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ON THE SIEGEL–TATUZAWA THEOREM FOR A CLASS OF L-FUNCTIONS

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Abstract. We consider an effective lower bound of the Siegel–Tatuzawa type for general L-functions with three standard assumptions. We further assume three hypotheses in this paper that are essential in developing our argument. Under these assumptions and hypotheses, we prove a theorem of Siegel–Tatuzawa type for general L-functions. In particular, we prove such a theorem for symmetric power L-functions under certain assumptions.

1. Introduction

Generally speaking, the value at $s = 1$ of twisted L-functions is deeply connected with real zeros of it. In the case of Dirichlet L-functions, the Siegel theorem asserts that for any $\varepsilon > 0$, there exists a positive non-effective constant $C(\varepsilon)$ such that $L(1, \chi) > C(\varepsilon)d^{-\varepsilon}$, where χ is a real primitive Dirichlet character and d is the conductor of it (see Davenport [**4**]). This implies that $L(\sigma, \chi)$ does not have real zeros for $\sigma > 1 - C'(\varepsilon)d^{-\varepsilon}$, where the constant $C'(\varepsilon)$ is positive and non-effective. In 1951, Tatuzawa proved that $C(\varepsilon)$ in the theorem of Siegel can be effective except for at most one real character (see [**16**]).

Let $\mathfrak{f}$ be a Maass form with respect to the Hecke congruence subgroup $\Gamma_0(N)$, which is an eigenfunction of the Laplacian with the eigenvalue λ, and F the adjoint square lift of $\mathfrak{f}$ in the sense of Gelbart–Jacquet [**5**]. Then it holds that

$$L(s, \mathfrak{f} \otimes \mathfrak{f}) = \zeta(s)L_N(s)L(s, F),$$

where $L(s, \mathfrak{f} \otimes \mathfrak{f})$ is the Rankin–Selberg L-function associated with $\mathfrak{f}$, $L(s, F)$ is the L-function attached to F and $L_N(s)$ is the product of bad Euler factors. In [**8**], Hoffstein and Lockhart proved the existence of an effective constant $c(\varepsilon) = c(\varepsilon, F) > 0$ for which

$$L(1, F) \geq c(\varepsilon)(\lambda N)^{-\varepsilon}$$

holds for all F with at most one exception. This is an analogue of the Siegel–Tatuzawa theorem. Hoffstein and Lockhart mentioned that their method can be applied to holomorphic cusp forms of weight k. In fact, a modification of their method shows that an analogue of the Siegel–Tatuzawa theorem holds for the Rankin–Selberg L-function $L_{f\otimes f}$ associated with a holomorphic cusp form f. Recently it has been proved that a much stronger result, which

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asserts the non-existence of the Siegel zero, is true for a few types of L-functions (see the appendix of Hoffstein and Lockhart [**8**], and also Banks [**2**], Hoffstein and Ramakrishnan [**9**] and Ramakrishnan and Wang [**14**]). For instance, the above L-function $L_{f\otimes f}$ does not have the Siegel zero.

In order to prove the theorems of Siegel–Tatuzawa type, a standard method is to introduce a suitable auxiliary Dirichlet series which is constructed by a product of several related L-functions. In the above cases, such auxiliary Dirichlet series (see Davenport [**4**] and Hoffstein and Lockhart [**8**]) have a simple pole at $s = 1$ and non-negative coefficients. These conditions have been essentially used in the known classical proofs of the theorems of Siegel–Tatuzawa type.

In general, it is often not so difficult to find an auxiliary Dirichlet series constructed by a product of some L-functions and has non-negative coefficients, therefore we may assume that hypothesis (H1) below holds. However, it is not always the case that the order of the pole of it at $s = 1$ is simple. In this paper we propose a more flexible condition (H2) below on the order of the pole, which is suitable for the purpose of discussing the matter of theorems of Siegel–Tatuzawa type. If the order of the pole is odd and satisfies (H2), then our argument is basically similar to that of Hoffstein and Lockhart [**8**]. However, the even order case requires us to introduce further new ideas to prove the theorems of Siegel–Tatuzawa type.

In literature, some auxiliary series with poles of even order have been introduced to show Siegel-type theorems. For example, in the case of Rankin–Selberg L-functions $L_{f\otimes g}$ associated with two cusp forms $f \neq g$, the first author introduced an auxiliary Dirichlet series for the proof of the analogue of Siegel's theorem for $L_{f\otimes g}$ (see [**10**]). It has non-negative coefficients, but the order of its pole is two. (Recently, Ramakrishnan and Wang proved that $L_{f\otimes g}$ does not have the Siegel zero in [**14**].) In the case of L-function L_f associated with a cusp form f, Golubeva and Fomenko introduced an auxiliary Dirichlet series for the proof of the analogue of Siegel's theorem for L_f (see [**6**]). It also has positive coefficients and a double pole at $s = 1$. (Hoffstein and Ramakrishnan proved L_f does not have the Siegel zero in [**9**].) Golubeva and Fomenko also considered an auxiliary Dirichlet series which has positive coefficients and a pole of order 4 in [**7**]. However, those auxiliary series seem to be not suitable for handling theorems of Siegel–Tatuzawa type.

In this paper, we introduce a new type of auxiliary Dirichlet series and develop a method of proving the theorem of Siegel–Tatuzawa type for general L-functions.

Let $s = \sigma + it$, let χ be a real Dirichlet character of the modulus d and let k be a positive integer. We consider the general L-functions $L_k(s, \chi)$ defined by the Euler product of the form

$$L_k(s,\chi) = \prod_{p:\text{prime}} \prod_{j=1}^{J(k)} \left(1 - \frac{a_k(j,p)\chi(p)}{p^s}\right)^{-1},$$

where $J(k)$ is a positive integer and the coefficients $a_k(j, p)$ are complex numbers with $|a_k(j, p)| \leq 1$ satisfying $|a_k(j, p)| = 1$ for almost all prime p. This converges absolutely for $\sigma > 1$. Throughout this paper we assume the following.

(A1) The L-function $L_k(s, \chi)$ can be continued meromorphically to the whole plane. It has a possible pole at $s = 1$ if the character χ is principal, while it is entire if χ is non-principal.

(A2) There exists an absolute constant $0 < \delta_k < 1/2$ such that for any $\varepsilon > 0$ the vertical estimate

$$L_k(s, \chi) \ll \exp(\exp(\varepsilon|t|)), \quad |t| \to \infty,$$

holds uniformly in $-\delta_k \le \sigma \le 1 + \delta_k$.

(A3) Let χ be primitive. Then there exists a natural number $N(k)$, real numbers $\alpha_\nu(k) > 0$ and complex numbers $\beta_\nu(k, \chi)$ $(1 \le \nu \le N(k))$ such that the functional equation

$$\widetilde{L_k}(s, \chi) = W_{k,\chi} \widetilde{L_k}(1 - s, \chi)$$

holds with

$$\widetilde{L_k}(s, \chi) = Q_{k,\chi}^s \prod_{\nu=1}^{N(k)} \Gamma(\alpha_\nu(k)s + \beta_\nu(k, \chi)) L_k(s, \chi),$$

where $Q_{k,\chi}$ is a real number satisfying $Q_{k,\chi} \ll d^{\gamma(k)}$ ($\gamma(k)$ is a natural number) and $W_{k,\chi}$ is a complex number with $|W_{k,\chi}| = 1$.

These assumptions are standard and are usually (sometimes conjecturally) known to be satisfied.

Remark 1. Here, we assume the Ramanujan type of condition on the magnitude of $a_k(j, p)$. However, the condition seems to be unnecessary in many cases. In this paper, we use the estimate (3) below which has been proved by Carletti *et al* in [**3**]. They proved it under the Ramanujan condition, but Molteni proved it under a different type of assumption instead of the Ramanujan condition (see [**11**]).

The plan of this paper is as follows. In Section 2 we give the statement of our main theorem (Theorem 1). For the preparation of the proof of it, we show some lemmas in Section 3, and we prove the main theorem in Section 4. In Section 5, we apply the main theorem to the case of symmetric power L-functions and show their Siegel–Tatuzawa theorem (Theorem 2).

2. Statement of the main theorem

We fix an L-function $L_1(s, \chi)$ satisfying assumptions (A1)–(A3), where χ is a real Dirichlet character. Our purpose is to prove an analogue of the Siegel–Tatuzawa theorem for $L_1(s, \chi)$. We assume that there exist suitable L-functions $L_k(s, \chi)$ $(2 \le k \le K)$ satisfying assumptions (A1)–(A3) and natural numbers e_k such that the auxiliary function

$$\Lambda(s, \chi) = \prod_{k=1}^{K} L_k(s, \chi)^{e_k}$$

satisfies hypotheses (H1) and (H2) below. For $\sigma > 1$, we have

$$\log \Lambda(s, \chi) = \sum_{p:\text{prime}} \sum_{h=1}^{\infty} \frac{\chi^h(p)}{hp^{hs}} \sum_{k=1}^{K} \sum_{j=1}^{J(k)} a_k(j, p)^h e_k. \tag{1}$$

We put $b(h, p) = \sum_{k=1}^{K} \sum_{j=1}^{J(k)} a_k(j, p)^h e_k$. Then we have the following hypotheses.

(H1) *Positivity*. The coefficients $b(h, p)$ are non-negative.

(H2) *Pole order condition*. In the case $\chi = \chi_0$ is the trivial character, we use the notation $\Lambda(s)$ in place of $\Lambda(s, \chi_0)$, and we denote by r the order of the pole of $\Lambda(s)$ at $s = 1$. Then $1 \leq r \leq e_1$.

Concerning $L_1(s, \chi)$, we add one more hypothesis.

(H3) *Zero-free region off the real axis*. There exists an effective positive constant C_1 such that $L_1(s, \chi)$ has no zeros in the region

$$\sigma > 1 - \frac{C_1}{\log d}$$

for $|t| \leq 1, t \neq 0$.

Remark 2. The hypothesis (H3) is automatically satisfied in many cases. In fact, let

$$L_1^*(s, \chi) = \prod_{p:\text{prime}} \prod_{j=1}^{J(1)} \left(1 - \frac{\overline{a}_1(j, p)\chi(p)}{p^s}\right)^{-1},$$

where $\overline{a}_1(j, p)$ is the complex conjugate of $a_1(j, p)$. If $L_1^*(s, \chi) = L_1(s, \chi)$ or, more generally, if the function $\Lambda(s, \chi)$ has $L_1^*(s, \chi)$ as one of the factors, we can show that there is an effective positive constant C_1 such that $L_1(s, \chi) \neq 0$ for

$$\sigma > 1 - \frac{C_1}{\log(d(|t| + 2))}$$

except for the real axis, by using the ordinary method under hypotheses (H1) and (H2). Therefore, we in particular find that $L_1(s, \chi)$ satisfies hypothesis (H3). This will be explained at the end of Section 3.

The hypotheses (H1)–(H3) are crucial in the proof of the Siegel–Tatuzawa theorem for $L_1(s, \chi)$. Under these hypotheses, we can prove the following result, which is the main theorem of the present paper.

THEOREM 1. *We denote by X the set of all real primitive Dirichlet characters. If we find L-functions $L_k(s, \chi)$ satisfying assumptions* (A1)–(A3) *and hypotheses* (H1)–(H3)*, then for any $\varepsilon > 0$, there exists an effective positive constant $C(\varepsilon)$ such that*

$$|L_1(1, \chi)| > \frac{C(\varepsilon)}{d^{\varepsilon}} \tag{2}$$

for any $\chi \in X$, except for at most one possible element of X. Here d is the conductor of χ.

Remark 3. The important point in the theorem is that the constant $C(\varepsilon)$ is effective. If we do not require its effectiveness, the inequality of the form (2) can be shown for any $\chi \in X$ without exception. This claim is an analogue of Siegel's theorem. As explained at the end of Section 4, this can be deduced easily from Theorem 1.

3. Preliminaries

First of all, we recall some known facts. Assumptions (A1)–(A3) imply that our $L_k(s, \chi)$ is an example of 'general L-functions' in the sense of Carletti *et al* [**3**] if χ is primitive.

Therefore, we have

$$L_k^{(n)}(1, \chi) \ll d^{\varepsilon} \tag{3}$$

for any non-negative integer n and an arbitrarily small number $\varepsilon > 0$ where χ is a non-principal Dirichlet character of the modulus d. This can be easily reduced to the case when χ is primitive, and follows from Theorem 2 in [**3**] for primitive χ. For $-1 \leq \sigma \leq 2$ it follows uniformly that

$$-\frac{L_k'}{L_k}(s, \chi) = \frac{m(k)}{s-1} - \sum_{|t-\gamma| \leq 1} \frac{1}{s-\rho} + O(\log(d(|t|+2))), \tag{4}$$

where the order of the pole of $L_k(s, \chi)$ at $s = 1$ is denoted by $m(k)$, $\rho = \beta + i\gamma$ runs over the zeros of $L_k(s, \chi)$ and χ is a Dirichlet character of the modulus d. This is again reduced to the primitive case, which is Lemma 4 in [**3**].

We put

$$\phi(s, \chi) = \Lambda(s)\Lambda(s, \chi),$$
$$\phi(s, \chi_1, \chi_2) = \Lambda(s)\Lambda(s, \chi_1)\Lambda(s, \chi_2)\Lambda(s, \chi_1\chi_2),$$

where the Dirichlet characters χ, χ_1 and χ_2 are primitive and real with the conductors $d > 1$, $d_1 > 1$ and $d_2 > 1$, respectively. Moreover $\chi_1\chi_2$ is not trivial. These functions are the analogue of classical auxiliary functions in the theory of the zeros of Dirichlet L-functions. A novelty of the present method is to introduce the new auxiliary function

$$\psi(s, \chi) = \zeta(s)\phi(s, \chi),$$

where $\zeta(s)$ is the Riemann zeta function. This function is helpful for the proof of the main theorem when there is no real zero of $L_1(s, \chi)$ near $s = 1$. The functions $\phi(s, \chi)$, $\phi(s, \chi_1, \chi_2)$ and $\psi(s, \chi)$ are convergent absolutely for $\sigma > 1$. We write their Dirichlet series expansion as follows:

$$\phi(s, \chi) = \sum_{n=1}^{\infty} \delta(n) n^{-s},$$
$$\phi(s, \chi_1, \chi_2) = \sum_{n=1}^{\infty} \tau(n) n^{-s},$$
$$\psi(s, \chi) = \sum_{n=1}^{\infty} \omega(n) n^{-s}.$$

From (1) we have

$$\log \phi(s, \chi) = \sum_{p:\text{prime}} \sum_{h=1}^{\infty} \frac{b(h, p)}{h}(1 + \chi^h(p))p^{-hs} \tag{5}$$

and

$$\log \phi(s, \chi_1, \chi_2) = \sum_{p:\text{prime}} \sum_{h=1}^{\infty} \frac{b(h, p)}{h}(1 + \chi_1^h(p))(1 + \chi_2^h(p))p^{-hs}, \tag{6}$$

hence hypothesis (H1) implies that $\delta(1) = \tau(1) = \omega(1) = 1$ and $\delta(n)$, $\tau(n)$ and $\omega(n)$ are non-negative real numbers for $n > 1$. We prove the following four lemmas on these functions. The residue of the function $f(s)$ at $s = a$ is denoted by $\mathrm{Res}_{s=a}(f(s))$.

LEMMA 1. *Let β_0 be a real number with $3/4 < \beta_0 < 1$ and ℓ be a sufficiently large natural number. Then there exists an effective constant $c_1 > 0$, independent of β_0, such that*

$$\frac{\phi(\beta_0, \chi)}{\ell!} + \operatorname*{Res}_{s=1-\beta_0}\left(\frac{\phi(s+\beta_0, \chi)d^{c_1 s}}{s(s+1)\cdots(s+\ell)}\right) \gg 1.$$

LEMMA 2. *The definitions of β_0 and ℓ are the same as in Lemma 1. Then there exists an effective constant $c_2 > 0$, independent of β_0, such that*

$$\frac{\phi(\beta_0, \chi_1, \chi_2)}{\ell!} + \operatorname*{Res}_{s=1-\beta_0}\left(\frac{\phi(s+\beta_0, \chi_1, \chi_2)(d_1 d_2)^{c_2 s}}{s(s+1)\cdots(s+\ell)}\right) \gg 1.$$

LEMMA 3. *The definitions of β_0 and ℓ are the same as in Lemma 1. Then there exists an effective constant $c_3 > 0$, independent of β_0, such that*

$$\frac{\psi(\beta_0, \chi)}{\ell!} + \operatorname*{Res}_{s=1-\beta_0}\left(\frac{\psi(s+\beta_0, \chi)d^{c_3 s}}{s(s+1)\cdots(s+\ell)}\right) \gg 1.$$

LEMMA 4. *There exists an effective positive constant c_4 such that $\phi(s, \chi_1, \chi_2)$ has at most r real zeros (counted with multiplicity) in the range*

$$1 - \frac{c_4}{\log d_1 d_2} < \sigma < 1.$$

Lemmas 1, 2 and 3 are analogues of Proposition 1.1 in [**8**]. We only show the proofs of Lemmas 1 and 4, because the proofs of Lemmas 2 and 3 are similar to that of Lemma 1.

Proof of Lemma 1. Let $x > 2$ be a non-integer. It is known that

$$\frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} \frac{x^s\,ds}{s(s+1)(s+2)\cdots(s+\ell)} = \begin{cases} \dfrac{1}{\ell!}\left(1-\dfrac{1}{x}\right)^{\ell} & \text{if } x > 1, \\ 0 & \text{if } 0 < x \le 1, \end{cases}$$

for any integer ℓ. By using the above formula, we have

$$\frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} \frac{\phi(s+\beta_0, \chi)x^s}{s(s+1)(s+2)\cdots(s+\ell)}\,ds = \sum_{n<x}\frac{\delta(n)}{n^{\beta_0}}\frac{1}{\ell!}\left(1-\frac{n}{x}\right)^{\ell} > \frac{1}{\ell! 2^{\ell}} \tag{7}$$

because $\delta(n)$ are non-negative and $\delta(1) = 1$. From assumptions (A2) and (A3) and the Phragmén–Lindelöf theorem, we obtain

$$L_k(\sigma + it, \chi) \ll d^{\gamma(k)(1+\varepsilon-\sigma)}(1+|t|)^{A(k)(1+\varepsilon-\sigma)}$$

and

$$L_k(\sigma + it) \ll (1+|t|)^{A(k)(1+\varepsilon-\sigma)} \quad (t \neq 0)$$

for $-\varepsilon < \sigma < 1+\varepsilon$, where $A(k) = \sum_{\nu=1}^{N(k)} \alpha_\nu(k)$. Hence,

$$\phi(s, \chi) \ll d^{\gamma(K)(1+\varepsilon-\sigma)}(1+|t|)^{2A(K)(1+\varepsilon-\sigma)},$$

where $\gamma(K) = \sum_{k=1}^{K}\gamma(k)e_k$ and $A(K) = \sum_{k=1}^{K} A(k)e_k$. From this estimate, the shifting of the path of integration in (7) to $\sigma = 1/2 - \beta_0$ shows that

$$\begin{aligned} &\frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} \frac{\phi(s+\beta_0, \chi)x^s}{s(s+1)(s+2)\cdots(s+\ell)}\,ds \\ &\quad = \frac{\phi(\beta_0, \chi)}{\ell!} + \operatorname*{Res}_{s=1-\beta_0}\left(\frac{\phi(s+\beta_0, \chi)x^s}{s(s+1)(s+2)\cdots(s+\ell)}\right) + O(d^{\gamma(K)(1/2+\varepsilon)}x^{1/2-\beta_0}) \end{aligned} \tag{8}$$

for all $\ell \geq \ell_0$, where $\ell_0 = \ell_0(A(K))$ is taken to be sufficiently large. Then results (7) and (8) imply that

$$\frac{\phi(\beta_0, \chi)}{\ell!} + \operatorname*{Res}_{s=1-\beta_0}\left(\frac{\phi(s+\beta_0, \chi)x^s}{s(s+1)\cdots(s+\ell)}\right) + O(d^{\gamma(K)(1/2+\varepsilon)}x^{1/2-\beta_0}) \geq \frac{1}{\ell! 2^\ell}. \tag{9}$$

We now take $x = d^{c_1}$. It is easy to find an effective constant $c_1 > 0$ such that the error term on the left-hand side of (9) is smaller than $(\ell! 2^{\ell+1})^{-1}$. We complete the proof of Lemma 1. □

Proof of Lemma 4. From (4) we have

$$-\frac{\phi'}{\phi}(s, \chi_1, \chi_2) = \frac{r}{s-1} - \sum_{|t-\gamma|\leq 1} \frac{1}{s-\rho} + O(\log(d_1 d_2(|t|+2))) \tag{10}$$

for $-1 \leq \sigma \leq 2$, where $\rho = \beta + i\gamma$ runs over the zeros of $\phi(s, \chi_1, \chi_2)$. We know that

$$-\frac{\phi'}{\phi}(\sigma, \chi_1, \chi_2) \geq 0 \tag{11}$$

for $\sigma > 1$ from (6). If we assume that there exist $r+1$ real zeros $\rho_1 \leq \rho_2 \leq \cdots \leq \rho_{r+1}$ of $\phi(s, \chi_1, \chi_2)$, then we obtain

$$0 \leq \frac{r}{\sigma-1} - \sum_{j=1}^{r+1} \frac{1}{\sigma-\rho_j} + A_1 \log d_1 d_2 \leq \frac{r}{\sigma-1} - \frac{r+1}{\sigma-\rho_1} + A_1 \log d_1 d_2$$

for $\sigma > 1$ by using (10) and (11), where A_1 is an effective positive constant. We put $0 < A_2 < 1/A_1$ and $\sigma = 1 + A_2/\log d_1 d_2$. Then it follows that there exists an effective positive constant c_4 such that

$$\rho_1 \leq 1 - \frac{c_4}{\log d_1 d_2}.$$

□

Lastly in this section we explain how to show the claim of Remark 2. This can be done quite similarly to the argument of Davenport [**4**, Section 14]. In fact, if $\rho = \beta + i\gamma$ is a zero of $L_1(s, \chi)$ with $|\gamma| \geq 1/2$, then, using (4), we can show

$$\frac{4e_1}{\sigma-\beta} \leq \frac{3r}{\sigma-1} + \Re\left(\frac{r}{\sigma-1+2i\gamma}\right) + O(\log(d(|\gamma|+2)))$$

for $\sigma > 1$, from which

$$\beta \leq 1 - \frac{C_1}{\log(d(|\gamma|+2))}$$

follows under hypothesis (H2). If $0 < |\gamma| < 1/2$, we use the fact that not only ρ but also $\overline{\rho} = \beta - i\gamma$ are the zeros of $\Lambda(s, \chi)$. This follows from the assumption of Remark 2. By using this fact we obtain

$$\begin{aligned}\frac{8e_1}{\sigma-\beta} &\leq \frac{3r}{\sigma-1} + \Re\left(\frac{r}{\sigma-1+2i\gamma}\right) + O(\log(d(|\gamma|+2))) \\ &\leq \frac{4r}{\sigma-1} + O(\log(d(|\gamma|+2))),\end{aligned}$$

which implies the desired result.

4. Proof of the main theorem

Proof of Theorem 1. We fix a sufficiently small real number ε_1 with $0 < \varepsilon_1 < 1/4$. We divide the argument into three cases.

Case 1. If r is an odd (respectively, even) number, we consider the case that $\phi(s, \chi)$ has an even (respectively, odd) number of real zeros in the range $1 - \varepsilon_1 \le \sigma \le 1$. We see that $\phi(\sigma, \chi)$ tends to $-\infty$ (respectively, $+\infty$) as $\sigma \to 1 - 0$ if r is an odd (respectively, even) number. Therefore, in any case we have $\phi(1 - \varepsilon_1, \chi) \le 0$. Taking $\beta_0 = 1 - \varepsilon_1$ in Lemma 1, we obtain

$$\operatorname*{Res}_{s=\varepsilon_1}\left(\frac{\phi(s + 1 - \varepsilon_1)d^{c_1 s}}{s(s+1)\cdots(s+\ell)}\right) \gg 1. \tag{12}$$

We write the Laurant expansion of $\phi(s, \chi)$ at $s = 1$ as

$$\phi(s, \chi) = \sum_{j=-r}^{\infty} \alpha_j (s-1)^j,$$

where $\alpha_{-r} \ne 0$. Then we see that

$$\operatorname*{Res}_{s=\varepsilon_1}\left(\frac{\phi(s + 1 - \varepsilon_1)d^{c_1 s}}{s(s+1)\cdots(s+\ell)}\right) = \sum_{\substack{-r\le j, 0\le m, n_0, \ldots, n_\ell \\ j+m+n_0+\cdots+n_\ell=-1}} \frac{(-1)^{n_0+\cdots+n_\ell}\alpha_j d^{c_1\varepsilon_1}(c_1 \log d)^m}{m!{\varepsilon_1}^{n_0+1}(\varepsilon_1+1)^{n_1+1}\cdots(\varepsilon_1+\ell)^{n_\ell+1}}. \tag{13}$$

From (12) and (13) we obtain

$$d^{c_1\varepsilon_1} \sum_{\substack{-r\le j,\ 0\le m, n_0, \ldots, n_\ell \\ j+m+n_0+\cdots+n_\ell=-1}} \alpha_j (\log d)^m \gg_{\varepsilon_1} 1. \tag{14}$$

Next we consider the coefficients α_j for $-r \le j \le -1$. We write the Laurant expansions at $s = 1$ of $\Lambda(s)$ and $L_k(s, \chi)$ as

$$\Lambda(s) = \sum_{j=-r}^{\infty} \lambda_j (s-1)^j$$

and

$$L_k(s, \chi) = \sum_{m_k=0}^{\infty} \frac{L_k^{(m_k)}(1, \chi)}{m_k!}(s-1)^{m_k},$$

respectively. We have

$$\begin{aligned}\phi(s, \chi) = &\left(\sum_{j=-r}^{\infty} \lambda_j (s-1)^j\right) \\ &\times \left\{\sum_{l_1=0}^{e_1} \binom{e_1}{e_1 - l_1} L_1(1, \chi)^{e_1 - l_1}\left(\sum_{m_1=1}^{\infty} \frac{L_1^{(m_1)}(1, \chi)}{m_1!}(s-1)^{m_1}\right)^{l_1}\right\} \\ &\times \prod_{k=2}^{K}\left(\sum_{m_k=0}^{\infty} \frac{L_k^{(m_k)}(1, \chi)}{m_k!}(s-1)^{m_k}\right)^{e_k}.\end{aligned} \tag{15}$$

Therefore, we see that the terms $\alpha_j(s-1)^j$, $-r \le j \le -1$, appear in the expansion of the following part of the right-hand side of (15):

$$\begin{aligned}
&\Bigg(\sum_{j=-r}^{-1} \lambda_j (s-1)^j\Bigg)\Bigg\{\sum_{l_1=0}^{r-1} \binom{e_1}{e_1-l_1} L_1(1,\chi)^{e_1-l_1}\Bigg(\sum_{m_1=1}^{r-1} \frac{L_1^{(m_1)}(1,\chi)}{m_1!}(s-1)^{m_1}\Bigg)^{l_1}\Bigg\}\\
&\quad\times \prod_{k=2}^{K}\Bigg(\sum_{m_k=0}^{r-1} \frac{L_k^{(m_k)}(1,\chi)}{m_k!}(s-1)^{m_k}\Bigg)^{e_k}\\
&\quad= L_1(1,\chi)\Bigg(\sum_{j=-r}^{-1} \lambda_j (s-1)^j\Bigg)\\
&\quad\times \Bigg\{\sum_{l_1=0}^{r-1} \binom{e_1}{e_1-l_1} L_1(1,\chi)^{e_1-l_1-1}\Bigg(\sum_{m_1=1}^{r-1} \frac{L_1^{(m_1)}(1,\chi)}{m_1!}(s-1)^{m_1}\Bigg)^{l_1}\Bigg\}\\
&\quad\times \prod_{k=2}^{K}\Bigg(\sum_{m_k=0}^{r-1} \frac{L_k^{(m_k)}(1,\chi)}{m_k!}(s-1)^{m_k}\Bigg)^{e_k}. \qquad (16)
\end{aligned}$$

Recalling hypothesis (H2), we see that $e_1 - l_1 - 1 \ge 0$. By using (3) and (16), we obtain that $\alpha_j \ll_{\varepsilon_1} |L_1(1,\chi)| d^{c\varepsilon_1}$ for $-r \le j \le -1$. Here, and in what follows, c is an effective positive constant, not necessarily the same at each occurrence. This estimate and (14) imply (2).

Case 2. If r is an even (respectively, odd) number, we consider the case that $\phi(s,\chi)$ has an even (respectively, odd) number of real zeros in the range $1-\varepsilon_1 \le \sigma \le 1$ and $L_1(s,\chi)$ has no real zeros in the same range. Then $\psi(s,\chi)$ also has an even (respectively, odd) number of real zeros if r is an even (respectively, odd) number. The order of the pole of $\psi(s,\chi)$ at $s=1$ is $r+1$, hence $\psi(\sigma,\chi)$ tends to $-\infty$ (respectively, $+\infty$) as $\sigma \to 1-0$ if r is an even (respectively, odd) number. Therefore, we have $\psi(1-\varepsilon_1,\chi) \le 0$. Hence, taking $\beta_0 = 1-\varepsilon_1$ in Lemma 3, we obtain

$$\operatorname*{Res}_{s=\varepsilon_1}\left(\frac{\psi(s+1-\varepsilon_1,\chi)d^{c_3 s}}{s(s+1)\cdots(s+\ell)}\right) \gg 1.$$

We write the Laurant expansion of $\psi(s,\chi)$ at $s=1$ as

$$\psi(s,\chi) = \sum_{j=-r-1}^{\infty} \alpha'_j (s-1)^j,$$

where $\alpha'_{-r-1} \ne 0$. We obtain that

$$d^{c_3\varepsilon_1} \sum_{\substack{-r-1\le j,\, 0\le m, n_0,\ldots n_\ell\\ j+m+n_0+\cdots+n_\ell=-1}} \alpha'_j (\log d)^m \gg_{\varepsilon_1} 1, \qquad (17)$$

by using the same argument as in the proof of (14). We write (near $s=1$)

$$\zeta(s)\Lambda(s) = \sum_{j=-r-1}^{\infty} \lambda'_j (s-1)^j.$$

It is clear that $\lambda'_{-r-1} \neq 0$. We have

$$\begin{aligned}
&\psi(s,\chi)\\
&=\Bigg(\sum_{j=-r-1}^{\infty}\lambda'_j(s-1)^j\Bigg)\Bigg\{\sum_{l_1=0}^{e_1}\binom{e_1}{e_1-l_1}L_1(1,\chi)^{e_1-l_1}\Bigg(\sum_{m_1=1}^{\infty}\frac{L_1^{(m_1)}(1,\chi)}{m_1!}(s-1)^{m_1}\Bigg)^{l_1}\Bigg\}\\
&\quad\times\prod_{k=2}^{K}\Bigg(\sum_{m_k=0}^{\infty}\frac{L_k^{(m_k)}(1,\chi)}{m_k!}(s-1)^{m_k}\Bigg)^{e_k}. \qquad (18)
\end{aligned}$$

We rearrange the right-hand side as follows:

$$\begin{aligned}
\psi(s,\chi)&=\Bigg(\sum_{j=-r-1}^{\infty}\lambda'_j(s-1)^j\Bigg)\Bigg\{\Bigg(\sum_{m_1=1}^{\infty}\frac{L_1^{(m_1)}(1,\chi)}{m_1!}(s-1)^{m_1}\Bigg)^{e_1}\\
&\qquad+\sum_{l_1=0}^{e_1-1}\binom{e_1}{e_1-l_1}L_1(1,\chi)^{e_1-l_1}\Bigg(\sum_{m_1=1}^{\infty}\frac{L_1^{(m_1)}(1,\chi)}{m_1!}(s-1)^{m_1}\Bigg)^{l_1}\Bigg\}\\
&\qquad\times\prod_{k=2}^{K}\Bigg(\sum_{m_k=0}^{\infty}\frac{L_k^{(m_k)}(1,\chi)}{m_k!}(s-1)^{m_k}\Bigg)^{e_k}\\
&=\Bigg(\sum_{j=e_1-r-1}^{\infty}\lambda'_{j-e_1}(s-1)^j\Bigg)\Bigg(\sum_{m_1=1}^{\infty}\frac{L_1^{(m_1)}(1,\chi)}{m_1!}(s-1)^{m_1-1}\Bigg)^{e_1}\\
&\qquad\times\prod_{k=2}^{K}\Bigg(\sum_{m_k=0}^{\infty}\frac{L_k^{(m_k)}(1,\chi)}{m_k!}(s-1)^{m_k}\Bigg)^{e_k}+\Bigg(\sum_{j=-r-1}^{\infty}\lambda'_j(s-1)^j\Bigg)L_1(1,\chi)\\
&\qquad\times\Bigg\{\sum_{l_1=0}^{e_1-1}\binom{e_1}{e_1-l_1}L_1(1,\chi)^{e_1-l_1-1}\Bigg(\sum_{m_1=1}^{\infty}\frac{L_1^{(m_1)}(1,\chi)}{m_1!}(s-1)^{m_1}\Bigg)^{l_1}\Bigg\}\\
&\qquad\times\prod_{k=2}^{K}\Bigg(\sum_{m_k=0}^{\infty}\frac{L_k^{(m_k)}(1,\chi)}{m_k!}(s-1)^{m_k}\Bigg)^{e_k}. \qquad (19)
\end{aligned}$$

Therefore, we see that the terms $\alpha'_j(s-1)^j$, $-r-1\leq j\leq -2$, appear in the expansion of the following part of the right-hand side of (18):

$$\begin{aligned}
&\Bigg(\sum_{j=-r-1}^{-2}\lambda'_j(s-1)^j\Bigg)L_1(1,\chi)\\
&\quad\times\Bigg\{\sum_{l_1=0}^{r-1}\binom{e_1}{e_1-l_1}L_1(1,\chi)^{e_1-l_1-1}\Bigg(\sum_{m_1=1}^{r-1}\frac{L_1^{(m_1)}(1,\chi)}{m_1!}(s-1)^{m_1}\Bigg)^{l_1}\Bigg\}\\
&\quad\times\prod_{k=2}^{K}\Bigg(\sum_{m_k=0}^{r-1}\frac{L_k^{(m_k)}(1,\chi)}{m_k!}(s-1)^{m_k}\Bigg)^{e_k}. \qquad (20)
\end{aligned}$$

Recalling hypothesis (H2), we see that $e_1-l_1-1\geq 0$. By using (3) and (20) we obtain that $\alpha'_j \ll_{\varepsilon_1} |L_1(1,\chi)|d^{c\varepsilon_1}$ for $-r-1\leq j\leq -2$. We next consider the term $\alpha'_{-1}(s-1)^{-1}$.

We see that it appears in the expansion of the following part of the right-hand side of (19):

$$\lambda'_{-r-1}(s-1)^{e_1-r-1}L'_1(1,\chi)^{e_1}\prod_{k=2}^{K}L_k(1,\chi)^{e_k}+\Bigg(\sum_{j=-r-1}^{-1}\lambda'_j(s-1)^j\Bigg)L_1(1,\chi)$$
$$\times\Bigg\{\sum_{l_1=0}^{r}\binom{e_1}{e_1-l_1}L_1(1,\chi)^{e_1-l_1-1}\Bigg(\sum_{m_1=1}^{r}\frac{L_1^{(m_1)}(1,\chi)}{m_1!}(s-1)^{m_1}\Bigg)^{l_1}\Bigg\}$$
$$\times\prod_{k=2}^{K}\Bigg(\sum_{m_k=0}^{r}\frac{L_k^{(m_k)}(1,\chi)}{m_k!}(s-1)^{m_k}\Bigg)^{e_k},\tag{21}$$

because we have that $e_1-r-1\geq -1$ from hypothesis (H2). By using (3) and (21), we see that $\alpha'_{-1}\ll |L_1(1,\chi)|d^{c\varepsilon_1}$, if $L'_1(1,\chi)\ll_{\varepsilon_1}|L_1(1,\chi)|d^{c\varepsilon_1}$ is true, the proof of which is given below. Then we obtain that $\alpha'_j\ll_{\varepsilon_1}|L_1(1,\chi)|d^{c\varepsilon_1}$ for $-r-1\leq j\leq -1$. These estimates and (17) imply (2).

Therefore, the remaining task is to prove that $L'_1(1,\chi)\ll_{\varepsilon_1}|L_1(1,\chi)|d^{c\varepsilon_1}$. From (4), we have

$$\frac{L'_1}{L_1}(1,\chi)=\sum_{\rho:\text{real}}\frac{1}{1-\rho}+\sum_{0<|\Im(\rho)|<1}\frac{1}{1-\rho}+O(\log d),\tag{22}$$

were ρ runs over the zeros of $L_1(s,\chi)$. We recall that there is no real zero of $L_1(s,\chi)$ in the range $1-\varepsilon_1\leq\sigma\leq 1$ by the assumption of Case 2. This implies that

$$\sum_{\rho:\text{real}}\frac{1}{1-\rho}\leq\sum_{\rho:\text{real}}\frac{1}{\varepsilon_1}\ll_{\varepsilon_1}\log d\tag{23}$$

by using Theorem 1 of Perelli [**13**]. Also we have

$$\sum_{0<|\Im(\rho)|<1}\frac{1}{1-\rho}\ll\sum_{0<|\Im(\rho)|<1}\frac{\log d}{C_1}\ll(\log d)^2\tag{24}$$

by using hypothesis (H3) and Theorem 1 of Perelli [**13**]. Consequently, we obtain that $L'_1(1,\chi)\ll_{\varepsilon_1}|L_1(1,\chi)|d^{c\varepsilon_1}$ from (22), (23) and (24).

Case 3. Finally we consider the case when r is an even (respectively, odd) number, $\phi(s,\chi)$ has an even (respectively, odd) number of real zeros in the range $1-\varepsilon_1\leq\sigma\leq 1$ and $L_1(s,\chi)$ has some real zeros in the same range. We denote by X^* the set of all real Dirichlet characters satisfying the assumptions of Case 3. There is a minimal conductor of the characters in X^* and we denote it by d_2. We fix a character χ_2 with modulus d_2 in X^*. We show the theorem for $L_1(s,\chi_1)$ with any character $\chi_1\in X^*$ and $\chi_1\neq\chi_2$. Let d_1 be the modulus of χ_1. Now we know that $L_1(\sigma,\chi_1)L_1(\sigma,\chi_2)$ has at least two real zeros ρ_1,ρ_2 in the range $1-\varepsilon_1\leq\sigma\leq 1$. We may assume $\rho_1\leq\rho_2$. Then $\phi(s,\chi_1,\chi_2)$ has at least $2e_1$ $(>r)$ real zeros. Taking $\beta_0=\rho_1$ in Lemma 2, we obtain

$$\operatorname*{Res}_{s=1-\rho_1}\Bigg(\frac{\phi(s+\rho_1,\chi_1,\chi_2)(d_1d_2)^{c_2s}}{s(s+1)\cdots(s+\ell)}\Bigg)\gg 1.$$

We write the Laurant expansion of $\phi(s,\chi_1,\chi_2)$ at $s=1$ as

$$\phi(s,\chi_1,\chi_2)=\sum_{j=-r}^{\infty}\alpha''_j(s-1)^j.$$

We have that $\varepsilon_1 \geq 1 - \rho_1 \geq c_4/(\log d_1 d_2)$ from Lemma 4 and the assumption of Case 3. By using these inequalities and the same argument as in the proof of (14), we obtain that

$$(d_1 d_2)^{c_2 \varepsilon_1} \sum_{\substack{-r \leq j, 0 \leq m, n_0, \ldots, n_\ell \\ j+m+n_0+\cdots+n_\ell=-1}} \alpha_j''(\log d_1 d_2)^{m+n_0+1} \gg_{\varepsilon_1} 1. \tag{25}$$

We have

$$\begin{aligned}
&\phi(s, \chi_1, \chi_2) \\
&= \left(\sum_{j=-r}^{\infty} \lambda_j (s-1)^j \right) \prod_{k_1=1}^{K} \left(\sum_{m_{k_1}=0}^{\infty} \frac{L_{k_1}^{(m_{k_1})}(1, \chi_1)}{m_{k_1}!} (s-1)^{m_{k_1}} \right)^{e_{k_1}} \\
&\quad \times \prod_{k_2=1}^{K} \left(\sum_{m_{k_2}=0}^{\infty} \frac{L_{k_2}^{(m_{k_2})}(1, \chi_2)}{m_{k_2}!} (s-1)^{m_{k_2}} \right)^{e_{k_2}} \prod_{k_3=1}^{K} \left(\sum_{m_{k_3}=0}^{\infty} \frac{L_{k_3}^{(m_{k_3})}(1, \chi_1\chi_2)}{m_{k_3}!} (s-1)^{m_{k_3}} \right)^{e_{k_3}} \\
&= \left(\sum_{j=-r}^{\infty} \lambda_j (s-1)^j \right) \left\{ \sum_{l_1=0}^{e_1} \binom{e_1}{e_1 - l_1} L_1(1, \chi)^{e_1 - l_1} \left(\sum_{m_1=1}^{\infty} \frac{L_1^{(m_1)}(1, \chi_1)}{m_1!} (s-1)^{m_1} \right)^{l_1} \right\} \\
&\quad \times \prod_{k_1=2}^{K} \left(\sum_{m_{k_1}=0}^{\infty} \frac{L_{k_1}^{(m_{k_1})}(1, \chi_1)}{m_{k_1}!} (s-1)^{m_{k_1}} \right)^{e_{k_1}} \prod_{k_2=1}^{K} \left(\sum_{m_{k_2}=0}^{\infty} \frac{L_{k_2}^{(m_{k_2})}(1, \chi_2)}{m_{k_2}!} (s-1)^{m_{k_2}} \right)^{e_{k_2}} \\
&\quad \times \prod_{k_3=1}^{K} \left(\sum_{m_{k_3}=0}^{\infty} \frac{L_{k_3}^{(m_{k_3})}(1, \chi_1\chi_2)}{m_{k_3}!} (s-1)^{m_{k_3}} \right)^{e_{k_3}}.
\end{aligned}$$

Therefore, we see that the terms $\alpha_j''(s-1)^j$, $-r \leq j \leq -1$, appear in the expansion of the following part of the right-hand side of the above formula:

$$\begin{aligned}
&\left(\sum_{j=-r}^{-1} \lambda_j (s-1)^j \right) L_1(1, \chi_1) \\
&\quad \times \left\{ \sum_{l_1=0}^{r-1} \binom{e_1}{e_1 - l_1} L_1(1, \chi)^{e_1 - l_1 - 1} \left(\sum_{m_1=1}^{r-1} \frac{L_1^{(m_1)}(1, \chi_1)}{m_1!} (s-1)^{m_1} \right)^{l_1} \right\} \\
&\quad \times \prod_{k_1=2}^{K} \left(\sum_{m_{k_1}=0}^{r-1} \frac{L_{k_1}^{(m_{k_1})}(1, \chi_1)}{m_{k_1}!} (s-1)^{m_{k_1}} \right)^{e_{k_1}} \prod_{k_2=1}^{K} \left(\sum_{m_{k_2}=0}^{r-1} \frac{L_{k_2}^{(m_{k_2})}(1, \chi_2)}{m_{k_2}!} (s-1)^{m_{k_2}} \right)^{e_{k_2}} \\
&\quad \times \prod_{k_3=1}^{K} \left(\sum_{m_{k_3}=0}^{r-1} \frac{L_{k_3}^{(m_{k_3})}(1, \chi_1\chi_2)}{m_{k_3}!} (s-1)^{m_{k_3}} \right)^{e_{k_3}}.
\end{aligned}$$

We see that $e_1 - l_1 - 1 \geq 0$ from hypothesis (H2), hence we have

$$\alpha_j'' \ll_{\varepsilon_1} |L_1(1, \chi_1)|(d_1 d_2)^{c\varepsilon_1} \ll_{\varepsilon_1} |L_1(1, \chi_1)|(d_1)^{c\varepsilon_1}$$

by using (3). This estimate and (25) imply (2). The proof of Theorem 1 is now complete. □

Here we mention how to prove the claim stated in Remark 3. We put $F(s) = \log \phi(s, \chi)$. We can write

$$F(s) = \sum_{n=1}^{\infty} \delta'(n) n^{-s} \tag{26}$$

in $\sigma > 1$ from (5), where $\delta'(n) \geq 0$. We prove that the Dirichlet series (26) has the finite abscissa of convergence σ_c. Suppose the contrary. Then the Dirichlet series (26) is convergent in the whole plane. This implies that $F(\sigma + it) = O(1)$ as $|t| \to \infty$. However, we have

$$\begin{aligned}\log \Lambda(s, \chi) = \sum_{k=1}^{K} e_k [\log W_{k,\chi} + (1-2s) \log Q_{k,\chi} + \sum_{\nu=1}^{N(k)} \{\log \Gamma(\alpha_\nu(k)(1-s) + \beta_\nu(k, \chi)) \\ - \log \Gamma(\alpha_\nu(k)s + \beta_\nu(k, \chi))\} + \log L_k(1-s, \chi)]\end{aligned}$$

by using the functional equation for $L_k(s, \chi)$. If $t \geq 2$ and $\sigma < 0$ we have

$$\begin{aligned}&\log \Gamma(\alpha_\nu(k)(1-s) + \beta_\nu(k, \chi)) - \log \Gamma(\alpha_\nu(k)s + \beta_\nu(k, \chi)) \\ &\quad = \log \Gamma(\alpha_\nu(k)(1-s) + \beta_\nu(k, \chi)) + \log \Gamma(1 - \alpha_\nu(k)s - \beta_\nu(k, \chi)) \\ &\qquad - \log \pi + \log \sin(\pi(\alpha_\nu(k)s + \beta_\nu(k, \chi))) \\ &\quad \sim -2i\alpha_\nu(k) t \log t\end{aligned}$$

by using Stirling's formula. Hence, $\log \Lambda(s, \chi) \sim -2iA(K)t \log t$. The same conclusion holds for $\log \Lambda(s)$, hence $F(s) \sim -4iA(K)t \log t$ for $t \geq 2$ and $\sigma < 0$. However, this gives a contradiction since $F(\sigma + it) = O(1)$. Hence, $F(s) = \sum_{n=1}^{\infty} \delta'(n) n^{-s}$ has the finite abscissa of convergence σ_c and hence has a singularity at $s = \sigma_c$ by Theorem 11.13 of Apostol [**1**]. Now suppose that $L_1(1, \chi) = 0$. Then the function $\phi(s, \chi)$ is entire because of (H2). Therefore, the above presence of the singularity at $s = \sigma_c$ shows that $\phi(\sigma_c, \chi) = 0$. However, we have $\phi(\sigma, \chi) \geq 1$ for $\sigma > \sigma_c$ since $\delta'(n) \geq 0$. We obtain a contradiction by letting $\sigma \to \sigma_c$. Hence, we see that $L_1(1, \chi) \neq 0$. From this fact and Theorem 1, Siegel's theorem for $L_1(s, \chi)$ follows immediately.

5. The symmetric power L-functions

In this final section we apply our main theorem to the case of symmetric power L-functions. Let

$$f(z) = \sum_{n=1}^{\infty} a(n) e^{2\pi i n z}$$

be a holomorphic cusp form, which is a newform of weight k and level N. Let us write for each prime p, which does not divide N,

$$a(p) = 2p^{(k-1)/2} \cos \theta_p.$$

For each integer $n \geq 0$, let

$$L_{\mathrm{sym},n}(s, \chi) = \prod_{p \nmid N} \prod_{j=0}^{n} \left(1 - \frac{\chi(p) e^{i\theta_p(n-2j)}}{p^s}\right)^{-1}.$$

Clearly, $L_{\mathrm{sym},n}(s, \chi)$ converges absolutely for $\sigma > 1$. It is, in fact, conjectured that each $L_{\mathrm{sym},n}(s, \chi)$ can be extended to an entire function for any $n \geq 1$ and χ. This is stronger than (A1). Here we suppose that this conjecture is true and the L-functions satisfy (A2) and (A3). In view of Serre [**15**], these assumptions are very natural. In the case $n = 0$, the L-function is equal to the Dirichlet L-function, hence the Siegel–Tatuzawa theorem in this case is classical. When $n \geq 1$, we divide the situation into two cases: n is even or n is odd. Put

$$\Lambda_{\mathrm{even}}(s, \chi) = L_{\mathrm{sym},0}(s, \chi)L_{\mathrm{sym},2}(s, \chi) \cdots L_{\mathrm{sym},2n}(s, \chi)$$

and

$$\begin{aligned}\Lambda_{\mathrm{odd}}(s, \chi) = (L_{\mathrm{sym},0}(s, \chi)L_{\mathrm{sym},1}(s, \chi)L_{\mathrm{sym},2}(s, \chi) \cdots L_{\mathrm{sym},2n+1}(s, \chi))^2 \\ \times L_{\mathrm{sym},2n+2}(s, \chi).\end{aligned}$$

Note that the case $\chi = \chi_0$ of these functions was introduced by Ram Murty (see [**12**, Proof of Theorem 3]). These functions satisfy (H1), (H2) and (H3). In fact, we can show

$$\log \Lambda_{\mathrm{even}}(s, \chi) = \sum_{\substack{p:\mathrm{prime} \\ p \nmid N}} \sum_{h=1}^{\infty} \frac{\chi^h(p)}{hp^{hs}} \left(\frac{\sin((n+1)\theta_p h)}{\sin(\theta_p h)} \right)^2$$

and

$$\log \Lambda_{\mathrm{odd}}(s, \chi) = \sum_{\substack{p:\mathrm{prime} \\ p \nmid N}} \sum_{h=1}^{\infty} \frac{\chi^h(p)}{hp^{hs}} \left(\frac{\sin((n+3/2)\theta_p h)}{\sin(\theta_p h/2)} \right)^2,$$

hence (H1) follows. Hypothesis (H2) trivially holds because we assume that $L_{\mathrm{sym},n}(s, \chi_0)$, $L_{\mathrm{sym},n}(s, \chi)$ are entire for $n \geq 1$, and (H3) follows from Remark 1. Therefore, the Siegel–Tatuzawa theorem for $L_{\mathrm{sym},2n}(s, \chi)$ and $L_{\mathrm{sym},2n+1}(s, \chi)$ follows from Theorem 1 under (A1)–(A3).

THEOREM 2. *Let X be the same as in Theorem 1. If the nth symmetric power L-function $L_{\mathrm{sym},n}(s, \chi)$ can be extended to an entire function* (*for any $n \geq 1$ and any χ*)*, and satisfies* (A2) *and* (A3)*, then there exists an effective constant $C(\varepsilon)$ such that*

$$|L_{\mathrm{sym},n}(1, \chi)| > \frac{C(\varepsilon)}{d^{\varepsilon}}$$

for any $\varepsilon > 0$, except for at most one possible element of X. Here d is the conductor of χ.

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