On value-relations, functional relations and singularities of Mordell-Tornheim and related triple zeta-functions

by

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1 Introduction

Let $\mathbb{N}$ be the set of natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{Z}$ the ring of rational integers, $\mathbb{Q}$ the field of rational numbers, $\mathbb{R}$ the field of real numbers and $\mathbb{C}$ the field of complex numbers.

The Mordell-Tornheim r-ple zeta-function

$$(1.1) \quad \zeta_{MT,r}(s_1,\ldots,s_r;s_{r+1}) = \sum_{m_1,\ldots,m_r=1}^{\infty} \frac{1}{m_1^{s_1}\cdots m_r^{s_r}(m_1+\cdots+m_r)^{s_{r+1}}}$$

was defined by the first-named author (see [5, 8]). This can be meromorphically continued to the whole space $\mathbb{C}^{r+1}$ and possible singularities of (1.1) can be determined explicitly (see [8, Theorem 1]).

Historically, in the 1950's, Tornheim considered the double series $\zeta_{MT,2}(p,q;r)$ $(p,q,r \in \mathbb{N})$, and gave some fascinating formulas (see [16]). A little later, Mordell independently studied the values $\zeta_{MT,2}(k,k;k)$ $(k \in \mathbb{N})$, and showed that $\zeta_{MT,2}(2p,2p;2p)$ can be written as $M_p \cdot \pi^{6p}$ for some constant $M_p \in \mathbb{Q}$ $(p \in \mathbb{N})$ (see [11]).

About 30 years later, Subbarao and Sitaramachandrarao gave an evaluation formula for $\zeta_{MT,2}(2p,2p;2p)$ ([14]). A little later, Zagier proved the following simple formula:

$$(1.2) \quad \zeta_{MT,2}(2p,2p;2p) = \frac{4}{3}\sum_{j=0}^{p}\binom{4p-2j-1}{2p-1}\zeta(2j)\zeta(6p-2j) \quad (p \in \mathbb{N}),$$

which is much simpler than the Subbarao-Sitaramachandrarao formula. As an analogue of (1.2), Huard, Williams and Zhang gave an evaluation formula for $\zeta_{MT,2}(2p+1,2p+1;2p+1)$ $(p \in \mathbb{N}_0)$:

$$(1.3) \quad \zeta_{MT,2}(2p+1,2p+1;2p+1) = -4\sum_{j=0}^{p}\binom{4p-2j+1}{2p}\zeta(2j)\zeta(6p-2j+3).$$

Recently, as interpolations of these formulas, the fourth-named author gave some functional relations for $\zeta_{MT,2}(s_1,s_2;s_3)$ (see [21, Theorem 4.5]).

More recently the second-named author proved functional relations for $\zeta_{MT,2}(s_1, s_2; s_3)$ by a different method ([12, Theorem 1.2]). His relations are

$$
\begin{aligned}
(1.4) \qquad & \zeta_{MT,2}(a, b; s) + (-1)^b \zeta_{MT,2}(b, s; a) + (-1)^a \zeta_{MT,2}(s, a; b) \\
& = \frac{2}{a!b!} \sum_{k=0}^{\max([a/2],[b/2])} \left\{ a \binom{b}{2k} + b \binom{a}{2k} \right\} \\
& \qquad \times (a + b - 2k - 1)!(2k)! \zeta(2k) \zeta(a + b + s - 2k)
\end{aligned}
$$

for $a, b \in \mathbb{N}$, where $[\,x\,]$ is the integer part of x. These have simpler forms than those in [21]. Note that (1.4) holds for all $s \in \mathbb{C}$ except for singularities of the both sides.

Furthermore, triple and more general multiple zeta values of the Mordell-Tornheim type have been studied. Actually Mordell considered the multiple series

$$
\sum_{m_1,\ldots,m_r=1}^{\infty} \frac{1}{m_1 m_2 \cdots m_r (m_1 + \cdots + m_r + a)}
$$

for $a > -r$, which can be regarded as a prototype of (1.1). Based on his work, Hoffman studied $\zeta_{MT,r}(1, 1, \ldots, 1; k)$ for $k \in \mathbb{N}$ and gave some relations between these values and the Euler-Zagier type of multiple zeta values (see [2, Section 4]). Markett independently wrote the value $\zeta_{MT,3}(1, 1, 1; k)$ ($k \in \mathbb{N}$) as a polynomial in the values of $\zeta(s)$ at positive integers with $\mathbb{Q}$-coefficients (see [4]). Recently the fourth-named author proved a certain property on the values of $\zeta_{MT,r}$ ([20, Theorem 1.1]), which is called the 'parity result' (for details, see Remark 4.8 in Section 4).

In the present paper, we mainly study the Mordell-Tornheim double and triple zeta-functions. In Section 2, we prove the key lemma (Lemma 2.1) in order to study double and triple series. As its applications, we confirm that the functional relations for $\zeta_{MT,2}$ of the fourth-named author coincide with those of the second-named author (1.4) (see Proposition 2.2), and consider some alternating double series. In Section 3, we give some relation formulas for the values of $\zeta_{MT,3}$ which can be regarded as triple analogues of (1.2) and (1.3) (see Theorem 3.1). In Section 4, we give some functional relations among triple zeta-functions, double zeta-functions and the Riemann zeta-function, which can be regarded as triple analogues of (1.4) (see Theorem 4.5). In Sections 5 and 6, we discuss analytic properties of triple zeta-functions appearing in Section 4. Actually in Section 6, we study more general $\zeta_{MT,r}$ and determine its true singularities (see Theorem 6.1).

2 The key lemma and its applications

Let $\zeta(s)$ be the Riemann zeta-function and

$$\phi(s)=\sum_{m=1}^{\infty}\frac{(-1)^m}{m^s}=\left(2^{1-s}-1\right)\zeta(s). \tag{2.1}$$

We recall that

$$\sum_{m=1}^{\infty}\frac{(-1)^m\cos(m\theta)}{m^{2k}}=\sum_{\nu=0}^{k}\ \phi(2k-2\nu)\ \frac{(-1)^{\nu}\theta^{2\nu}}{(2\nu)!}; \tag{2.2}$$

$$\sum_{m=1}^{\infty}\frac{(-1)^m\sin(m\theta)}{m^{2l+1}}=\sum_{\nu=0}^{l}\ \phi(2l-2\nu)\ \frac{(-1)^{\nu}\theta^{2\nu+1}}{(2\nu+1)!} \tag{2.3}$$

for $k\in\mathbb{N}$, $l\in\mathbb{N}_0$ and $\theta\in(-\pi,\pi)\subset\mathbb{R}$ (see, for example, [17, Lemma 2]). Note that $\phi(0)=\zeta(0)=-\frac{1}{2}$. Since the both sides of (2.2) and of (2.3) are continuous for $\theta\in[-\pi,\pi]$ in the case $k,l\in\mathbb{N}$, we can let $\theta\to\pi$ on the both sides of (2.2) and of (2.3), respectively. Then, by $\cos(n\pi)=(-1)^n$ and $\sin(n\pi)=0$ $(n\in\mathbb{Z})$, we have

$$\zeta(2k)=\sum_{\nu=0}^{k}\ \phi(2k-2\nu)\ \frac{(-1)^{\nu}\pi^{2\nu}}{(2\nu)!}; \tag{2.4}$$

$$0=\sum_{\nu=0}^{l}\ \phi(2l-2\nu)\ \frac{(-1)^{\nu}\pi^{2\nu+1}}{(2\nu+1)!} \tag{2.5}$$

for $k,l\in\mathbb{N}$.

Now we prove the following lemma which is a key when we consider the rearrangement of sums appearing in relation formulas for double and triple zeta values.

Lemma 2.1. *For arbitrary functions $f,g:\mathbb{N}_0\to\mathbb{C}$ and $a\in\mathbb{N}$, we have*

$$\sum_{\substack{j=0\\ j\equiv a \ (\mathrm{mod}\ 2)}}^{a}\ \phi(a-j)\sum_{\mu=0}^{[j/2]}f(j-2\mu)\frac{(-1)^{\mu}\pi^{2\mu}}{(2\mu)!}=\sum_{\rho=0}^{[a/2]}\zeta(2\rho)f(a-2\rho), \tag{2.6}$$

and

$$\sum_{\substack{j=0\\ j\equiv a \ (\mathrm{mod}\ 2)}}^{a}\ \phi(a-j)\ \sum_{\mu=0}^{[(j-1)/2]}\ g(j-2\mu)\frac{(-1)^{\mu}\pi^{2\mu}}{(2\mu+1)!}=-\frac{1}{2}g(a). \tag{2.7}$$

Proof. On the left-hand side of (2.6), we change the running indices j and μ to ρ and ν by $j=a+2\mu-2\rho$ and $\mu=\nu$ $(\leq\rho)$. By $0\leq j\leq a$ and

$0 \le \mu \le [j/2]$, we have $\rho = \mu + (a-j)/2 \ge \mu = \nu$ and $0 \le 2\rho \le a$, namely $0 \le \rho \le [a/2]$. Hence, by $a - j = 2\rho - 2\nu$ and $j - 2\mu = a - 2\rho$, we see that the left-hand side of (2.6) is

$$\sum_{\rho=1}^{[a/2]} \sum_{\nu=0}^{\rho} \phi(2\rho - 2\nu) f(a - 2\rho) \frac{(-1)^\nu \pi^{2\nu}}{(2\nu)!} + \phi(0) f(a).$$

By (2.4) and $\phi(0) = \zeta(0) = -\frac{1}{2}$, this is equal to the right-hand side of (2.6). Thus we obtain the proof of (2.6). Similarly, changing the running indices j and μ to ρ and ν by $j = a + 2\mu - 2\rho$ and $\nu = \mu \ \le \rho$ on the left-hand side of (2.7), and using (2.5), we can see that (2.7) holds. This completes the proof.

As a direct application of this lemma, we obtain the following proposition which implies that the result (1.4) of the second-named author ([12]) essentially coincides with that of the fourth-named author ([21]).

Proposition 2.2. *For $a, b \in \mathbb{N}$,*

$$\begin{aligned} \zeta_{MT,2}(a,b;s) &+ (-1)^a \zeta_{MT,2}(a,s;b) + (-1)^b \zeta_{MT,2}(b,s;a) \\ &= 2 \sum_{\rho=0}^{\max([a/2],[b/2])} \left\{ \binom{a+b-2\rho-1}{a-1} + \binom{a+b-2\rho-1}{b-1} \right\} \\ &\quad \times \zeta(2\rho)\zeta(s+a+b-2\rho) \end{aligned} \tag{2.8}$$

holds for all $s \in \mathbb{C}$ except for singularities of the both sides of (2.8).

Proof. The fourth-named author gave the functional relations

$$\begin{aligned} \zeta_{MT,2}(a,b;s) &+ (-1)^a \zeta_{MT,2}(a,s;b) + (-1)^b \zeta_{MT,2}(b,s;a) \\ = 2 & \sum_{\substack{j=0 \\ j \equiv a \ (\mathrm{mod}\ 2)}}^{a} \phi(a-j) \sum_{\mu=0}^{[j/2]} \frac{(i\pi)^{2\mu}}{(2\mu)!} \binom{b-1+j-2\mu}{j-2\mu} \zeta(b+j+s-2\mu) \\ - 4 & \sum_{\substack{j=0 \\ j \equiv a \ (\mathrm{mod}\ 2)}}^{a} \phi(a-j) \sum_{\mu=0}^{[(j-1)/2]} \frac{(i\pi)^{2\mu}}{(2\mu+1)!} \sum_{\substack{\nu=0 \\ \nu \equiv b \ (2)}}^{b} \zeta(b-\nu) \\ \times & \binom{\nu-1+j-2\mu}{j-2\mu-1} \zeta(\nu+j+s-2\mu) \end{aligned} \tag{2.9}$$

for $a, b \in \mathbb{N}$ and $s \in \mathbb{C}$ (see [21, Theorem 4.5]). Applying (2.6) and (2.7) to (2.9) with

$$f(X) = \binom{b-1+X}{X} \zeta(b+s+X);$$

$$g(X) = \sum_{\substack{\nu=0 \\ \nu \equiv b \ (2)}}^{b} \zeta(b-\nu) \binom{\nu-1+X}{X-1} \zeta(\nu+s+X),$$

we have

$$\begin{aligned}
&\zeta_{MT,2}(a,b;s)+(-1)^a\zeta_{MT,2}(a,s;b)+(-1)^b\zeta_{MT,2}(b,s;a)\\
&=2\sum_{\rho=0}^{[a/2]}\binom{a+b-2\rho-1}{a-2\rho}\zeta(s+a+b-2\rho)\\
&\quad+2\sum_{\substack{\nu=0\\ \nu\equiv b\ (2)}}^{b}\zeta(b-\nu)\binom{\nu-1+a}{a-1}\zeta(s+\nu+a).
\end{aligned}$$

By putting $\nu=b-2\rho$ in the second summation on the right-hand side, we obtain (2.8).

We can easily check that

$$\frac{2}{m!n!}m\binom{n}{2k}(m+n-2k-1)!(2k)!=2\binom{m+n-2k-1}{m-1} \tag{2.10}$$

for $k,m,n\in\mathbb{N}_0$. Substituting (2.10) in the cases $(m,n)=(a,b)$ and (b,a) into (1.4), we obtain (2.8). Thus we showed that (1.4), (2.8) and (2.9) are all equivalent.

Remark 2.3. Putting $a=b=2p$ and $2p+1$ in (2.8), we can obtain (1.2) and (1.3). Hence (2.8) can be regarded as a continuous interpolation of both (1.2) and (1.3).

Lemma 2.1 is also useful for the study of

$$\phi_2(s_1,s_2;s_3)=\sum_{m,n=1}^{\infty}\frac{(-1)^{m+n}}{m^{s_1}n^{s_2}(m+n)^{s_3}}; \tag{2.11}$$

$$\psi_2(s_1,s_2;s_3)=\sum_{m,n=1}^{\infty}\frac{(-1)^{n}}{m^{s_1}n^{s_2}(m+n)^{s_3}} \tag{2.12}$$

for $s_1,s_2,s_3\in\mathbb{C}$. The fourth-named author proved that

(2.13)

$$\begin{aligned}
\phi_2(2p+1,2p+1;2p+1)&=2\sum_{\rho=0}^{p}\binom{4p+1-2\rho}{2p}\phi(2\rho)\phi(6p+3-2\rho)\\
-4\sum_{j=0}^{p}\phi(2j)\Big\{&\sum_{\nu=0}^{p}\phi(2p-2\nu)\sum_{\mu=0}^{\nu}\binom{2p+1-2j+2\nu-2\mu}{2\nu-2\mu}\\
&\times\zeta(4p+3-2j+2\nu-2\mu)\frac{(-1)\mu\pi^{2\mu}}{(2\mu+1)!}\Big\}
\end{aligned}$$

for $p\in\mathbb{N}_0$ (see [19, Theorem 3.4]). Applying (2.7) to (2.13) with $a=2p+1$ and

$$g(t)=\binom{2p-2j+t}{t-1}\zeta(4p+2-2j+t),$$

we have

$$\begin{aligned}\phi_2(2p+1,2p+1;2p+1) =& 2\sum_{\rho=0}^{p}\binom{4p+1-2\rho}{2p}\phi(2\rho)\phi(6p+3-2\rho)\\ &+2\sum_{j=0}^{p}\phi(2j)\binom{4p+1-2j}{2p}\zeta(6p+3-2j).\end{aligned}$$

Since $\phi(s)=(2^{1-s}-1)\zeta(s)$, we can rewrite (2.13) as follows.

Proposition 2.4. *For $p\in\mathbb{N}_0$,*

$$\begin{aligned}&\phi_2(2p+1,2p+1;2p+1)\\ &\quad=2^{-6p}\sum_{j=0}^{p}\binom{4p+1-2j}{2p}\left(2^{2j-1}-1\right)\zeta(2j)\zeta(6p+3-2j).\end{aligned}\tag{2.14}$$

On the other hand, from [13, Theorem 3.1] and (2.10), we have

Proposition 2.5. *For $p\in\mathbb{N}_0$,*

$$\begin{aligned}&\phi_2(2p+1,2p+1;2p+1)-2\psi_2(2p+1,2p+1;2p+1)\\ &\quad=4\sum_{j=0}^{p}\binom{4p+1-2j}{2p}\left(2^{2j-2-6p}-1\right)\zeta(2j)\zeta(6p+3-2j).\end{aligned}\tag{2.15}$$

Hence, combining (2.14) and (2.15), we have

Proposition 2.6. *For $p\in\mathbb{N}_0$,*

(2.16)

$$\begin{aligned}&\psi_2(2p+1,2p+1;2p+1)\\ &\quad=2^{-6p-1}\sum_{j=0}^{p}\binom{4p+1-2j}{2p}\left(2^{6p+2}-2^{2j-1}-1\right)\zeta(2j)\zeta(6p+3-2j).\end{aligned}$$

Example 2.7. By (2.16), for example, we obtain

$$\begin{aligned}\psi_2(5,5;5)&=-\frac{2064195}{16384}\zeta(15)+\frac{573335}{8192}\zeta(2)\zeta(13)+\frac{81875}{8192}\zeta(4)\zeta(11),\\ \psi_2(7,7;7)&=-\frac{899676921}{524288}\zeta(21)+\frac{242220363}{262144}\zeta(2)\zeta(19)\\ &\qquad+\frac{22019907}{131072}\zeta(4)\zeta(17)+\frac{7339801}{524288}\zeta(6)\zeta(15).\end{aligned}$$

These formulas do not coincide with those in [18, Example 3.7]. In fact, the fourth-named author made some mistakes in calculating the formulas in [18, Example 3.7]. The above formulas are surely correct.

3 Relation formulas for triple zeta values

In this section, we prove relation formulas for $\zeta_{MT,3}(k,k,k;k)$ for $k \in \mathbb{N}$, which are the triple analogues of (1.2) and (1.3). The method to prove these formulas is similar to that introduced in the previous work of the fourth-named author (see [17, 20]). Combining that method with Lemma 2.1, we can obtain the following simple expressions like (1.2) and (1.3).

Theorem 3.1. *For $p \in \mathbb{N}$,*

$$\begin{aligned}(3.1)\quad &\zeta_{MT,3}(2p,2p,2p;2p)\\ &= 4\sum_{\nu=1}^{p}\binom{2\nu+2p-1}{2p-1}\zeta(2p-2\nu)\\ &\quad\times\{\zeta_{MT,2}(2p,2p;2p+2\nu)-\zeta_{MT,2}(2p+2\nu,2p;2p)\}-\zeta(4p)^2,\end{aligned}$$

and for $p \in \mathbb{N}_0$,

$$\begin{aligned}(3.2)\quad &\zeta_{MT,3}(2p+1,2p+1,2p+1;2p+1)\\ &= -4\sum_{\nu=0}^{p}\binom{2\nu+2p-1}{2p}\zeta(2p-2\nu)\\ &\quad\times\Big\{\zeta_{MT,2}(2p+1,2p+1;2p+2\nu+2)\\ &\qquad\qquad+\zeta_{MT,2}(2p+2\nu+2,2p+1;2p+1)\Big\}+\zeta(4p+2)^2.\end{aligned}$$

Example 3.2. By Theorem 3.1, for example, we can obtain

$$\begin{aligned}\zeta_{MT,3}(1,1,1;1) &= \frac{12}{5}\zeta(2)^2=\frac{1}{15}\pi^4;\\ \zeta_{MT,3}(2,2,2;2) &= 6\{\zeta_{MT,2}(4,2;2)-\zeta_{MT,2}(2,2;4)\}-\zeta(4)^2;\\ \zeta_{MT,3}(3,3,3;3) &= -12\zeta(2)\{\zeta_{MT,2}(3,3;4)+\zeta_{MT,2}(4,3;3)\}\\ &\quad+20\{\zeta_{MT,2}(3,3;6)+\zeta_{MT,2}(6,3;3)\}+\zeta(6)^2.\end{aligned}$$

Using (2.8), we can rewrite

$$\zeta_{MT,3}(2,2,2;2)=\frac{1}{11340}\pi^8-9\zeta_{MT,2}(2,2;4).$$

Note that it has not been proven yet that $\zeta_{MT,2}(2,2;4)$ is expressed by means of the values of $\zeta(s)$.

Now we give some preparations for the proof of Theorem 3.1. Fix any

$p \in \mathbb{N}$. By (2.2), we have

(3.3)
$$\sum_{l,m=1}^{\infty} \frac{(-1)^{l+m}\cos((l+m)\theta)}{l^{2p}m^{2p}} \left\{ \sum_{n=1}^{\infty} \frac{(-1)^n \cos(n\theta)}{n^{2p}} - \sum_{j=0}^{p} \phi(2p-2j)\, \frac{(-1)^j \theta^{2j}}{(2j)!} \right\}$$
$$- \sum_{l=1}^{\infty} \frac{(-1)^l \sin(l\theta)}{l^{2p}} \left\{ \sum_{m=1}^{\infty} \frac{(-1)^m \cos(m\theta)}{m^{2p}} - \sum_{j=0}^{p} \phi(2p-2j)\, \frac{(-1)^j \theta^{2j}}{(2j)!} \right\}$$
$$\times \sum_{n=1}^{\infty} \frac{(-1)^n \sin(n\theta)}{n^{2p}}$$
$$- \left\{ \sum_{l=1}^{\infty} \frac{(-1)^l \cos(l\theta)}{l^{2p}} - \sum_{j=0}^{p} \phi(2p-2j)\, \frac{(-1)^j \theta^{2j}}{(2j)!} \right\}$$
$$\times \sum_{m=1}^{\infty} \frac{(-1)^m \sin(m\theta)}{m^{2p}} \sum_{n=1}^{\infty} \frac{(-1)^n \sin(n\theta)}{n^{2p}} = 0$$

for $\theta \in (-\pi, \pi)$. Using the addition formulas for $\sin x$ and $\cos x$, we can rewrite (3.3) as

(3.4)
$$\sum_{l,m,n=1}^{\infty} \frac{(-1)^{l+m+n}\cos((l+m+n)\theta)}{l^{2p}m^{2p}n^{2p}}$$
$$+ \sum_{j=0}^{p} \phi(2p-2j)\, \frac{(-1)^j \theta^{2j}}{(2j)!}$$
$$\times \left\{ \sum_{l,m=1}^{\infty} \frac{(-1)^{l+m}\cos((l-m)\theta)}{l^{2p}m^{2p}} - 2 \sum_{l,m=1}^{\infty} \frac{(-1)^{l+m}\cos((l+m)\theta)}{l^{2p}m^{2p}} \right\}$$
$$= \sum_{l,m,n=1}^{\infty} \frac{(-1)^{l+m+n}\cos((l+m+n)\theta)}{l^{2p}m^{2p}n^{2p}}$$
$$+ \sum_{j=0}^{p} \phi(2p-2j)\, \frac{(-1)^j \theta^{2j}}{(2j)!}$$
$$\times \left\{ \sum_{\substack{l,m=1 \\ l\neq m}}^{\infty} \frac{(-1)^{l+m}\cos((l-m)\theta)}{l^{2p}m^{2p}} - 2 \sum_{l,m=1}^{\infty} \frac{(-1)^{l+m}\cos((l+m)\theta)}{l^{2p}m^{2p}} \right\}$$
$$+ \zeta(4p) \sum_{l=1}^{\infty} \frac{(-1)^l \cos(l\theta)}{l^{2p}} = 0$$

for $\theta \in (-\pi, \pi)$, using (2.2) again. This implies, by integrating the both sides by parts repeatedly, that

(3.5)

$$\sum_{l,m,n=1}^{\infty} \frac{(-1)^{l+m+n}\sin((l+m+n)\theta)}{l^{2p}m^{2p}n^{2p}(l+m+n)^{2d+1}}$$

$$+\sum_{j=0}^{p}\ \phi(2p-2j)\ \sum_{\nu=0}^{2j}\binom{2d+2j-\nu}{2j-\nu}\frac{(-1)^{\nu}\theta^{\nu}}{\nu!}$$

$$\times\left\{\sum_{\substack{l,m=1\\ l\neq m}}^{\infty}\frac{(-1)^{l+m}\sin^{(\nu)}((l-m)\theta)}{l^{2p}m^{2p}(l-m)^{2d+2j+1-\nu}}-2\sum_{l,m=1}^{\infty}\frac{(-1)^{l+m}\sin^{(\nu)}((l+m)\theta)}{l^{2p}m^{2p}(l+m)^{2d+2j+1-\nu}}\right\}$$

$$+\zeta(4p)\sum_{l=1}^{\infty}\frac{(-1)^{l}\sin(l\theta)}{l^{2p+2d+1}}=\sum_{\rho=0}^{d}C_{2d-2\rho}(2p)\frac{(-1)^{\rho}\theta^{2\rho+1}}{(2\rho+1)!}\quad(\theta\in(-\pi,\pi))$$

for $d\in\mathbb{N}_0$, and

(3.6)

$$\sum_{l,m,n=1}^{\infty} \frac{(-1)^{l+m+n}\cos((l+m+n)\theta)}{l^{2p}m^{2p}n^{2p}(l+m+n)^{2e}}$$

$$+\sum_{j=0}^{p}\ \phi(2p-2j)\ \sum_{\nu=0}^{2j}\binom{2e-1+2j-\nu}{2j-\nu}\frac{(-1)^{\nu}\theta^{\nu}}{\nu!}$$

$$\times\left\{\sum_{\substack{l,m=1\\ l\neq m}}^{\infty}\frac{(-1)^{l+m}\cos^{(\nu)}((l-m)\theta)}{l^{2p}m^{2p}(l-m)^{2e+2j-\nu}}-2\sum_{l,m=1}^{\infty}\frac{(-1)^{l+m}\cos^{(\nu)}((l+m)\theta)}{l^{2p}m^{2p}(l+m)^{2e+2j-\nu}}\right\}$$

$$+\zeta(4p)\sum_{l=1}^{\infty}\frac{(-1)^{l}\cos(l\theta)}{l^{2p+2e}}=\sum_{\rho=0}^{e}C_{2e-2\rho}(2p)\frac{(-1)^{\rho}\theta^{2\rho}}{(2\rho)!}\quad(\theta\in(-\pi,\pi))$$

for $e\in\mathbb{N}_0$, where $\{C_{2\nu}(2p)\,|\,\nu\in\mathbb{N}_0\}$ are constants which are determined inductively, $f^{(\nu)}(x)$ denotes the νth derivative of $f(x)$ and $f^{(\nu)}(\alpha):=f^{(\nu)}(x)\big|_{x=\alpha}$ for $f(x)=\sin x,\ \cos x$. Note that the left-hand side of (3.5) (resp. (3.6)) is an odd (resp. even) function, hence each coefficient of $\theta^{2\rho}$ (resp. $\theta^{2\rho+1}$) on the right-hand side of (3.5) (resp. (3.6)) is equal to 0.

Since the both sides of (3.5) and of (3.6) are continuous for $\theta\in[-\pi,\pi]$, (3.5) and (3.6) hold for $\theta=\pi$. Note that, by putting $h=l-m$ (resp. $k=m-l$) if $l>m$ (resp. $l<m$), we have, for example,

(3.7)
$$\sum_{\substack{l,m=1\\ l\neq m}}^{\infty}\frac{1}{l^{2p}m^{2p}(l-m)^{2d+2j-2\mu}}$$
$$=\sum_{h,m=1}^{\infty}\frac{1}{h^{2d+2j-2\mu}m^{2p}(h+m)^{2p}}+\sum_{k,l=1}^{\infty}\frac{1}{k^{2d+2j-2\mu}l^{2p}(k+l)^{2p}}$$
$$=2\zeta_{MT,2}(2d+2j-2\mu,2p;2p).$$

Hence, letting $\theta \to \pi$ on the both sides of (3.5) and of (3.6), we have

$$\begin{aligned}(3.8)\quad & \sum_{j=0}^{p} \phi(2p-2j) \sum_{\mu=0}^{j-1} \binom{2d+2j-2\mu-1}{2j-2\mu-1} \frac{(-1)^\mu \pi^{2\mu+1}}{(2\mu+1)!} \\ & \qquad \times \left\{ \zeta_{MT,2}(2d+2j-2\mu, 2p; 2p) - \zeta_{MT,2}(2p,2p;2d+2j-2\mu) \right\} \\ & = \sum_{\mu=0}^{d} C_{2d-2\mu}(2p) \frac{(-1)^\mu \pi^{2\mu+1}}{(2\mu+1)!}\end{aligned}$$

and

(3.9)

$$\begin{aligned}& \zeta_{MT,3}(2p,2p,2p;2e) + 2\sum_{j=0}^{p} \phi(2p-2j) \sum_{\mu=0}^{j} \binom{2e+2j-2\mu-1}{2j-2\mu} \frac{(-1)^\mu \pi^{2\mu}}{(2\mu)!} \\ & \times \left\{ \zeta_{MT,2}(2e+2j-2\mu, 2p; 2p) - \zeta_{MT,2}(2p,2p;2e+2j-2\mu) \right\} \\ & + \zeta(4p)\zeta(2p+2e) = \sum_{\mu=0}^{e} C_{2e-2\mu}(2p) \frac{(-1)^\mu \pi^{2\mu}}{(2\mu)!}.\end{aligned}$$

Applying Lemma 2.1 to (3.8) and (3.9) with $a = 2p$ and

$$\begin{aligned} f(x) &= \binom{2e+x-1}{x} \left\{ \zeta_{MT,2}(2e+x, 2p; 2p) - \zeta_{MT,2}(2p,2p;2e+x) \right\}; \\ g(x) &= \binom{2d+x-1}{x-1} \left\{ \zeta_{MT,2}(2d+x, 2p; 2p) - \zeta_{MT,2}(2p,2p;2d+x) \right\},\end{aligned}$$

we can rewrite (3.8) and (3.9) as

$$\begin{aligned}(3.10)\quad & \binom{2d+2p-1}{2p-1} \left\{ \zeta_{MT,2}(2d+2p, 2p; 2p) - \zeta_{MT,2}(2p,2p;2d+2p) \right\} \\ & = \sum_{\mu=0}^{d} C_{2d-2\mu}(2p) \frac{(-1)^\mu \pi^{2\mu}}{(2\mu+1)!}\end{aligned}$$

and

$$\begin{aligned}(3.11)\quad & \zeta_{MT,3}(2p,2p,2p;2e) + 2\sum_{\xi=0}^{p} \zeta(2\xi) \binom{2e+2p-2\xi-1}{2p-2\xi} \\ & \times \left\{ \zeta_{MT,2}(2e+2p-2\xi, 2p; 2p) - \zeta_{MT,2}(2p,2p;2e+2p-2\xi) \right\} \\ & + \zeta(4p)\zeta(2p+2e) = \sum_{\mu=0}^{e} C_{2e-2\mu}(2p) \frac{(-1)^\mu \pi^{2\mu}}{(2\mu)!}\end{aligned}$$

for $d, e \in \mathbb{N}_0$.

Now we recall the following.

Lemma 3.3 ([21] Lemma 4.4). *Let* $\{\alpha_{2d}\}_{d\in\mathbb{N}_0}$, $\{\beta_{2d}\}_{d\in\mathbb{N}_0}$, $\{\gamma_{2d}\}_{d\in\mathbb{N}_0}$ *be sequences satisfying that*

$$\alpha_{2d}=\sum_{j=0}^{d}\gamma_{2d-2j}\frac{(-1)^j\pi^{2j}}{(2j)!},\ \ \beta_{2d}=\sum_{j=0}^{d}\gamma_{2d-2j}\frac{(-1)^j\pi^{2j}}{(2j+1)!}$$

for any $d\in\mathbb{N}_0$. *Then*

$$\alpha_{2d}=-2\sum_{\nu=0}^{d}\beta_{2\nu}\zeta(2d-2\nu)$$

for any $d\in\mathbb{N}_0$.

Proof of Theorem 3.1 Applying Lemma 3.3 to (3.10) and (3.11) with $d=e$, and put $\nu=p-\xi$ in (3.11), we have

(3.12)

$$\begin{aligned}&\zeta_{MT,3}(2p,2p,2p;2d)\\&\ +2\sum_{\nu=0}^{p}\zeta(2p-2\nu)\binom{2d+2\nu-1}{2\nu}\Big\{\zeta_{MT,2}(2d+2\nu,2p;2p)-\zeta_{MT,2}(2p,2p;2\nu+2p)\Big\}\\&\ +\zeta(4p)\zeta(2p+2d)\\&=4\sum_{\nu=0}^{d}\zeta(2d-2\nu)\binom{2\nu+2p-1}{2p-1}\Big\{\zeta_{MT,2}(2\nu+2p,2p;2p)-\zeta_{MT,2}(2p,2p;2\nu+2p)\Big\}.\end{aligned}$$

In particular when $d=p$, we obtain (3.1).

As well as (3.3), from (2.2), we have the relation

(3.13)

$$\begin{aligned}&\sum_{l,m=1}^{\infty}\frac{(-1)^{l+m}\cos((l+m)\theta)}{l^{2p+1}m^{2p+1}}\\&\qquad\times\left\{\sum_{n=1}^{\infty}\frac{(-1)^n\sin(n\theta)}{n^{2p+1}}-\sum_{j=0}^{p}\ \phi(2p-2j)\ \frac{(-1)^j\theta^{2j+1}}{(2j+1)!}\right\}\\&\ +\sum_{l=1}^{\infty}\frac{(-1)^l\cos(l\theta)}{l^{2p+1}}\left\{\sum_{m=1}^{\infty}\frac{(-1)^m\sin(m\theta)}{m^{2p+1}}-\sum_{j=0}^{p}\ \phi(2p-2j)\ \frac{(-1)^j\theta^{2j+1}}{(2j+1)!}\right\}\\&\qquad\times\sum_{n=1}^{\infty}\frac{(-1)^n\cos(n\theta)}{n^{2p+1}}\\&\ -\left\{\sum_{l=1}^{\infty}\frac{(-1)^l\sin(l\theta)}{l^{2p+1}}-\sum_{j=0}^{p}\ \phi(2p-2j)\ \frac{(-1)^j\theta^{2j+1}}{(2j+1)!}\right\}\\&\qquad\times\sum_{m=1}^{\infty}\frac{(-1)^m\cos(m\theta)}{m^{2p+1}}\sum_{n=1}^{\infty}\frac{(-1)^n\cos(n\theta)}{n^{2p+1}}=0\end{aligned}$$

for $\theta \in (-\pi, \pi)$. Then, by the same argument as mentioned above, we can prove (3.2). This completes the proof of Theorem 3.1.

4 Functional relations for triple zeta-functions

The aim of this section is to give some functional relations for triple zeta-functions. These can be regarded as triple analogues of (2.8). For this aim, we consider analytic properties of $\zeta_{MT,3}(s_1, s_2, s_3; s_4)$ and

$$G(s_1, s_2, s_3, s_4) := \sum_{\substack{k,l,m,n=1 \\ k+l=m+n}}^{\infty} \frac{1}{k^{s_1} l^{s_2} m^{s_3} n^{s_4}}. \tag{4.1}$$

First we enumerate the following two theorems. The proofs of these theorems will be given in the following sections. In fact, we first give the proof of Theorem 4.2 in the next section. Next we generalize Theorem 4.1 to the result on $\zeta_{MT,r}$ for any $r \geq 3$ (see Theorem 6.1) and give the proof of this generalized result in Section 6.

Theorem 4.1. $\zeta_{MT,3}(s_1, s_2, s_3; s_4)$ *can be continued meromorphically to* $\mathbb{C}^4$*, and the singularities are located only on the subsets of* $\mathbb{C}^4$ *defined by one of the following equations:*

$$s_j + s_4 = 1 - l \quad (1 \leq j \leq 3;\ l \in \mathbb{N}_0); \tag{4.2}$$

$$s_j + s_k + s_4 = 2 - l \quad (1 \leq j < k \leq 3;\ l \in \mathbb{N}_0); \tag{4.3}$$

$$s_1 + s_2 + s_3 + s_4 = 3, \tag{4.4}$$

all of which are true singularities.

Theorem 4.2. $G(s_1, s_2, s_3, s_4)$ *can be continued meromorphically to* $\mathbb{C}^4$*, and the singularities are located only on the subsets of* $\mathbb{C}^4$ *defined by one of the following equations:*

$$s_j + s_k = 1 - l \quad (j = 1, 2;\ k = 3, 4;\ l \in \mathbb{N}_0); \tag{4.5}$$

$$s_h + s_j + s_k = 2 - l \quad (1 \leq h < j < k \leq 4;\ l \in \mathbb{N}_0); \tag{4.6}$$

$$s_1 + s_2 + s_3 + s_4 = 3, \tag{4.7}$$

all of which are true singularities.

Based on these results, we give some functional relations for triple zeta-functions mentioned above. In the rest of this section, we use the same notation as in [12] and generalize Proposition 2.2 to the case of triple zeta-functions. The method used in this section can be regarded as a triple analogue of that introduced by the second-named author in [12].

We denote by $B_j(x)$ the jth Bernoulli polynomial defined by

$$\frac{te^{xt}}{e^t-1}=\sum_{j=0}^{\infty}B_j(x)\frac{t^j}{j!}\qquad (|t|<2\pi).$$

It is known (see [1, p. 266, (22) and p. 267, (24)]) that

$$B_{2j}:=B_{2j}(0)=(-1)^{j+1}2(2j)!(2\pi)^{-2j}\zeta(2j)\qquad (j\in\mathbb{N}),\tag{4.8}$$

$$B_j(x-[x])=-\frac{j!}{(2\pi i)^j}\lim_{K\to\infty}\sum_{\substack{k=-K\\k\neq 0}}^{K}\frac{e^{2\pi ikx}}{k^j}\qquad (j\in\mathbb{N}).\tag{4.9}$$

Hence, for $k\in\mathbb{Z}$, $j\in\mathbb{N}$ we have

$$\int_0^1 e^{-2\pi ikx}B_j(x)\,dx=\begin{cases}0 & (k=0);\\ -(2\pi ik)^{-j}j! & (k\neq 0).\end{cases}\tag{4.10}$$

It follows from [1, p. 276 19.(b)] that for $p+q\geq 2$,

(4.11)

$$\begin{aligned}&B_p(x)B_q(x)\\ &=\sum_{k=0}^{\max([p/2],[q/2])}\left\{p\binom{q}{2k}+q\binom{p}{2k}\right\}\frac{B_{2k}B_{p+q-2k}(x)}{p+q-2k}-(-1)^p\frac{p!q!}{(p+q)!}B_{p+q}.\end{aligned}$$

Using these facts, we obtain the following theorems.

Theorem 4.3. *For $a,b\in\mathbb{N}$,*

$$\begin{aligned}&(-1)^b\zeta_{MT,3}(b,s_3,s_4;a)+(-1)^a\zeta_{MT,3}(s_3,s_4,a;b)+G(a,b,s_3,s_4)\\ &\quad=\frac{2}{a!\,b!}\sum_{k=0}^{\max([a/2],[b/2])}\left\{a\binom{b}{2k}+b\binom{a}{2k}\right\}(a+b-2k-1)!(2k)!\\ &\qquad\qquad\times\zeta(2k)\zeta_{MT,2}(s_3,s_4;a+b-2k)\end{aligned}\tag{4.12}$$

holds for $s_3,s_4\in\mathbb{C}$ except for singularities of the both sides of (4.12).

Proof. For $\Re(s_3)>1$, $\Re(s_4)>1$, we have

$$\begin{cases}\displaystyle\lim_{K\to\infty}\int_0^1\sum_{k=1}^{K}\frac{e^{2\pi ikx}}{k^a}\sum_{l=1}^{K}\frac{e^{2\pi ilx}}{l^b}\sum_{m=1}^{K}\frac{e^{2\pi imx}}{m^{s_3}}\sum_{n=1}^{K}\frac{e^{2\pi inx}}{n^{s_4}}dx=0;\\ \displaystyle\lim_{K\to\infty}\int_0^1\sum_{k=-K}^{-1}\frac{e^{2\pi ikx}}{k^a}\sum_{l=1}^{K}\frac{e^{2\pi ilx}}{l^b}\sum_{m=1}^{K}\frac{e^{2\pi imx}}{m^{s_3}}\sum_{n=1}^{K}\frac{e^{2\pi inx}}{n^{s_4}}dx=(-1)^a\zeta_{MT,3}(b,s_3,s_4;a);\\ \displaystyle\lim_{K\to\infty}\int_0^1\sum_{k=1}^{K}\frac{e^{2\pi ikx}}{k^a}\sum_{l=-K}^{-1}\frac{e^{2\pi ilx}}{l^b}\sum_{m=1}^{K}\frac{e^{2\pi imx}}{m^{s_3}}\sum_{n=1}^{K}\frac{e^{2\pi inx}}{n^{s_4}}dx=(-1)^b\zeta_{MT,3}(s_3,s_4,a;b);\\ \displaystyle\lim_{K\to\infty}\int_0^1\sum_{k=-K}^{-1}\frac{e^{2\pi ikx}}{k^a}\sum_{l=-K}^{-1}\frac{e^{2\pi ilx}}{l^b}\sum_{m=1}^{K}\frac{e^{2\pi imx}}{m^{s_3}}\sum_{n=1}^{K}\frac{e^{2\pi inx}}{n^{s_4}}dx=(-1)^{a+b}G(a,b,s_3,s_4).\end{cases}$$

Therefore we have

$$\begin{aligned}&(-1)^a\zeta_{MT,3}(b,s_3,s_4;a)+(-1)^b\zeta_{MT,3}(s_3,s_4,a;b)+(-1)^{a+b}G(a,b,s_3,s_4)\\&=\int_0^1 \lim_{K\to\infty}\sum_{\substack{k=-K\\k\neq 0}}^{K}\frac{e^{2\pi ikx}}{k^a}\sum_{\substack{l=-K\\l\neq 0}}^{K}\frac{e^{2\pi ilx}}{l^b}\sum_{m=1}^{K}\frac{e^{2\pi imx}}{m^{s_3}}\sum_{n=1}^{K}\frac{e^{2\pi inx}}{n^{s_4}}dx.\end{aligned}$$

Changing the order of limitation and integration is justified by bounded convergence. In fact, we need to treat the case $a=1$ or $b=1$ carefully. For this case, we know that $\sum_{m=1}^{\infty}\sin(2\pi mx)/m$ is boundedly convergent for $x>0$ (see [15, p. 15]).

By using (4.8), (4.9), (4.10) and (4.11), we obtain (4.12) in this region. By Theorems 4.1 and 4.2, we see that (4.12) holds for all $a,b\in\mathbb{N}$, and $s_3,s_4\in\mathbb{C}$ except for singularities of the both sides of (4.12).

Theorem 4.4. *For $a,b\in\mathbb{N}$,*

(4.13)

$$\begin{aligned}&\zeta_{MT,3}(s_4,a,b;s_3)+(-1)^aG(a,s_3,b,s_4)\\&\quad+(-1)^bG(s_3,b,a,s_4)+(-1)^{a+b}\zeta_{MT,3}(a,b,s_3;s_4)\\&\quad=\frac{2}{a!\,b!}\sum_{k=0}^{\max([a/2],[b/2])}\left\{a\binom{b}{2k}+b\binom{a}{2k}\right\}(a+b-2k-1)!(2k)!\\&\qquad\qquad\times\zeta(2k)\zeta_{MT,2}(a+b-2k,s_4;s_3)\\&\quad+\frac{2(-1)^{a+b}}{a!\,b!}\sum_{k=0}^{\max([a/2],[b/2])}\left\{a\binom{b}{2k}+b\binom{a}{2k}\right\}(a+b-2k-1)!(2k)!\\&\qquad\qquad\times\zeta(2k)\zeta_{MT,2}(a+b-2k,s_3;s_4)\\&\quad+(-1)^{a+1}\frac{(2\pi i)^{a+b}B_{a+b}}{(a+b)!}\zeta(s_3+s_4)\end{aligned}$$

holds for $s_3,s_4\in\mathbb{C}$ except for singularities of the both sides of (4.13).

Proof. Assume $\Re(s_3)>1$ and $\Re(s_4)>1$. Then we have

$$\begin{cases} \displaystyle\lim_{K\to\infty}\int_0^1 \sum_{k=1}^{K}\frac{e^{2\pi ikx}}{k^a}\sum_{l=1}^{K}\frac{e^{2\pi ilx}}{l^b}\sum_{m=1}^{K}\frac{e^{-2\pi imx}}{m^{s_3}}\sum_{n=1}^{K}\frac{e^{2\pi inx}}{n^{s_4}}dx=\zeta_{MT,3}(s_4,a,b;s_3); \\ \displaystyle\lim_{K\to\infty}\int_0^1 \sum_{k=-K}^{-1}\frac{e^{2\pi ikx}}{k^a}\sum_{l=1}^{K}\frac{e^{2\pi ilx}}{l^b}\sum_{m=1}^{K}\frac{e^{-2\pi imx}}{m^{s_3}}\sum_{n=1}^{K}\frac{e^{2\pi inx}}{n^{s_4}}dx=(-1)^aG(a,s_3,b,s_4); \\ \displaystyle\lim_{K\to\infty}\int_0^1 \sum_{k=1}^{K}\frac{e^{2\pi ikx}}{k^a}\sum_{l=-K}^{-1}\frac{e^{2\pi ilx}}{l^b}\sum_{m=1}^{K}\frac{e^{-2\pi imx}}{m^{s_3}}\sum_{n=1}^{K}\frac{e^{2\pi inx}}{n^{s_4}}dx=(-1)^bG(s_3,b,a,s_4); \\ \displaystyle\lim_{K\to\infty}\int_0^1 \sum_{k=-K}^{-1}\frac{e^{2\pi ikx}}{k^a}\sum_{l=-K}^{-1}\frac{e^{2\pi ilx}}{l^b}\sum_{m=1}^{K}\frac{e^{-2\pi imx}}{m^{s_3}}\sum_{n=1}^{K}\frac{e^{2\pi inx}}{n^{s_4}}dx \\ \qquad =(-1)^{a+b}\zeta_{MT,3}(a,b,s_3;s_4). \end{cases}$$

Therefore we can prove Theorem 4.4 in the same way as in the proof of Theorem 4.3.

We define $K_1(a,b,s_3,s_4)$ and $K_2(a,b,s_3,s_4)$ by the right-hand side of (4.12) and (4.13) respectively. By the preceding theorems, we obtain the following theorem which essentially includes not only Theorem 3.1 but also the assertion in the triple case given in [20].

Theorem 4.5. *For $a,b,c\in\mathbb{N}$,*

$$\begin{aligned} \zeta_{MT,3}(a,b,c;s)&-(-1)^{b+c}\zeta_{MT,3}(b,c,s;a) \\ &-(-1)^{c+a}\zeta_{MT,3}(c,s,a;b)-(-1)^{a+b}\zeta_{MT,3}(s,a,b;c) \\ &=(-1)^{a+b}K_2(a,b,c,s)-(-1)^bK_1(a,c,b,s)-(-1)^aK_1(c,b,a,s) \end{aligned} \tag{4.14}$$

holds for $s\in\mathbb{C}$ except for singularities of the both sides of (4.14).

Proof. By (4.12), we have

$$G(a,b,c,s)=K_1(a,b,c,s)-(-1)^b\zeta_{MT,3}(b,c,s;a)-(-1)^a\zeta_{MT,3}(c,s,a;b).$$

By exchanging the order of variables, we have

$$\begin{aligned} G(a,c,b,s)&=K_1(a,c,b,s)-(-1)^c\zeta_{MT,3}(b,c,s;a)-(-1)^a\zeta_{MT,3}(s,a,b;c); \\ G(c,b,a,s)&=K_1(c,b,a,s)-(-1)^b\zeta_{MT,3}(s,a,b;c)-(-1)^c\zeta_{MT,3}(c,s,a;b). \end{aligned}$$

Substituting these relations into (4.13), we obtain (4.14).

We denote by $M(a,b,c,s)$ the right-hand side of (4.14). We prove the following explicit formulas for $\zeta_{MT,3}(a,b,c;d)$.

Theorem 4.6. *For $a, b, c, d \in \mathbb{N}$ with $a+b+c+d \in 2\mathbb{N}$,*

$$\zeta_{MT,3}(a,b,c;d) = \frac{1}{4}\Big\{ M(a,b,c,d) - (-1)^{b+c} M(b,c,d,a) - (-1)^{a+c} M(c,d,a,b) - (-1)^{a+b} M(d,a,b,c) \Big\}. \tag{4.15}$$

Proof. By changing variables in (4.14), we have

$$\begin{aligned} M(b,c,d,a) =& \zeta_{MT,3}(b,c,d;a) - (-1)^{c+d}\zeta_{MT,3}(c,d,a;b) \\ &- (-1)^{d+b}\zeta_{MT,3}(d,a,b;c) - (-1)^{b+c}\zeta_{MT,3}(a,b,c;d); \\ M(c,d,a,b) =& \zeta_{MT,3}(c,d,a;b) - (-1)^{d+a}\zeta_{MT,3}(d,a,b;c) \\ &- (-1)^{a+c}\zeta_{MT,3}(a,b,c;d) - (-1)^{c+d}\zeta_{MT,3}(b,c,d;a); \\ M(d,a,b,c) =& \zeta_{MT,3}(d,a,b;c) - (-1)^{a+b}\zeta_{MT,3}(a,b,c;d) \\ &- (-1)^{b+d}\zeta_{MT,3}(b,c,d;a) - (-1)^{d+a}\zeta_{MT,3}(c,d,a;b). \end{aligned}$$

Multiply $(-1)^{b+c}$, $(-1)^{a+c}$, $(-1)^{a+b}$ on the both sides of above three equations, respectively, and sum them up. Then, by using (4.14) in the case $s = d$, we obtain (4.15).

Example 4.7. Put $(a,b,c) = (1,1,1)$ in (4.14). Then we obtain

$$\begin{aligned} &\zeta_{MT,3}(1,1,1;s) - 3\zeta_{MT,3}(s,1,1;1) + 6\zeta_{MT,2}(1,2;s) \\ &\quad + 6\zeta_{MT,2}(s,2;1) - 6\zeta(2)\zeta(s+1) + 12\zeta(s+3) = 0, \end{aligned}$$

which was essentially given by the first-named and the fourth-named authors (see [10, Example 6.1]). Similarly, putting $(a,b,c) = (2,2,2)$ in (4.14), we obtain

$$\begin{aligned} &\zeta_{MT,3}(2,2,2;s) - 3\zeta_{MT,3}(2,2,s;2) \\ &= 6\left\{2\zeta_{MT,2}(2,s;4) - \zeta_{MT,2}(4,s;2) - \zeta_{MT,2}(2,4;s)\right\} \\ &\quad + 4\zeta(2)\left\{\zeta_{MT,2}(2,2;s) - \zeta_{MT,2}(2,s;2)\right\} + 2\zeta(4)\zeta(s+2). \end{aligned}$$

In particular when $s = 2$, we obtain the formula for $\zeta_{MT,3}(2,2,2;2)$ given in Example 3.2. Put $(a,b,c,d) = (1,1,1,3)$ in (4.15). Then we obtain

$$\zeta_{MT,3}(1,1,1;3) = -6\zeta(3)^2 + \frac{23}{2520}\pi^6,$$

which can also be obtained from Hoffman's result in [2, Corollary 4.2] and Markett's result in [4, Corollary 4.3].

Remark 4.8. In [20], the fourth-named author proved, in a different way, that for $k_1, \ldots, k_{r+1} \in \mathbb{N}$, $\zeta_{MT,r}(k_1, \ldots, k_r; k_{r+1})$ can be expressed as a rational linear combination of products of the values of $\zeta_{MT,j}$ $(j < r)$ at positive integers if r and $\sum_{j=1}^{r+1} k_j$ are of different parity. This fact is sometimes called the 'parity result' for the Mordell-Tornheim zeta values. From this fact, we know that $\zeta_{MT,3}(a, b, c; d)$ $(a, b, c, d \in \mathbb{N})$ can be expressed as a rational linear combination of products of $\zeta_{MT,2}(p, q; r)$ and $\zeta(s)$ when $a+b+c+d \in 2\mathbb{N}$. Therefore we can interpret the results in Theorem 4.6 as concrete formulas which represent the parity result for $\zeta_{MT,3}$.

The method in this section can be applied to more general situation, which will be discussed elsewhere.

5 Analytic properties of certain triple zeta-functions

In this section, we aim to give the proof of Theorem 4.2. For this aim, we mainly use the method established by the first-named author (see [5, 6, 7, 8]).

Let

$$H(s_1, s_2, s_3, s_4) := \sum_{k,l,m=1}^{\infty} \frac{1}{k^{s_1} m^{s_2} (k+l)^{s_3} (l+m)^{s_4}}. \tag{5.1}$$

By (4.1), we have

$$\begin{aligned} G(s_1, s_2, s_3, s_4) &= \sum_{N=1}^{\infty} \sum_{\substack{k,l=1 \\ k+l=N}}^{\infty} \frac{1}{k^{s_1} l^{s_2}} \sum_{\substack{m,n=1 \\ m+n=N}}^{\infty} \frac{1}{m^{s_3} n^{s_4}} \\ &= \sum_{k,m=1}^{\infty} \frac{1}{k^{s_1} m^{s_3}} \sum_{N > \max(k,m)} \frac{1}{(N-k)^{s_2} (N-m)^{s_4}}. \end{aligned} \tag{5.2}$$

We separate the right-hand side of (5.2) as $\sum_{k<m} + \sum_{k=m} + \sum_{k>m}$. Then the first term and the third term are equal to $H(s_1, s_4, s_3, s_2)$ and $H(s_3, s_2, s_1, s_4)$, respectively. The second term is equal to $\zeta(s_1 + s_3)\zeta(s_2 + s_4)$. Hence we have

$$\begin{aligned} G(s_1, s_2, s_3, s_4) &= \zeta(s_1 + s_3)\zeta(s_2 + s_4) \\ &\quad + H(s_1, s_4, s_3, s_2) + H(s_3, s_2, s_1, s_4). \end{aligned} \tag{5.3}$$

Therefore we need to consider $H(s_1, s_2, s_3, s_4)$. Actually, $H(s_1, s_2, s_3, s_4)$ is equal to $\zeta_{\mathfrak{sl}(4)}(s_1, 0, s_2, s_3, s_4, 0)$, where $\zeta_{\mathfrak{sl}(4)}(s_1, s_2, s_3, s_4, s_5, s_6)$ is the Witten zeta-function associated with $\mathfrak{sl}(4)$ (see [9]) defined by

(5.4)

$$\zeta_{\mathfrak{sl}(4)}(s_1,s_2,s_3,s_4,s_5,s_6)=\sum_{l,m,n=1}^{\infty}\frac{1}{l^{s_1}m^{s_2}n^{s_3}(l+m)^{s_4}(m+n)^{s_5}(l+m+n)^{s_6}}.$$

First we prove the following lemma. Though this is regarded as a special case of [9, Theorem 3.5], we can prove this lemma more simply by considering a simple integral representation of $H(s_1,s_2,s_3,s_4)$ (see (5.15)).

Lemma 5.1. *The function $H(s_1,s_2,s_3,s_4)$ can be continued meromorphically to $\mathbb{C}^4$, and all of its singularities are located on the subsets of $\mathbb{C}^4$ defined by one of the equations:*

$$s_1+s_3=1-l \quad (l\in\mathbb{N}_0); \tag{5.5}$$

$$s_2+s_4=1-l \quad (l\in\mathbb{N}_0); \tag{5.6}$$

$$s_3+s_4=1-l \quad (l\in\mathbb{N}_0); \tag{5.7}$$

$$s_1+s_3+s_4=2-l \quad (l\in\mathbb{N}_0); \tag{5.8}$$

$$s_2+s_3+s_4=2-l \quad (l\in\mathbb{N}_0); \tag{5.9}$$

$$s_1+s_2+s_3+s_4=3, \tag{5.10}$$

all of which are true singularities.

Proof. We use the same notation as in the proof of [8, Theorem 1]. We recall the Mellin-Barnes formula

$$(1+\lambda)^{-s}=\frac{1}{2\pi i}\int_{(c)}\frac{\Gamma(s+z)\Gamma(-z)}{\Gamma(s)}\lambda^z dz, \tag{5.11}$$

where $\Re s>0$, $|\arg\lambda|<\pi$, $\lambda\neq 0$, $c\in\mathbb{R}$ with $-\Re s<c<0$, $i=\sqrt{-1}$ and the path (c) of integration is the vertical line $\Re z=c$.

Assume $s_j\in\mathbb{C}$ with $\Re s_j>1$ $(j=1,2,3,4)$. Then $H(s_1,s_2,s_3,s_4)$ is convergent absolutely. Let $(k+l)^{-s_3}=l^{-s_3}\left(1+\frac{k}{l}\right)^{-s_3}$ in (5.1), and substitute (5.11) with $\lambda=k/l$ into (5.1). Assume $-\Re s_3<c<0$. Then we have

$$\begin{aligned}(5.12)\quad & H(s_1,s_2,s_3,s_4)\\ &=\frac{1}{2\pi i}\int_{(c)}\frac{\Gamma(s_3+z)\Gamma(-z)}{\Gamma(s_3)}\sum_{k=1}^{\infty}\frac{1}{k^{s_1}}\sum_{l,m=1}^{\infty}\frac{1}{l^{s_3}m^{s_2}(l+m)^{s_4}}\left(\frac{k}{l}\right)^z dz\\ &=\frac{1}{2\pi i}\int_{(c)}\frac{\Gamma(s_3+z)\Gamma(-z)}{\Gamma(s_3)}\zeta(s_1-z)\zeta_{MT,2}(s_2,s_3+z;s_4)dz.\end{aligned}$$

Note that, by the assumptions $\Re s_j>1$ $(1\le j\le 4)$ and $-\Re s_3<c<0$, we see that each series in the integrand on the second member of (5.12) is convergent absolutely. By [5, Theorem 1], the singularities of $\zeta_{MT,2}(s_1,s_2;s_3)$

are located only on the subsets of $\mathbb{C}^3$ defined by one of the equations:

$$s_1+s_3=1-l,\ s_2+s_3=1-l\ (l\in\mathbb{N}_0),\ s_1+s_2+s_3=2.$$

Hence, by considering singularities of $\Gamma(s)$, $\zeta(s)$ and $\zeta_{MT,2}(s_1,s_2;s_3)$, we see that the singularities of the integrand of (5.12) are determined by $z=-s_3-l$, $z=l$, $z=s_1-1$, $s_2+s_4=1-l$, $z=1-s_3-s_4-l$ $(l\in\mathbb{N}_0)$ and $z=2-s_2-s_3-s_4$.

Now we shift the path $\Re z=c$ to $\Re z=M-\varepsilon$ for a sufficiently large $M\in\mathbb{N}$ and a sufficiently small positive $\varepsilon\in\mathbb{R}$. Then all the relevant singularities are $z=l$ $(0\le l\le M-1)$ and $z=s_1-1$. Counting their residues, and using the relations

$$\frac{(-1)^l}{l!}\frac{\Gamma(s+l)}{\Gamma(s)}=(-1)^l\binom{s+l-1}{l}=\binom{-s}{l}, \tag{5.13}$$

and

$$\Gamma(-l-\delta)=\frac{\Gamma(1-\delta)}{(-\delta)\cdots(-l-\delta)}=-\frac{(-1)^l}{l!}\left(\frac{1}{\delta}+O(1)\right)\quad(\delta\to 0) \tag{5.14}$$

for $l\in\mathbb{N}_0$, we have

(5.15)

$$\begin{aligned}H(s_1,s_2,s_3,s_4)=&\frac{\Gamma(s_1+s_3-1)\Gamma(1-s_1)}{\Gamma(s_3)}\zeta_{MT,2}(s_2,s_1+s_3-1;s_4)\\&+\sum_{k=0}^{M-1}\binom{-s_3}{k}\zeta(s_1-k)\zeta_{MT,2}(s_2,s_3+k;s_4)\\&+\frac{1}{2\pi i}\int_{(M-\varepsilon)}\frac{\Gamma(s_3+z)\Gamma(-z)}{\Gamma(s_3)}\zeta(s_1-z)\zeta_{MT,2}(s_2,s_3+z;s_4)dz,\end{aligned}$$

because $\mathrm{Res}_{z=s_1-1}\,\zeta(s_1-z)=-1$ and

$$\frac{\Gamma(s_3+k)}{\Gamma(s_3)}\operatorname*{Res}_{z=k}\Gamma(-z)=-\binom{-s_3}{k}.$$

Since M can be taken arbitrarily large, (5.15) implies the meromorphic continuation of $H(s_1,s_2,s_3,s_4)$ to $\mathbb{C}^4$.

Fix a point $(s_1,s_2,s_3,s_4)\in\mathbb{C}^4$. Among the above list of singularities of the integrand of (5.12), only the family $s_2+s_4=1-l$ $(l\in\mathbb{N}_0)$ is independent of z. Therefore, by choosing a sufficiently large M, we may assume that the integral term on the right-hand side of (5.15) is holomorphic, except $s_2+s_4=1-l$ $(l\in\mathbb{N}_0)$, around the fixed (s_1,s_2,s_3,s_4). Also we see that the singularities of the first term on the right-hand side of (5.15) are determined by $s_1+s_3=1-l$, $s_1=1+l$, $s_2+s_4=1-l$, $s_1+s_3+s_4=2-l$,

$s_1+s_2+s_3+s_4=3$ ($l\in\mathbb{N}_0$), those of the second term are determined by $s_1=1+k$, $s_2+s_4=1-l$, $s_3+s_4=1-(k+l)$, $s_2+s_3+s_4=2-k$ ($0\le k\le M-1$; $l\in\mathbb{N}_0$). Using the symmetricity

$$H(s_2,s_1,s_4,s_3)=H(s_1,s_2,s_3,s_4), \tag{5.16}$$

we see that $s_1=1+l$ is not a singularity of $H(s_1,s_2,s_3,s_4)$ because $s_2=1+l$ is not a singularity. On the other hand, we see that

$$\zeta_{MT,2}(s_1,-l;s_2)\not\equiv 0 \quad (l\in\mathbb{N}_0), \tag{5.17}$$

because

$$\zeta_{MT,2}(l+2,-l;l+2)=\sum_{m,n=1}^{\infty}\frac{n^l}{m^{l+2}(m+n)^{l+2}}>0, \tag{5.18}$$

which is convergent absolutely. Hence, from (5.17), we see that $s_1+s_3=1-l$ is not cancelled with the factor of $\zeta_{MT,2}$ in the first term on the right-hand side of (5.15). Hence we find that (5.5), (5.8)-(5.10) determine true singularities because these equations come from only one term on the right-hand side of (5.15). The singularity $s_3+s_4=1-l$ comes from the terms, corresponding to $0\le k\le l$, in the sum part on the right-hand side of (5.15); but these are not cancelled, because the residues coming from different terms have different order with respect to s_3. Hence (5.7) also gives true singularities. Furthermore, combining (5.16) with the fact that (5.5) determines a true singularity as mentioned above, we can conclude that (5.6) also determines a true singularity. Thus we obtain the proof of Lemma 5.1.

Remark 5.2. From [9, Theorem 3.5], we see that the list of singularities of $H(s_1,s_2,s_3,s_4)=\zeta_{\mathfrak{sl}(4)}(s_1,0,s_2,s_3,s_4,0)$ are given by (5.5)-(5.10) and

$$s_1+s_2+s_3+s_4=2-l \quad (l\in\mathbb{N}_0), \tag{5.19}$$

though (5.19) does not appear in Lemma 5.1. In fact, we can check that (5.19) does not determine the singularity of $H(s_1,s_2,s_3,s_4)$ as follows. The possible singularity (5.19) comes from [9, (3.43)], the singularities of $\zeta_{\mathfrak{sl}(4)}$, which come from

$$\begin{aligned}S_1=&\frac{\Gamma(s_3+s_5+s_6+n-1)\Gamma(1-s_3-s_5-n)}{\Gamma(s_6)}\\&\times\zeta_{MT,2}(s_1,s_2-n;s_3+s_4+s_5+s_6+n-1)\end{aligned}$$

corresponding to $s_1+(s_3+s_4+s_5+s_6+n-1)=1-l$ ($l\in\mathbb{N}_0$). These are indeed true singularities of $\zeta_{\mathfrak{sl}(4)}(s_1,s_2,s_3,s_4,s_5,s_6)$. However, in the above argument, we consider the case $(s_2,s_6)=(0,0)$. Hence these singularities are cancelled with $\Gamma(s_6)$ as $s_6\to 0$. Thus (5.19) does not determine a singularity of $H(s_1,s_2,s_3,s_4)$.

From Lemma 5.1, we give the proof of Theorem 4.2, namely determine the true singularities of $G(s_1, s_2, s_3, s_4)$.

Proof of Theorem 4.2 The meromorphic continuation of G comes from that of H and of $\zeta(s)$. From Lemma 5.1, true singularities of $H(s_1, s_4, s_3, s_2)$ are determined by $s_1 + s_3 = 1 - l$, $s_2 + s_4 = 1 - l$, $s_2 + s_3 = 1 - l$, $s_1+s_2+s_3 = 2-l$, $s_2+s_3+s_4 = 2-l$ and $s_1+s_2+s_3+s_4 = 3$ ($l \in \mathbb{N}_0$), and those of $H(s_3, s_2, s_1, s_4)$ are determined by $s_1 + s_3 = 1 - l$, $s_2 + s_4 = 1 - l$, $s_1+s_4 = 1-l$, $s_1+s_3+s_4 = 2-l$, $s_1+s_2+s_4 = 2-l$ and $s_1+s_2+s_3+s_4 = 3$. Furthermore those of $\zeta(s_1+s_3)\zeta(s_2+s_4)$ are determined by $s_1+s_3 = 1$ and $s_2+s_4 = 1$. Hence we have only to check that $s_1+s_3 = 1-l$, $s_2+s_4 = 1-l$ and $s_1 + s_2 + s_3 + s_4 = 3$ determine true singularities.

Using the relation $G(s_1, s_2, s_3, s_4) = G(s_2, s_1, s_3, s_4) = G(s_1, s_2, s_4, s_3)$ and from the fact that $s_1 + s_4 = 1 + l$ determines a true singularity as mentioned above, we can conclude that $s_1 + s_3 = 1 - l$ and $s_2 + s_4 = 1 - l$ also determine true singularities.

On the other hand, this kind of argument using symmetry is not enough to prove the fact that $s_1 + s_2 + s_3 + s_4 = 3$ determines a true singularity. Hence we have to give more detailed consideration. From [5, (5.3)], we have

$$\begin{aligned}(5.20)\qquad \zeta_{MT,2}(s_1, s_2; s_3) = {} & \frac{\Gamma(s_2+s_3-1)\Gamma(1-s_2)}{\Gamma(s_3)}\zeta(s_1+s_2+s_3-1) \\ & + \sum_{k=0}^{M-1}\binom{-s_3}{k}\zeta(s_1+s_3+k)\zeta(s_2-k) \\ & + \frac{1}{2\pi i}\int_{(M-\varepsilon)}\frac{\Gamma(s_3+z)\Gamma(-z)}{\Gamma(s_3)}\zeta(s_1+s_3+z)\zeta(s_2-z)dz.\end{aligned}$$

Therefore the singular part of $\zeta_{MT,2}(s_2, s_1 + s_3 - 1; s_4)$ corresponding to $s_1+s_2+s_3+s_4 = 3$ comes from $\zeta(s_1+s_2+s_3+s_4-2)$. Hence the relevant singular part of $H(s_1, s_4, s_3, s_2)$ is

$$\begin{aligned}(5.21)\qquad & \frac{\Gamma(s_1+s_3-1)\Gamma(1-s_1)}{\Gamma(s_3)}\cdot\frac{\Gamma(s_1+s_2+s_3-2)\Gamma(2-s_1-s_3)}{\Gamma(s_2)} \\ & \times\zeta(s_1+s_2+s_3+s_4-2).\end{aligned}$$

Similarly, the relevant singular part of $H(s_3, s_2, s_1, s_4)$ is

$$\begin{aligned}(5.22)\qquad & \frac{\Gamma(s_1+s_3-1)\Gamma(1-s_3)}{\Gamma(s_1)}\cdot\frac{\Gamma(s_1+s_3+s_4-2)\Gamma(2-s_1-s_3)}{\Gamma(s_4)} \\ & \times\zeta(s_1+s_2+s_3+s_4-2).\end{aligned}$$

Therefore the relevant singular part of $G(s_1,s_2,s_3,s_4)$ is

(5.23)

$$\begin{aligned}&\Gamma(s_1+s_3-1)\Gamma(2-s_1-s_3)\zeta(s_1+s_2+s_3+s_4-2)\\&\quad\times\left\{\frac{\Gamma(1-s_1)\Gamma(s_1+s_2+s_3-2)}{\Gamma(s_2)\Gamma(s_3)}+\frac{\Gamma(1-s_3)\Gamma(s_1+s_3+s_4-2)}{\Gamma(s_1)\Gamma(s_4)}\right\}.\end{aligned}$$

If we substitute $s_4=3-s_1-s_2-s_3$ into the part of the curly parentheses in (5.23), then we find that it is equal to

$$\frac{\Gamma(1-s_1)\Gamma(s_1+s_2+s_3-2)}{\Gamma(s_2)\Gamma(s_3)}+\frac{\Gamma(1-s_3)\Gamma(1-s_2)}{\Gamma(s_1)\Gamma(3-s_1-s_2-s_3)}.$$

We can check that this quantity is $\not\equiv 0$, by observing the situation when $\Im s_1\to\infty$, or by observing the value at (s_1,s_2,s_3) with $s_2=s_3=1/2$ and $s_1\to 1$. Thus we see that $s_1+s_2+s_3+s_4=3$ determines a true singularity. This completes the proof of Theorem 4.2. (Note that we will give another expression of (5.23) in Remark 6.3.)

Remark 5.3. In the proof of Theorem 4.2, we concluded that $s_1+s_3=1+l$ ($l\in\mathbb{N}_0$) gives a true singularity by the argument using symmetry of indices and the fact that $s_1+s_4=1+l$ determines a true singularity. On the other hand, we can directly prove this fact as follows.

Singularities determined by $s_1+s_3=1-l$ ($l\in\mathbb{N}_0$) come from $\Gamma(s_1+s_3-1)$ in the second term on the right-hand side of (5.12). Suppose $l\in\mathbb{N}$. Then the relevant singular part of $G(s_1,s_2,s_3,s_4)$ is

$$\begin{aligned}\text{(5.24)}\qquad&\Gamma(s_1+s_3-1)\left\{\frac{\Gamma(1-s_1)}{\Gamma(s_3)}\zeta_{MT,2}(s_4,s_1+s_3-1;s_2)\right.\\&\qquad\left.+\frac{\Gamma(1-s_3)}{\Gamma(s_1)}\zeta_{MT,2}(s_2,s_1+s_3-1;s_4)\right\}.\end{aligned}$$

If we substitute $s_3=1-s_1-l$ into the part of the curly parentheses, then we have

$$\begin{aligned}\text{(5.25)}\qquad&\frac{\Gamma(1-s_1)}{\Gamma(1-s_1-l)}\zeta_{MT,2}(s_4,-l;s_2)+\frac{\Gamma(s_1+l)}{\Gamma(s_1)}\zeta_{MT,2}(s_2,-l;s_4)\\&\quad=s_1(s_1+1)\cdots(s_1+l-1)\\&\qquad\times\left\{\zeta_{MT,2}(s_4,-l;s_2)+(-1)^l\zeta_{MT,2}(s_2,-l;s_4)\right\}\not\equiv 0.\end{aligned}$$

In fact, if l is even, then by putting $s_2=s_4=l+2$ and using (5.18), we see that (5.25) holds. If l is odd, then by putting $s_2=l+2$ and $s_4=l+3$ we have

$$\zeta_{MT,2}(l+3,-l;l+2)-\zeta_{MT,2}(l+2,-l;l+3)=\zeta_{MT,2}(l+3,-l-1;l+3)>0,$$

hence (5.25) holds. This implies that $s_1 + s_3 = 1 - l$ ($l \in \mathbb{N}$) determine true singularities.

Suppose $l = 0$. Then the relevant singular part of $G(s_1, s_2, s_3, s_4)$ is (5.20) plus $\zeta(s_1 + s_3)\zeta(s_2 + s_4)$ which can be written as

$$\begin{aligned} &\frac{1}{s_1 + s_3 - 1}\left\{\zeta(s_2 + s_4) + \frac{\Gamma(1-s_1)}{\Gamma(s_3)}\zeta_{MT,2}(s_4, s_1 + s_3 - 1; s_2)\right. \\ &\quad \left. + \frac{\Gamma(1-s_3)}{\Gamma(s_1)}\zeta_{MT,2}(s_2, s_1 + s_3 - 1; s_4)\right\} + O(1). \end{aligned} \tag{5.26}$$

If we substitute $s_3 = 1 - s_1$ into the part of the curly parentheses, then we have

$$\zeta(s_2 + s_4) + \zeta_{MT,2}(s_2, 0; s_4) + \zeta_{MT,2}(s_4, 0; s_2) = \zeta(s_2)\zeta(s_4) \not\equiv 0.$$

This implies that $s_1 + s_3 = 1$ determines a true singularity.

6 True singularities of $\zeta_{MT,r}$ and some remarks

In this section, we consider further applications of the method used in Section 5.

First we determine true singularities of $\zeta_{MT,r}(s_1, \ldots, s_r; s_{r+1})$. Actually, in [8, Theorem 1], the first-named author showed that $\zeta_{MT,r}(s_1, \ldots, s_r; s_{r+1})$ can be continued meromorphically to $\mathbb{C}^{r+1}$ and gave the list ((6.1) below) of the *possible* singularities. By combining this method with our present method, we can determine true singularities of $\zeta_{MT,r}$ as follows. Note that the case $r = 3$ of this theorem coincides with Theorem 4.1.

Theorem 6.1. *The function $\zeta_{MT,r}(s_1, \ldots, s_r; s_{r+1})$ can be continued meromorphically to $\mathbb{C}^{r+1}$ and their singularities are given by one of the following equations:*

$$\left\{ \begin{array}{l} (s_j - 1) + s_{r+1} = -l \quad (1 \le j \le r,\ l \in \mathbb{N}_0); \\ (s_{j_1} - 1) + (s_{j_2} - 1) + s_{r+1} = -l \quad (1 \le j_1 < j_2 \le r,\ l \in \mathbb{N}_0); \\ \quad \cdots\cdots \\ \displaystyle\sum_{\nu=1}^{r-1}(s_{j_\nu} - 1) + s_{r+1} = -l \quad (1 \le j_1 < \cdots < j_{r-1} \le r, l \in \mathbb{N}_0); \\ s_1 + s_2 + \cdots + s_{r+1} = r, \end{array} \right. \tag{6.1}$$

all of which are true singularities.

Proof. We will prove this theorem by induction on $r \ge 1$.

In the case $r=1$, we see that $\zeta_{MT,1}(s_1;s_2)=\zeta(s_1+s_2)$. Hence $s_1+s_2=1$ only determines a singularity of $\zeta_{MT,1}(s_1;s_2)$. Thus we have the assertion. Actually the case $r=2$ has also been proved in [5, Theorem 1].

Assume that the assertion in the case of $r-1$ $(r>1)$ holds, and consider the case of r. From [8, (3.2)], we have

(6.2)

$$\begin{aligned}
&\zeta_{MT,r}(s_1,\ldots,s_r;s_{r+1})\\
&\quad=\frac{\Gamma(s_r+s_{r+1}-1)\Gamma(1-s_r)}{\Gamma(s_{r+1})}\zeta_{MT,r-1}(s_1,\ldots,s_{r-1};s_r+s_{r+1}-1)\\
&\quad+\sum_{k=0}^{M-1}\binom{-s_{r+1}}{k}\zeta_{MT,r-1}(s_1,\ldots,s_{r-1};s_{r+1}+k)\zeta(s_r-k)\\
&\quad+\frac{1}{2\pi i}\int_{(M-\varepsilon)}\frac{\Gamma(s_{r+1}+z)\Gamma(-z)}{\Gamma(s_{r+1})}\zeta_{MT,r-1}(s_1,\ldots,s_{r-1};s_{r+1}+z)\zeta(s_r-z)dz,
\end{aligned}$$

where $M\,(\in\mathbb{N})$ is a sufficiently large number and $\varepsilon\,(\in\mathbb{R})$ is a sufficiently small positive number. Since M can be taken arbitrarily large, (6.2) implies the meromorphic continuation of $\zeta_{MT,r}(s_1,\ldots,s_r;s_{r+1})$ to $\mathbb{C}^{r+1}$, by the assumption of induction.

Now we take a sufficiently large M which satisfies that the third term on the right-hand side of (6.2) is holomorphic on a certain neighbourhood of $(s_1,\ldots,s_{r+1})$. Indeed, we can take it by the assumption of induction. Then, by the assumption again, we see that singularities of the first term on the right-hand side of (6.2) are determined by

$$\sum_{j\in J}(s_j-1)+(s_r+s_{r+1}-1)=-l\quad(l\in\mathbb{N}_0);\tag{6.3}$$

$$\sum_{j=1}^{r-1}(s_j-1)+(s_r+s_{r+1}-1)=0;\tag{6.4}$$

$$s_r+s_{r+1}=1-l\quad(l\in\mathbb{N}_0);\tag{6.5}$$

$$s_r=1+l\quad(l\in\mathbb{N}_0),\tag{6.6}$$

where $J(\neq\emptyset)$ runs over all proper subsets of $\{1,2,\ldots,r-1\}$. Similarly, we see that singularities of the second term on the right-hand side of (6.2) are determined by

$$\sum_{j\in J}(s_j-1)+(s_{r+1}+k)=-l\quad(k,l\in\mathbb{N}_0);\tag{6.7}$$

$$\sum_{j=1}^{r-1}(s_j-1)+(s_{r+1}+k)=0\quad(k\in\mathbb{N}_0);\tag{6.8}$$

$$s_r-k=1\quad(k\in\mathbb{N}_0)\tag{6.9}$$

for J as above.

First we claim that (6.6), namely (6.9) is not a singularity of $\zeta_{MT,r}$. In fact, since $s_1 = 1 + l$ ($l \in \mathbb{N}_0$) is not singular because of $r > 1$, we see from the symmetry of indices

$$\zeta_{MT,r}(s_1, \ldots, s_r; s_{r+1}) = \zeta_{MT,r}(s_r, s_1, \ldots, s_{r-1}; s_{r+1})$$

that $s_r = 1 + l$ ($l \in \mathbb{N}_0$) is also not singular. Note that this fact has been already obtained from [8, (3.3)], by checking the cancellation directly. However the above argument is much simpler than that in [8].

Next we claim that (6.3), (6.4) and (6.5) determine true singularities of $\zeta_{MT,r}$. In fact, these come from only the first term on the right-hand side of (6.2). Furthermore (6.3) and (6.4) are not cancelled with the Gamma factors, hence determine true singularities. On the other hand, we need to check whether (6.5) is cancelled with the factor of $\zeta_{MT,r-1}$ or not. For this checking, we claim that

$$\zeta_{MT,r}(s_1, \ldots, s_r; -l) \not\equiv 0 \tag{6.10}$$

for $l \in \mathbb{N}_0$. Actually, as well as (5.17) and (5.18), this fact comes from

$$\zeta_{MT,r}(l+2, \ldots, l+2; -l) = \sum_{m_1,\ldots,m_r=1}^{\infty} \frac{(m_1 + \cdots + m_r)^l}{m_1^{l+2} \cdots m_r^{l+2}}$$
$$= \sum_{\substack{k_1,\ldots,k_r \in \mathbb{N}_0 \\ k_1+\cdots+k_r=l}} \frac{l!}{k_1! \cdots k_r!} \zeta(l+2-k_1) \cdots \zeta(l+2-k_r) > 0.$$

From these facts, we see that (6.5) is not cancelled with the factor of $\zeta_{MT,r-1}$, hence (6.5) determines a true singularity.

Lastly we consider (6.7) and (6.8). In fact, from the symmetry of indices and by the fact that (6.3) determines a true singularity, we see that (6.7) and (6.8) also determine true singularities. Thus, from the above consideration, we see that the assertion in the case of r holds. By induction on r, we obtain the proof of Theorem 6.1.

Remark 6.2. Here we give an alternating proof of (6.10) by induction on $r \in \mathbb{N}$. In the case $r = 1$, it is obvious. Hence we assume that (6.10) holds for $r - 1$ ($r > 1$), and prove the case of r. Put $s_r = -l$ in (6.2). Then the first and the third terms on the right-hand side of (6.2) vanish because of the Gamma factor. Therefore we have

$$\begin{aligned} &\zeta_{MT,r}(s_1, \ldots, s_r; -l) \\ &\quad = \sum_{k=0}^{M-1} \binom{l}{k} \zeta_{MT,r-1}(s_1, \ldots, s_{r-1}; k-l) \zeta(s_r - k). \end{aligned} \tag{6.11}$$

We see that as a set of meromorphic functions, $\{\zeta(s-k) \mid k \in \mathbb{N}_0\}$ are linearly independent over $\mathbb{C}$. In fact, we have only to consider each pole of $\zeta(s-k)$ $(k \in \mathbb{N}_0)$. From the assumption of induction, we have

$$\zeta_{MT,r-1}(s_1, ..., s_{r-1}; -l) \neq 0 \tag{6.12}$$

for some $(s_1, ..., s_{r-1}) \in \mathbb{C}^{r-1}$. If we regard (6.11) as a linear relation for the functions in s_r, then (6.12) implies that the coefficient of $\zeta(s_r)$ does not vanish. Hence we see that $\zeta_{MT,r}(s_1, ..., s_r; -l) \not\equiv 0$. Thus we have the assertion.

Remark 6.3. Since (6.4) is not cancelled with the Gamma factor, we can prove that the singular part of $\zeta_{MT,r}(s_1, \ldots, s_r; s_{r+1})$ relevant to (6.4) can be written as

$$\frac{\Gamma(1-s_1)\cdots\Gamma(1-s_r)}{(s_1+\cdots+s_{r+1}-r)\Gamma(s_{r+1})} + O(1) \tag{6.13}$$

as $s_1 + \cdots + s_{r+1} \to r$, by induction on r. In fact, in the case $r = 1$, we see that (6.13) is

$$\frac{\Gamma(1-s_1)}{(s_1+s_2-1)\Gamma(s_2)} + O(1) = \frac{1}{s_1+s_2-1} + O(1), \tag{6.14}$$

which coincides with the singular part of $\zeta_{MT,1}(s_1; s_2)(= \zeta(s_1+s_2))$ relevant to $s_1 + s_2 - 1$. Hence we have the assertion in the case $r = 1$. Assume that the case of $r - 1$ holds. Then the singular part of $\zeta_{MT,r-1}(s_1, \ldots, s_{r-1}; s_r + s_{r+1} - 1)$ relevant to (6.4) is

$$\frac{\Gamma(1-s_1)\cdots\Gamma(1-s_{r-1})}{(s_1+\cdots+s_{r+1}-r)\Gamma(s_r+s_{r+1}-1)} + O(1). \tag{6.15}$$

By substituting (6.15) into (6.2), we immediately obtain the assertion in the case of r.

Applying (6.13) in the case $r = 2$ to (5.15), we see that (5.21) can be written as

$$\begin{aligned} &\frac{\Gamma(s_1+s_3-1)\Gamma(1-s_1)}{\Gamma(s_3)} \frac{\Gamma(1-s_4)\Gamma(2-s_1-s_3)}{(s_1+s_2+s_3+s_4-3)\Gamma(s_2)} + O(1) \\ &\quad = -\frac{\Gamma(1-s_1)\Gamma(1-s_2)\Gamma(1-s_3)\Gamma(1-s_4)}{s_1+s_2+s_3+s_4-3} \frac{\sin(\pi s_2)\sin(\pi s_3)}{\pi \sin(\pi(s_1+s_3))} + O(1), \end{aligned} \tag{6.16}$$

because $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$. Similarly, we see that (5.22) can be written as

$$-\frac{\Gamma(1-s_1)\Gamma(1-s_2)\Gamma(1-s_3)\Gamma(1-s_4)}{s_1+s_2+s_3+s_4-3} \frac{\sin(\pi s_4)\sin(\pi s_1)}{\pi \sin(\pi(s_1+s_3))} + O(1). \tag{6.17}$$

Since it can be elementarily shown that

$$\begin{aligned}&\sin(\pi s_1)\sin(\pi s_4)+\sin(\pi s_2)\sin(\pi s_3)-\sin(\pi(s_1+s_2))\sin(\pi(s_1+s_3))\\&\quad=\{\sin(\pi s_4)-\sin(\pi(s_1+s_2+s_3))\}\sin(\pi s_1),\end{aligned}$$

we have

$$\begin{aligned}&\sin(\pi s_1)\sin(\pi s_4)+\sin(\pi s_2)\sin(\pi s_3)\\&\qquad=\sin(\pi(s_1+s_2))\sin(\pi(s_1+s_3))+O(s_1+s_2+s_3+s_4-3).\end{aligned}\tag{6.18}$$

Hence, using (6.18), we see that (5.23), that is the singular part of $G(s_1,s_2,s_3,s_4)$ relevant to (5.10), can be obtained as (6.16) plus (6.17), namely

$$\begin{aligned}&-\frac{\Gamma(1-s_1)\Gamma(1-s_2)\Gamma(1-s_3)\Gamma(1-s_4)}{s_1+s_2+s_3+s_4-3}\frac{\sin(\pi s_2)\sin(\pi s_3)+\sin(\pi s_4)\sin(\pi s_1)}{\pi\sin(\pi(s_1+s_3))}\\&=-\frac{\Gamma(1-s_1)\Gamma(1-s_2)\Gamma(1-s_3)\Gamma(1-s_4)}{s_1+s_2+s_3+s_4-3}\frac{\sin(\pi(s_1+s_2))}{\pi}+O(1).\end{aligned}\tag{6.19}$$

From this expression it is obvious that (6.19), that is (5.23), is indeed singular at $s_1+s_2+s_3+s_4=3$.

We conclude this paper with a comment on the Witten multiple zeta-function (5.4) associated with $\mathfrak{sl}(4)$. From (5.4), we see that

$$\zeta_{\mathfrak{sl}(4)}(s_1,s_2,s_3,s_4,s_5,s_6)=\zeta_{\mathfrak{sl}(4)}(s_3,s_2,s_1,s_5,s_4,s_6).\tag{6.20}$$

In [9, Section 4], it was shown that true singularities of $\zeta_{\mathfrak{sl}(4)}(s_1,s_2,s_3,s_4,s_5,s_6)$ are determined by

$$s_1+s_4+s_6=1-l\quad(l\in\mathbb{N}_0);\tag{6.21}$$

$$s_2+s_4+s_5+s_6=1-l\quad(l\in\mathbb{N}_0);\tag{6.22}$$

$$s_3+s_5+s_6=1-l\quad(l\in\mathbb{N}_0);\tag{6.23}$$

$$s_1+s_2+s_4+s_5+s_6=2-l\quad(l\in\mathbb{N}_0);\tag{6.24}$$

$$s_1+s_3+s_4+s_5+s_6=2-l\quad(l\in\mathbb{N}_0);\tag{6.25}$$

$$s_2+s_3+s_4+s_5+s_6=2-l\quad(l\in\mathbb{N}_0);\tag{6.26}$$

$$s_1+s_2+s_3+s_4+s_5+s_6=3.\tag{6.27}$$

Using (6.20), we see that (6.23) and (6.26) determine true singularities from the fact that (6.21) and (6.24) do so. This argument for (6.23) and (6.26) is much simpler than the original method in [9]. Hence we can see that this kind of argument using symmetry is convenient for checking whether singularities are true or not. On the other hand, the method used in the latter part of the proof of Theorem 4.2 and in Remarks 5.3 and 6.3 is convenient for getting explicit information about singularities. Therefore it seems that we should use these two methods properly case by case.

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