote by $L_l^\alpha(u)$ the Laguerre polynomial of degree l with parameter α, given explicitly by the formula

$$L_l^\alpha(u) = \sum_{j=0}^{l} (-1)^j \binom{\alpha+l}{l-j} \frac{u^j}{j!}.$$

$L_l^\alpha(u)$ satisfies the orthogonality relation

$$\int_0^\infty L_l^\alpha(u) L_{l'}^\alpha(u) e^{-u} u^\alpha \mathrm{d}u = \begin{cases} 0 & \text{if } l \neq l', \\ h_{\alpha,l} & \text{if } l = l', \end{cases} \tag{3.2}$$

where $h_{\alpha,l} = (\alpha+l)!/l!$. Moreover the following identities are well known (see, e.g., [57, Chapter 5.1]):

$$L_l^\alpha(u) = \sum_{j=0}^{l} \binom{\alpha-\beta+j-1}{j} L_{l-j}^\beta(u), \tag{3.3}$$

$$u^l = l! \sum_{j=0}^{l} (-1)^j \binom{l+\alpha}{l-j} L_j^\alpha(u). \tag{3.4}$$

These identities on Laguerre polynomials will be used in the arguments in Section 3.3. We simply write $L_l(u)$ for $L_l^0(u)$.

Lemma 3.2. *Let p, s, s' be integers with $0 \leq s' \leq s$, $0 \leq p-1 \leq s-s' \leq 2p-1$. Assume that $(\{R_1, \ldots, R_p\}, \{W_1 R_1^{s'}, \ldots, W_p R_p^{s'}\})$ forms a quadrature formula of degree $s-s'$ with respect to $u^{s'} e^{-u}$. Then $\{R_1, \ldots, R_p\}$ is the set of zeros of*

$$L_p^{s'}(u) + c_1 L_{p-1}^{s'}(u) + \cdots + c_{2p-1+s'-s} L_{s-s'+1-p}^{s'}(u)$$

for some real numbers $c_1, \ldots, c_{2p-1+s'-s}$.

Proof. Let $\pi(u) = (u-R_1)\cdots(u-R_p)$. Then there exist some real numbers $c_1, \cdots, c_p$ such that

$$\pi(u) = (-1)^p p! \Big\{ L_p^{s'}(u) + c_1 L_{p-1}^{s'}(u) + \cdots + c_p L_0^{s'}(u) \Big\}.$$

By using (3.2) and the assumption, for every $j = 1, \ldots, s-s'-p$, we have

$$\begin{aligned} 0 = \sum_{i=1}^{p} (W_i R_i^{s'}) \pi(R_i) L_j^{s'}(R_i) &= \int_0^\infty \pi(u) L_j^{s'}(u) u^{s'} e^{-u} \mathrm{d}u \\ &= c_j (-1)^p p! \int_0^\infty L_j^{s'}(u)^2 u^{s'} e^{-u} \mathrm{d}u = c_j (-1)^p p! \cdot h_{s',j}, \end{aligned}$$

which completes the proof. □

3.2 Two dimensional tight Euclidean designs

The purpose of this section is to specify the tight Euclidean t-designs on p concentric spheres. Bajnok [2] gave the examples of tight Euclidean t-designs on p concentric spheres for all $p \leq [t/4]+1$. These examples were discussed in [24, 61] to investigate a necessary condition for the existence of minimal cubature formulae of degree $4k+1$ for a bivariate Gaussian integral. We also discuss Euclidean t-designs with t other than $4k+1$. In each case of t, we will study the tight Euclidean t-designs on p concentric spheres for all $p \leq [t/4]+1$. After listing up the properties of tight Euclidean designs, we give these proof.

Theorem 3.3. *Let X be a tight Euclidean $(4k+1)$-design on p concentric spheres. Assume $(0,0) \in X$. Then the following hold:*

(i) $2 \leq p \leq k+1$.

(ii) *Each $X_i \neq \{(0,0)\}$ is a tight spherical $(4k-2p+5)$-design and a regular $(4k-2p+6)$-gon.*

(iii) *Let $w_1 = w(\xi)$ for $\xi \in X_1$. Then*

$$w_i = \frac{r_1^{4k-2p+6}}{r_i^{4k-2p+6}} \frac{\prod_{j=2}^{i-1}(r_1^2 - r_j^2)\prod_{j=i+1}^{p-1}(r_1^2 - r_j^2)}{\prod_{j=2}^{i-1}(r_j^2 - r_i^2)\prod_{j=i+1}^{p-1}(r_i^2 - r_j^2)} w_1,$$

where $w_i = w(\xi)$ for $\xi \in X_i$, $i \neq 1$ and $\xi \neq (0,0)$.

(iv) *The angle between the line passing through the origin and a vertex of X_i and the line passing through the origin and a vertex of X_{i+1} is* (an odd integer) $\times \pi/(4k-2p+6)$.

(v) *If $p = k+1$, then X is a tight Euclidean $(4k+1)$-design of $\mathbb{R}^2$.*

Theorem 3.4. *Let X be a tight Euclidean $(4k+1)$-design on p concentric spheres. Assume $(0,0) \notin X$, and $p \leq k+1$. Then the following hold:*

(i) *Each X_i is a tight spherical $(4k-2p+3)$-design and a regular $(4k-2p+4)$-gon.*

(ii) *Let $w_1 = w(\xi)$ for $\xi \in X_1$. Then*

$$w_i = \frac{r_1^{4k-2p+4}}{r_i^{4k-2p+4}} \frac{\prod_{j=2}^{i-1}(r_1^2 - r_j^2)\prod_{j=i+1}^{p}(r_1^2 - r_j^2)}{\prod_{j=2}^{i-1}(r_j^2 - r_i^2)\prod_{j=i+1}^{p}(r_i^2 - r_j^2)} w_1,$$

where $w_i = w(\xi)$ for $\xi \in X_i$, $i \neq 1$.

(iii) *The angle between the line passing through the origin and a vertex of* X_i *and the line passing through the origin and a vertex of* X_{i+1} *is* (an odd integer) $\times \pi/(4k-2p+4)$.

(iv) *If* $p = k+1$, *then* X *is a tight Euclidean* $(4k+1)$*-design of* $\mathbb{R}^2$.

Theorem 3.5. *Let* X *be a tight Euclidean* $(4k+3)$*-design on* p *concentric spheres. Assume* $p \leq k+1$. *Then* X *is antipodal and* $(0,0) \notin X$. *Moreover the following hold:*

(i) *Each* X_i *is a tight spherical* $(4k-2p+5)$*-design and a regular* $(4k-2p+6)$*-gon.*

(ii) *Let* $w_1 = w(\xi)$ *for* $\xi \in X_1$. *Then*

$$w_i = \frac{r_1^{4k-2p+6}}{r_i^{4k-2p+6}} \frac{\prod_{j=2}^{i-1}(r_1^2 - r_j^2) \prod_{j=i+1}^{p}(r_1^2 - r_j^2)}{\prod_{j=2}^{i-1}(r_j^2 - r_i^2) \prod_{j=i+1}^{p}(r_i^2 - r_j^2)} w_1,$$

where $w_i = w(\xi)$ *for* $\xi \in X_i$, $i \neq 1$.

(iii) *The angle between the line passing through the origin and a vertex of* X_i *and the line passing through the origin and a vertex of* X_{i+1} *is* (an odd integer) $\times \pi/(4k-2p+6)$.

(iv) *If* $p = k+1$, *then* X *is a tight Euclidean* $(4k+3)$*-design of* $\mathbb{R}^2$.

Theorem 3.6. *Let* X *be a tight Euclidean* $4k$*-design on* p *concentric spheres. Assume* $(0,0) \in X$. *Then the following hold:*

(i) $p \leq k+1$ *and each* X_i *is a tight spherical* $(4k-2p+4)$*-design and a regular* $(4k-2p+5)$*-gon.*

(ii) *Let* $w_1 = w(\xi)$ *for* $\xi \in X_1$. *Then*

$$w_i = \frac{r_1^{4k-2p+5}}{r_i^{4k-2p+5}} \frac{\prod_{j=2}^{i-1}(r_1^2 - r_j^2) \prod_{j=i+1}^{p-1}(r_1^2 - r_j^2)}{\prod_{j=2}^{i-1}(r_j^2 - r_i^2) \prod_{j=i+1}^{p-1}(r_i^2 - r_j^2)} w_1,$$

where $w_i = w(\xi)$ *for* $\xi \in X_i$, $i \neq 1$ *and* $\xi \neq (0,0)$.

(iii) *The angle between the line passing through the origin and a vertex of* X_i *and the line passing through the origin and a vertex of* X_{i+1} *is* (an odd integer) $\times \pi/(4k-2p+5)$.

(iv) *If* $p = k+1$, *then* X *is a tight Euclidean* $4k$*-design of* $\mathbb{R}^2$.

Theorem 3.7. *Let* X *be a tight Euclidean* $4k$*-design on* p *concentric spheres. Assume* $(0,0) \notin X$ *and* $p \leq k+1$. *Then the following hold:*

(i) *Each* X_i *is a tight spherical* $(4k-2p+2)$*-design and a regular* $(4k-2p+3)$*-gon.*

(ii) *Let* $w_1 = w(\xi)$ *for* $\xi \in X_1$*. Then*

$$w_i = \frac{r_1^{4k-2p+3} \prod_{j=2}^{i-1}(r_1^2 - r_j^2) \prod_{j=i+1}^{p-1}(r_1^2 - r_j^2)}{r_i^{4k-2p+3} \prod_{j=2}^{i-1}(r_j^2 - r_i^2) \prod_{j=i+1}^{p-1}(r_i^2 - r_j^2)} w_1,$$

where $w_i = w(\xi)$ *for* $\xi \in X_i$*,* $i \neq 1$*.*

(iii) *The angle between the line passing through the origin and a vertex of* X_i *and the line passing through the origin and a vertex of* X_{i+1} *is* (an odd integer) $\times \pi/(4k-2p+3)$*.*

(iv) *If* $p = k+1$*, then* X *is a tight Euclidean* $4k$*-design of* $\mathbb{R}^2$*.*

Theorem 3.8. *Let* X *be a tight Euclidean* $(4k+2)$*-design on* p *concentric spheres. Assume* $(0,0) \in X$*. Then the following hold:*

(i) $p \leq k+1$*, hence* X *is not a tight Euclidean* $(4k+2)$*-design of* $\mathbb{R}^2$*.*

(ii) *Each* $X_i \neq \{(0,0)\}$ *is a spherical tight* $(4k-2p+6)$*-design and a regular* $(4k-2p+7)$*-gon.*

(iii) *Let* $w_1 = w(\xi)$ *for* $\xi \in X_1$*. Then*

$$w_i = \frac{r_1^{4k-2p+7} \prod_{j=2}^{i-1}(r_1^2 - r_j^2) \prod_{j=i+1}^{p-1}(r_1^2 - r_j^2)}{r_i^{4k-2p+7} \prod_{j=2}^{i-1}(r_j^2 - r_i^2) \prod_{j=i+1}^{p-1}(r_i^2 - r_j^2)} w_1,$$

where $w_i = w(\xi)$ *for* $\xi \in X_i$*,* $i \neq 1$ *and* $\xi \neq (0,0)$*.*

(iv) *The angle between the line passing through the origin and a vertex of* X_i *and the line passing through the origin and a vertex of* X_{i+1} *is* (an odd integer) $\times \pi/(4k-2p+7)$*.*

Theorem 3.9. *Let* X *be a tight Euclidean* $(4k+2)$*-design on* p *concentric spheres. Assume* $(0,0) \notin X$ *and* $p \leq k+1$*. Then the following hold:*

(i) *Each* X_i *is a tight spherical* $(4k-2p+4)$*-design and a regular* $(4k-2p+5)$*-gon.*

(ii) *Let* $w_1 = w(\xi)$ *for* $\xi \in X_1$*. Then*

$$w_i = \frac{r_1^{4k-2p+5} \prod_{j=2}^{i-1}(r_1^2 - r_j^2) \prod_{j=i+1}^{p-1}(r_1^2 - r_j^2)}{r_i^{4k-2p+5} \prod_{j=2}^{i-1}(r_j^2 - r_i^2) \prod_{j=i+1}^{p-1}(r_i^2 - r_j^2)} w_1,$$

where $w_i = w(\xi)$ *for* $\xi \in X_i$*,* $i \neq 1$*.*

(iii) *The angle between the line passing through the origin and a vertex of* X_i *and the line passing through the origin and a vertex of* X_{i+1} *is* (an odd integer) $\times \pi/(4k-2p+5)$.

(iv) *If* $p = k+1$*, then* X *is a tight Euclidean* $(4k+2)$*-design of* $\mathbb{R}^2$.

In the following we will give the proof for the theorems given above. The following proposition is useful.

Proposition 3.10. *Let* $A_1, \ldots, A_m$ *be real numbers. Let* $B_1, B_2, \ldots, B_m$ *with* $B_1 > B_2 > \cdots > B_m$ *be positive real numbers. Assume*

$$\sum_{i=1}^{m} A_i B_i^j = 0$$

holds for $j = 0, 1, \ldots, m-2$*. Then* A_i *is uniquely determined by the following formulas:*

$$A_i = (-1)^{i-1} \frac{\prod_{j=2}^{i-1}(B_1 - B_j)\prod_{j=i+1}^{m}(B_1 - B_j)}{\prod_{j=2}^{i-1}(B_j - B_i)\prod_{j=i+1}^{m}(B_i - B_j)} A_1$$

for $i = 2, \ldots, m$.

Proof. Let M, $\boldsymbol{a}$ and $\boldsymbol{b}$ be the following $(m-1)\times(m-1)$ matrix and vectors in $\mathbb{R}^{m-1}$:

$$M = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ B_2 & B_3 & \cdots & B_m \\ \vdots & \vdots & \vdots & \vdots \\ B_2^{m-2} & B_3^{m-2} & \cdots & B_m^{m-2} \end{bmatrix}, \quad \boldsymbol{a} = \begin{bmatrix} A_2 \\ A_3 \\ \vdots \\ A_m \end{bmatrix}, \quad \boldsymbol{b}_i = \begin{bmatrix} 1 \\ B_i \\ \vdots \\ B_i^{m-2} \end{bmatrix}.$$

Then we have $M\boldsymbol{a} = -A_1\boldsymbol{b}_1$. Since $R_2, \ldots, R_m$ are distinct, the Vandermonde matrix M is nonsingular. Hence Cramer's formula gives $A_i = -A_1 \det M_{i-1}/\det M$, where $M_{i-1} = [\boldsymbol{b}_2, \boldsymbol{b}_3, \ldots, \boldsymbol{b}_{i-1}, \boldsymbol{b}_1, \boldsymbol{b}_{i+1}, \ldots, \boldsymbol{b}_m]$. Then M_{i-1} is also a Vandermonde matrix and nonsingular.

$$\det M = (-1)^{(m-1)(m-2)/2} \prod_{2\le l<j\le m} (B_l - B_j), \tag{3.5}$$

$$\begin{aligned} \det M_{i-1} &= (-1)^{i-2}|\boldsymbol{b}_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_{i-1}, \boldsymbol{b}_{i+1}, \ldots, \boldsymbol{b}_m| \\ &= (-1)^{i-2}(-1)^{(m-1)(m-2)/2} \prod_{\substack{1\le l<j\le m \\ l,j\ne i}} (B_l - B_j). \end{aligned} \tag{3.6}$$

Then (3.5) and (3.6) imply the formula for A_i. □

Proof of Theorem 3.3. Note that $2 \le p$. If $p \ge k+2$, then $X\backslash\{(0,0)\}$ is a Euclidean $(4k+1)$-design supported by a union S' of $p-1$ concentric spheres. Since $p-1 \ge k+1$ we must have $\dim(\mathcal{P}^*_{2k}(S')) = \dim(\mathcal{P}^*_{2k}(S)) = \dim(\mathcal{P}^*_{2k}(\mathbb{R}^2))$ and $|X| \ge 2\dim(\mathcal{P}^*_{2k}(S')) + 1 > 2\dim(\mathcal{P}^*_{2k}(S)) - 1$. This is a contradiction. Hence $p \le k+1$. If $p = k+1$, then $\dim(\mathcal{P}^*_{2k}(S)) = \dim(\mathcal{P}^*_{2k}(\mathbb{R}^2))$ and X is a tight Euclidean $(4k+1)$-design of $\mathbb{R}^2$. This implies Theorem 3.3(5). Next, Theorem 2.23 implies that X is antipodal and weight function w is centrally symmetric. Then Theorem 2.33 implies that each $X_i \ne \{(0,0)\}$ is a spherical $(4k-2p+5)$-design. Hence $|X_i| \ge 2(2k-p+3)$ holds. If $p=2$, then $X\backslash\{(0,0)\}$ is a tight spherical $(4k+1)$-design and Theorem 3.3 holds. Hence we may assume $p \ge 3$. Since $\dim(\mathcal{P}^*_{2k}(S)) = 1 + \sum_{i=0}^{p-2}\binom{2k-2i+1}{2k-2i} = 1 + (p-1)(2k-p+3)$, we have $|X| = 1 + 2(p-1)(2k-p+3)$. Hence we must have $|X_i| = 2(2k-p+3)$ for $X_i \ne \{0\}$. Therefore each $X_i \ne \{(0,0)\}$ is a tight spherical $(4k-2p+5)$-design and a regular $(4k-2p+6)$-gon. This proves Theorem 3.3 (2). Equation (3.2) in [5] implies $|\{\langle \xi, \eta\rangle \mid \xi \in X_i,\ \eta \in X_j\}|$ is at most $2k+1$ for any $X_i, X_j \ne \{(0,0)\}$ with $i \ne j$. On the other hand $4k-2p+6 \ge 2(k+2)$. Let $m = 4k-2p+6$ and let us consider the points in $\mathbb{R}^2$ using the complex number plane $\mathbb{C}$. Then we can express

$$X_i = \{r_i\zeta_i\zeta^l \mid 0 \le l \le m-1\}$$

where $\zeta = \exp(2\pi\sqrt{-1}/m)$ and $\zeta_i \in \mathbb{C}$ with $|\zeta_i| = 1$. We may assume $\zeta_1 = 1$. Let $1 < i \le p-1$. Since $|\{\langle \xi,\eta\rangle \mid \xi \in X_1, \eta \in X_i\}| \le 2k+1$ and $m \ge 2(k+2)$, the $(p-1)$ regular $(4k-2p+6)$-gons $X_1, \ldots, X_{p-1}$ cannot be at a general position. We must have $\zeta_i = 1$ or $\zeta_i = \exp(\pi\sqrt{-1}/m)$. Next, let $z = x + y\sqrt{-1}$. Then it is well known that the vector space $\mathrm{Harm}_l(\mathbb{R}^2)$ is spanned by $\{\mathrm{Re}(z^l), \mathrm{Im}(z^l)\}$. Hence Theorem 2.17 implies that X is a Euclidean $(4k+1)$-design if and only if

$$\sum_{i=1}^{p-1} w_i \sum_{l=0}^{m-1} r_i^{2j}(r_i\zeta_i\zeta^l)^\nu = 0$$

holds for any $1 \le \nu \le 4k+1$, and $0 \le j \le (4k+1-\nu)/2$. Since for any ν satisfying $m \not| \nu$ we have $\sum_{l=0}^{m-1}(\zeta^s)^l = 0$ we must only need to check the case when ν is a multiple of m. Since $\nu \le 4k+1$ and $m \ge 2(k+2)$, we only need to check when $\nu = m$. Hence we have

$$\sum_{i=1}^{p-1} w_i \sum_{l=0}^{m-1} r_i^{2j}(r_i\zeta_i\zeta^l)^m = 0.$$

This implies

$$\sum_{i=1}^{p-1} w_i r_i^{2j+m}\epsilon_i = 0$$

for $j = 0, 1, \ldots, p-3$ $(= (4k-m)/2)$, where $\epsilon_1 = \zeta_1^m = 1, \epsilon_i = \zeta_i^m (= 1 \text{ or } -1)$. Let $A_i = w_i r_i^m \epsilon_i$. Then $A_1 = w_1 r_1^m > 0$ and

$$\sum_{i=1}^{p-1} A_i r_i^{2j} = 0, \quad \text{for } j = 0, 1, \ldots, p-3.$$

Since $w_1, \ldots, w_p > 0$, $A_1 > 0$, Proposition 3.10 implies $(-1)^{i-1} A_i > 0$ and $\epsilon_i = (-1)^{i-1}$. This completes the proof. □

Proof of Theorem 3.4. By assumption $|X| = 2\dim(\mathcal{P}_{2k}^*(S)) = 2\sum_{i=0}^{p-1}(2k-2i+1) = 2p(2k-p+2)$. Since $p \le k+1$, Theorem 2.24 implies that X is antipodal. Hence Theorem 2.33 implies that each X_i is a spherical $(4k-2p+3)$-design. Hence $|X_i| \ge 2(2k-p+2)$. Hence we must have $|X_i| = 2(2k-p+2)$ and X_i is a tight spherical $(4k-2p+3)$-design, i.e., a regular $2(2k-p+2)$-gon. Let $m = 2(2k-p+2)$. Then by a similar argument given in the proof of Theorem 3.3 will complete the proof. □

Proof of Theorem 3.5. By assumption $|X| = 2\dim(\mathcal{P}_{2k+1}^*(S))$. Since $2k+1$ is an odd number, X is antipodal. Hence X cannot contain the origin. If $p \le k+1$, then $|X| = 2\dim(\mathcal{P}_{2k+1}^*(S)) = 2\sum_{i=0}^{p-1}\binom{2k+2-2i}{2k+1-2i} = 2p(2k-p+3)$. On the other hand, as a similar argument as before implies that each X_i is a spherical $(4k-2p+5)$-design. Then we must have $|X_i| \ge 4k-2p+6$. Therefore we must have $|X_i| = 4k-2p+6$ and X_i is a tight spherical $(4k+3)$-design and regular $(4k-2p+6)$-gon. Then a similar argument as before will complete the proof. □

Proof of Theorem 3.6. Let $S_p = \{(0,0)\}$. Then $X \backslash \{(0,0)\}$ is a Euclidean $4k$-design supported by the union $S' = S_1 \cup \cdots \cup S_{p-1}$ of $p-1$ concentric spheres. If $p \ge k+2$, then we have $\dim(\mathcal{P}_{2k}(S)) = \dim(\mathcal{P}_{2k}(S')) = \dim(P_{2k}(\mathbb{R}^n)) = \binom{2k+2}{2k}$. Then $\dim(\mathcal{P}_{2k}(S)) - 1 = |X \backslash \{0\}| \ge \dim(\mathcal{P}_{2k}(S'))$. This is a contradiction. Hence $p \le k+1$. Then $|X| = \dim(\mathcal{P}_{2k}(S)) = 1 + \sum_{i=0}^{2p-3}(2k-i+1) = 1 + (4k-2p+5)(p-1)$. Moreover each X_i is a spherical $(4k-2p+4)$-design. Hence we must have $|X_i| = 4k-2p+5$ for $i \ne p$. Hence X_i is a tight spherical $(4k-2p+4)$-design and a regular $(4k-2p+5)$-gon. A similar argument will complete the proof. □

Proof of Theorem 3.7. If $p \le k+1$, then $|X| = \dim(\mathcal{P}_{2k}(S)) = \sum_{i=0}^{2p-1}(2k-i+1) = (4k+2p+3)p$. Since X_i is a spherical $(4k-2p+2)$-design, we have $|X_i| \ge 4k-2p+3$. Hence $|X_i| = 4k-2p+3$, X_i is a tight spherical $(4k-2p+2)$-design and a regular

$(4k-2p+3)$-gon. A similar argument will complete the proof. □

Proof of Theorem 3.8. By assumption $|X| = \dim(\mathcal{P}_{2k+1}(S))$. Let $S_p = \{(0,0)\}$. A similar argument given in the proof of Theorem 3.6 implies $p \le k+1$. Since $p \le k+1$ we have $|X| = \dim(\mathcal{P}_{2k+1}(S)) = 1 + \sum_{i=0}^{2p-3}(2k-i+2) = 1 + (4k-2p+7)(p-1) < \dim(\mathcal{P}_{2k+1}(\mathbb{R}^2))$. Hence X is not a tight Euclidean $(4k+2)$-design of $\mathbb{R}^2$. Since $X_i \neq \{(0,0)\}$ is a spherical $(4k-2p+6)$-design, we must have $|X_i| = 4k-2p+7$ for $i \neq p$. Hence $X_i \neq \{(0,0)\}$ is a tight spherical $(4k-2p+6)$-design and a regular $(4k-2p+7)$-gon. A similar argument as before will complete the proof. □

Proof of Theorem 3.9. By assumption $|X| = \dim(\mathcal{P}_{2k+1}(S)) = \sum_{i=0}^{2p-1}(2k-i+2) = (4k-2p+5)p$. If $p = k+1$, then $\dim(\mathcal{P}_{2k+1}(S)) = \dim(\mathcal{P}_{2k+1}(\mathbb{R}^2))$ and X is a tight Euclidean $(4k+2)$-design of $\mathbb{R}^2$. Each X_i is a spherical $(4k-2p+4)$-design. Hence each X_i is a spherical tight $(4k-2p+4)$-design and a regular $(4k-2p+5)$-gon. A similar argument will complete the proof. □

3.3 Minimal cubature formulae of degree $2e+1$

The purpose of this section is to prove the following theorem.

Theorem 3.11. (i) *For any positive integer k, there exists no minimal cubature formula of degree $4k+1$ for the bivariate Gaussian integral $\mathcal{I}$ whose points are supported by $k+1$ concentric spheres without the origin.*
(ii) *For any positive integer k, there exists no minimal cubature formula of degree $4k+3$ for the bivariate Gaussian integral $\mathcal{I}$ whose points are supported by $k+1$ concentric spheres.*

Remark 3.12. By the antipodality of minimal cubature formula (see Theorem 2.6), there exists no minimal cubature formula of degree $4k+3$ for the bivariate Gaussian integral $\mathcal{I}$ whose points are supported by $k+1$ concentric spheres including the origin.

In order to prove Theorem 3.11, we give necessary conditions for the existence of minimal cubature formula of degree $4k+1$ and $4k+3$, respectively.

Proposition 3.13. (1) *Let k be a positive integer. Assume that (X, w) forms a minimal cubature formula of degree $4k+1$ for the bivariate Gaussian integral $\mathcal{I}$ which does not include the origin. and that X is supported by a union of $k+1$ concentric spheres S_i with radius r_i. Then the following hold:*

(i) $\{R_1, \ldots, R_{k+1}\}$ *is the set of zeros of* $L_{k+1}(u) + c_1 L_k(u)$ *for some real number* c_1.

(ii) *With the same* c_1 *as in (i),*

$$L_{k+1}(u) + c_1 L_k(u)$$
$$= \begin{cases} \frac{a!(a+1)!}{(2a+1)!} L_a^{2a+1}(u)\Big(L_{a+1}^{2a+1}(u) + \gamma_1 L_a^{2a+1}(u) + \gamma_2 L_{a-1}^{2a+1}(u)\Big) & \text{if } k = 2a, \\ \frac{((a+1)!)^2}{(2a+2)!}\Big(L_{a+1}^{2a+2}(u) + \gamma_1 L_a^{2a+2}(u)\Big)\Big(L_{a+1}^{2a+2}(u) + \gamma_2 L_a^{2a+2}(u)\Big) & \text{if } k = 2a+1, \end{cases}$$

where γ_1, γ_2 *are some real numbers.*

(2) *Let* k *be a positive integer. Assume that there exists a minimal cubature formula of degree* $4k+3$ *for the bivariate Gaussian integral* $\mathcal{I}$ *and that* X *is supported by a union of* $k+1$ *concentric spheres* S_i *with* $r_i = \sqrt{R_i}$. *Then the following hold:*

(i) $\{R_1, \ldots, R_{k+1}\}$ *is the set of zeros of* $L_{k+1}(u)$.

(ii)

$$L_{k+1}(u)$$
$$= \begin{cases} \frac{a!(a+1)!}{(2a+1)!} L_a^{2a+2}(u)\Big(L_{a+1}^{2a+2}(u) + \gamma_1 L_a^{2a+2}(u) + \gamma_2 L_{a-1}^{2a+2}(u)\Big) & \text{if } k = 2a, \\ \frac{((a+1)!)^2}{(2a+2)!}\Big(L_{a+1}^{2a+3}(u) + \gamma_1 L_a^{2a+3}(u)\Big)\Big(L_{a+1}^{2a+3}(u) + \gamma_2 L_a^{2a+3}(u)\Big) & \text{if } k = 2a+1, \end{cases}$$

where γ_1, γ_2 *are some real numbers.*

Proof. ((1) - (i)) By using Proposition 2.20 and Theorem 3.4, a minimal cubature formula of degree $4k+1$ whose points are supported by $S_1, \cdots, S_{k+1}$ without the origin has the form

$$Q[f] = \sum_{j=1}^{k+1} \frac{W_j}{2k+2} \sum_{l=0}^{2k+1} f\Big(\sqrt{R_j}\cos\Big(\frac{j+2l}{2k+2}\pi\Big), \sqrt{R_j}\sin\Big(\frac{j+2l}{2k+2}\pi\Big)\Big).$$

For any $\alpha = 0, \ldots, 2k$, let $f(x,y) = (x^2+y^2)^\alpha$. By using (3.1), we have

$$\mathcal{I}[(x^2+y^2)^\alpha] = \int_0^\infty u^\alpha e^{-u} \mathrm{d}u = \sum_{j=1}^{k+1} W_j R_j^\alpha = Q[(x^2+y^2)^\alpha]. \tag{3.7}$$

Since these equations can be viewed as a quadrature formula of degree $2k$ with respect to e^{-u}, we obtain the desired result by applying Lemma 3.2 with $p = k+1, s = 2k, s' = 0$.
((1) - (ii)) For distinct non-negative integers α, β with $\alpha + \beta \le 4k+1$, let $f(x,y) = (x+\sqrt{-1}y)^\alpha (x-\sqrt{-1}y)^\beta$. Then we have

$$\mathcal{I}[(x+\sqrt{-1}y)^\alpha(x-\sqrt{-1}y)^\beta] = Q[(x+\sqrt{-1}y)^\alpha(x-\sqrt{-1}y)^\beta]. \tag{3.8}$$

It is easy to see that

$$\mathcal{I}[(x+\sqrt{-1}y)^{\alpha}(x-\sqrt{-1}y)^{\beta}]=0.$$

Hence it follows by (3.8) that

$$\begin{aligned}0=&\mathcal{I}[(x+\sqrt{-1}y)^{\alpha}(x-\sqrt{-1}y)^{\beta}]=Q[(x+\sqrt{-1}y)^{\alpha}(x-\sqrt{-1}y)^{\beta}]\\=&\sum_{j=1}^{k+1}\frac{W_j}{2k+2}\sum_{l=0}^{2k+1}R_j^{(\alpha+\beta)/2}\Big(\cos\Big(\frac{(j+2l)(\alpha-\beta)}{2k+2}\pi\Big)+\sqrt{-1}\sin\Big(\frac{(j+2l)(\alpha-\beta)}{2k+2}\pi\Big)\Big)\\=&\sum_{j=1}^{k+1}\frac{W_j}{2k+2}R_j^{(\alpha+\beta)/2}e^{\sqrt{-1}j\pi(\alpha-\beta)/(2k+2)}\sum_{l=0}^{2k+1}e^{\sqrt{-1}l\pi(\alpha-\beta)/(k+1)}.\end{aligned}\tag{3.9}$$

If $2k+2$ does not divide $\alpha-\beta$, then this equation is automatically satisfied. So it remains to consider the case that $\alpha=2k+2+m,\beta=m$, where $m=0,\ldots,k-1$. Since it holds that

$$0=\sum_{j=1}^{k+1}W_jR_j^{k+1+m}e^{\sqrt{-1}j\pi}=\sum_{j=1}^{k+1}(-1)^j(W_jR_j^{k+1})R_j^m,\tag{3.10}$$

we have

$$\sum_{\substack{j=1\\ j:\ \text{odd}}}^{k+1}(W_jR_j^{k+1})R_j^m=\sum_{\substack{j=1\\ j:\ \text{even}}}^{k+1}(W_jR_j^{k+1})R_j^m.\tag{3.11}$$

On the other hand, by using Lemma 3.1 (i), we have

$$\sum_{j=1}^{k+1}(W_jR_j^{k+1})R_j^m=\int_0^\infty u^m\cdot u^{k+1}e^{-u}\mathrm{d}u,\quad m=0,\ldots,k-1.\tag{3.12}$$

By combining (3.11) and (3.12), we obtain

$$\int_0^\infty u^m\cdot u^{k+1}e^{-u}\mathrm{d}u=\begin{cases}\displaystyle\sum_{\substack{j=1\\ j:\ \text{odd}}}^{k+1}(2W_jR_j^{k+1})R_j^m,\\ \displaystyle\sum_{\substack{j=1\\ j:\ \text{even}}}^{k+1}(2W_jR_j^{k+1})R_j^m.\end{cases}\tag{3.13}$$

For $k\geq 1$ this implies that the R_i, i being even, are the points of a quadrature formula (3.13) of degree $k-1$ with respect to $u^{k+1}e^{-u}$. Similarly the R_i, i being odd, are the points of a quadrature formula (3.13) of degree $k-1$ with respect to $u^{k+1}e^{-u}$. Thus by applying Lemma 3.2 to Lemma 3.1 (i) and (3.13), we obtain the assertion (ii).

((2) - (i)) By Proposition 2.20 and Theorem 3.5, a minimal cubature formula of degree $4k+3$ whose points are supported by $S_1, \cdots, S_{k+1}$ has the form

$$Q[f] = \sum_{j=1}^{k+1} \frac{W_j}{2k+4} \sum_{l=0}^{2k+3} f\Big(\sqrt{R_j}\cos\Big(\frac{j+2l}{2k+4}\pi\Big), \sqrt{R_j}\sin\Big(\frac{j+2l}{2k+4}\pi\Big)\Big).$$

For any $\alpha = 0, \ldots, 2k+1$, let $f(x,y) = (x^2+y^2)^\alpha$. By using (3.1), we have

$$\mathcal{I}[(x^2+y^2)^\alpha] = \int_0^\infty u^\alpha e^{-u} \mathrm{d}u = \sum_{j=1}^{k+1} W_j R_j^\alpha = Q[(x^2+y^2)^\alpha]. \tag{3.14}$$

Since these equations can be viewed as a quadrature formula of degree $2k+1$ with respect to e^{-u}, we obtain the desired result by applying Lemma 3.2 with $p = k+1, s = 2k+1, s' = 0$.
((2) - (ii)) For distinct non-negative integers α, β with $\alpha + \beta \leq 4k+3$, let $f(x,y) = (x+\sqrt{-1}y)^\alpha (x-\sqrt{-1}y)^\beta$. By the same calculation as in (3.9), we obtain

$$\sum_{j=1}^{k+1} \frac{W_j}{2k+4} R_j^{(\alpha+\beta)/2} e^{\sqrt{-1}j\pi(\alpha-\beta)/(2k+4)} \sum_{l=0}^{2k+3} e^{\sqrt{-1}l\pi(\alpha-\beta)/(k+2)} = 0.$$

If $2k+4$ does not divide $\alpha - \beta$, then this equation is automatically satisfied. So it remains to consider the case that $\alpha = 2k+4+m, \beta = m$, where $m = 0, \ldots, k-1$. Since it holds that

$$0 = \sum_{j=1}^{k+1} W_j R_j^{k+2+m} e^{\sqrt{-1}j\pi} = \sum_{j=1}^{k+1} (-1)^j (W_j R_j^{k+2}) R_j^m, \tag{3.15}$$

we have

$$\sum_{\substack{j=1 \\ j:\ \text{odd}}}^{k+1} (W_j R_j^{k+2}) R_j^m = \sum_{\substack{j=1 \\ j:\ \text{even}}}^{k+1} (W_j R_j^{k+2}) R_j^m. \tag{3.16}$$

On the other hand, by using Lemma 3.1 (ii), we have

$$\sum_{j=1}^{k+1} (W_j R_j^{k+2}) R_j^m = \int_0^\infty u^m \cdot u^{k+2} e^{-u} \mathrm{d}u, \quad m = 0, \ldots, k-1. \tag{3.17}$$

By combining (3.16) and (3.17), we obtain

$$\int_0^\infty u^m \cdot u^{k+2} e^{-u} \mathrm{d}u = \begin{cases} \displaystyle\sum_{\substack{j=1 \\ j:\ \text{odd}}}^{k+1} (2W_j R_j^{k+2}) R_j^m, \\ \displaystyle\sum_{\substack{j=1 \\ j:\ \text{even}}}^{k+1} (2W_j R_j^{k+2}) R_j^m. \end{cases} \tag{3.18}$$

For $k \geq 1$ this implies that the R_i, i being even, are the points of a quadrature formula (3.18) of degree $k-1$ with respect to $u^{k+2}e^{-u}$. Similarly the R_i, i being odd, are the points of a quadrature formula (3.18) of degree $k-1$ with respect to $u^{k+2}e^{-u}$. Thus by applying Lemma 3.2 to Lemma 3.1 (i) and (3.18), we obtain the assertion (ii). □

Lemma 3.14. *Let γ, δ be integers such that $\delta \geq \gamma \geq 1$. Then the following identities hold:*

(i) $\displaystyle\sum_{j=0}^{\gamma} \frac{(\delta-j)!}{(\gamma-j)!} = \frac{(\delta+1)!}{\gamma!(\delta-\gamma+1)}.$

(ii) $\displaystyle\sum_{j=0}^{\gamma} j \cdot \frac{(\delta-j)!}{(\gamma-j)!} = \frac{(\delta+1)!}{(\gamma-1)!(\delta-\gamma+1)(\delta-\gamma+2)}.$

(iii) $\displaystyle\sum_{j=0}^{\gamma} j^2 \cdot \frac{(\delta-j)!}{(\gamma-j)!} = \frac{(\delta+1)!(\delta+\gamma+1)}{(\gamma-1)!(\delta-\gamma+1)(\delta-\gamma+2)(\delta-\gamma+3)}.$

Proof. The assertion (i) follows by

$$\sum_{j=0}^{\gamma} \frac{(\delta-j)!}{(\gamma-j)!} = (\delta-\gamma)! \sum_{j=0}^{\gamma} \binom{\delta-\gamma+j}{\delta-\gamma} = (\delta-\gamma)! \binom{\delta+1}{\delta-\gamma+1}.$$

The assertion (ii) or (iii) can be similarly shown. □

Proof of Theorem 3.11 (i). Assume that there exists a minimal cubature formula of degree $4k+1$ for $\mathcal{I}$ whose points are supported by $k+1$ concentric spheres without the origin. The proof consists of two parts.

First we consider the case $k = 2a+1,\ a \geq 0$. By contrasting the coefficients of u^k (or u^{k-1}, resp.) on both sides of equation in Proposition 3.13 (1) (ii), we obtain

$$0 = 2c_1 - (\gamma_1+\gamma_2) - (2a+2), \tag{3.19}$$

$$0 = -2(2a+1)^2 c_1 + (a+1)\gamma_1\gamma_2 + (6a^2+8a+3)(\gamma_1+\gamma_2) + (a+1)(10a^2+16a+7). \tag{3.20}$$

Moreover if $a \geq 1$, by contrasting the coefficients of u^{k-2}, we obtain

$$\begin{aligned} 0 = & -6a(2a+1)^2 c_1 + 3(3a+2)(3a^2+4a+2)(\gamma_1+\gamma_2) \\ & + 3(a+1)(3a+2)\gamma_1\gamma_2 + 2(a+1)(19a^3+46a^2+40a+12). \end{aligned} \tag{3.21}$$

By solving (3.19), (3.20) and (3.21), we have

$$\{\gamma_1, \gamma_2\} = \left\{ -\frac{4a^3+16a^2+19a+6 \pm \sqrt{36+84a-11a^2-136a^3-84a^4-16a^5-8a^6}}{6a(a+1)} \right\}. \tag{3.22}$$

On the other hand, it follows from (3.2) and Proposition 3.13 (1) (ii) that

$$\begin{aligned}
&\int_0^\infty \Big(L_{2a+2}(u)+c_1L_{2a+1}(u)\Big)u^{2a+2}e^{-u}\mathrm{d}u \\
&=\frac{((a+1)!)^2}{(2a+2)!}\int_0^\infty \Big(L_{a+1}^{2a+2}(u)+\gamma_1L_a^{2a+2}(u)\Big)\Big(L_{a+1}^{2a+2}(u)+\gamma_2L_a^{2a+2}(u)\Big)u^{2a+2}e^{-u}\mathrm{d}u \\
&=\frac{((a+1)!)^2}{(2a+2)!}\Big\{\frac{(3a+3)!}{(a+1)!}+\gamma_1\gamma_2\frac{(3a+2)!}{a!}\Big\}. \qquad (3.23)
\end{aligned}$$

By (3.2) and (3.4), the left hand side of (3.23) can be calculated as

$$\begin{aligned}
&\int_0^\infty \Big(L_{2a+2}(u)+c_1L_{2a+1}(u)\Big)\Big((2a+2)!\sum_{j=0}^{2a+2}(-1)^j\binom{2a+2}{2a+2-j}L_j(u)\Big)e^{-u}\mathrm{d}u \\
&=(2a+2)!\int_0^\infty (L_{2a+2}(u))^2e^{-u}\mathrm{d}u-(2a+2)!(2a+2)c_1\int_0^\infty (L_{2a+1}(u))^2e^{-u}\mathrm{d}u \\
&=(2a+2)!-(2a+2)(2a+2)!c_1. \qquad (3.24)
\end{aligned}$$

By combining (3.23) and (3.24), we obtain

$$0=2(a+1)((2a+2)!)^2c_1+(a+1)(a+1)!(3a+2)!(3+\gamma_1\gamma_2)-((2a+2)!)^2. \qquad (3.25)$$

Substituting γ_1,γ_2 of (3.22) into (3.25) leads to the following equation:

$$0=(a^3-2a^2-8a-3)((2a+2)!)^2+(a^4+5a^3+20a^2+22a+6)(a+1)!(3a+2)!.$$

However, it is easily checked that the right side of this equation is greater than 0, which is a contradiction. It remains to consider the case where $a=0$. In this case, by solving (3.19), (3.20), (3.25) for $\gamma_1\neq -2$, we have

$$c_1=\frac{\gamma_1^2+2\gamma_1-1}{2(2+\gamma_1)},\qquad \gamma_2=-\frac{5+2\gamma_1}{2+\gamma_1},$$

and there exists no solution for $\gamma_1=-2$. In this case we can explicitly compute the roots R_1,R_2 of $L_2(u)+c_1L_1(u)$ as follows:

$$R_1=\frac{1+\gamma_1}{2+\gamma_1},\qquad R_2=3+\gamma_1.$$

By Lemma 3.1 (i), $(\{R_1,R_2\},\{W_1,W_2\})$ forms a quadrature formula of degree 2 with respect to e^{-u}. By using this formula, we see that

$$W_1=\frac{(\gamma_1+2)^2}{1+(\gamma_1+2)^2},\qquad W_2=\frac{1}{1+(\gamma_1+2)^2}.$$

This contradicts (3.10), since $\sum_{j=1}^{2}(-1)^j(W_jR_j^2) = 4(\gamma_1+2)/\{1+(\gamma_1+2)^2\} \neq 0$.

Next we consider the case $k = 2a, a \geq 1$. By contrasting the coefficients of u^k (or u^{k-1}, resp.) on both sides of equation in Proposition 3.13 (1) (ii), we obtain

$$0 = (2a+1)c_1 - (a+1)\gamma_1 - (2a^2+2a+1), \tag{3.26}$$

$$0 = 4a(2a+1)c_1 - 2(a+1)(3a+1)\gamma_1 - (a+1)\gamma_2 - (10a^3+16a^2+13a+3). \tag{3.27}$$

Moreover if $a \geq 2$, by contrasting the coefficients of u^{k-2}, we obtain

$$\begin{aligned} 0 = 3(2a-1)^2(2a+1)c_1 - 3(a+1)(3a-1)(3a+1)\gamma_1 - 3(a+1)(3a-1)\gamma_2 \\ - 38a^4 - 54a^3 - 38a^2 + 12a + 7. \end{aligned} \tag{3.28}$$

On the other hand, if $a \geq 1$, it follows from (3.2), (3.3) and Proposition 3.13 (1) (ii) that

$$\begin{aligned} 0 &= \int_0^\infty \Big(L_{2a+1}(u) + c_1 L_{2a}(u)\Big) u^{2a-1}e^{-u}\mathrm{d}u \\ &= \frac{a!(a+1)!}{(2a+1)!}\int_0^\infty L_a^{2a+1}(u)\Big(L_{a+1}^{2a+1}(u) + \gamma_1 L_a^{2a+1}(u) \\ &\qquad\qquad + \gamma_2 L_{a-1}^{2a+1}(u)\Big)u^{2a-1}e^{-u}\mathrm{d}u \\ &= \frac{a!(a+1)!}{(2a+1)!}\Bigg\{\sum_{j=0}^{a}(j+1)(j+2)\int_0^\infty (L_{a-j}^{2a-1}(u))^2\cdot u^{2a-1}e^{-u}\mathrm{d}u \\ &\qquad + \gamma_1\sum_{j=0}^{a}(j+1)^2\int_0^\infty (L_{a-j}^{2a-1}(u))^2\cdot u^{2a-1}e^{-u}\mathrm{d}u \\ &\qquad + \gamma_2\sum_{j=0}^{a-1}(j+1)(j+2)\int_0^\infty (L_{a-1-j}^{2a-1}(u))^2\cdot u^{2a-1}e^{-u}\mathrm{d}u\Bigg\} \\ &= \frac{a!(a+1)!}{(2a+1)!}\Bigg\{\sum_{j=0}^{a}(j+1)(j+2)\frac{(3a-1-j)!}{(a-j)!} \\ &\qquad + \gamma_1\sum_{j=0}^{a}(j+1)^2\frac{(3a-1-j)!}{(a-j)!} + \gamma_2\sum_{j=0}^{a-1}(j+1)(j+2)\frac{(3a-2-j)!}{(a-1-j)!}\Bigg\}. \end{aligned}$$

Thus by Lemma 3.14, we have

$$0 = (2a+1)\gamma_1 + a\gamma_2 + 3a + 2. \tag{3.29}$$

Solving (3.26), (3.27) and (3.29), we have

$$\gamma_1 = \frac{2(-1+3a^2+a^3)}{1-2a^2}, \quad \gamma_2 = \frac{1-2a-8a^2-4a^3}{1-2a^2}. \tag{3.30}$$

Substituting γ_1, γ_2 of (3.30) into (3.28) leads to the following equation:

$$0 = (a-1)(a+1)(2a^4 + 6a^2 + 6a + 1).$$

However it is easily checked that if $a \geq 2$, the right side of this equation is greater than 0, which is a contradiction. It remains to consider the case where $a = 1$. In this case, by solving (3.26), (3.27) and (3.29), we obtain $c_1 = -7/3$. By using Proposition 3.13 (1) (i), we can explicitly compute the roots R_1, R_2, R_3 of $L_3(u)+c_1L_2(u)$: $R_1 = 4, R_2 = -1+\sqrt{3}, R_3 = -1-\sqrt{3}$. This is a contradiction since $R_3 < 0$. □

Proof of Theorem 3.11 (ii). Assume that there exists a minimal cubature formula of degree $4k+3$ for $\mathcal{I}$ whose points are supported by $k+1$ concentric spheres. The proof consists of two parts.

First we consider the case $k = 2a+1, a \geq 0$. By contrasting the coefficients of u^k (or u^{k-1}, resp.) on both sides of equation in Proposition 3.13 (2) (ii), we obtain

$$0 = \gamma_1 + \gamma_2 + 2(a+2). \tag{3.31}$$

$$0 = \gamma_1\gamma_2 + 2(3a+2)(\gamma_1 + \gamma_2) + 2(7 + 14a + 5a^2). \tag{3.32}$$

By solving (3.31) and (3.32), we obtain

$$\{\gamma_1, \gamma_2\} = \{-2 - a \pm \sqrt{2 - a^2}\}.$$

Since γ_2 is a real number, it holds that $a \in \{0, 1\}$. When $a = 1$, we obtain $\gamma_1 = -3 \mp 1$ and $\gamma_2 = -3 \pm 1$. This is impossible since the coefficient of u on the left side of the equation in Proposition 3.13 (2) (ii) is -4, whereas the coefficient of u on the right side is -2. It remains to consider the case when $a = 0$. In this case by using Proposition 3.13 (i), we can explicitly compute the roots R_1, R_2 of $L_2(u)$: $R_1 = 2+\sqrt{2}, R_2 = 2-\sqrt{2}$. Thus by using Lemma 3.1 (i), we obtain $W_1 = (2-\sqrt{2})/4, W_2 = (2+\sqrt{2})/4$. Then it holds that $\sum_{j=1}^{2}(-1)^j(W_jR_j^3) = -4\sqrt{2} \neq 0$, which contradicts (3.15).

Next we consider the case where $k = 2a,\ a \geq 1$. As in the previous case, we contrast the coefficients of u^k (or u^{k-1}, resp.) on both sides of equation in Proposition 3.13 (2) (ii) and then solve the corresponding equations. Then we have

$$\gamma_1 = -2(a+1), \quad \gamma_2 = 2a(a+1).$$

It follows from (3.2), (3.3) and Proposition 3.13 (2) (ii) that

$$\begin{aligned}
0 &= \int_0^\infty L_{2a+1}(u)u^{2a}e^{-u}\mathrm{d}u \\
&= \frac{a!(a+1)!}{(2a+1)!}\int_0^\infty L_a^{2a+2}(u)\Big(L_{a+1}^{2a+2}(u)+\gamma_1 L_a^{2a+2}(u)+\gamma_2 L_{a-1}^{2a+2}(u)\Big)u^{2a}e^{-u}\mathrm{d}u \\
&= \frac{a!(a+1)!}{(2a+1)!}\Bigg\{\sum_{j=0}^{a}(j+1)(j+2)\int_0^\infty (L_{a-j}^{2a}(u))^2\cdot u^{2a}e^{-u}\mathrm{d}u \\
&\qquad +\sum_{j=0}^{a}(j+1)^2\int_0^\infty (L_{a-j}^{2a}(u))^2\cdot u^{2a}e^{-u}\mathrm{d}u \\
&\qquad +\sum_{j=0}^{a-1}(j+1)(j+2)\int_0^\infty (L_{a-1-j}^{2a}(u))^2\cdot u^{2a}e^{-u}\mathrm{d}u\Bigg\} \\
&= \frac{a!(a+1)!}{(2a+1)!}\Bigg\{\sum_{j=0}^{a}(j+1)(j+2)\cdot\frac{(3a-j)!}{(a-j)!} \\
&\qquad +\sum_{j=0}^{a}(j+1)^2\cdot\frac{(3a-j)!}{(a-j)!}+\sum_{j=0}^{a-1}(j+1)^2\cdot\frac{(3a-1-j)!}{(a-1-j)!}\Bigg\}.
\end{aligned}$$

Thus by using Lemma 3.14, we have

$$0 = 6(1+a)+(3+4a)\gamma_1+2a\gamma_2.$$

By substituting γ_1, γ_2 to this equation and noting that $a \geq 1$, we have $a = 2$. However in this case, the coefficient of u^2 on the left side of the equation in Proposition 3.13 (2) (ii) is 5, whereas the coefficient of u^2 on the right side is 21/5, which is a contradiction. □

The following result follows by Theorem 3.11 and the result of Verlinden and Cools [61] and Cools and Schmid [24].

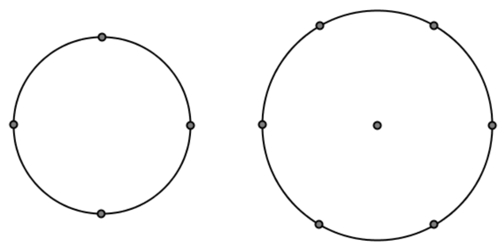


Figure 3.1: Minimal cubature formulae of degree 3 and 5 for $\mathcal{I}$

Corollary 3.15. *Let e be a positive integer. Then there exists a minimal cubature formula of degree $2e+1$ for the bivariate Gaussian integral $\mathcal{I}$ whose points are supported by $k+1$ concentric spheres only when $e = 1, 2$.*

Remark 3.16. The points of the minimal cubature formula of degree 3 are supported by a unit circle S^1 and each weight is equals to $1/4$. Moreover, the points of the minimal cubature formula of degree 5 are supported by the origin and a circle with radius $\sqrt{2}$. The weight of the origin is equal to $1/2$, and the others are equal to $1/12$ (see also Figure 3.1). In Appendix A.1, we also give a list of tight Gaussian $(2e+1)$-designs of $\mathbb{R}^n$, that is, minimal cubature formulae of degree $2e+1$ for multivariate Gaussian integrals and almost tight Gaussian $(2e+1)$-designs of $\mathbb{R}^n$.

Chapter 4

Minimal cubature formulae for spherically symmetric integrals

In this chapter, we study the existence and nonexistence of minimal cubature formulae for spherically symmetric integrals. In particular, we will restrict ourselves to a minimal cubature formula X of degree $4k+1$ for the spherically symmetric integral $\mathcal{I}$ given by (1.5) which includes the origin, that is, the number of points in the formula attains the equality $|X| = 2\dim(\mathcal{P}_e^*(\mathbb{R}^n)) - 1$. Since the integral $\mathcal{I}$ is also centrally symmetric, Theorem 2.6 implies that, if there exists a minimal cubature formula for the integral $\mathcal{I}$ of degree $4k+1$ which includes the origin, then it has the form

$$\mathcal{I}[f] = w(\mathbf{0})f(\mathbf{0}) + \sum_{\xi \in X' \setminus \{\mathbf{0}\}} w(\xi)\{f(\xi) + f(-\xi)\}, \tag{4.1}$$

where $X = X' \cup (-X')$ with $X' \cap (-X') = \{\mathbf{0}\}$.

In the two dimensional case, a large number of minimal cubature formulae have been found (see [22, 56, 63] and Section 3.2). However, the minimal cubature formulae on higher dimensional spaces are not the case. Taylor [58] proved that there exists no minimal cubature formula of degree $2e$ on S^{n-1} for $n \geq 3$ and $e \geq 3$. Moreover, Noskov and Schmid [50] showed that there exists no minimal cubature formula of degree 5, for the integral over the n-dimensional unit ball B^n with respect to $W \equiv 1$ which includes the origin, except for some specific values of n. On the other hand, they also found a minimal cubature formula of degree 5 for the integral over B^7 or B^{23} with respect to $W \equiv 1$ which includes the origin.

The aim of this chapter is to study the existence and nonexistence of minimal cubature formulae of degree $4k+1$ for spherically symmetric integrals which includes the origin. For this purpose, we combine the theories on Euclidean designs with that on reproducing kernels. In Section 4.1, we briefly review necessary notions and results in these two theories. In Section 4.2, we specify the general structure of minimal cubature formulae of degree $4k + 1$ for spherically symmetric integrals. It is shown that, if there exists a minimal cubature formula of degree $4k+1$ which includes the origin, then the points are distributed into $k+1$ concentric spheres, and the weights in the formula are constant on each concentric sphere.

We particularly focus on the cases of degrees 5 and 9. In Section 4.3, it is shown that the existence of a minimal cubature formula of degree 5 which includes the origin, a minimal cubature formula of degree 4 which includes the origin, a tight spherical 5-design, and a tight spherical 4-design are all equivalent. Since it is known (see [40, 54]) that there exists a tight spherical 5-design on S^6 or S^{22}, we obtain a 7- or 23-dimensional minimal cubature formula of degree 5 for any spherically symmetric integral, respectively. In Section 4.4, it is shown that there exists no minimal cubature formula of degree 9 which includes the origin for spherically symmetric integral such as the Gaussian weight on $\mathbb{R}^n$, the radial exponential weight on $\mathbb{R}^n$ and the ultraspherical weight on B^n.

4.1 Reproducing kernels for cubature formulae and Euclidean designs

Letting $\Omega = \{\boldsymbol{x} \in \mathbb{R}^n \mid q_1 \leq \|\boldsymbol{x}\| < q_2\}$, $0 \leq q_1 < q_2 \leq \infty$ and W be a radial weight function of $\|\boldsymbol{x}\|$, we consider the spherically symmetric integral $\mathcal{I}$ given by (1.5). To study a minimal cubature formula of odd degree for $\mathcal{I}$, we use the modified reproducing kernels for orthonormal polynomials with respect to the inner product

$$\langle f, g\rangle_{\mathcal{I}} = \frac{1}{V}\int_{\Omega} f(\boldsymbol{x})g(\boldsymbol{x})W(\|\boldsymbol{x}\|)\mathrm{d}\boldsymbol{x}.$$

We denote by $\Pi_l(\Omega)$ the subspace of $\mathcal{P}_l(\Omega)$ consisting of all polynomials of degree l and $r_l^n = \dim(\Pi_l(\Omega))$. Let $\{P_{l,i} \mid 1 \leq i \leq r_l^d\}$ be an orthonormal basis of $\Pi_l(\Omega)$ with respect to the inner product $\langle \cdot, \cdot\rangle_{\mathcal{I}}$. Setting $\mathbb{P}_l = (P_{l,1}, \ldots, P_{l,r_l^n})$, we define the kernel function $\tilde{K}_t(\boldsymbol{x}, \boldsymbol{y})$ by

$$\tilde{K}_t(\boldsymbol{x}, \boldsymbol{y}) = \sum_{\substack{0\leq l\leq t \\ l\equiv t \pmod 2}} \mathbb{P}_l(\boldsymbol{x})\mathbb{P}_l^{\mathrm{T}}(\boldsymbol{y}), \quad \boldsymbol{x}, \boldsymbol{y} \in \Omega,$$

and call it *the t-th modified reproducing kernel of $\mathcal{P}_t^*(\Omega)$ for $\mathcal{I}$*. Note that

$$f(\boldsymbol{y}) = \frac{1}{V}\int_{\Omega} f(\boldsymbol{x})\tilde{K}_t(\boldsymbol{x}, \boldsymbol{y})W(\|\boldsymbol{x}\|)\mathrm{d}\boldsymbol{x}$$

for every $f \in \mathcal{P}_t^*(\mathbb{R}^n)$. The *$t$-th reproducing kernel of $\mathcal{P}_t(\Omega)$ for $\mathcal{I}$* takes the form $K_t(\boldsymbol{x}, \boldsymbol{y}) = \tilde{K}_t(\boldsymbol{x}, \boldsymbol{y}) + \tilde{K}_{t-1}(\boldsymbol{x}, \boldsymbol{y})$. Since $\tilde{K}_t(\boldsymbol{x}, \boldsymbol{y})$ is a polynomial of $\boldsymbol{x}$ and $\boldsymbol{y}$, we may regard that it is defined on $\mathbb{R}^n \times \mathbb{R}^n$ (see, e.g., [29]).

The following theorems due to Mysovskikh [44, 47] show close relationships between minimal formulae of odd degree for $\mathcal{I}$ and the modified reproducing kernel for $\mathcal{I}$.

Theorem 4.1. *Assume that there exists a cubature formula (1.2) of degree $4k+1$ for the spherically symmetric integral $\mathcal{I}$ which includes the origin. This formula is minimal if and only if*

(i) $\tilde{K}_{2k}(\xi, \xi') = 0, \qquad \xi, \xi' \in X,\ \xi \neq \pm\xi'$, *and*

(ii) $w(\mathbf{0}) = \tilde{K}_{2k}(\mathbf{0}, \mathbf{0})^{-1}, \quad w(\xi) = \tilde{K}_{2k}(\xi, \xi)^{-1}/2, \quad \xi \in X \setminus \{\mathbf{0}\}$.

Theorem 4.2. *Assume that there exists a cubature formula (1.2) of degree $2e$ for the spherically symmetric integral $\mathcal{I}$. This formula is minimal if and only if*

(i) $K_e(\xi, \xi') = 0, \quad \xi, \xi' \in X,\ \xi \neq \xi'$, *and*

(ii) $w(\xi) = K_e(\xi, \xi)^{-1}, \quad \xi \in X$.

We should note here that Mysovskikh [44, 47] showed Theorem 4.1 for centrally symmetric integrals and Theorem 4.2 for the integrals given by (1.1).

In order to calculate the t-th modified reproducing kernel for the spherically symmetric integral $\mathcal{I}$, we choose an orthonormal basis of $\mathcal{P}_t^*(\Omega)$. For a nonnegative integer l, let $\{\phi_{l,i}(\boldsymbol{x}) \mid 1 \leq i \leq \dim \mathrm{Harm}_l(\mathbb{R}^n)\}$ be an orthonormal basis of $\mathrm{Harm}_l(\mathbb{R}^n)$ with respect to the inner product

$$(f,\ g) = \frac{1}{|S^{n-1}|} \int_{S^{n-1}} f(\boldsymbol{x}) g(\boldsymbol{x}) \mathrm{d}\sigma(\boldsymbol{x}).$$

For a nonnegative integer l, let $\{g_{l,j}(\|\boldsymbol{x}\|^2) \mid 0 \leq j \leq [(t-l)/2]\}$ be the set of polynomials on Ω of total degree $2j$ such that

$$\frac{1}{\int_{q_1}^{q_2} u^{n-1} W(u) \mathrm{d}u} \int_p^q g_{l,j}(u^2) g_{l,j'}(u^2) u^{2l+n-1} W(u) \mathrm{d}u = \delta_{j,j'}. \tag{4.2}$$

Noting that $1, \|\boldsymbol{x}\|^2, \cdots, \|\boldsymbol{x}\|^{2[(t-l)/2]}$ are linearly independent in $\mathcal{P}_{2[(t-l)/2]}(\Omega)$ and by applying the Gram-Schmidt method, we can construct such polynomials. Hence the set

$$\left\{ g_{l,j}(\|\boldsymbol{x}\|^2) \phi_{l,i}(\boldsymbol{x}) \mid 0 \leq l + 2j \leq t,\ l \equiv t \pmod 2,\ 1 \leq i \leq \dim \mathrm{Harm}_l(\mathbb{R}^n) \right\}$$

gives an orthonormal basis of $\mathcal{P}_t^*(\Omega)$ with respect to the inner product $\langle \cdot, \cdot \rangle_{\mathcal{I}}$ (see [8, 27]).

Lemma 4.3. *The t-th modified reproducing kernel for the spherically symmetric integral $\mathcal{I}$ is given as follows:*

$$\tilde{K}_t(\boldsymbol{x},\boldsymbol{y}) = \begin{cases} \dfrac{1}{n-2} \displaystyle\sum_{\substack{0\le l+2j\le t,\\ l\equiv t \pmod 2}} (n+2l-2)(\|\boldsymbol{x}\|\|\boldsymbol{y}\|)^l \\ \quad \times g_{l,j}(\|\boldsymbol{x}\|^2) g_{l,j}(\|\boldsymbol{y}\|^2) C_l^{(n-2)/2}\Big(\langle \dfrac{\boldsymbol{x}}{\|\boldsymbol{x}\|}, \dfrac{\boldsymbol{y}}{\|\boldsymbol{y}\|}\rangle\Big) & \text{if } \boldsymbol{x},\boldsymbol{y} \in \mathbb{R}^n \setminus \{\boldsymbol{0}\}, \\ \sum_{0\le j\le \frac{t}{2}} g_{0,j}(\|\boldsymbol{x}\|^2) g_{0,j}(\|\boldsymbol{y}\|^2) & \text{if } \boldsymbol{x}=\boldsymbol{0} \text{ and } t\equiv 0 \pmod 2, \\ 0 & \text{if } \boldsymbol{x}=\boldsymbol{0} \text{ and } t\equiv 1 \pmod 2. \end{cases}$$

Proof. By the addition formula for the surface harmonics $\{\phi_{l,i}\}_i$ (see [30, pp.242-243]), it follows that for $\boldsymbol{x},\boldsymbol{y} \in \mathbb{R}^n \setminus \{\boldsymbol{0}\}$,

$$\begin{aligned} &\tilde{K}_t(\boldsymbol{x},\boldsymbol{y}) \\ &= \sum_{\substack{0\le l+2j\le t,\\ l\equiv t \pmod 2}} \sum_{i=1}^{\dim \mathrm{Harm}_l(\mathbb{R}^n)} g_{l,j}(\|\boldsymbol{x}\|^2)\phi_{l,i}(\boldsymbol{x}) g_{l,j}(\|\boldsymbol{y}\|^2)\phi_{l,i}(\boldsymbol{y}) \\ &= \sum_{\substack{0\le l+2j\le t,\\ l\equiv t \pmod 2}} (\|\boldsymbol{x}\|\|\boldsymbol{y}\|)^l g_{l,j}(\|\boldsymbol{x}\|^2) g_{l,j}(\|\boldsymbol{y}\|^2) \sum_{i=1}^{\dim \mathrm{Harm}_l(\mathbb{R}^n)} \phi_{l,i}\Big(\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}\Big)\phi_{l,i}\Big(\frac{\boldsymbol{y}}{\|\boldsymbol{y}\|}\Big) \\ &= \sum_{\substack{0\le l+2j\le t,\\ l\equiv t \pmod 2}} (\|\boldsymbol{x}\|\|\boldsymbol{y}\|)^l g_{l,j}(\|\boldsymbol{x}\|^2) g_{l,j}(\|\boldsymbol{y}\|^2) \cdot \frac{n+2l-2}{n-2} C_l^{(n-2)/2}\Big(\langle \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}, \frac{\boldsymbol{y}}{\|\boldsymbol{y}\|}\rangle\Big), \end{aligned}$$

which completes the proof. □

Let (X,w) be a Euclidean t-design supported by a union S of p concentric spheres, and we denote $\mathcal{I}_{\mathrm{E}}$ the associated integral with the Euclidean design (X,w), that is,

$$\mathcal{I}_{\mathrm{E}}[f] = \sum_{i=1}^{p} \frac{W_i}{|S_i|}\int_{S_i} f(\boldsymbol{x})\mathrm{d}\sigma_i(\boldsymbol{x}) = \sum_{\xi\in X} w(\xi)f(\xi). \tag{4.3}$$

Similar to Lemma 4.3, we can also calculate the t-th modified reproducing kernel of $\mathcal{P}_t^*(S)$ for $\mathcal{I}_{\mathrm{E}}$, as a polynomial of $\|\boldsymbol{x}\|^2$, $\|\boldsymbol{y}\|^2$ and $\langle \boldsymbol{x}/\|\boldsymbol{x}\|, \boldsymbol{y}/\|\boldsymbol{y}\|\rangle$, under the restriction that t is an even integer and $\boldsymbol{0}\notin S$. This restriction is required for the argument in Section 4.4.

For a nonnegative integer l, let $\{g_{l,j}(\|\boldsymbol{x}\|^2) \mid 0\le j\le \min\{p-1, [(t-l)/2]\}\}$ be the set of polynomials on X of total degree $2j$ such that

$$\sum_{\boldsymbol{x}\in X} w(\boldsymbol{x})\|\boldsymbol{x}\|^{2l} g_{l,j}(\|\boldsymbol{x}\|^2) g_{l,j'}(\|\boldsymbol{x}\|^2) = \delta_{j,j'}. \tag{4.4}$$

Since the functions $1, \|\boldsymbol{x}\|^2, \cdots, \|\boldsymbol{x}\|^{2\min\{p-1,\, [(t-l)/2]\}}$ are linearly independent in $\mathcal{P}_{2\min\{p-1,\, [(t-l)/2]\}}(X)$, we can construct the polynomials $\{g_{l,j}\}$ by applying the Gram-Schmidt method. Hence the set

$$\Big\{ g_{l,j}(\|\boldsymbol{x}\|^2)\phi_{l,i}(\boldsymbol{x}) \mid 0 \le l \le t,\ l \equiv t \pmod 2, \\ 0 \le j \le \min\Big\{p-1, \Big[\frac{t-l}{2}\Big]\Big\},\ 1 \le i \le \dim \mathrm{Harm}_l(\mathbb{R}^n) \Big\}$$

gives an orthonormal basis of $\mathcal{P}_t^*(S)$ with respect to the inner product

$$\langle f, g \rangle_{\mathcal{I}_{\mathrm{E}}} = \sum_{i=1}^{p} \frac{W_i}{|S_i|} \int_{S_i} f(\boldsymbol{x}) g(\boldsymbol{x}) \mathrm{d}\sigma_i(\boldsymbol{x}).$$

By the same argument as that used in the proof of Lemma 4.3, we obtain the following modified reproducing kernel for $\mathcal{I}_{\mathrm{E}}$. By using this reproducing kernel, the usual inner products among the points in a minimal cubature formula of degree 9 will be determined by the reproducing kernel in Lemma 4.18.

Lemma 4.4. *Under the same set-up as those in the above paragraph, the $2k$-th modified reproducing kernel of $\mathcal{P}_{2k}^*(S)$ for $\mathcal{I}_{\mathrm{E}}$ is given as follows:*

$$\tilde{K}_{2k}(\boldsymbol{x}, \boldsymbol{y}) \\ = \begin{cases} \dfrac{1}{n-2} \displaystyle\sum_{\substack{0 \le l \le 2k,\ l \equiv 0 \pmod 2, \\ 0 \le j \le \min\{p-1,\, [(2k-l)/2]\}}} (n+2l-2)(\|\boldsymbol{x}\|\|\boldsymbol{y}\|)^l \\ \quad \times g_{l,j}(\|\boldsymbol{x}\|^2) g_{l,j}(\|\boldsymbol{y}\|^2) C_l^{(n-2)/2}\Big(\langle \dfrac{\boldsymbol{x}}{\|\boldsymbol{x}\|}, \dfrac{\boldsymbol{y}}{\|\boldsymbol{y}\|} \rangle\Big) & \text{if } \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n \setminus \{\boldsymbol{0}\}, \\ \sum_{0 \le j \le \min\{p-1, k\}} g_{0,j}(\|\boldsymbol{x}\|^2) g_{0,j}(\|\boldsymbol{y}\|^2) & \text{if } \boldsymbol{x} = \boldsymbol{0}. \end{cases}$$

Remark 4.5. Let us consider the case $p = 1$ in Lemma 4.4. Then $\tilde{K}_{2k}(\boldsymbol{x}, \boldsymbol{y})$ is invariant under the n-dimensional orthogonal group $O(n)$. From Theorem 4.1 (ii) and Lemma 4.4, we see that, if there exists a minimal cubature formula of degree $4k+1$ for the unnormalized surface measure σ on S^{n-1}, then the weights in the formula are a constant on the sphere. Hence, by using a well known result of [17], we can prove that there exists no minimal cubature formula of degree $4k+1$ for σ on S^{n-1} for $k \ge 2$ and $n \ge 4$. This is an analogue of Taylor's theorem [58] which mentions nonexistence of minimal cubature formulae of degree $2k$ for σ on S^{n-1} for $k \ge 3$ and $n \ge 3$.

4.2 General structure of a minimal formula of degree $4k+1$

This section is concerned with the general structure of minimal cubature formulae of degree $4k+1$ for spherically symmetric integrals which includes the origin. We first prove the following lemma needed later.

Lemma 4.6. *For a nonnegative integer l and a positive integer m, let*

$$A_l^{(m)} = \begin{pmatrix} \eta_l & \eta_{l+2} & \cdots & \eta_{l+2(m-1)} \\ \eta_{l+2} & \eta_{l+4} & \cdots & \eta_{l+2m} \\ \vdots & \vdots & \ddots & \vdots \\ \eta_{l+2(m-1)} & \eta_{l+2m} & \cdots & \eta_{l+4(m-1)} \end{pmatrix},$$

where $\eta_l = \int_p^q u^{n+l-1} W(u) \mathrm{d}u$. Then it holds that $\det A_l^{(m)} > 0$.

Proof. $A_l^{(m)}$ is the Gram matrix of the inner product $\langle f, g \rangle_l = \int_p^q f(u)g(u)u^{l+n-1}W(u)\mathrm{d}u$. Then the result follows from the assumption that $W(u)$ is positive. □

Verlinden and Cools [61] showed that, if there exists the two dimensional minimal cubature formula of degree $4k+1$ which includes the origin, then the points are distributed into $k+1$ concentric spheres including the origin. The following proposition is a generalization of this result for higher dimensional cases.

Proposition 4.7. *If there exists a minimal cubature formula of degree $4k+1$ for the spherically symmetric integral $\mathcal{I}$ which includes the origin, then the points in the formula are supported by $k+1$ concentric spheres including the origin.*

Proof. Let X be the points in a minimal cubature formula of degree $4k+1$ supported by a union of p concentric spheres S_i with radius r_i. Then for each l with $0 \le l \le 2k$, we have

$$\frac{\eta_{2l}}{\eta_0} = \sum_{i=1}^{p} W_i r_i^{2l}. \tag{4.5}$$

Assume $p \le k$. By using (4.5) and noting that $\mathbf{0} \in X$, we have

$$\det A_2^{(k)} = \eta_0^k \sum_{1 \le i_1, i_2, \cdots, i_k \le p} \begin{vmatrix} W_{i_1} r_{i_1}^2 & W_{i_2} r_{i_2}^4 & \cdots & W_{i_k} r_{i_k}^{2k} \\ W_{i_1} r_{i_1}^4 & W_{i_2} r_{i_2}^6 & \cdots & W_{i_k} r_{i_k}^{2k+2} \\ \vdots & \vdots & & \vdots \\ W_{i_1} r_{i_1}^{2k} & W_{i_2} r_{i_2}^{2k+2} & \cdots & W_{i_k} r_{i_k}^{4k-2} \end{vmatrix}$$

$$= \eta_0^k \sum_{1 \le i_1, i_2, \cdots, i_k \le p-1} \begin{vmatrix} W_{i_1} r_{i_1}^2 & W_{i_2} r_{i_2}^4 & \cdots & W_{i_k} r_{i_k}^{2k} \\ W_{i_1} r_{i_1}^4 & W_{i_2} r_{i_2}^6 & \cdots & W_{i_k} r_{i_k}^{2k+2} \\ \vdots & \vdots & & \vdots \\ W_{i_1} r_{i_1}^{2k} & W_{i_2} r_{i_2}^{2k+2} & \cdots & W_{i_k} r_{i_k}^{4k-2} \end{vmatrix}.$$

Here we observe that $\det A_2^{(k)} = 0$. In fact, by the assumption, there exist at least two same numbers in each k-tuple $T = (i_1, \ldots, i_k) \in \{1, \ldots, p-1\}^k$. Without loss of generality, we may let $i_j = i_{j'}$. Then the j-th column of the determinant corresponding to T is a multiple of the j'-th column. This implies that every determinant in the last summation takes 0, and so does $\det A_2^{(k)}$. However, this is a contradiction by Lemma 4.6 and hence $p \ge k+1$. Next we show that $p \le k+1$. Since $\tilde{K}_{2k}(\mathbf{0}, \boldsymbol{y})$ is a polynomial of $\|\boldsymbol{y}\|^2$ of degree k by Lemma 4.3, Theorem 4.1 (i) implies that $X \setminus \{\mathbf{0}\}$ is supported by at most k concentric spheres. □

We also obtain the above lemma by using Theorem 2.18 and 2.29.

The following two theorems imply the structure of a minimal cubature formula of degree $4k+1$ for the spherically symmetric integral $\mathcal{I}$ which includes the origin.

Theorem 4.8. *Assume that there exists a minimal cubature formula (X, w) for $\mathcal{I}$ of degree $4k+1$ which includes the origin. Then the points X_i on each concentric sphere without the origin are similar to a spherical $(2k+3)$-design, that is, each $\tilde{X}_i$ is a spherical $(2k+3)$-design.*

Proof. By using Theorem 2.18 and Proposition 4.7, (X, w) is a tight Euclidean $(4k+1)$-design supported by $k+1$ concentric spheres which includes the origin. Thus Theorem 2.33 implies the desired result. □

Theorem 4.9. *Assume that there exists a minimal cubature formula (X, w) of degree $4k+1$ for the spherically symmetric integral $\mathcal{I}$ which includes the origin. Then it has the form*

$$\mathcal{I}[f] = w_{k+1} f(\mathbf{0}) + \sum_{i=1}^{k} w_i \sum_{\xi \in X_i'} \{f(\xi) + f(-\xi)\}, \tag{4.6}$$

where $w_i = w(\xi)$ for $\xi \in X_i$ and $X_i = X_i' \cup (-X_i')$ with $X_i' \cap (-X_i') = \emptyset$ for $i = 1, \ldots, k$.

Proof. By Lemma 4.3, the $2k$-th modified reproducing kernel $\tilde{K}_{2k}(\boldsymbol{x}, \boldsymbol{y})$ for $\mathcal{I}$ can be regarded as a polynomial of $\|\boldsymbol{x}\|^2, \|\boldsymbol{y}\|^2, \langle \boldsymbol{x}/\|\boldsymbol{x}\|, \boldsymbol{y}/\|\boldsymbol{y}\| \rangle$ and thus $\tilde{K}_{2k}(\boldsymbol{x}, \boldsymbol{y})$ is invariant under $O(n)$. Therefore by Theorem 4.1 (ii), the weights in a cubature formula are constant on each concentric sphere. Since X is antipodal by Theorem 2.6 and supported by $k+1$ concentric spheres including the origin by Proposition 4.7, we obtain the desired result. □

By using the above theorem, we can reprove the result in [50].

Corollary 4.10. *Assume that there exists a minimal cubature formula of degree* 5 *for the integral over* B_n *with respect to* $W \equiv 1$ *which includes the origin Then it has the form*

$$\mathcal{I}[f] = w_2 f(\mathbf{0}) + w_1 \sum_{\xi \in X_1'} \{f(\xi) + f(-\xi)\}.$$

4.3 Minimal cubature formulae of degree 4 and 5

The main purpose of this section is to prove the following theorem.

Theorem 4.11. *The following statements are equivalent:*

(i) *There exists an* n*-dimensional minimal cubature formula of degree* 5 *for any spherically symmetric integral which includes the origin.*

(ii) *There exists an* $(n-1)$*-dimensional minimal cubature formula of degree* 4 *for any spherically symmetric integral which includes the origin.*

(iii) *There exists a spherical tight* 5*-design on* S^{n-1}*.*

(iv) *There exists a spherical tight* 4*-design on* S^{n-2}*.*

We first prepare two lemmas.

Lemma 4.12. *Let* $\tilde{K}_2(\boldsymbol{x}, \boldsymbol{y})$ *be the* 2*-nd modified reproducing kernel for the spherically symmetric integral* $\mathcal{I}$*. Then it holds that*

$$\tilde{K}_2(\boldsymbol{x}, \boldsymbol{y}) = a_1 + a_2 \langle \boldsymbol{x}, \boldsymbol{y} \rangle^2 + a_3(\|\boldsymbol{x}\|^2 + \|\boldsymbol{y}\|^2) + a_4 \|\boldsymbol{x}\|^2 \|\boldsymbol{y}\|^2, \tag{4.7}$$

$$K_2(\boldsymbol{x}, \boldsymbol{y}) = \tilde{K}_2(\boldsymbol{x}, \boldsymbol{y}) + \frac{n\eta_0}{\eta_2} \langle \boldsymbol{x}, \boldsymbol{y} \rangle, \tag{4.8}$$

where

$$a_1 = \frac{\eta_0 \eta_4}{\det A_0^{(2)}},\ a_2 = \frac{n(n+2)\eta_0}{2\eta_4},\ a_3 = -\frac{\eta_0\eta_2}{\det A_0^{(2)}},\ a_4 = \frac{\eta_0\{(n+2)\eta_2^2 - n\eta_0\eta_4\}}{2\eta_4 \det A_0^{(2)}}.$$

Proof. By Lemma 4.6, it holds that $\det A_0^{(2)} > 0$. Let

$$g_{0,0}(\|\boldsymbol{x}\|^2) \equiv 1, \quad g_{0,1}(\|\boldsymbol{x}\|^2) = \frac{\eta_0}{\sqrt{\det A_0^{(2)}}} \Big(\|\boldsymbol{x}\|^2 - \frac{\eta_2}{\eta_0} \Big),$$

$$g_{1,0}(\|\boldsymbol{x}\|^2) \equiv \sqrt{\frac{\eta_0}{\eta_2}}, \quad g_{2,0}(\|\boldsymbol{x}\|^2) \equiv \sqrt{\frac{\eta_0}{\eta_4}}.$$

Then it is easily checked that these polynomials satisfy (4.2). Hence the results follow from (2.5) and Lemma 4.3. □

Proposition 4.13. (1) *Assume that there exists a minimal cubature formula* (X, w) *of degree* 5 *for the spherically symmetric integral* $\mathcal{I}$ *which includes the origin. Then it holds that*

(i) $w(\mathbf{0}) = \dfrac{\eta_0\eta_4 - \eta_2^2}{\eta_0\eta_4}$, $\quad w(\xi) = \dfrac{\eta_2^2}{n(n+1)\eta_0\eta_4}$, $\quad \xi \in X \setminus \{\mathbf{0}\}$, *and*

(ii) $\|\xi\|^2 = \dfrac{\eta_4}{\eta_2}$, $\quad \xi \in X \setminus \{\mathbf{0}\}$.

(2) *Assume that there exists a minimal cubature formula* (X, w) *of degree* 4 *for the spherically symmetric integral* $\mathcal{I}$ *which includes the origin. Then it holds that*

(i) $w(\mathbf{0}) = \dfrac{\eta_0\eta_4 - \eta_2^2}{\eta_0\eta_4}$, $\quad w(\xi) = \dfrac{2\eta_2^2}{n(n+3)\eta_0\eta_4}$, $\quad \xi \in X \setminus \{\mathbf{0}\}$, *and*

(ii) $\|\xi\|^2 = \dfrac{\eta_4}{\eta_2}$, $\quad \xi \in X \setminus \{\mathbf{0}\}$.

Proof. (1) Since the 2-nd modified reproducing kernel $\tilde{K}_2(\boldsymbol{x}, \boldsymbol{y})$ for $\mathcal{I}$ is given by (4.7) in Lemma 4.12, by using Theorem 4.1, we obtain the desired result.
(2) Since the 2-nd reproducing kernel $K_2(\boldsymbol{x}, \boldsymbol{y})$ for $\mathcal{I}$ is given by (4.8) in Lemma 4.12, by using Theorem 4.2, we obtain the desired result. □

Proof of Theorem 4.11. ((iii) ⇔ (iv)). This is a well known fact; see [43, p.620].
((i) ⇔ (iii)). Assume that $X = \{\xi_1, \dots, \xi_{n(n+1)}\} \subset S^{n-1}$ forms a spherical tight 5-design. In view of Proposition 4.13 (1), let $w_1 = w(\xi_i)$ for any nonzero point ξ_i and $w_2 = w(0)$. Let $\xi_l = (\xi_{1,l}, \dots, \xi_{n,l})$. Then it follows that

$$\mathcal{I}[x_i^2] = \frac{\eta_2}{d\eta_0} = \mathcal{I}_{\mathrm{E}}[x_i^2]\frac{\eta_2}{\eta_0} = \left(\frac{1}{n(n+1)}\sum_{l=1}^{n(n+1)} \xi_{i,l}^2\right)\frac{\eta_2}{\eta_0} = w_1 \sum_{l=1}^{n(n+1)} \left(\sqrt{\frac{\eta_4}{\eta_2}}\xi_{i,l}\right)^2.$$

A similar computation can be done for the other nonvanishing moments $1, x_i^4, x_i^2 x_j^2$ for $\mathcal{I}$. We thus conclude $\{\mathbf{0}\} \cup \{\sqrt{\frac{\eta_4}{\eta_2}}\xi \mid \xi \in X\}$ forms the points in a minimal formula of degree 5 for $\mathcal{I}$. The converse direction also holds.
((ii) ⇔ (iv)). In the above argument, replace the use of Proposition 4.13 (1) by that of Proposition 4.13 (2). □

The above theorem imply the generalization of the result of [50].

Corollary 4.14. *Let* X *be a finite set in* S^{n-1}. *Then* X *is a spherical tight* 5*-design on* S^{n-1} *if and only if* $\{\mathbf{0}\} \cup \{\sqrt{\frac{n+2}{n+4}}\xi \mid \xi \in X\}$ *forms the points in a minimal cubature formula for the constant weight* $W \equiv 1$ *on* B^n *of degree* 5 *which includes the origin.*

By use of results of Bannai and Damerell [17] and Bannai, Munemasa and Venkov [19], the bound in Theorem 2.8 (ii) for a spherical 5-design X on S^{n-1} is improved as follows:

$$|X| \geq \begin{cases} 2\dim(\mathcal{P}_2^*(S^{n-1})) & \text{if } n = 1, 2, 3,\ n = (2l+1)^2 - 2,\ l \geq 5, \\ 2\dim(\mathcal{P}_2(S^{n-1})) + 1 & \text{otherwise.} \end{cases} \tag{4.9}$$

Corollary 4.15. (i) *There exists an n-dimensional minimal cubature formula of degree* 5 *for any spherically symmetric integral which includes the origin for $n \in \{1, 2, 3, 7, 23\}$.*
(ii) *There exists an n-dimensional minimal cubature formula of degree* 4 *for any spherically symmetric integral which includes the origin for $n \in \{1, 2, 6, 22\}$.*
(iii) *There exists no n-dimensional minimal cubature formula for any spherically symmetric integral of degree* 5 *which includes the origin for any positive integer n with possible exceptions that $n = 1,\ 2,\ 3,\ (2m+1)^2 - 2,\ m \geq 1$.*

Proof. (i) Since there exists tight spherical 5-design on S^{n-1} for $n \in \{1, 2, 3\}$ and $n = 7$ (see [54]) and $n = 23$ (see [40]), by using Theorem 4.11, we obtain the desired result. (ii) By using (i) and Theorem 4.11, we obtain the desired result. (iii) Since there exists no tight spherical 5-design on S^{n-1} for any positive integer n with possible exceptions that $n = 1,\ 2,\ 3,\ (2m+1)^2 - 2,\ m \geq 1$, by applying Theorem 4.11, we obtain the desired result. □

We exhibit the 7-dimensional minimal cubature formula of degree 5 for any spherically symmetric integral explicitly in Table 4.1, each of which corresponds to the unique combinatorial 2-(7, 3, 1) design.

$$\begin{array}{lllllll} (0, & 0, & 0, & 0, & 0, & 0, & 0), \\ (\pm a, & \pm a, & 0, & \pm a, & 0, & 0, & 0), \\ (0, & \pm a, & \pm a, & 0, & \pm a, & 0, & 0), \\ (0, & 0, & \pm a, & \pm a, & 0, & \pm a, & 0), \\ (0, & 0, & 0, & \pm a, & \pm a, & 0, & \pm a), \\ (\pm a, & 0, & 0, & 0, & \pm a, & \pm a, & 0), \\ (0, & \pm a, & 0, & 0, & 0, & \pm a, & \pm a), \\ (\pm a, & 0, & \pm a, & 0, & 0, & 0, & \pm a), \end{array} \quad \text{where } a = \sqrt{\frac{\eta_4}{3\eta_2}}.$$

Table 4.1: 7-dimensional minimal cubature formula of degree 5

More generally, there is a close relationship between combinatorial t-designs and orthogonal arrays, and cubature formulae whose points are invariant under reflection group

([39, 62]). In Table 4.2, classical spherically symmetric integrals for which a minimal cubature formula of degree 5 which includes the origin has been already found are listed.

(Ω, W)	$n = 7$	$n = 23$
$(S^{n-1}, 1)$	[54, 56]	[40]
$([q_1, q_2) \times S^{n-1}, 1)$	Corollary 4.15	Corollary 4.15
$(B^n, 1)$	[53, 56]	[50]
$(\mathbb{R}^n, e^{-\|\boldsymbol{x}\|^2})$	[56, 62]	Corollary 4.15
$(\mathbb{R}^n, e^{-\|\boldsymbol{x}\|})$	Corollary 4.15	Corollary 4.15

Table 4.2: Minimal cubature formulae of degree 5

In the two dimensional case, Cools and Schmid [24] presented criteria of the existence of a minimal cubature formula of degree $4k+1$ for particular types of circularly symmetric integrals which includes the origin and also showed that for such integrals there exists no minimal cubature formula of degree $4k+1$ which includes the origin for any integer $k \geq 2$ ([24]). Here we have studied the existence problem of minimal cubature formulae in higher dimensional cases. Corollary 4.15 gives an answer to this problem in the degree five. In the next section we will discuss the same problem in the case of degree nine.

4.4 Minimal cubature formulae of degree 9

The purpose of this section is prove the following theorem.

Theorem 4.16. (i) *For $n \geq 2$ and $\alpha > 0$, there exists no minimal cubature formula of degree 9 for the integral over $\mathbb{R}^n$ with respect to $W(\|\boldsymbol{x}\|) = e^{-\alpha\|\boldsymbol{x}\|^2}$ which includes origin.*
(ii) *For $n \geq 2$ and $\beta > 0$, there exists no minimal cubature formula of degree 9 for the integral over $\mathbb{R}^n$ with respect to $W(\|\boldsymbol{x}\|) = e^{-\beta\|\boldsymbol{x}\|}$ which includes the origin.*
(iii) *For given $\mu \geq 0$ and sufficiently large n, there exists no minimal cubature formula of degree 9 for the integral over B^n with respect to $W(\|\boldsymbol{x}\|) = (1 - \|\boldsymbol{x}\|^2)^{\mu-1/2}$ which includes the origin. In particular, if $\mu \in \{0, 1/2, 1\}$, for any $n \geq 2$, there exists no minimal formula of degree 9 for the integral over B^n with respect to $W(\|\boldsymbol{x}\|) = (1-\|\boldsymbol{x}\|^2)^{\mu-1/2}$ which includes the origin.*

In order to prove this theorem, we need some necessary conditions for the existence of minimal formulae of degree 9 for the spherically symmetric integral $\mathcal{I}$.

Proposition 4.17. *Assume that there exists a minimal formula of degree* 9 *for the spherically symmetric integral* $\mathcal{I}$ *which includes the origin. Then,*

$$\frac{(n-1)n(n+1)(n+6)(2\eta_6^3-3\eta_4\eta_6\eta_8+\eta_2\eta_8^2)}{24\eta_8\sqrt{(\eta_2\eta_8-\eta_4\eta_6)^2-4(\eta_2\eta_6-\eta_4^2)(\eta_4\eta_8-\eta_6^2)}}$$

is an integer.

Proof. By Lemma 4.6, $\det A_0^{(2)}$, $\det A_4^{(2)}$, $\det A_0^{(3)}$ are all positive. Let

$$\begin{aligned}
g_{0,0}(\|\boldsymbol{x}\|^2) \equiv& 1, \quad g_{2,0}(\|\boldsymbol{x}\|^2) \equiv \sqrt{\frac{\eta_0}{\eta_4}}, \quad g_{4,0}(\|\boldsymbol{x}\|^2) \equiv \sqrt{\frac{\eta_0}{\eta_8}},\\
g_{0,1}(\|\boldsymbol{x}\|^2) =& \frac{\eta_0}{\sqrt{\det A_0^{(2)}}}\Big(\|\boldsymbol{x}\|^2-\frac{\eta_2}{\eta_0}\Big),\ g_{2,1}(\|\boldsymbol{x}\|^2) = \sqrt{\frac{\eta_0\eta_4}{\det A_4^{(2)}}}\Big(\|\boldsymbol{x}\|^2-\frac{\eta_6}{\eta_4}\Big),\\
g_{0,2}(\|\boldsymbol{x}\|^2) =& \frac{\sqrt{\eta_0}\Big((\eta_0\eta_4-\eta_2^2)\|\boldsymbol{x}\|^4-(\eta_0\eta_6-\eta_2\eta_4)\|\boldsymbol{x}\|^2+(\eta_2\eta_6-\eta_4^2)\Big)}{\sqrt{\det A_0^{(2)}\det A_0^{(3)}}}.
\end{aligned}$$

Then it is easily checked that these polynomials satisfy (4.2). Hence by use of (2.5) and Lemma 4.3, the 4-th modified reproducing kernel $\tilde{K}_4(\boldsymbol{x},\boldsymbol{y})$ for the integral $\mathcal{I}$ is computed explicitly; see Appendix A.2. By using Theorem 4.9, for $f\equiv 1$, it holds that

$$|X_1| = \frac{1-w_3-(2\dim(\mathcal{P}_4^*(\mathbb{R}^n))-2)w_2}{w_1-w_2}.$$

Moreover, since w_1, w_2, w_3 can be calculated by Theorem 4.1 (ii), the above equation is equal to

$$|X_1| = \dim(\mathcal{P}_4^*(\mathbb{R}^n))-1+\frac{(n-1)n(n+1)(n+6)(2\eta_6^3-3\eta_4\eta_6\eta_8+\eta_2\eta_8^2)}{24\eta_8\sqrt{(\eta_2\eta_8-\eta_4\eta_6)^2-4(\eta_2\eta_6-\eta_4^2)(\eta_4\eta_8-\eta_6^2)}},$$

which completes the proof. □

Proposition 4.18. *Assume that there exists a minimal cubature formula* (X,w) *of degree* 9 *for the spherically symmetric integral* $\mathcal{I}$ *which includes the origin. Then the following statements follow:*

(i) $\{\langle\xi,\eta\rangle \mid \xi\in\tilde{X}_1, \eta\in\tilde{X}_2\}$

$$= \Big\{\pm\sqrt{\frac{6+3n+\sqrt{6(n+1)(n+2)}}{(n+2)(n+4)}}, \pm\sqrt{\frac{6+3n-\sqrt{6(n+1)(n+2)}}{(n+2)(n+4)}}\Big\}, \tag{4.10}$$

(ii) $\tilde{X}_1$ *and* $\tilde{X}_2$ *are disjoint.*

Proof. (i) By Proposition 4.7 and Theorem 2.18, X is a tight Euclidean 9-design supported by 3 concentric spheres which includes the origin. Moreover, since $X \setminus \{\mathbf{0}\}$ is also a Euclidean 9-design by Theorem 2.17, $X_1 \cup X_2$ is a tight Euclidean 9-design supported by two concentric spheres. It is easy to see that the polynomials

$$\begin{aligned} g_{l,0}(\|\boldsymbol{x}\|^2) \equiv & \frac{1}{\sqrt{r_1^{2l} w(X_1) + r_2^{2l} w(X_2)}}, \quad l = 0, 2, 4, \\ g_{0,1}(\|\boldsymbol{x}\|^2) = & \frac{(w(X_1) + w(X_2))\|\boldsymbol{x}\|^2 - r_1^2 w(X_1) - r_2^2 w(X_2)}{(r_1^2 - r_2^2)\sqrt{w(X_1)w(X_2)(w(X_1) + w(X_2))}}, \\ g_{2,1}(\|\boldsymbol{x}\|^2) = & \frac{(r_1^4 w(X_1) + r_2^4 w(X_2))\|\boldsymbol{x}\|^2 - r_1^6 w(X_1) - r_2^6 w(X_2)}{r_1^2 r_2^2 (r_1^2 - r_2^2)\sqrt{w(X_1)w(X_2)(r_1^4 w(X_1) + r_2^4 w(X_2))}} \end{aligned}$$

satisfy (4.4). Then by Theorem 4.1 (i) and Lemma 4.4, it holds that for $\xi \in \tilde{X}_1, \eta \in \tilde{X}_2$,

$$3 - 6(n+2)\langle \xi, \eta \rangle^2 + (n+2)(n+4)\langle \xi, \eta \rangle^4 = 0.$$

Thus the result follows by solving this equation.
(ii) By using (4.10), we may check that $1 \notin \{\langle \xi, \eta \rangle \mid \xi \in \tilde{X}_1,\ \eta \in \tilde{X}_2\}$. □

For $\alpha \in \{\langle x, y \rangle \mid x, y \in \tilde{X}_1 \cup \tilde{X}_2\}$ and $\xi, \eta \in \tilde{X}_1 \cup \tilde{X}_2$, we let

$$\begin{aligned} p^{(0)}_{\alpha,\beta}(\xi, \eta) = & |\{\zeta \in \tilde{X}_1 \cup \tilde{X}_2 \mid \langle \xi, \zeta \rangle = \alpha,\ \langle \zeta, \eta \rangle = \beta\}|, \\ p^{(i)}_{\alpha,\beta}(\xi, \eta) = & |\{\zeta \in \tilde{X}_i \mid \langle \xi, \zeta \rangle = \alpha,\ \langle \zeta, \eta \rangle = \beta\}|, \\ v^{(0)}_{\alpha}(\xi) = & |\{\zeta \in \tilde{X}_1 \cup \tilde{X}_2 \mid \langle \xi, \zeta \rangle = \alpha\}|, \\ v^{(i)}_{\alpha}(\xi) = & |\{\zeta \in \tilde{X}_i \mid \langle \xi, \zeta \rangle = \alpha\}|. \end{aligned}$$

Then we have $p^{(i)}_{\alpha,\alpha}(\xi, \xi) = v^{(i)}_{\alpha}(\xi)$ and $p^{(i)}_{\alpha,\beta}(\xi, \xi) = 0$ if $\alpha \neq \beta$.

The following result was obtained in a private communication with Eiichi Bannai and Etsuko Bannai.

Proposition 4.19. *Assume that there exists a minimal cubature formula of degree* 9 *for the spherically symmetric integral* $\mathcal{I}$ *which includes the origin.*
(i) *For* $\xi \in \tilde{X}_2$ *and* $\alpha, \beta \in \{\langle \boldsymbol{x}, \boldsymbol{y} \rangle \mid \boldsymbol{x} \in \tilde{X}_1,\ \boldsymbol{y} \in \tilde{X}_2\}$ *with* $\beta > \alpha > 0$, *it holds that*

$$\begin{aligned} v^{(1)}_{\alpha}(\xi) = & \frac{|X_1|\big(3n(n+1) - (n-2)\sqrt{6(n+1)(n+2)}\big)}{12n(n+1)}, \\ v^{(1)}_{\beta}(\xi) = & \frac{|X_1|\big(3n(n+1) + (n-2)\sqrt{6(n+1)(n+2)}\big)}{12n(n+1)}. \end{aligned}$$

(ii) *If $n \geq 3$, then $6(n+1)(n+2)$ is a square integer, that is,*

$$n = \frac{1}{2}\Big(\sum_{l=0}^{[k/2]} \binom{k}{2l} 5^{k-2l} 24^l - 3\Big)$$

for some integer $k \geq 2$.

Proof. (i) By Theorem 4.8, $\tilde{X}_1$ and $\tilde{X}_2$ are spherical 7-designs, and so is $\tilde{X}_1 \cup \tilde{X}_2$. Hence it follows that for each $l = 0, 1, \ldots, 7$ and $\xi \in \tilde{X}_2$,

$$\begin{aligned}
&\sum_{\alpha \in \{\langle \boldsymbol{x},\boldsymbol{y}\rangle | \boldsymbol{x} \in \tilde{X}_1,\, \boldsymbol{y} \in \tilde{X}_2\}} \alpha^l v_\alpha^{(1)}(\xi) \\
&= \sum_{\alpha \in \{\langle \boldsymbol{x},\boldsymbol{y}\rangle | \boldsymbol{x} \in \tilde{X}_1,\, \boldsymbol{y} \in \tilde{X}_2\}} \alpha^l v_\alpha^{(1)}(\xi) + \sum_{\alpha \in \{\langle \boldsymbol{x},\boldsymbol{y}\rangle | \boldsymbol{x},\boldsymbol{y} \in \tilde{X}_2,\, \boldsymbol{x} \neq \boldsymbol{y}\}} \alpha^l v_\alpha^{(2)}(\xi) \\
&\quad - \sum_{\alpha \in \{\langle \boldsymbol{x},\boldsymbol{y}\rangle | \boldsymbol{x},\boldsymbol{y} \in \tilde{X}_2,\, \boldsymbol{x} \neq \boldsymbol{y}\}} \alpha^l v_\alpha^{(2)}(\xi) \\
&= \Bigg(\sum_{\alpha \in \{\langle \boldsymbol{x},\boldsymbol{y}\rangle | \boldsymbol{x} \in \tilde{X}_1,\, \boldsymbol{y} \in \tilde{X}_2\}} \alpha^l v_\alpha^{(1)}(\xi) + \sum_{\substack{\alpha \in \{\langle \boldsymbol{x},\boldsymbol{y}\rangle | \boldsymbol{x},\boldsymbol{y} \in \tilde{X}_1 \cup \tilde{X}_2,\, \boldsymbol{x} \neq \boldsymbol{y}\} \\ \setminus \{\langle \boldsymbol{x},\boldsymbol{y}\rangle | \boldsymbol{x} \in \tilde{X}_1,\, \boldsymbol{y} \in \tilde{X}_2\}}} \alpha^l v_\alpha^{(1)}(\xi) \Bigg) \\
&\quad + \Bigg(\sum_{\alpha \in \{\langle \boldsymbol{x},\boldsymbol{y}\rangle | \boldsymbol{x},\boldsymbol{y} \in \tilde{X}_2,\, \boldsymbol{x} \neq \boldsymbol{y}\}} \alpha^l v_\alpha^{(2)}(\xi) + \sum_{\substack{\alpha \in \{\langle \boldsymbol{x},\boldsymbol{y}\rangle | \boldsymbol{x},\boldsymbol{y} \in \tilde{X}_1 \cup \tilde{X}_2,\, \boldsymbol{x} \neq \boldsymbol{y}\} \\ \setminus \{\langle \boldsymbol{x},\boldsymbol{y}\rangle | \boldsymbol{x},\boldsymbol{y} \in \tilde{X}_2,\, \boldsymbol{x} \neq \boldsymbol{y}\}}} \alpha^l v_\alpha^{(2)}(\xi) \Bigg) \\
&\quad - \sum_{\alpha \in \{\langle \boldsymbol{x},\boldsymbol{y}\rangle | \boldsymbol{x},\boldsymbol{y} \in \tilde{X}_2,\, \boldsymbol{x} \neq \boldsymbol{y}\}} \alpha^l v_\alpha^{(2)}(\xi) \\
&= \sum_{\alpha \in \{\langle \boldsymbol{x},\boldsymbol{y}\rangle | \boldsymbol{x},\boldsymbol{y} \in \tilde{X}_1 \cup \tilde{X}_2,\, \boldsymbol{x} \neq \boldsymbol{y}\}} \alpha^l v_\alpha^{(1)}(\xi) + \sum_{\alpha \in \{\langle x,y\rangle | \boldsymbol{x},\boldsymbol{y} \in \tilde{X}_1 \cup \tilde{X}_2,\, \boldsymbol{x} \neq \boldsymbol{y}\}} \alpha^l v_\alpha^{(2)}(\xi) \\
&\quad - \sum_{\alpha \in \{\langle \boldsymbol{x},\boldsymbol{y}\rangle | \boldsymbol{x},\boldsymbol{y} \in \tilde{X}_2,\, \boldsymbol{x} \neq \boldsymbol{y}\}} \alpha^l v_\alpha^{(2)}(\xi) \\
&= \sum_{\alpha \in \{\langle \boldsymbol{x},\boldsymbol{y}\rangle | \boldsymbol{x},\boldsymbol{y} \in \tilde{X}_1 \cup \tilde{X}_2,\, \boldsymbol{x} \neq \boldsymbol{y}\}} \alpha^l v_\alpha^{(0)}(\xi) - \sum_{\alpha \in \{\langle \boldsymbol{x},\boldsymbol{y}\rangle | \boldsymbol{x},\boldsymbol{y} \in \tilde{X}_2,\, \boldsymbol{x} \neq \boldsymbol{y}\}} \alpha^l v_\alpha^{(2)}(\xi) \\
&= \sum_{\alpha,\beta \in \{\langle \boldsymbol{x},\boldsymbol{y}\rangle | \boldsymbol{x},\boldsymbol{y} \in \tilde{X}_1 \cup \tilde{X}_2,\, \boldsymbol{x} \neq \boldsymbol{y}\}} \alpha^l p_{\alpha,\beta}^{(0)}(\xi,\xi) - \sum_{\alpha,\beta \in \{\langle \boldsymbol{x},\boldsymbol{y}\rangle | \boldsymbol{x},\boldsymbol{y} \in \tilde{X}_2,\, \boldsymbol{x} \neq \boldsymbol{y}\}} \alpha^l p_{\alpha,\beta}^{(2)}(\xi,\xi) \\
&= |X_1 \cup X_2| f_{l,0} - |X_2| f_{l,0},
\end{aligned}$$

where we have used Theorem 2.12 for the last equality. Moreover, since X_1 and X_2 are disjoint by Proposition 4.18 (ii), the last expression is equal to

$$\sum_{\alpha \in \{\langle x,y\rangle | x \in \tilde{X}_1,\, y \in \tilde{X}_2\}} \alpha^l v_\alpha^{(1)}(\xi) = |X_1| f_{l,0}. \tag{4.11}$$

Note that for any $\alpha \in \mathbb{R}$, $v_\alpha(\xi) = v_{-\alpha}(\xi)$. Then by using (2.6), we can extract no information from (4.11) for $l = 1, 3, 5, 7$ and hence (4.11) can be reduced to the following four equations:

$$2\alpha^l v_\alpha^{(1)}(\xi) + 2\beta^l v_\beta^{(1)}(\xi) = |X_1| f_{l,0}, \quad l = 0, 2, 4, 6, \quad \beta > \alpha > 0. \tag{4.12}$$

Thus in view of (2.6) and Proposition 4.18 (i), the result follows by solving (4.12).
(ii) By (i), $6(n+1)(n+2)$ is a square integer, or equivalently, the equation $(2n+3)^2 - 24y^2 = 1$ holds for some integer y. Let $x = 2n+3$. Then the unit theorem implies that $x^2 - 24y^2 = 1$ has an integer root if and only if $x + \sqrt{24}y = (5 + \sqrt{24})^k$ for some integer k. Hence we have $x = 2n + 3 = \sum_{l=0}^{[k/2]} \binom{k}{2l} 5^{k-2l} 24^l$. □

We are now in a position to complete the proof of Theorem 4.16.

Proof of Theorem 4.16. (i) Assume that there exists a minimal cubature formula of degree 9 for the integral over $\mathbb{R}^n$ with respect to $W(\|\boldsymbol{x}\|) = e^{-\alpha\|\boldsymbol{x}\|^2}$ which includes the origin. By Proposition 4.17,

$$\frac{(n-1)n(n+1)\sqrt{2(n+4)}}{12}$$

is an integer and so is $\sqrt{2(n+4)}$. This is equivalent to the fact that $n + 4 = 2l^2$ has an integer root. Therefore d is an even integer. This is impossible by Proposition 4.19 (ii).
(ii) Assume that there exists a minimal cubature formula of degree 9 for the integral over $\mathbb{R}^n$ with respect to $W(\|\boldsymbol{x}\|) = e^{-\beta\|\boldsymbol{x}\|}$ which includes the origin. By Proposition 4.17,

$$\frac{(n-1)n(n+1)(n+4)(7n+39)\sqrt{(n+4)(n+5)}}{12(n+7)\sqrt{2(8n^3+102n^2+424n+579)}} \tag{4.13}$$

is an integer and so is

$$\frac{(n+4)^3(n-1)^2n^2(n+1)^2(7n+39)^2(n+5)}{288 \cdot (n+7)^2(8n^3+102n^2+424n+579)}.$$

By using the Euclidean algorithm, it is shown that $n+7$ is a divisor of $10^2 \cdot 8^2 \cdot 7^2 \cdot 6^2 \cdot 3^3 \cdot 2$ greater than 7. Then by a simple computation, it is easy to check that (4.13) is not an integer for all possible n.
(iii) Let $\mu \geq 0$. Assume that there exists a minimal cubature formula of degree 9 for the integral over B^n with respect to $W(\|\boldsymbol{x}\|) = (1 - \|x\|^2)^{\mu-1/2}$ which includes the origin. By Proposition 4.17,

$$\frac{(n-1)n(n+1)(n+4)(n+4\mu+8)}{12\sqrt{2(n+4)(2\mu+3)(n+2\mu+5)}} \tag{4.14}$$

is an integer and hence $\mu = \gamma/\delta$ is a rational number. Obviously the square

$$\frac{(n-1)^2 n^2 (n+1)^2 (n+4)(\delta n + 4\gamma + 8\delta)^2}{12^2 \cdot 2(2\gamma + 3\delta)(\delta n + 2\gamma + 5\delta)} \tag{4.15}$$

is an integer. By use of the Euclidean algorithm, $\delta n + 2\gamma + 5\delta$ is a divisor of $(\delta + 2\gamma)(3\delta + 2\gamma)^2(4\delta + 2\gamma)^2(5\delta + 2\gamma)^2(6\delta + 2\gamma)^2$, which implies that there exists no minimal cubature formula for sufficiently large n.

A further investigation will be done for $\mu \in \{0, 1/2, 1\}$. First we consider the case $\mu = 0$. In this case (4.15) is transformed as

$$\frac{(n-1)^2 n^2 (n+1)^2 (n+8)^2 (n+4)}{864(n+5)}.$$

Thus $n + 5$ is a divisor of $6^2 \cdot 5^2 \cdot 4^2 \cdot 3^2$ greater than 5. Then by a simple computation, it is easy to check that (4.14) is not an integer for all n with the only possible exception that $n = 20$. By Proposition 4.19 (ii), there exists no 20-dimensional minimal cubature formula.

Secondly we consider the case $\mu = 1/2$. Through an argument similar to that use in the previous paragraph, (4.14) is not an integer for all n with the only possible exception that $n = 94$. By Proposition 4.19 (ii), there exists no 94-dimensional minimal cubature formula.

Thirdly we consider the case $\mu = 1$. A similar argument done in the previous two cases shows that (4.14) is not an integer for all d with two possible exceptions that $n = 23, 238$. By Proposition 4.19 (ii), there exists no 238-dimensional minimal cubature formula. When $n = 23$, we note that $|X_1| = 25852$. Then by Proposition 4.19 (i), we have $v_\alpha^{(1)}(\xi) = 3091/2$, which is not an integer. □

Chapter 5

Minimal cubature formulae on multiple spheres

In this chapter, we present a new 4-dimensional tight Euclidean 5-design on 3 concentric spheres which does not include the origin. As far as the author knows, this is the first and the only known example of tight Euclidean t-designs with $\mathbf{0} \notin X, t \geq 4, n \geq 4$ and $p \geq 3$. Explicit forms of tight Euclidean designs with $\mathbf{0} \notin X, t \geq 4, n \geq 3$ and $p \geq 2$ known so far are listed in Table 5.1, which has been updated from the recent survey paper [11]. We also show that our new design does not have an algebraic structure called coherent configuration, and the other known tight Euclidean designs have.

5.1 Tight Euclidean 5-design on 3 concentric spheres

The set of all transpositions of coordinates in $\mathbb{R}^n$ forms a finite group H, called the Weyl group of type A_{n-1}, $n \geq 2$. Let ρ be the involution on $\mathbb{R}^n$ defined by $\boldsymbol{x}^\rho = -\boldsymbol{x}$ and G be the semidirect product of H and $\langle \rho \rangle$. Then it is obvious that G is a subgroup of the Weyl group of type B_n, say $W(B_n)$, and so the order of G is $2 \cdot n!$. For $\boldsymbol{x} \in \mathbb{R}^n$, we denote by $\mathrm{Orb}_G(\boldsymbol{x})$ the orbit of $\boldsymbol{x}$ under G, that is, $\mathrm{Orb}_G(\boldsymbol{x}) = \{\boldsymbol{x}^\tau \mid \tau \in G\}$.

Proposition 5.1. *There exists a 4-dimensional tight Euclidean 5-design (X, w) on 3 concentric spheres which has the form*

$$\sum_{\xi \in X} w(\xi) f(\xi) = z_6 \sum_{\xi \in \mathrm{Orb}_G((z_1,z_1,z_1,z_1))} f(\xi) + z_7 \sum_{\xi \in \mathrm{Orb}_G((z_2,z_2,z_2,z_3))} f(\xi) + z_8 \sum_{\xi \in \mathrm{Orb}_G((z_4,z_4,z_5,z_5))} f(\xi), \tag{5.1}$$

where $z_1, \cdots, z_8$ *are positive real numbers such that*

$$2z_1^2 - 3z_2^2 > 0, \quad -3z_2 = z_3, \quad z_4 = \frac{(\sqrt{2}-1)z_1z_2}{\sqrt{2z_1^2 - 3z_2^2}},$$
$$z_5 = -\frac{(\sqrt{2}+1)z_1z_2}{\sqrt{2z_1^2 - 3z_2^2}}, \quad z_6 = \frac{3z_2^4}{4(7z_1^4 - 18z_1^2z_2^2 + 15z_2^4)},$$
$$z_7 = \frac{z_1^4}{8(7z_1^4 - 18z_1^2z_2^2 + 15z_2^4)}, \quad z_8 = \frac{(2z_1^2 - 3z_2^2)^2}{8(7z_1^4 - 18z_1^2z_2^2 + 15z_2^4)}.$$

Proof. The result follows by checking that equation (2.7) holds for any $f \in \{1,\ x_i^2,\ x_ix_j,\ x_i^4,\ x_i^3x_j,\ x_i^2x_j^2,\ x_i^2x_jx_k,\ x_1x_2x_3x_4\}$. □

We close this section by presenting a list of tight Euclidean designs with $\mathbf{0} \notin X,\ t \geq 4,\ n \geq 3,\ p \geq 2$ in Table 5.1 below. "CC" means whether the design has the structure of a coherent configuration. "Ref." means reference information on the existence of tight Euclidean designs for given $t,\ n,\ p$. We emphasize that No. 5, No. 9 are the only known examples of the tight Euclidean designs with $\mathbf{0} \notin X,\ t \geq 4,\ n \geq 3,\ p \geq 3$.

No.	t	n	p	CC	Ref.
1	4	4	2	○	[6]
2	4	5	2	○	[6]
3	4	6	2	○	[6]
4	5	3	2	○	[5]
5	5	4	3	×	Proposition 5.1
6	5	5	2	○	[5]

No.	t	n	p	CC	Ref.
7	5	6	2	○	[5]
8	6	22	2	○	[14]
9	7	3	3	○	[3]
10	7	4	2	○	[3]
11	7	7	2	○	[13]

Table 5.1: Multidimensional tight Euclidean designs with $\mathbf{0} \notin X,\ t \geq 4,\ n \geq 3,\ p \geq 2$

5.2 Structures of Euclidean designs in Table 5.1

A natural question is whether all known tight Euclidean designs in Table 5.1 have the structure of a coherent configuration. By Corollary 2.36, we see that any tight Euclidean t-design on 2 concentric spheres has the structure of a coherent configuration. It thus remains to investigate the designs of No. 5 and No. 9 in Table 5.1.

Proposition 5.2. (i) *The design contained in Proposition 5.1 or No. 5 in Table 5.1 does not have the structure of a coherent configuration.*
(ii) *The design of No. 9 found by Bajnok [3] has the structure of a coherent configuration.*

Proof. (i) With the same notations as in Proposition 5.1, let $\xi_1 = (-z_4, -z_4, -z_5, -z_5)$, $\xi_2 = (-z_4, -z_5, -z_4, -z_5)$, $\xi_3 = (z_5, z_5, z_4, z_4) \in \mathrm{Orb}_G((z_4, z_4, z_5, z_5))$. Then we have $\langle \xi_1, \xi_2 \rangle / \|\xi_1\|^2 = \langle \xi_1, \xi_3 \rangle / \|\xi_1\|^2 = 1/3$, and

$$\left| \left\{ \eta \in \mathrm{Orb}_G((z_2, z_2, z_2, z_3)) \;\middle|\; \tfrac{\langle \xi_1, \eta \rangle}{\|\xi_1\| \|\eta\|} = \tfrac{\sqrt{2}}{3}, \tfrac{\langle \eta, \xi_2 \rangle}{\|\eta\| \|\xi_2\|} = \tfrac{\sqrt{2}}{3} \right\} \right| = 2,$$

$$\left| \left\{ \eta \in \mathrm{Orb}_G((z_2, z_2, z_2, z_3)) \;\middle|\; \tfrac{\langle \xi_1, \eta \rangle}{\|\xi_1\| \|\eta\|} = \tfrac{\sqrt{2}}{3}, \tfrac{\langle \eta, \xi_3 \rangle}{\|\eta\| \|\xi_3\|} = \tfrac{\sqrt{2}}{3} \right\} \right| = 4,$$

which implies the assertion.
(ii) It is known ([3, Corollary 9]) that the design of No. 9 is of the form

$$\sum_{\xi \in X} w(\xi) f(\xi) = y_4 \sum_{x \in \mathrm{Orb}_{W(B_3)}((y_1, 0, 0))} f(\xi) + y_5 \sum_{x \in \mathrm{Orb}_{W(B_3)}((y_2, y_2, 0))} f(\xi)$$
$$+ y_6 \sum_{x \in \mathrm{Orb}_{W(B_3)}((y_3, y_3, y_3))} f(\xi),$$

where $y_1, \ldots, y_6$ are some positive real numbers such that

$$y_2 = \sqrt{\frac{3y_1^2 + 2\rho^2}{10}} \cdot \frac{y_1}{\rho}, \quad y_3 = \sqrt{\frac{3y_1^2 + 2\rho^2}{15}},$$
$$y_5 = \frac{100\rho^6 y_4}{(3y_1^2 + 2\rho^2)^3}, \qquad y_6 = \frac{675 y_1^6 y_4}{8(3y_1^2 + 2\rho^2)^3}, \quad \rho > 0.$$

Let $X_1 = \mathrm{Orb}_{W(B_3)}((y_1, 0, 0)), X_2 = \mathrm{Orb}_{W(B_3)}((y_2, y_2, 0)), X_3 = \mathrm{Orb}_{W(B_3)}((y_3, y_3, y_3))$. Then by Theorem 2.35, it suffices to show that $p(\alpha^{(v)}_{\lambda,\nu}, \alpha^{(\omega)}_{\nu,\mu}, \boldsymbol{x}, \boldsymbol{y})$ is independent of $(\boldsymbol{x}, \boldsymbol{y}) \in R_{\lambda,\mu,u}$ for each $(\lambda, \nu, \mu) \in \{1, 2, 3\}^3$ such that $s_{\lambda,\nu} + s_{\nu,\mu} - \delta_{\lambda,\nu} - \delta_{\nu,\mu} > 5$. Namely, we may consider the following cases: $(\lambda, \nu, \mu) = (1, 2, 1), (1, 2, 2), (1, 2, 3), (2, 1, 2), (2, 2, 2), (2, 3, 2), (3, 2, 2), (3, 2, 3)$.

First, we consider the case $(\lambda, \nu, \mu) = (1, 2, 1)$. Let $X_1 = \{\xi_1, \ldots, \xi_6\}$, where

$$\xi_1 = (0, 0, -y_1), \quad \xi_2 = (0, 0, y_1), \quad \xi_3 = (0, -y_1, 0),$$
$$\xi_4 = (0, y_1, 0), \qquad \xi_5 = (-y_1, 0, 0), \; \xi_6 = (y_1, 0, 0)$$

and $X_2 = \{\eta_1, \ldots, \eta_{12}\}$, where

$$\eta_1 = (0, -y_2, -y_2), \quad \eta_2 = (0, -y_2, y_2), \quad \eta_3 = (0, y_2, -y_2), \quad \eta_4 = (0, y_2, y_2),$$
$$\eta_5 = (-y_2, 0, -y_2), \quad \eta_6 = (-y_2, 0, y_2), \quad \eta_7 = (-y_2, -y_2, 0), \; \eta_8 = (-y_2, y_2, 0),$$
$$\eta_9 = (y_2, 0, -y_2), \quad \eta_{10} = (y_2, 0, y_2), \quad \eta_{11} = (y_2, -y_2, 0), \quad \eta_{12} = (y_2, y_2, 0),$$

We denote the elements of $A_{1,1},\ A_{1,2},\ A_{2,1}$ by

$$\begin{aligned} &\alpha_{1,1}^{(0)} = 1, \quad \alpha_{1,1}^{(1)} = 0, \quad \alpha_{1,1}^{(2)} = -1, \\ &\alpha_{1,2}^{(1)} = \frac{1}{\sqrt{2}}, \ \alpha_{1,2}^{(2)} = 0, \quad \alpha_{1,2}^{(3)} = -\frac{1}{\sqrt{2}}, \\ &\alpha_{2,1}^{(1)} = \frac{1}{\sqrt{2}}, \ \alpha_{2,1}^{(2)} = 0, \quad \alpha_{2,1}^{(3)} = -\frac{1}{\sqrt{2}}. \end{aligned}$$

For example, we will check that $p(\alpha_{1,2}^{(v)}, \alpha_{2,1}^{(\omega)}, \boldsymbol{x}, \boldsymbol{y})$ is independent of $(\boldsymbol{x}, \boldsymbol{y}) \in R_{1,1,2} = \{(\boldsymbol{x}, \boldsymbol{y}) \in X_1 \times X_1 \mid \langle \boldsymbol{x}, \boldsymbol{y} \rangle = \alpha_{1,1}^{(2)}\}$. For this purpose, we list up the pairs $(\langle \eta_i, \xi_j \rangle, \langle \xi_k, \eta_i \rangle)$ in Table 5.2.

	(ξ_1,ξ_2)	(ξ_2,ξ_1)	(ξ_3,ξ_4)	(ξ_4,ξ_3)	(ξ_5,ξ_6)	(ξ_6,ξ_5)
η_1	$(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}})$	$(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$	$(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}})$	$(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$	$(0,0)$	$(0,0)$
η_2	$(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$	$(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}})$	$(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}})$	$(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$	$(0,0)$	$(0,0)$
η_3	$(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}})$	$(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$	$(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$	$(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}})$	$(0,0)$	$(0,0)$
η_4	$(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$	$(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}})$	$(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$	$(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}})$	$(0,0)$	$(0,0)$
η_5	$(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}})$	$(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$	$(0,0)$	$(0,0)$	$(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}})$	$(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$
η_6	$(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$	$(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}})$	$(0,0)$	$(0,0)$	$(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}})$	$(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$
η_7	$(0,0)$	$(0,0)$	$(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}})$	$(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$	$(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}})$	$(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$
η_8	$(0,0)$	$(0,0)$	$(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$	$(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}})$	$(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}})$	$(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$
η_9	$(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}})$	$(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$	$(0,0)$	$(0,0)$	$(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$	$(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}})$
η_{10}	$(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$	$(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}})$	$(0,0)$	$(0,0)$	$(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$	$(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}})$
η_{11}	$(0,0)$	$(0,0)$	$(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}})$	$(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$	$(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$	$(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}})$
η_{12}	$(0,0)$	$(0,0)$	$(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$	$(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}})$	$(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$	$(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}})$

Table 5.2: Table of $p(\alpha_{1,2}^{(v)}, \alpha_{2,1}^{(\omega)}, \boldsymbol{x}, \boldsymbol{y})$

As is easily checked in Table 5.2, each of $(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$, $(0,0)$, $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ appears four times in every column. That is, it holds that $p(\alpha_{1,2}^{(1)}, \alpha_{2,1}^{(3)}, \xi_i, \xi_j) = p(\alpha_{1,2}^{(2)}, \alpha_{2,1}^{(2)}, \xi_i, \xi_j) = p(\alpha_{1,2}^{(3)}, \alpha_{2,1}^{(1)}, \xi_i, \xi_j) = 4$ for any $(\xi_i, \xi_j) \in R_{1,1,2}$. Similarly, we can check that $p(\alpha_{1,2}^{(v)}, \alpha_{2,1}^{(\omega)}, \boldsymbol{x}, \boldsymbol{y})$ is independent of $(\boldsymbol{x}, \boldsymbol{y}) \in R_{1,1,0}$ or $(\boldsymbol{x}, \boldsymbol{y}) \in R_{1,1,1}$, respectively.

Similarly, for the remaining seven cases of (λ, ν, μ), we can check that all $p(\alpha_{\lambda,\nu}^{(v)}, \alpha_{\nu,\mu}^{(\omega)}, \boldsymbol{x}, \boldsymbol{y})$ is independent of $(\boldsymbol{x}, \boldsymbol{y}) \in R_{\lambda,\mu,u}$. $\square$

Remark 5.3. (i) In this chapter, we have constructed a new 4-dimensional tight Euclidean 5-design supported by 3 concentric spheres. Bajnok [3] showed that a union of generalized regular hyperoctahedra in $\mathbb{R}^n$ forms Euclidean 3-, 5- and 7-designs. A generalized regular hyperoctahedron in $\mathbb{R}^n$ is the orbit of $\sum_{i=1}^{k} \boldsymbol{e}_i$ under $W(B_n)$. Bajnok's method is algebraically interesting, but in most cases Euclidean designs obtained from his method could have large number of points, since the order of $W(B_n)$ is very large (i.e., $2^n \cdot n!$). To overcome this problem, we focused on the subgroup G of $W(B_n)$ in Proposition 5.1. It should also be noted that each point of the Euclidean design in Proposition 5.1 is the linear combination of a generalized regular hyperoctahedron and its complement. This is a simple generalization of the concept of a generalized regular hyperoctahedron.
(ii) As far as the authors know, the design of No. 5 in Table 5.1 is the only known example of tight Euclidean designs with $\mathbf{0} \notin X, t \geq 4, n \geq 4$ and $p \geq 3$. Moreover, among all known tight Euclidean designs, including the two dimensional designs by [2], the design of No. 5 only does not have the structure of a coherent configuration. From the algebraic viewpoint, it might be interesting to find some algebraic concept which characterizes all tight Euclidean designs.

Chapter 6

Concluding remarks

Combining the theories on Euclidean designs in algebraic combinatorics and on orthogonal polynomials, we have studied the existence and nonexistence of minimal cubature formulae for spherically symmetric integrals.

In Chapter 3, we have studied the existence and nonexistence of minimal cubature formulae of degree $2e + 1$ for a bivariate Gaussian integral whose points are supported by $[e/2] + 1$ concentric spheres without the origin. By specifying the two dimensional tight Euclidean t-designs supported by p concentric spheres with $p \leq [t/4] + 1$, we have given necessary conditions for the existence of minimal cubature formula of degree $2e + 1$ for a bivariate Gaussian integral in terms of Laguerre polynomials. Thus, we have shown that there exists minimal cubature formula of degree $2e + 1$ whose points are supported by $[e/2] + 1$ concentric spheres without the origin only when $e = 1$.

The necessary conditions may also be applied to the existence and nonexistence of cubature formula of degree $2e + 1$ for other circularly symmetric integrals. So we obtain the following problem.

Problem 1. *Generalize the necessary conditions in Proposition 3.13 for other circularly symmetric integrals, and determine whether there exist minimal cubature formulae or not.*

In this thesis, we have not dealed with the existence of minimal cubature formulae of degree $2e$ since we do not have similar necessary conditions to the case of odd degree. Hence, we have also to study the following problem.

Problem 2. *Find necessary conditions for the existence of minimal cubature formulae of degree $2e$ for the Gaussian integral in terms of Laguerre polynomials.*

For the case when the number of concentric spheres is larger than $[t/4] + 1$, we have no classification of tight Euclidean designs on multiple concentric spheres. So we need to study the following problem.

Problem 3. *Determine the shapes of two dimensional tight Euclidean t-designs on p concentric spheres with $p \geq [t/4] + 2$.*

It is shown in [63] that, for each e, the integral over $\Omega = \{(x, y) \mid 1 \leq x^2 + y^2 < \infty\}$ with respect to $W(\sqrt{x^2 + y^2}) = \sqrt{x^2 + y^2 - 1}/(x^2 + y^2)^{e+2}$ admits minimal cubature formulae of both degree $2e - 1$ and degree $2e$. Hence we have the following problem.

Problem 4. *Find other circularly symmetric integrals which admit minimal cubature formulae in order to obtain a sufficient condition for the existence of minimal cubature formulae.*

In Chapter 4, we have studied the existence of minimal cubature formula of degree $4k+1$ for spherically symmetric integrals which includes the origin. Combining the theories on tight Euclidean designs and that on reproducing kernels, we have specified the minimal cubature formulae of degree $4k + 1$ which includes the origin in general. By applying this result in the case of $k = 1$, we showed the equivalence of minimal cubature formula of degree 5 and 4 which includes the origin and a spherical tight 5- and 4-design. Moreover, considering the case $k = 2$, we derived some necessary conditions. By increasing the number of concentric spheres, it seems difficult to derive and check the necessary conditions for the existence of minimal cubature formulae. For example, a minimal cubature formula of degree 13 which includes the origin are supported by 4 concentric spheres. Even in this case, the situation is complex.

Problem 5. *Find another type of necessary condition for minimal cubature formulae of degree* 13 *for spherically symmetric integrals, and determine whether there exists minimal cubature formulae of degree* 13 *or not.*

The existence problem of minimal cubature formula of degree t for spherically symmetric integrals has a deep connection the existence of tight Euclidean t-design. Hence we have the following problem.

Problem 6. *Find necessary condition for the existence of tight Euclidean* 13*-design of* $\mathbb{R}^n$*, and determine whether there exists tight Euclidean* 13*-design of* $\mathbb{R}^n$ *or not.*

In Chapter 5, we have constructed a 4-dimensional tight Euclidean 5-design supported by 3 concentric spheres. In order to construct this design, we have focused on the subgroup G of the Weyl group of type B_n in Proposition 5.1. We hope that the present approach will be formulated so as to obtain more minimal cubature formulae and tight Euclidean designs.

Problem 7. *Find suitable groups for constructing minimal cubature formulae or cubature formulae with smaller number of points.*

Moreover, the design obtained in Section 5.1 serves the only example known so far, which does not have the structure of a coherent configuration. The other known examples of tight Euclidean designs have it. Hence, in order to classify tight Euclidean designs on p concentric spheres, we have to find other algebraic structures.

Problem 8. *Find an algebraic concept which characterizes all the tight Euclidean designs.*

Appendix A

Appendix to Chapters 3 and 4

A.1 Examples of tight and almost tight Gaussian designs

In this section, for convenient to use, we present a list of all known tight Gaussian t-design supported by $[\frac{t}{4}]+1$ almost tight Gaussian t-designs on $[\frac{t}{4}]+2$ concentric spheres for $t \geq 4$ (see also [13, 31, 49, 56]).

Tight Gaussian 4-designs of of $\mathbb{R}^n$. Tight Gaussian 4-designs supported by 2 concentric spheres are classified by Bannai and Bannai [8].
(i) Let $X = X_1 \cup \{\mathbf{0}\}$ be a tight Gaussian 4-design of $\mathbb{R}^n$ which includes the origin. Then $\tilde{X}_1$ is a spherical 4-design. Let $X_2 = \{\mathbf{0}\}$ and $w_i = w(\xi)$ for $\xi \in X_i, i = 1, 2$. We list the these values in the following table. See Figure A.1 for the precise structure of the two dimensional case.

n	$\lvert X_1\rvert$	α	r_1	w_1	w_2
2	5	α	$\sqrt{\frac{2}{\alpha}}$	$\frac{1}{10}$	$\frac{1}{2}$
6	27	α	$\frac{2}{\sqrt{\alpha}}$	$\frac{1}{36}$	$\frac{1}{4}$
22	275	α	$2\sqrt{\frac{3}{\alpha}}$	$\frac{1}{300}$	$\frac{1}{12}$

(ii) Let $X = X_1 \cup X_2$ be a tight Gaussian 4-design of $\mathbb{R}^2$ which does not include the origin. Then $n = 2$. Each $\tilde{X}_i$ is a regular simplex. Let $\xi \in X_i, i = 1, 2$. We list the these values in the following table. See Figure A.1 for the precise structure of X.

n	$\lvert X_i\rvert,\ i = 1, 2$	α	r_1	r_2	w_1	w_2
2	3	α	$\frac{\sqrt{5}+1}{\alpha\sqrt{2}}$	$\frac{\sqrt{5}-1}{\alpha\sqrt{2}}$	$\frac{1}{6} - \frac{\sqrt{5}}{15}$	$\frac{1}{6} + \frac{\sqrt{5}}{15}$

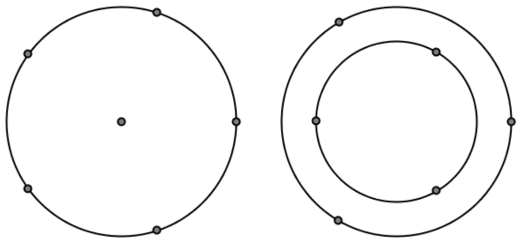


Figure A.1: Tight Gaussian 4-designs of $\mathbb{R}^2$ supported by 2 concentric spheres

Almost tight Gaussian 4-designs of $\mathbb{R}^n$. Let $X = X_1 \cup X_2 \cup \{\mathbf{0}\}$ be an almost tight Gaussian 4-design of $\mathbb{R}^n$. Then $n = 4$ or 5 and X is similar to a tight Euclidean 4-design given in [6]. Let $1 \in \{r_1, r_2\}$. Then the value of $\alpha > 0$ and the weight function w is determined uniquely. Let $X_3 = \{\mathbf{0}\}$ and $w_i = w(\xi)$ for $\xi \in X_i$, $i = 1, 2, 3$. We list these values in the following table. Please refer [6] for the precise structure of X.

n	$\lvert X_1\rvert$	$\lvert X_2\rvert$	α	r_1	r_2	w_1	w_2	w_3
4	5	10	12	1	$\frac{1}{\sqrt{6}}$	$\frac{1}{300}$	$\frac{9}{100}$	$\frac{1}{12}$
4	9	6	4	$\sqrt{2}$	1	$\frac{1}{12}$	$\frac{1}{36}$	$\frac{1}{4}$
5	15	6	$\frac{5}{2}$	$\sqrt{\frac{8}{5}}$	1	$\frac{1}{18}$	$\frac{1}{36}$	$\frac{1}{4}$

Tight Gaussian 5-designs of $\mathbb{R}^n$. (i) Let $X = X_1 \cup \{\mathbf{0}\}$ be a tight Gaussian 5-design of $\mathbb{R}^n$ which includes the origin. Then $\tilde{X}_1$ is a spherical 5-design. Let $X_2 = \{\mathbf{0}\}$ and $w_i = w(\xi)$ for $\xi \in X_i, i = 1, 2$. We list the values in the following table. See Figure A.2 for the precise structure of the two dimensional case.

n	$\lvert X_1\rvert$	α	r_1	w_1	w_2
2	6	α	$\sqrt{\frac{2}{\alpha}}$	$\frac{1}{12}$	$\frac{1}{2}$
3	12	α	$\sqrt{\frac{5}{\alpha}}$	$\frac{1}{20}$	$\frac{2}{5}$
7	56	α	$\frac{3}{\sqrt{2\alpha}}$	$\frac{1}{72}$	$\frac{2}{9}$
23	552	α	$\frac{5}{\sqrt{2\alpha}}$	$\frac{1}{600}$	$\frac{2}{25}$

(ii) Let $X = X_1 \cup X_2$ be a tight Gaussian 5-design of $\mathbb{R}^n$ which does not include the origin. Then $n = 3, 5$ or 6 and X is the tight Euclidean design 5-design of $\mathbb{R}^n$ given in [5]. Let $1 \in \{r_1, r_2\}$. Then the value of $\alpha > 0$ and the weight function w is determined uniquely. Let $w_i = w(\xi)$ for $\xi \in X_i$, $i = 1, 2$. We list these values in the following table. Please refer

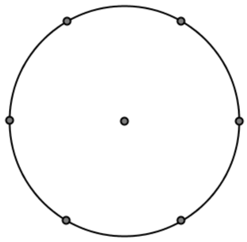


Figure A.2: Tight Gaussian 5-design which includes the origin

[5] for the precise structure of X.

n	$\lvert X_1\rvert$	$\lvert X_2\rvert$	α	r_1	r_2	w_1	w_2
3	8	6	$\frac{5}{4}$	$\sqrt{6}$	1	$\frac{1}{200}$	$\frac{4}{25}$
5	20	12	$\frac{5(5\pm3\sqrt{2})}{2}$	$\sqrt{\frac{3(3\mp2\sqrt{2})}{5}}$	1	$\frac{1}{88\mp48\sqrt{2}}$	$\frac{1}{1032\pm720\sqrt{2}}$
6	32	12	2	$\sqrt{3}$	1	$\frac{1}{128}$	$\frac{1}{16}$

Almost tight Gaussian 5-design of $\mathbb{R}^n$. Let $X = X_1 \cup X_2 \cup \{\mathbf{0}\}$ be an almost tight Gaussian 5-design of $\mathbb{R}^n$. Then $n = 2$. $X \setminus \{\mathbf{0}\}$ is a tight Euclidean 5-design of $\mathbb{R}^2$ and X_1 and X_2 are squares. Let $1 \in \{r_1, r_2\}$, $X_3 = \{\mathbf{0}\}$ and $w_i = w(\xi)$ for $\xi \in X_i, i = 1, 2, 3$. Then we list the parameters in the following. See Figure A.2 for the precise structure of X.

n	$\lvert X_i\rvert, i = 1, 2$	α	r_1	r_2	w_1	w_2	w_3
2	4	$\frac{r^2+1}{r^2}$	1	$r \neq 1$	$\frac{r^4}{4(r^2+1)^2}$	$\frac{1}{4(r^2+1)^2}$	$\frac{2r^2}{(r^2+1)^2}$

Almost tight Gaussian 6-design of $\mathbb{R}^2$. Let $X = X_1 \cup X_2 \cup \{\mathbf{0}\}$ and $X \setminus \{\mathbf{0}\}$ be a tight Euclidean 6-design of $\mathbb{R}^2$. Let $1 \in \{r_1, r_2\}$, $X_3 = \{\mathbf{0}\}$ and $w_i = w(\xi)$ for $\xi \in X_i, i = 1, 2, 3$. Then the following parameters give X a tight Gaussian 6-design structure. See Figure A.2 for the precise structure of X.

n	$\lvert X_i\rvert, i = 1, 2$	α	r_1	r_2	w_0	w_1	w_2
2	5	6	1	$\frac{1}{2}$	$\frac{1}{270}$	$\frac{16}{135}$	$\frac{7}{18}$

Almost tight Gaussian 7-designs of $\mathbb{R}^n$. Let $X = X_1 \cup X_2 \cup \{\mathbf{0}\}$ be an almost tight Gaussian 7-design of $\mathbb{R}^n$. Then $n = 2, 4, 7$ and $X \setminus \{\mathbf{0}\}$ is one of tight Euclidean 7-designs of $\mathbb{R}^n$ given in [12]. Let $1 \in \{r_1, r_2\}$, $X_3 = \{\mathbf{0}\}$ $w_i = w(\xi)$ for $\xi \in X_i, i = 1, 2, 3$. Then we list the parameters in the following. See Figure A.2 for the precise structure of the two dimensional case and see [12] for the other cases.

n	$\|X_1\|$	$\|X_2\|$	α	r_1	r_2	w_1	w_2	w_3
2	6	6	$6 \pm 3\sqrt{2}$	1	$\sqrt{2} \mp 1$	$\frac{10 \mp 7\sqrt{2}}{216}$	$\frac{10 \pm 7\sqrt{2}}{216}$	$\frac{4}{9}$
4	24	24	$6 \pm 2\sqrt{3}$	1	$\frac{\sqrt{2}(\sqrt{3} \mp 1)}{2}$	$\frac{9 \mp 5\sqrt{3}}{576}$	$\frac{9 \pm 5\sqrt{3}}{576}$	$\frac{1}{4}$
7	56	126	$\frac{27 \pm 12\sqrt{3}}{2}$	1	$\frac{3 \mp \sqrt{3}}{3}$	$\frac{75}{10648} \mp \frac{97\sqrt{3}}{23958}$	$\frac{5}{1331} \pm \frac{37\sqrt{3}}{23958}$	$\frac{16}{121} \pm \frac{35\sqrt{3}}{1089}$

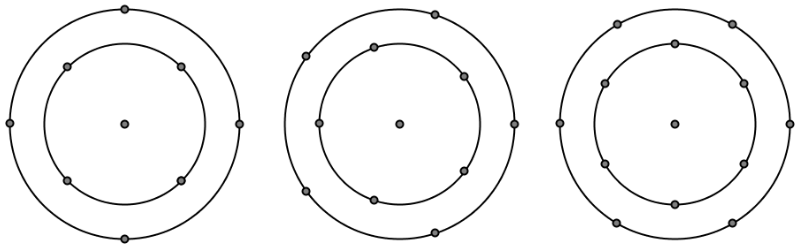


Figure A.3: Almost tight Gaussian $5, 6, 7$-designs of $\mathbb{R}^2$ supported by 3 concentric spheres

Almost tight Gaussian 8-design of $\mathbb{R}^2$. Let $X = X_1 \cup X_2 \cup X_3 \cup \{\mathbf{0}\}$ and $X \setminus \{\mathbf{0}\}$ be a tight Euclidean 8-design of $\mathbb{R}^2$. Let $X_4 = \{\mathbf{0}\}$ and $w_i = w(\xi)$ for $\xi \in X_i, i = 1, 2, 3, 4$. Then the following parameters give X an almost tight Gaussian 8-design structure. See Figure A.3 for the precise structure of X.

n	$\|X_i\|, 1 \leq i \leq 3$	α	r_1	r_2	r_3	w_1	w_2	w_3	w_4
2	5	1	$\sqrt{6}$	$\frac{\sqrt{78}+\sqrt{6}}{6}$	$\frac{\sqrt{78}-\sqrt{6}}{6}$	$\frac{1}{540}$	$\frac{37}{540} - \frac{53\sqrt{13}}{3510}$	$\frac{37}{540} + \frac{53\sqrt{13}}{3510}$	$\frac{11}{36}$

Almost tight Gaussian 9-design of $\mathbb{R}^2$. Let $X = X_1 \cup X_2 \cup X_3 \cup \{\mathbf{0}\}$ and $X \setminus \{\mathbf{0}\}$ be a tight Euclidean 9-design of $\mathbb{R}^2$. Let $X_4 = \{\mathbf{0}\}$. Then the following parameters give X an almost tight Gaussian 9-design structure. See [31] and Figure A.3 for the precise structure of X.

n	$\|X_i\|, 1 \leq i \leq 3$	α	r_1	r_2	r_3	w_1	w_2	w_3	w_4
2	6	1	$2\sqrt{\frac{6-\sqrt{15}}{7}}$	2	$2\sqrt{\frac{6+\sqrt{15}}{7}}$	$\frac{605-151\sqrt{15}}{11520}$	$\frac{605+151\sqrt{15}}{11520}$	$\frac{1}{128}$	$\frac{31}{96}$

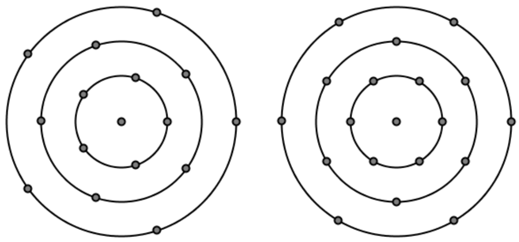


Figure A.4: Almost tight Gaussian $8, 9$-designs of $\mathbb{R}^2$ supported by 4 concentric spheres

A.2 The 4-th modified reproducing kernel

The 4-th modified reproducing kernel for the spherically symmetric integral $\mathcal{I}$ given by (1.5) is given as follows:

$$\begin{aligned}\tilde{K}_4(\boldsymbol{x}, \boldsymbol{y}) =& a_1 + a_2\langle \boldsymbol{x}, \boldsymbol{y}\rangle^2 + a_3\langle \boldsymbol{x}, \boldsymbol{y}\rangle^4 + a_4(\|\boldsymbol{x}\|^2 + \|\boldsymbol{y}\|^2) \\ &+ a_5\|\boldsymbol{x}\|^2\|\boldsymbol{y}\|^2 + a_6(\|\boldsymbol{x}\|^2 + \|\boldsymbol{y}\|^2)\langle \boldsymbol{x}, \boldsymbol{y}\rangle^2 + a_7\|\boldsymbol{x}\|^2\|\boldsymbol{y}\|^2\langle \boldsymbol{x}, \boldsymbol{y}\rangle^2 \\ &+ a_8(\|\boldsymbol{x}\|^4 + \|\boldsymbol{y}\|^4) + a_9\|\boldsymbol{x}\|^2\|\boldsymbol{y}\|^2(\|\boldsymbol{x}\|^2 + \|\boldsymbol{y}\|^2) + a_{10}\|\boldsymbol{x}\|^4\|\boldsymbol{y}\|^4, \quad \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n,\end{aligned}$$

where

$$\begin{aligned} a_1 &= \eta_0 \frac{\det A_4^{(2)}}{\det A_0^{(3)}}, \quad a_2 = \eta_0\eta_8 \frac{n(n+2)}{2\det A_4^{(2)}}, \quad a_3 = \frac{\eta_0 n(n+2)(n+4)(n+6)}{24\eta_8}, \\ a_4 &= -\frac{\eta_0(\eta_2\eta_8 - \eta_4\eta_6)}{\det A_0^{(3)}}, \quad a_5 = \frac{\eta_0\{2(\eta_2\eta_8 - \eta_4\eta_6)^2 - n\eta_8 \det A_0^{(3)}\}}{2\det A_4^{(2)} \det A_0^{(3)}}, \\ a_6 &= -\eta_0\eta_6 \frac{n(n+2)}{2\det A_4^{(2)}}, \quad a_7 = -\frac{n(n+2)\eta_0\{(n+4)\eta_4\eta_8 - (n+6)\eta_6^2\}}{4\eta_8 \det A_4^{(2)}}, \\ a_8 &= \eta_0 \frac{\det A_2^{(2)}}{\det A_0^{(3)}}, \quad a_9 = \frac{\eta_0\{-2\det A_2^{(2)}(\eta_2\eta_8 - \eta_4\eta_6) + n\eta_6 \det A_0^{(3)}\}}{2\det A_4^{(2)} \det A_0^{(3)}}, \end{aligned}$$

$$\begin{aligned} a_{10} =& \frac{\eta_0}{8\eta_8 \det A_4^{(2)} \det A_0^{(3)}} \Big(n(n+6)\eta_6^2(\eta_0\eta_6^2 - 2\eta_2\eta_4\eta_6 + \eta_4^3) \\ &- n(n+4)\eta_4\eta_6\eta_8(\eta_0\eta_6 - \eta_2\eta_4) + (n+4)\eta_8 \det A_2^{(2)} \Big((n-2)\eta_4^2 + 2\eta_2\eta_6\Big) \\ &+ n\eta_8 \det A_0^{(2)} \Big((n+2)\eta_4\eta_8 - (n+4)\eta_6^2\Big)\Big). \end{aligned}$$

Remark A.1. When $\Omega = B^n$ and $W(\boldsymbol{x}) = (1 - \|\boldsymbol{x}\|^2)^{\mu-1/2}$ with $\mu \geq 0$, we have

$$\begin{aligned}
\tilde{K}_4(\boldsymbol{x}, \boldsymbol{y}) = F_{\mu,n}\Big\{ &\frac{(n+2)(n+4)}{(n+2\mu+5)(n+2\mu+7)} + \frac{2(n+6)(2\mu+3)}{n+2\mu+7}\langle \boldsymbol{x}, \boldsymbol{y}\rangle^2 \\
&+ \frac{(2\mu+1)(2\mu+3)}{3}\langle \boldsymbol{x}, \boldsymbol{y}\rangle^4 - \frac{2(n+4)}{n+2\mu+7}(\|\boldsymbol{x}\|^2 + \|\boldsymbol{y}\|^2) + \frac{2(2n+2\mu+11)}{n+2\mu+7}\|\boldsymbol{x}\|^2\|\boldsymbol{y}\|^2 \\
&- 2(2\mu+3)(\|\boldsymbol{x}\|^2 + \|\boldsymbol{y}\|^2)\langle \boldsymbol{x}, \boldsymbol{y}\rangle^2 + 2(2\mu+3)\|\boldsymbol{x}\|^2\|\boldsymbol{y}\|^2\langle \boldsymbol{x}, \boldsymbol{y}\rangle^2 + \|\boldsymbol{x}\|^4 + \|\boldsymbol{y}\|^4 \\
&- 2\|\boldsymbol{x}\|^2\|\boldsymbol{y}\|^2(\|\boldsymbol{x}\|^2 + \|\boldsymbol{y}\|^2) + \|\boldsymbol{x}\|^4\|\boldsymbol{y}\|^4\Big\}, \qquad \text{(A.1)}
\end{aligned}$$

where $F_{\mu,n} = 8^{-1}(2\mu+1)^{-1}(2\mu+3)^{-1}(n+2\mu+1)(n+2\mu+3)(n+2\mu+5)(n+2\mu+7)$.

We observe that (A.1) can also be obtained from the Xu's compact formula ([64]). By using this compact formula, Xu [65] obtained the new lower bound for the number of points of cubature formula for the integral over B^n.

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