THE ANALYTIC CONTINUATION AND THE ASYMPTOTIC BEHAVIOUR OF CERTAIN MULTIPLE ZETA-FUNCTIONS III

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Dedicated to Professor Akio Fujii,
my supervisor in my graduate course days,
on the occasion of his sixtieth birthday

1. Introduction

Let r be a positive integer, $s_1, \ldots, s_r$ be complex variables, $\alpha_1, \ldots, \alpha_r$, $w_1, \ldots,$ w_r be complex parameters, and consider the r-ple zeta-function

$$\begin{aligned}
&\zeta_r((s_1, \ldots, s_r); (\alpha_1, \ldots, \alpha_r), (w_1, \ldots, w_r)) \\
&= \sum_{m_1=0}^{\infty} \cdots \sum_{m_r=0}^{\infty} (\alpha_1 + m_1 w_1)^{-s_1} (\alpha_2 + m_1 w_1 + m_2 w_2)^{-s_2} \\
&\qquad \times \cdots \times (\alpha_r + m_1 w_1 + \cdots + m_r w_r)^{-s_r}. \qquad (1.1)
\end{aligned}$$

If $\alpha_j + m_1 w_1 + \cdots + m_j w_j = 0$ for some j and some $(m_1, \ldots, m_j)$, then the corresponding terms are to be removed from the sum.

This type of multiple zeta-functions was introduced by the author [11] [12], as a simultaneous generalization of both Barnes multiple zeta-functions (the case $s_1 = \cdots = s_{r-1} = 0$ in (1.1)) and Euler-Zagier sums (the case $\alpha_j = j$, $w_j = 1$ for $1 \leq j \leq r$ in (1.1)). Let ℓ be a fixed line on the complex plane $\mathbf{C}$ crossing the origin. Then ℓ divides $\mathbf{C}$ into three parts; two open half-planes and ℓ itself. Let $H(\ell)$ be one of those half-planes, and assume that

$$w_j \in H(\ell) \qquad (1 \leq j \leq r). \qquad (1.2)$$

In [11] it has been proved that, under assumption (1.2), the series (1.1) is convergent absolutely in the region

$$\mathcal{A}_r = \{(s_1, \ldots, s_r) \in \mathbf{C}^r \mid \Re(s_{r-k+1} + \cdots + s_r) > k \quad (1 \leq k \leq r)\} \qquad (1.3)$$

uniformly in any compact subset of $\mathcal{A}_r$. In [12], the meromorphic continuation of (1.1) to the whole space $\mathbf{C}^r$ has been shown in the special case when $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_r$, and $w_j = 1$ for all j.

Then in [13] [14], the meromorphic continuation of (1.1) has been studied in a more general setting. In [13], besides (1.2), we further assume

$$\alpha_j \in H(\ell) \qquad (1 \leq j \leq r) \qquad (1.4)$$

and

$$\alpha_{j+1}-\alpha_j\in H(\ell)\qquad (1\leq j\leq r-1). \tag{1.5}$$

Note that $\alpha_j+m_1w_1+\cdots+m_jw_j\in H(\ell)$ always holds under (1.2) and (1.4). In [13], the meromorphic continuation of (1.1) to the whole space $\mathbf{C}^r$ has been established under the above assumptions. Moreover in the same paper, the asymptotic expansion of $\zeta_r((s_1,\ldots,s_r);(\alpha_1,\ldots,\alpha_r),(w_1,\ldots,w_r))$ with respect to w_r when $|w_r|\to 0$ has been proved. In [14], the meromorphic continuation of (1.1) has been proved just under assumption (1.2), without assuming (1.4), (1.5).

In the present paper we return to the situation in [13]; therefore hereafter we assume (1.2), (1.4) and (1.5). The main purpose of the present paper is to study the asymptotic behaviour of $\zeta_r((s_1,\ldots,s_r);(\alpha_1,\ldots,\alpha_r),(w_1,\ldots,w_r))$ when $|w_r|\to\infty$. In the next section we will prove a formula, which describes the behaviour of ζ_r when $|w_r|\to\infty$. In Sections 3 and 4 we will study an important special case, that is the class of Barnes multiple zeta-functions. Then in Section 5 we will deduce an asymptotic expansion formula for multiple gamma-functions. Finally in Section 6 we will give some comments on Shintani's theory for Hecke L-functions of totally real number fields. In particular we will show that there is a recursive structure in the family of Shintani multiple zeta-functions.

2. The general asymptotic formula

The purpose of this section is to prove an asymptotic expansion formula for general $\zeta_r((s_1,\ldots,s_r);(\alpha_1,\ldots,\alpha_r),(w_1,\ldots,w_r))$, under assumptions (1.2), (1.4) and (1.5). Let

$$\zeta(s)=\sum_{m=1}^{\infty}m^{-s},\qquad \zeta(s,a)=\sum_{m=0}^{\infty}(m+a)^{-s}$$

be the Riemann zeta and the Hurwitz zeta-function (with parameter a) respectively, where

$$(m+a)^{-s}=\exp(-s\log(m+a)),\qquad -\pi<\arg(m+a)\leq\pi.$$

Let $\mathbf{N}$ be the set of positive integers, $\mathbf{N}_0$ the set of non-negative integers, and put

$$\binom{s}{k}=\begin{cases}s(s-1)\cdots(s-k+1)/k! & \text{if } k\in\mathbf{N},\\ 1 & \text{if } k=0.\end{cases}$$

The case $r=1$ is easy. In this case the series (1.1) is

$$\zeta_1(s_1;\alpha_1,w_1)=\sum_{m=0}^{\infty}(\alpha_1+mw_1)^{-s_1}=w_1^{-s_1}\sum_{m=0}^{\infty}\left(m+\frac{\alpha_1}{w_1}\right)^{-s_1} \tag{2.1}$$

similarly to (3.6) of [13]. The right-hand side of (2.1) is equal to $w_1^{-s_1}\zeta(s_1,\alpha_1/w_1)$. When a is a non-zero complex number with $|a|<1$, we have

$$\zeta(s,a)=a^{-s}+\sum_{k=0}^{\infty}\binom{-s}{k}\zeta(s+k)a^k. \tag{2.2}$$

When $0<a<1$, this is due to Mikolás [16]. Or it can be proved by letting $N\to\infty$ in Lemma 6 of Katsurada-Matsumoto [8]. Lemma 6 of [8] is stated only for real positive a, but it is valid also for complex a. Substituting (2.2) into the right-hand side of (2.1), we obtain the expansion formula for $\zeta_1(s_1;\alpha_1,w_1)$ which is valid when $|w_r|$ is large.

Now we assume $r\geq 2$, and write $s_j=\sigma_j+it_j$ $(1\leq j\leq r)$. Let $N\geq 2$ be a positive integer, and at first assume

$$\sigma_j\geq 0 \quad (1\leq j\leq r-1), \qquad \sigma_r>N+r-1. \tag{2.3}$$

Then $(s_1,\ldots,s_r)\in\mathcal{A}_r$, hence the series (1.1) is convergent absolutely. Using Mellin-Barnes integral formula ((3.1) of [13]), similarly to (3.7) of [13], we can show

$$\begin{aligned}&\zeta_r((s_1,\ldots,s_r);(\alpha_1,\ldots,\alpha_r),(w_1,\ldots,w_r))\\&=\frac{1}{2\pi i}\int_{(c)}\frac{\Gamma(s_r+z)\Gamma(-z)}{\Gamma(s_r)}\zeta_{r-1}((s_1,\ldots,s_{r-2},s_{r-1}+s_r+z);\\&\quad(\alpha_1,\ldots,\alpha_{r-1}),(w_1,\ldots,w_{r-1}))\,\zeta\left(-z,\frac{\alpha_r-\alpha_{r-1}}{w_r}\right)w_r^z dz,\end{aligned} \tag{2.4}$$

where the path of integration is the vertical line $\Re z=c$. In order to use the Mellin-Barnes formula, it is necessary to assume $-\sigma_r<c<0$. Moreover, to ensure the convergence of two zeta factors on the right-hand side of (2.4), we require $\sigma_{r-k}+\cdots+\sigma_{r-1}+\sigma_r+c>k$ $(1\leq k\leq r-1)$ and $c<-1$. Therefore, comparing with (2.3), a suitable choice is $c=-N+\varepsilon$ with a small $\varepsilon>0$.

We list up the singularities of the integrand on the right-hand side of (2.4). The poles of the gamma factors are

(A) $z=-s_r-n_1$ $(n_1\in\mathbf{N}_0)$,
(B) $z=n$ $(n\in\mathbf{N}_0)$.

The Hurwitz zeta-function gives a pole at

(C) $z=-1$.

Lastly, according to Theorem 1 of [13], the possible singularities of ζ_{r-1} in the integrand are

(D) $z=-s_{r-j}-\cdots-s_r+j-n_j$ $(n_j\in\mathbf{N}_0,\ 2\leq j\leq r-1)$,
(E) $z=-s_{r-1}-s_r+1$.

From (2.3) and the choice $c = -N + \varepsilon$ we see that the poles (B) and (C) are on the right of the path of integration, while all the other poles are on the left.

Now let $(s_1^0, \ldots, s_r^0)$ be any point in $\mathbf{C}^r$, and we aim at proving an asymptotic formula of $\zeta_r((s_1, \ldots, s_r); (\alpha_1, \ldots, \alpha_r), (w_1, \ldots, w_r))$ at the point $(s_1^0, \ldots, s_r^0)$. Write $s_j^0 = \sigma_j^0 + it_j^0$ $(1 \le j \le r)$, and assume that

$$0,\ t_r^0,\ t_{r-1}^0 + t_r^0,\ \ldots,\ t_1^0 + \cdots + t_r^0 \quad \text{are all different.} \tag{2.5}$$

This condition especially implies (in view of Theorem 1 of [13]) that ζ_r is not singular at $(s_1^0, \ldots, s_r^0)$. Let $s_j^* = \max\{\sigma_j^0, 0\} + it_j^0$ $(1 \le j \le r-1)$ and $s_r^* = N + r - 1 + \eta + it_r^0$, where η is a small positive number. Then $(s_1, \ldots, s_r) = (s_1^*, \ldots, s_r^*)$ satisfies (2.3), hence (2.4) holds for $(s_1, \ldots, s_r) = (s_1^*, \ldots, s_r^*)$. Since $t_r^0,\ t_{r-1}^0 + t_r^0,\ \ldots,\ t_1^0 + \cdots + t_r^0$ are all different from 0 by (2.5), we can deform the path $\Re z = c$ of (2.4) to obtain the new path $\mathcal{C}$ from $c - i\infty$ to $c + i\infty$, such that all the half-lines

$$L_j = \{\sigma - i(t_{r-j}^0 + \cdots + t_r^0) \mid \sigma \le -\sigma_{r-j}^0 - \cdots - \sigma_r^0 + j\} \qquad (0 \le j \le r-1) \tag{2.6}$$

are on the left of $\mathcal{C}$, while the poles (B) and (C) still remain on the right of $\mathcal{C}$ (see Figure 1). Then we have

$$\begin{aligned} &\zeta_r((s_1, \ldots, s_r); (\alpha_1, \ldots, \alpha_r), (w_1, \ldots, w_r)) \\ &= \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{\Gamma(s_r + z)\Gamma(-z)}{\Gamma(s_r)} \zeta_{r-1}((s_1, \ldots, s_{r-2}, s_{r-1} + s_r + z); \\ &\quad (\alpha_1, \ldots, \alpha_{r-1}), (w_1, \ldots, w_{r-1}))\, \zeta\left(-z, \frac{\alpha_r - \alpha_{r-1}}{w_r}\right) w_r^z dz. \end{aligned} \tag{2.7}$$

The next step is to move $(s_1, \ldots, s_r)$ from $(s_1^*, \ldots, s_r^*)$ to $(s_1^0, \ldots, s_r^0)$ with keeping the values of imaginary parts of each s_j. The poles (A), (D) and (E) of the integrand move along the half-lines L_j $(0 \le j \le r-1)$, hence they do not cross the contour $\mathcal{C}$ during this procedure. To carry out this procedure, it is necessary to show that the integral is always convergent during the procedure. For this purpose, we need one more assumption. Let

$$\rho(a, w) = \max\{|\arg a|, |\arg w|\}.$$

We assume

$$\rho(\alpha_i - \alpha_{i-1}, w_i) + \rho(\alpha_j - \alpha_{j-1}, w_j) < \pi \quad (1 \le i < j \le r) \tag{2.8}$$

(where $\alpha_0 = 0$), which is necessary to use Theorem 4 of [13]. Write $z = x + iy$. Applying Stirling's formula to gamma factors, Theorem 4 (iii) of [13] to ζ_{r-1}, and Lemma 2 of [13] to the factor

$$\zeta\left(-z, \frac{\alpha_r - \alpha_{r-1}}{w_r}\right) w_r^z$$

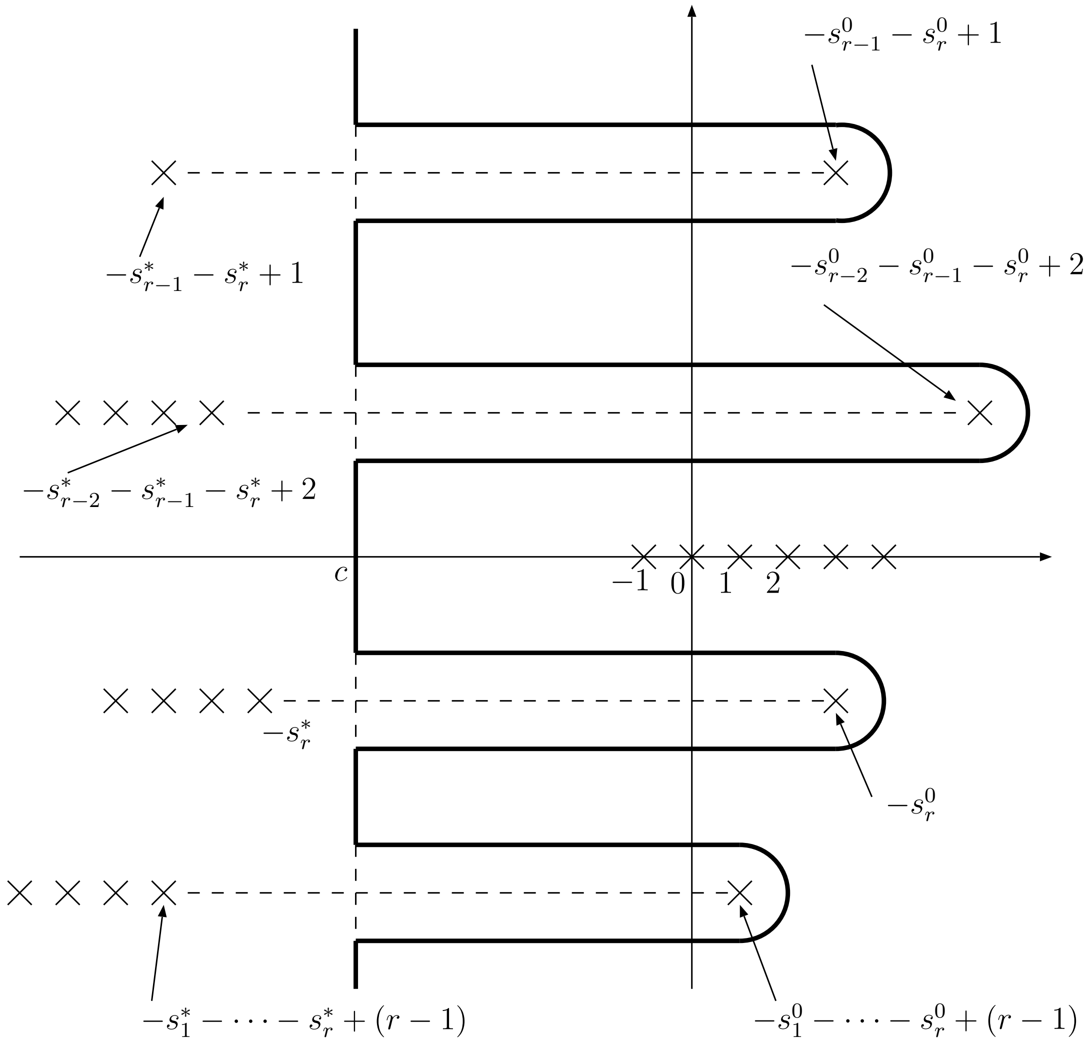

Fig.1

in the integrand, we find that the integrand is

$$\begin{aligned}
&\ll \exp\left(\frac{\pi}{2}(|t_r| - |t_r + y| - |y|)\right)(|t_r + y| + 1)^{\sigma_r + x - 1/2} \\
&\quad\times (|y| + 1)^{-x-1/2}(|t_r| + 1)^{-\sigma_r + 1/2} \\
&\quad\times \left\{\sum_{j=1}^{r-1}(|t_{r-1} + t_r + y| + 1)^{f(j)}\exp(|t_{r-1} + t_r + y|\rho(\alpha_j - \alpha_{j-1}, w_j))\right\} \\
&\quad\times |w_r|^x(|y| + 1)^{\max\{0,1+x\}+\varepsilon}\exp(|y|\rho(\alpha_r - \alpha_{r-1}, w_r)), \qquad (2.9)
\end{aligned}$$

where $f(j)$, $f(r-1)$ are certain positive numbers depending on $\sigma_1, \ldots, \sigma_r, x, \varepsilon$. Therefore, under assumption (2.8), we see that the integral is convergent absolutely during the above procedure. Hence (2.7) with $(s_1, \ldots, s_r) = (s^0_1, \ldots, s^0_r)$

is established. The idea of the above argument of analytic continuation was first appeared in [15].

Next we deform the contour $\mathcal{C}$ back to the original path $\Re z = c = -N+\varepsilon$. During this deformation we meet several poles, whose residues we should count. Because of assumption (2.5), none of the poles (A), (D), (E) for $(s_1,\ldots,s_r) = (s_1^0,\ldots,s_r^0)$ coincides with each other. Hence we can easily show inductively, by using (4.4) of [13], that all of those poles are (at most) simple. We denote the residues of $\zeta_{r-1}((s_1,\ldots,s_{r-2},s_{r-1}+s_r+z);(\alpha_1,\ldots,\alpha_{r-1}),(w_1,\ldots,w_{r-1}))$ at the poles (D) and (E) by

$$R_j^D(n_j;(s_1,\ldots,s_{r-2});(\alpha_1,\ldots,\alpha_{r-1}),(w_1,\ldots,w_{r-1}))$$

($n_j \in \mathbf{N}_0$, $2\le j\le r-1$) and

$$R^E((s_1,\ldots,s_{r-2});(\alpha_1,\ldots,\alpha_{r-1}),(w_1,\ldots,w_{r-1})),$$

respectively. (These do not depend on s_{r-1}, s_r.) Then the residues of the integrand on the right-hand side of (2.7) with $(s_1,\ldots,s_r)=(s_1^0,\ldots,s_r^0)$ at the poles (A), (D), (E) are

$$X^A(n_1)w_r^{-s_r^0-n_1},\quad X_j^D(n_j)w_r^{-s_{r-j}^0-\cdots-s_r^0+j-n_j},\quad X^E w_r^{-s_{r-1}^0-s_r^0+1},$$

respectively, where

$$\begin{aligned}
X^A(n_1) =& X^A(n_1;(s_1^0,\ldots,s_r^0),(\alpha_1,\ldots,\alpha_r),(w_1,\ldots,w_r))\\
=&\binom{-s_r^0}{n_1}\zeta_{r-1}((s_1^0,\ldots,s_{r-2}^0,s_{r-1}^0-n_1);(\alpha_1,\ldots,\alpha_{r-1}),(w_1,\ldots,w_{r-1}))\\
&\times\zeta\left(s_r^0+n_1,\frac{\alpha_r-\alpha_{r-1}}{w_r}\right),
\end{aligned}$$

$$\begin{aligned}
X_j^D(n_j) =& X_j^D(n_j;(s_1^0,\ldots,s_r^0),(\alpha_1,\ldots,\alpha_r),(w_1,\ldots,w_r))\\
=&\frac{\Gamma(-s_{r-j}^0-\cdots-s_{r-1}^0+j-n_j)\Gamma(s_{r-j}^0+\cdots+s_r^0-j+n_j)}{\Gamma(s_r^0)}\\
&\times R_j^D(n_j;(s_1^0,\ldots,s_{r-2}^0);(\alpha_1,\ldots,\alpha_{r-1}),(w_1,\ldots,w_{r-1}))\\
&\times\zeta\left(s_{r-j}^0+\cdots+s_r^0-j+n_j,\frac{\alpha_r-\alpha_{r-1}}{w_r}\right),
\end{aligned}$$

and

$$\begin{aligned}
X^E =& X^E((s_1^0,\ldots,s_r^0),(\alpha_1,\ldots,\alpha_r),(w_1,\ldots,w_r))\\
=&\frac{\Gamma(1-s_{r-1}^0)\Gamma(s_{r-1}^0+s_r^0-1)}{\Gamma(s_r^0)}\\
&\times R^E((s_1^0,\ldots,s_{r-2}^0);(\alpha_1,\ldots,\alpha_{r-1}),(w_1,\ldots,w_{r-1}))\\
&\times\zeta\left(s_{r-1}^0+s_r^0-1,\frac{\alpha_r-\alpha_{r-1}}{w_r}\right).
\end{aligned}$$

The poles of the form (A) are on the right of $\Re z = c$ when $-\sigma_r^0 - n_1 > -N + \varepsilon$, that is,

$$0 \le n_1 < N - \sigma_r^0 - \varepsilon. \tag{2.10}$$

Similarly, the poles of the form (D) are on the right of $\Re z = c$ when

$$0 \le n_j < N - (\sigma_{r-j}^0 + \cdots + \sigma_r^0) + j - \varepsilon \qquad (2 \le j \le r-1). \tag{2.11}$$

The poles of the form (E) are on the right of $\Re z = c$ when

$$-\sigma_{r-1}^0 - \sigma_r^0 + 1 > -N + \varepsilon. \tag{2.12}$$

We choose $N = N(\sigma_1^0, \ldots, \sigma_r^0)$ sufficiently large such that (2.12) is valid and the right-hand sides of (2.10) and (2.11) are positive. Let $\lambda_j^0 = \sigma_{r-j}^0 + \cdots + \sigma_r^0$, and write $\lambda_j^0 = [\lambda_j^0] + \{\lambda_j^0\}$, where $[\lambda_j^0]$ is an integer and $0 \le \{\lambda_j^0\} < 1$. Similarly we write $\sigma_r^0 = [\sigma_r^0] + \{\sigma_r^0\}$. If we choose $\varepsilon = \varepsilon(\sigma_1^0, \ldots, \sigma_r^0) > 0$ sufficiently small for which $\{\lambda_j^0\} + \varepsilon < 1$ $(2 \le j \le r-1)$ and $\{\sigma_r^0\} + \varepsilon < 1$, then from (2.10) and (2.11) we have

$$0 \le n_1 \le N - [\sigma_r^0] - 1, \quad 0 \le n_j \le N - [\lambda_j^0] + j - 1 \ \ (2 \le j \le r-1).$$

Therefore

$$\begin{aligned}
&\zeta_r((s_1^0, \ldots, s_r^0); (\alpha_1, \ldots, \alpha_r), (w_1, \ldots, w_r)) \\
&\quad = \sum_{n_1=0}^{N-[\sigma_r^0]-1} X^A(n_1) w_r^{-s_r^0 - n_1} \\
&\quad + \sum_{j=2}^{r-1} \sum_{n_j=0}^{N-[\lambda_j^0]+j-1} X_j^D(n_j) w_r^{-s_{r-j}^0 - \cdots - s_r^0 + j - n_j} + X^E w_r^{-s_{r-1}^0 - s_r^0 + 1} \\
&\quad + \frac{1}{2\pi i} \int_{(-N+\varepsilon)} \frac{\Gamma(s_r^0 + z)\Gamma(-z)}{\Gamma(s_r^0)} \zeta_{r-1}((s_1^0, \ldots, s_{r-2}^0, s_{r-1}^0 + s_r^0 + z); \\
&\qquad (\alpha_1, \ldots, \alpha_{r-1}), (w_1, \ldots, w_{r-1})) \, \zeta\left(-z, \frac{\alpha_r - \alpha_{r-1}}{w_r}\right) w_r^z dz.
\end{aligned} \tag{2.13}$$

Lastly we estimate the integral term on the right-hand side of (2.13). The integrand is estimated as (2.9) with $x = -N + \varepsilon$. Hence the integral is convergent and $O(|w_r|^{-N+\varepsilon})$. Therefore we now obtain the following result.

Theorem 1. *Let $r \geq 2$, and assume* (1.2), (1.4), (1.5) *and* (2.8). *Let* $(s_1^0, \ldots, s_r^0)$ *be any point in* $\mathbf{C}^r$ *satisfying* (2.5). *Then the formula*

$$\begin{aligned}
&\zeta_r((s_1^0, \ldots, s_r^0); (\alpha_1, \ldots, \alpha_r), (w_1, \ldots, w_r)) \\
&= \sum_{n_1=0}^{N-[\sigma_r^0]-1} X^A(n_1) w_r^{-s_r^0-n_1} + \sum_{j=2}^{r-1} \sum_{n_j=0}^{N-[\lambda_j^0]+j-1} X_j^D(n_j) w_r^{-s_{r-j}^0-\cdots-s_r^0+j-n_j} \\
&\quad + X^E w_r^{-s_{r-1}^0-s_r^0+1} + O(|w_r|^{-N+\varepsilon}) \qquad (2.14)
\end{aligned}$$

holds for any sufficiently large N and sufficiently small $\varepsilon > 0$.

Remark 1 Let $\theta \in (-\pi, \pi]$ be the number determined by

$$H(\ell) = \left\{ w \in \mathbf{C} \setminus \{0\} \;\middle|\; \theta - \frac{\pi}{2} < \arg w < \theta + \frac{\pi}{2} \right\}.$$

Then, without assumption (2.8), we can prove a formula similar to (2.14), but with replacing w_r by $w_r e^{-i\theta}$, by the method developed in the last section of [13]. Moreover, it is also not difficult to obtain a formula without assumptions (1.4) and (1.5), if we follow the argument in [14].

Remark 2 In Theorem 4 of [13], we have shown an upper bound estimate of ζ_r, valid uniformly in any vertical strip. Such an estimate is not necessary if we only want to prove the analytic continuation and the asymptotic expansion for small $|w_r|$, because we can prove those results by the "right-shift" argument of the path of integration, and during this procedure the factor ζ_{r-1} in the integrand is always in the domain of absolute convergence (see [14]). However, as we have seen in this section, the estimate of Theorem 4 of [13] is essentially necessary to study the behaviour of ζ_r when $|w_r|$ is large.

Remark 3 Formula (2.14) is *not* the asymptotic expansion with respect to w_r in the strict sense, because the Hurwitz zeta factors in the coefficients $X^A(n_1)$, $X_j^D(n_j)$, X^E still include w_r. But when $|w_r|$ is large, we can substitute formula (2.2) of Mikolás into those factors to obtain the asymptotic expansion. A special case is the situation when

$$\alpha_r - \alpha_{r-1} = b w_r \qquad (2.15)$$

holds with a constant b satisfying $|\arg b| < \pi$. Then $\zeta(s, (\alpha_r - \alpha_{r-1})/w_r) = \zeta(s, b)$ is independent of w_r, so (2.14) itself is the asymptotic expansion with respect to w_r in the strict sense.

Remark 4 This remark is an additional comment for [13]. In [13] we have discussed the asymptotic behaviour of ζ_r when $|w_r|$ is small. We have stated asymptotic expansion formulas (Theorems 2 and 5 of [13]) only under the above condition (2.15). However we can relax the condition to

$$\alpha_r - \alpha_{r-1} = b_1 w_r + b_2 \tag{2.16}$$

with two constants b_1, b_2. In this case we have

$$\zeta\left(s, \frac{\alpha_r - \alpha_{r-1}}{w_r}\right) = \zeta\left(s, b_1 + \frac{b_2}{w_r}\right).$$

Hence we can apply Katsurada's asymptotic formula (Theorem 1 of [7]) to obtain the asymptotic expansion of $\zeta(s,(\alpha_r-\alpha_{r-1})/w_r)$, hence the asymptotic expansion of ζ_r, when $|w_r|$ is small.

3. The case of Barnes multiple zeta-functions

One of the originators of analytic thoery of multiple zeta-functions is Barnes, who introduced his r-ple zeta-function

$$\zeta_{B,r}(s;\alpha,(w_1,\ldots,w_r)) = \sum_{m_1=0}^{\infty}\cdots\sum_{m_r=0}^{\infty}(\alpha + m_1w_1 + \cdots + m_rw_r)^{-s} \tag{3.1}$$

in [1] ($r=2$) and [2] (any r). As mentioned in Section 1, this is the special case $s_1 = \cdots = s_{r-1} = 0$, $s_r = s$ and $\alpha_r = \alpha$ in (1.1). It is now well known that Barnes multiple zeta-functions are very important objects in number theory, hence it is natural to study this case in detail.

Let $r \geq 2$, N be a large positive integer, and $\alpha \in H(\ell)$. Assume (1.2), and also assume

$$\rho(\alpha, w_i) + \rho(\alpha, w_j) < \pi \qquad (1 \leq i < j \leq r-1), \tag{3.2}$$

$$\rho(\alpha, w_i) + |\arg w_r| < \pi \qquad (1 \leq i \leq r-1). \tag{3.3}$$

Choose the parameters $\alpha_1, \ldots, \alpha_r$ by

$$\alpha_j = \frac{j}{r-1}\alpha \ \ (1 \leq j \leq r-2), \quad \alpha_{r-1} = \alpha, \quad \alpha_r = \alpha + w_r. \tag{3.4}$$

Then (1.4), (1.5) and (2.8) are satisfied. Hence we can apply the argument in the preceding section to the present case.

Let $s_1 = \cdots = s_{r-1} = 0$, $s_r = s$ and write $s = \sigma + it$. When $\sigma > N + r - 1$, we have $(0,\ldots,0,s) \in \mathcal{A}_r$, and (2.4) in the present case can be rewritten as

$$\begin{aligned}
&\zeta_{B.r}(s;\alpha + w_r,(w_1,\ldots,w_r)) \\
&= \zeta_r((0,\ldots,0,s);(\alpha/(r-1),\ldots,\alpha,\alpha+w_r),(w_1,\ldots,w_r)) \\
&= \frac{1}{2\pi i}\int_{(c)}\frac{\Gamma(s+z)\Gamma(-z)}{\Gamma(s)}\zeta_{r-1}((0,\ldots,0,s+z); \\
&\quad (\alpha/(r-1),\ldots,\alpha),(w_1,\ldots,w_{r-1}))\,\zeta(-z)w_r^z dz,
\end{aligned} \tag{3.5}$$

because $(\alpha_r-\alpha_{r-1})/w_r=1$. The poles (A), (B) and (C) of the integrand are the same as in Section 2. The poles (D) and (E) are coming from the factor ζ_{r-1}, which is now

$$\begin{aligned}&\zeta_{r-1}((0,\ldots,0,s+z);(\alpha/(r-1),\ldots,\alpha),(w_1,\ldots,w_{r-1}))\\&\quad=\zeta_{B,r-1}(s+z;\alpha,(w_1,\ldots,w_{r-1})).\end{aligned}\tag{3.6}$$

The poles of Barnes multiple zeta-functions were studied by Barnes himself. Barnes [2] proved that (3.1) can be continued meromorphically to the whole complex plane, and holomorphic except for the poles of order 1 at $s=k$ $(1\leq k\leq r)$. Hence the poles of (3.6) are at

(F) $z=-s+k$ $(1\leq k\leq r-1)$,

which are on the left of the path of integration $\Re z=c=-N+\varepsilon$.

Let $s^0=\sigma^0+it^0\in\mathbf{C}$ with $t^0\neq 0$. Then we can define a path $\mathcal{C}'$ from $c-i\infty$ to $c+i\infty$ such that

$$\{\sigma-it^0\mid\sigma\leq-\sigma^0+r-1\}$$

is on the left of $\mathcal{C}'$ while (B) and (C) are on the right of $\mathcal{C}'$ (see Figure 2, where $s^*=N+r-1+\eta+it^0$). Then, by an argument similar to that in the preceding section, we can continue (3.1) to $s=s^0$, and we have

$$\begin{aligned}&\zeta_{B,r}(s^0;\alpha+w_r,(w_1,\ldots,w_r))\\&=\sum_{n=0}^{N-[\sigma^0]-1}Y^A(n;s^0)w_r^{-s^0-n}+\sum_{k=1}^{r-1}Y^F(k;s^0)w_r^{-s^0+k}\\&+\frac{1}{2\pi i}\int_{(-N+\varepsilon)}\frac{\Gamma(s^0+z)\Gamma(-z)}{\Gamma(s^0)}\zeta_{B,r-1}(s^0+z;\alpha,(w_1,\ldots,w_{r-1}))\,\zeta(-z)w_r^zdz,\end{aligned}\tag{3.7}$$

where

$$\begin{aligned}Y^A(n;s^0)=&Y^A(n;s^0;\alpha,(w_1,\ldots,w_{r-1}))\\=&\binom{-s^0}{n}\zeta_{B,r-1}(-n;\alpha,(w_1,\ldots,w_{r-1}))\zeta(s^0+n),\end{aligned}$$

$$\begin{aligned}Y^F(k;s^0)=&Y^F(k;s^0;\alpha,(w_1,\ldots,w_{r-1}))\\=&\frac{(k-1)!\Gamma(s^0-k)}{\Gamma(s^0)}R_{r-1,k}(\alpha)\zeta(s^0-k),\end{aligned}$$

and

$$R_{r-1,k}(\alpha)=R_{r-1,k}(\alpha,(w_1,\ldots,w_{r-1}))$$

is the residue of $\zeta_{B,r-1}(s^0+z;\alpha,(w_1,\ldots,w_{r-1}))$ as a function in z at $z=-s^0+k$, that is, the residue of $\zeta_{B,r-1}(s;\alpha,(w_1,\ldots,w_{r-1}))$ as a function in s at $s=k$.

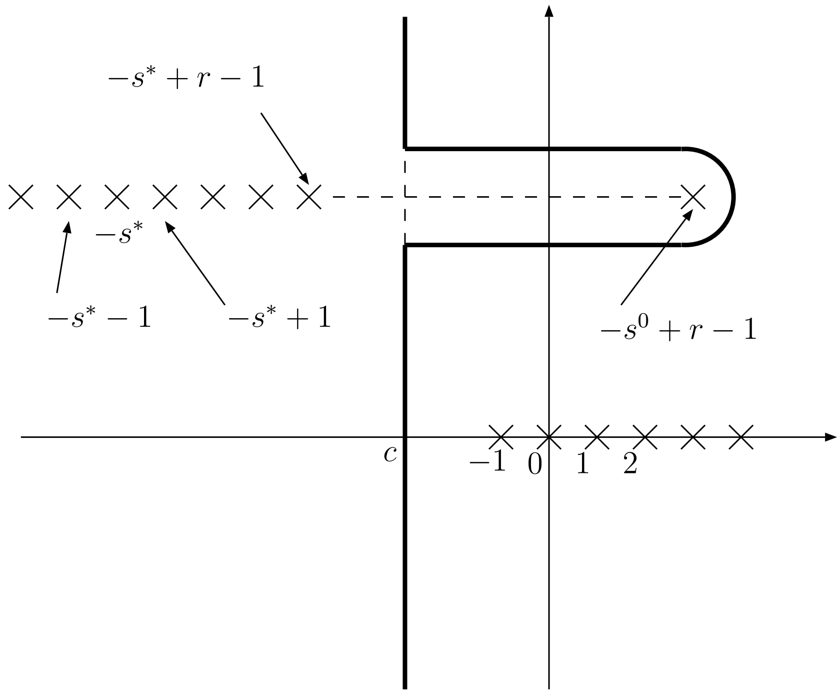

Fig.2

Combining (3.7) with the relation

$$\begin{aligned}&\zeta_{B,r}(s^0;\alpha+w_r,(w_1,\ldots,w_r))\\&=\zeta_{B,r}(s^0;\alpha,(w_1,\ldots,w_r))-\zeta_{B,r-1}(s^0;\alpha,(w_1,\ldots,w_{r-1}))\end{aligned}\tag{3.8}$$

(formula (4.9) of [13]), we have

$$\begin{aligned}&\zeta_{B,r}(s^0;\alpha,(w_1,\ldots,w_r))\\&=\zeta_{B,r-1}(s^0;\alpha,(w_1,\ldots,w_{r-1}))\\&+\sum_{n=0}^{N-[\sigma^0]-1}Y^A(n;s^0)w_r^{-s^0-n}+\sum_{k=1}^{r-1}Y^F(k;s^0)w_r^{-s^0+k}\\&+\frac{1}{2\pi i}\int_{(-N+\varepsilon)}\frac{\Gamma(s^0+z)\Gamma(-z)}{\Gamma(s^0)}\zeta_{B,r-1}(s^0+z;\alpha,(w_1,\ldots,w_{r-1}))\,\zeta(-z)w_r^z dz.\end{aligned}\tag{3.9}$$

The value of $R_{r-1,k}(\alpha)$ was evaluated by Barnes. Let u be a complex variable, and define $A_{r,k}(u)$ and $T_{r,n}(u)$ by the following Laurent expansion:

$$(-1)^r ze^{-uz}\prod_{k=1}^r\left(1-e^{-w_kz}\right)^{-1}=\sum_{k=1}^r(-1)^kA_{r,k}(u)z^{1-k}+\sum_{n=1}^\infty(-1)^{n-1}T_{r,n}(u)z^n.\tag{3.10}$$

The n-th r-ple Bernoulli polynomial $S_{r,n}(u)=S_{r,n}(u;(w_1,\ldots,w_r))$ is defined by the properties $S_{r,n}(0)=0$ and

$$\frac{1}{n!}\frac{d}{du}S_{r,n}(u)=T_{r,n}(u).$$

Then it is not difficult to see that

$$A_{r,k}(u)=S_{r,1}^{(k+1)}(u)\qquad(1\leq k\leq r),\tag{3.11}$$

where the right-hand side means the $(k+1)$-th derivative of $S_{r,1}(u)$ with respect to u (Section 3 of Barnes [2]). Since $S_{r,1}(u)$ is a polynomial of degree $r+1$ in u (Section 7 of [2]), $A_{r,k}(u)$ is a polynomial of degree $r-k$ in u. Barnes proved (in Section 28 of [2]) that the residue $R_{r,k}(\alpha)$ of $\zeta_{B,r}(s;\alpha,(w_1,\ldots,w_r))$ at $s=k$ is

$$R_{r,k}(\alpha)=\frac{(-1)^{k+r}S_{r,1}^{(k+1)}(\alpha)}{(k-1)!},\tag{3.12}$$

hence

$$Y^F(k;s^0)=\frac{(-1)^{k+r-1}}{(s^0-1)\cdots(s^0-k)}S_{r-1,1}^{(k+1)}(\alpha)\zeta(s^0-k).\tag{3.13}$$

The integral term on the right-hand side of (3.9) is $O(|w_r|^{-N+\varepsilon})$, as in the preceding section. Therefore we now arrive at the following theorem.

Theorem 2. *Let $r\geq 2$, $\alpha\in H(\ell)$, and assume* (1.2), (3.2) *and* (3.3). *Let s^0 be any complex number whose imaginary part is not 0. Then the formula*

$$\begin{aligned}&\zeta_{B,r}(s^0;\alpha,(w_1,\ldots,w_r))\\&=\zeta_{B,r-1}(s^0;\alpha,(w_1,\ldots,w_{r-1}))\\&\quad+\sum_{k=1}^{r-1}\frac{(-1)^{k+r-1}}{(s^0-1)\cdots(s^0-k)}S_{r-1,1}^{(k+1)}(\alpha)\zeta(s^0-k)w_r^{-s^0+k}\\&\quad+\sum_{n=0}^{N-[\sigma^0]-1}\binom{-s^0}{n}\zeta_{B,r-1}(-n;\alpha,(w_1,\ldots,w_{r-1}))\zeta(s^0+n)w_r^{-s^0-n}\\&\quad+O(|w_r|^{-N+\varepsilon})\end{aligned}\tag{3.14}$$

holds for any sufficiently large N and sufficiently small $\varepsilon>0$.

Remark Since the choice (3.4) satisfies (2.15) (with $b=1$), the above formula (3.14) gives the asymptotic expansion with respect to w_r in the strict sense.

4. Special values of Barnes multiple zeta-functions at integer points

Theorem 2 in the preceding section requires the assumption $t^0 \neq 0$. Hence it excludes an important special case that $s^0 = h$ is an integer. However, formula (3.9) actually gives the analytic continuation of $\zeta_{B,r}(s^0;\alpha,(w_1,\ldots,w_r))$ to a neighbourhood of any point s^0 satisfying $t^0 \neq 0$. Therefore, putting $s^0 = h+it^0$ in (3.9) and letting $t^0 \to 0$, we can obtain explicit information on the behaviour of $\zeta_{B,r}$ around $s^0 = h$.

(I) *Non-positive case.* First consider the case of non-positive integers. Let $s^0 \to -h$, $h \in \mathbf{N}_0$. Then $Y^A(h+1;s^0)$ has a singular factor $\zeta(s^0+h+1)$, but this singularity is cancelled by the binomial coefficient factor

$$\binom{-s^0}{h+1} = (-s^0)(-s^0-1)\cdots(-s^0-h)/(h+1)!$$

which vanishes at $s^0 = -h$. The result of the cancellation is that

$$\lim_{s^0\to -h} Y^A(h+1;s^0) = -\frac{1}{h+1}\zeta_{B,r-1}(-h-1;\alpha,(w_1,\ldots,w_{r-1})). \tag{4.1}$$

The binomial factor is also equal to 0 at $s^0 = -h$ for any $n \geq h+2$, hence

$$Y^A(n;-h) = 0 \qquad (n \geq h+2). \tag{4.2}$$

Since $\Gamma(s^0)$ has a pole at $s^0 = -h$, the integral term on the right-hand side of (3.9) also vanishes. Therefore, letting $s^0 \to -h$ in (3.9) we obtain

Theorem 3. *Under the same assumptions as in Theorem 2, we have*

$$\begin{aligned}
&\zeta_{B,r}(-h;\alpha,(w_1,\ldots,w_r))\\
&= \zeta_{B,r-1}(-h;\alpha,(w_1,\ldots,w_{r-1}))\\
&+ \sum_{k=1}^{r-1}\frac{(-1)^{r-1}}{(h+1)\cdots(h+k)}S_{r-1,1}^{(k+1)}(\alpha)\zeta(-h-k)w_r^{h+k}\\
&- \frac{1}{h+1}\zeta_{B,r-1}(-h-1;\alpha,(w_1,\ldots,w_{r-1}))w_r^{-1}\\
&+ \sum_{n=0}^{h}\binom{h}{n}\zeta_{B,r-1}(-n;\alpha,(w_1,\ldots,w_{r-1}))\zeta(-h+n)w_r^{h-n}
\end{aligned} \tag{4.3}$$

for any $h \in \mathbf{N}_0$.

On the other hand, Barnes proved

$$\zeta_{B,r}(-h;\alpha,(w_1,\ldots,w_r)) = \frac{(-1)^r}{h+1}S'_{r,h+1}(\alpha) \tag{4.4}$$

for any $h \in \mathbf{N}_0$ (Sections 22 and 30 of [2]). Hence (4.3) may be rewritten as follows.

Proposition 1. *For any $h \in \mathbf{N}_0$, we have*

$$S'_{r,h+1}(\alpha) = -S'_{r-1,h+1}(\alpha) - \sum_{k=1}^{r-1} \frac{1}{(h+2)\cdots(h+k)} S^{(k+1)}_{r-1,1}(\alpha)\zeta(-h-k)w_r^{h+k}$$
$$+ \frac{1}{h+2} S'_{r-1,h+2}(\alpha)w_r^{-1} - \sum_{n=0}^{h} \binom{h+1}{n+1} S'_{r-1,n+1}(\alpha)\zeta(-h+n)w_r^{h-n}. \tag{4.5}$$

This is an explicit formula for $S'_{r,h+1}(\alpha)$ in terms of $(r-1)$-ple Bernoulli polynomials. The special case $h = 0$ of (4.5) is

$$S'_{r,1}(\alpha) = -\frac{1}{2} S'_{r-1,1}(\alpha) - \sum_{k=1}^{r-1} \frac{1}{k!} S^{(k+1)}_{r-1,1}(\alpha)\zeta(-k)w_r^k + \frac{1}{2} S'_{r-1,2}(\alpha)w_r^{-1}. \tag{4.6}$$

(II) *Positive case.* Next consider the situation when $s^0 \to h$, where $h \in \mathbf{N}$, $1 \le h \le r$. The function $\zeta_{B,r}$ has a pole of order 1 at $s^0 = h$, and we write the Laurent expansion as

$$\zeta_{B,r}(h+\delta; \alpha, (w_1, \ldots, w_r))$$
$$= R_{r,h}(\alpha)\frac{1}{\delta} + C^0_{r,h}(\alpha) + C^1_{r,h}(\alpha)\delta + C^2_{r,h}(\alpha)\delta^2 + \cdots .$$

The factor $\zeta(s^0+n)$ in $Y^A(n; s^0)$ is singular at $s^0 = h$ only if $h = 1$ and $n = 0$. In this case the residue of $Y^A(0; s^0)w_r^{-s^0}$ at $s^0 = h = 1$ is

$$\zeta_{B,r-1}(0; \alpha, (w_1, \ldots, w_{r-1}))w_r^{-1} = (-1)^{r-1} S'_{r-1,1}(\alpha)w_r^{-1}. \tag{4.7}$$

Next we determine when $Y^F(k; s^0)$ is singular at $s^0 = h$. From (3.13) we see that

(i) if $1 \le h \le r-1$, then the factor $((s^0-1)\cdots(s^0-k))^{-1}$ for $h \le k \le r-1$ is singular at $s^0 = h$,

(ii) if $2 \le h \le r$, then the factor $\zeta(s^0-k)$ for $k = h-1$ is singular at $s^0 = h$.

Therefore, when $h = r$ (≥ 2), the singularity appears only from (ii). Putting $s^0 = r + \delta$, we have

$$Y^F(r-1; s^0)w_r^{-s^0+r-1}$$
$$= \frac{1}{(1+\delta)(2+\delta)\cdots(r-1+\delta)} S^{(r)}_{r-1,1}(\alpha)\zeta(1+\delta)w_r^{-1-\delta}$$
$$= \frac{1}{(r-1)!} S^{(r)}_{r-1,1}(\alpha)w_r^{-1}\left\{\frac{1}{\delta} - \left(1 + \frac{1}{2} + \cdots + \frac{1}{r-1} - \gamma + \log w_r\right) + \cdots\right\},$$

where γ is Euler's constant. Hence the residue of the right-hand side of (3.9) at $s^0 = r$ is

$$\frac{1}{(r-1)!} S^{(r)}_{r-1,1}(\alpha) w_r^{-1}. \tag{4.8}$$

On the other hand, by (3.12), the residue of the left-hand side of (3.9) at $s^0 = r$ is

$$R_{r,r}(\alpha) = \frac{1}{(r-1)!} S^{(r+1)}_{r,1}(\alpha). \tag{4.9}$$

Since

$$S^{(r+1)}_{r,1}(\alpha) = \frac{1}{w_1 w_2 \cdots w_r} \tag{4.10}$$

(Section 7 of [2]), it is clear that (4.9) coincides with (4.8). Concerning the constant term of the Laurent expansion at $s^0 = r$, we have

$$\begin{aligned} C^0_{r,r}(\alpha) &= \zeta_{B,r-1}(r; \alpha, (w_1, \ldots, w_{r-1})) \\ &- \frac{1}{(r-1)!} S^{(r)}_{r-1,1}(\alpha) w_r^{-1} \left(1 + \frac{1}{2} + \cdots + \frac{1}{r-1} - \gamma + \log w_r\right) \\ &+ \sum_{k=1}^{r-2} Y^F(k; r) w_r^{-r+k} + \sum_{N=0}^{N-r-1} Y^A(n; r) w_r^{-r-n} + O(|w_r|^{-N+\varepsilon}). \end{aligned} \tag{4.11}$$

When $2 \le h \le r-1$, we should count the residues of both (i) and (ii). The first term $\zeta_{B,r-1}(s^0; \alpha, (w_1, \ldots, w_{r-1}))$ on the right-hand side of (3.9) is also singular. Hence the residue of the right-hand side of (3.9) in this case is

$$\begin{aligned} &\frac{(-1)^{h+r-1}}{(h-1)!} S^{(h+1)}_{r-1,1}(\alpha) + \frac{(-1)^{h+r}}{(h-1)!} S^{(h)}_{r-1,1}(\alpha) w_r^{-1} \\ &+ \sum_{k=h}^{r-1} \frac{(-1)^{h+r-1}}{(h-1)!(k-h)!} S^{(k+1)}_{r-1,1}(\alpha) \zeta(h-k) w_r^{-h+k}. \end{aligned} \tag{4.12}$$

When $h = 1$, the singularity of the right-hand side of (3.9) appears in the terms $\zeta_{B,r-1}(s^0; \alpha, (w_1, \ldots, w_{r-1}))$, $Y^A(0; s^0)$ (with the residue (4.7)), and (i). The resulting residue coincides with the case $h = 1$ of (4.12). Therefore (4.12) gives the residue of the right-hand side of (3.9) for $1 \le h \le r-1$. This is to be equal to the residue of the left-hand side of (3.9), which is

$$R_{r,h}(\alpha) = \frac{(-1)^{h+r} S^{(h+1)}_{r,1}(\alpha)}{(h-1)!}.$$

Hence we obtain the following formula.

Proposition 2. *For $1 \leq h \leq r-1$, we have*

$$S_{r,1}^{(h+1)}(\alpha) = -S_{r-1,1}^{(h+1)}(\alpha) + S_{r-1,1}^{(h)}(\alpha) w_r^{-1} - \sum_{k=h}^{r-1} \frac{1}{(k-h)!} S_{r-1,1}^{(k+1)}(\alpha)\zeta(h-k) w_r^{-h+k}. \tag{4.13}$$

This identity is related with the behaviour of $\zeta_{B,r}$ at positive integer points, but it is interesting to note that this identity can be deduced from (4.6), which is related with the behaviour of $\zeta_{B,r}$ at $s=0$. In fact, differentiating the both sides of (4.6) h-times with respect to α, we have

$$\begin{aligned} S_{r,1}^{(h+1)}(\alpha) &= -\frac{1}{2} S_{r-1,1}^{(h+1)}(\alpha) + \frac{1}{2} S_{r-1,2}^{(h+1)}(\alpha) w_r^{-1} \\ &\quad - \sum_{k=1}^{r-1} \frac{1}{k!} S_{r-1,1}^{(k+h+1)}(\alpha)\zeta(-k) w_r^{k} \\ &= -\frac{1}{2} S_{r-1,1}^{(h+1)}(\alpha) + \frac{1}{2} S_{r-1,2}^{(h+1)}(\alpha) w_r^{-1} \\ &\quad - \sum_{k=h+1}^{r+h-1} \frac{1}{(k-h)!} S_{r-1,1}^{(k+1)}(\alpha)\zeta(h-k) w_r^{-h+k}. \end{aligned} \tag{4.14}$$

Barnes proved

$$S''_{r-1,n+1}(\alpha) = (n+1) S'_{r-1,n}(\alpha) \tag{4.15}$$

for any n (Section 7 of [2]). Putting $n=1$ in (4.15) and differentiating $(h-1)$-times, we have

$$S_{r-1,2}^{(h+1)}(\alpha) = 2 S_{r-1,1}^{(h)}(\alpha). \tag{4.16}$$

Also, since $S_{r-1,1}(\alpha)$ is a polynomial of degree r in α, we see that $S_{r-1,1}^{(k+1)}(\alpha) = 0$ for $k \geq r$. Therefore from (4.14) we have

$$\begin{aligned} S_{r,1}^{(h+1)}(\alpha) &= -\frac{1}{2} S_{r-1,1}^{(h+1)}(\alpha) + S_{r-1,1}^{(h)}(\alpha) w_r^{-1} \\ &\quad - \sum_{k=h+1}^{r-1} \frac{1}{(k-h)!} S_{r-1,1}^{(k+1)}(\alpha)\zeta(h-k) w_r^{-h+k}, \end{aligned}$$

which implies (4.13).

Note that, when $h=r-1$ and $h=r-2$, formula (4.13) can be checked directly by using Barnes' formulas (4.10),

$$S_{r,1}^{(r)}(\alpha) = \frac{\alpha}{w_1 \cdots w_r} - \frac{w_1 + \cdots + w_r}{2 w_1 \cdots w_r},$$

and

$$
\begin{aligned}
S_{r,1}^{(r-1)}(\alpha) &= \frac{\alpha^2}{2w_1\cdots w_r} - \frac{(w_1+\cdots+w_r)\alpha}{2w_1\cdots w_r} \\
&+ \frac{1}{12w_1\cdots w_r}\left\{(w_1^2+\cdots+w_r^2) + 3\sum_{1\le i<j\le r} w_i w_j\right\}
\end{aligned}
$$

which are proved in Section 7 of [2].

It is also possible to deduce an asymptotic expansion formula similar to (4.11) for $C_{r,h}^0(\alpha)$ $(1\le h\le r-1)$.

5. An asymptotic expansion for multiple gamma-functions

The major purpose of Barnes' paper [2] is to develop the theory of the r-ple gamma-function $\Gamma_r(\alpha;(w_1,\ldots,w_r))$. It is defined by

$$
\log\frac{\Gamma_r(\alpha;(w_1,\ldots,w_r))}{\rho_r(w_1,\ldots,w_r)} = \zeta_{B,r}'(0;\alpha;(w_1,\ldots,w_r)), \tag{5.1}
$$

where $\zeta_{B,r}'$ denotes the derivative of $\zeta_{B,r}$ with respect to s, and

$$
\log\rho_r(w_1,\ldots,w_r) = -\lim_{\alpha\to 0}\Big(\zeta_{B,r}'(0;\alpha;(w_1,\ldots,w_r)) + \log\alpha\Big). \tag{5.2}
$$

When $\alpha = z+\alpha_0$, where α_0 is fixed and $|z|$ is large, the asymptotic expansion for $\log\Gamma_r$ with respect to z is discussed by Barnes [2] himself, and also by Katayama and Ohtsuki [6]. In the case of double gamma-functions, some asymptotic expansions were studied also by Billingham and King [3], and by Ferreira and López [4].

In this paper we prove an asymptotic expansion for $\log\Gamma_r$ with respect to w_r when $|w_r|$ is large. Define by

$$
B_{r,n}(w_1,\ldots,w_r) = \frac{1}{n}\frac{d}{du}S_{r,n}(u;(w_1,\ldots,w_r))\bigg|_{u=0} \tag{5.3}
$$

the n-th r-ple Bernoulli number.

Theorem 4. *For any sufficiently large positive integer N, we have*

$$
\begin{aligned}
&\log\Gamma_r(\alpha,(w_1,\ldots,w_r))\\
&=\log\Gamma_{r-1}(\alpha,(w_1,\ldots,w_{r-1}))\\
&+(-1)^{r-1}\sum_{k=1}^{r-1}\frac{1}{k!}\left(S_{r-1,1}^{(k+1)}(\alpha)-S_{r-1,1}^{(k+1)}(0)\right)\\
&\quad\times\left\{\left(\left(1+\frac{1}{2}+\cdots+\frac{1}{k}\right)\zeta(-k)+\zeta'(-k)\right)w_r^k-\zeta(-k)w_r^k\log w_r\right\}\\
&+\frac{1}{2}\left\{\zeta_{B,r-1}(0;\alpha,(w_1,\ldots,w_{r-1}))-(-1)^{r-1}B_{r-1,1}(w_1,\ldots,w_{r-1})\right\}\\
&\qquad\qquad\times(\log w_r-\log 2\pi)\\
&+\left\{\zeta_{B,r-1}(-1;\alpha,(w_1,\ldots,w_{r-1}))-(-1)^{r-1}B_{r-1,2}(w_1,\ldots,w_{r-1})\right\}\\
&\qquad\qquad\times(w_r^{-1}\log w_r-\gamma w_r^{-1})\\
&+\sum_{n=2}^{N-1}\frac{(-1)^n}{n}\{\zeta_{B,r-1}(-n;\alpha,(w_1,\ldots,w_{r-1}))\\
&\quad-(-1)^{r-1}B_{r-1,n+1}(w_1,\ldots,w_{r-1})\Big\}\zeta(n)w_r^{-n}\\
&+O(|w_r|^{-N}),
\end{aligned}
\tag{5.4}
$$

where γ is Euler's constant.

When $r=2$, the above type of asymptotic expansion was first proved in [9], and then in an improved form in [10]. The above theorem gives a generalization of those results.

Proof of Theorem 4. From (3.9) we have

$$
\begin{aligned}
&\zeta'_{B,r}(0;\alpha,(w_1,\ldots,w_r))\\
&=\zeta'_{B,r-1}(0;\alpha,(w_1,\ldots,w_{r-1}))\\
&\quad+\sum_{k=1}^{r-1}\left\{\frac{d}{ds}Y^F(k;s)\Big|_{s=0}w_r^k-Y^F(k;0)w_r^k\log w_r\right\}\\
&\quad+\sum_{n=0}^{N-1}\left\{\frac{d}{ds}Y^A(n;s)\Big|_{s=0}w_r^{-n}-Y^A(n;0)w_r^{-n}\log w_r\right\}\\
&\quad+\frac{1}{2\pi i}\int_{(-N+\varepsilon)}\frac{d}{ds}\left\{\frac{\Gamma(s+z)}{\Gamma(s)}\zeta_{B,r-1}(s+z;\alpha,(w_1,\ldots,w_{r-1}))\right\}\Bigg|_{s=0}\\
&\qquad\qquad\times\Gamma(-z)\zeta(-z)w_r^z dz.
\end{aligned}
\tag{5.5}
$$

By direct computations from (3.13) we have

$$
Y^F(k;0)=\frac{(-1)^{r-1}}{k!}S_{r-1,1}^{(k+1)}(\alpha)\zeta(-k)
\tag{5.6}
$$

and

$$\frac{d}{ds}Y^F(k;s)\bigg|_{s=0} = \frac{(-1)^{r-1}}{k!}S^{(k+1)}_{r-1,1}(\alpha)\left(\left(1+\frac{1}{2}+\cdots+\frac{1}{k}\right)\zeta(-k)+\zeta'(-k)\right). \tag{5.7}$$

Also we have

$$Y^A(n;0) = \begin{cases} -\frac{1}{2}\zeta_{B,r-1}(0;\alpha,(w_1,\ldots,w_{r-1})) & (n=0), \\ -\zeta_{B,r-1}(-1;\alpha,(w_1,\ldots,w_{r-1})) & (n=1), \\ 0 & (n\geq 2). \end{cases} \tag{5.8}$$

In fact, the second and the third formulas are the special case $h=0$ of (4.1) and (4.2), respectively, and the first formula is immediate from the definition and the fact $\zeta(0)=-1/2$.

Next, we have

$$\begin{aligned}\frac{d}{ds}Y^A(n;s) = &\left\{\frac{d}{ds}\binom{-s}{n}\right\}\zeta_{B,r-1}(-n;\alpha,(w_1,\ldots,w_{r-1}))\zeta(s+n) \\ &+\binom{-s}{n}\zeta_{B,r-1}(-n;\alpha,(w_1,\ldots,w_{r-1}))\zeta'(s+n)\end{aligned} \tag{5.9}$$

and

$$\begin{aligned}\frac{d}{ds}\binom{-s}{n} &= 0 \quad (n=0); \qquad = -1 \quad (n=1); \\ &= \frac{(-1)^n}{n!}\{(s+1)(s+2)\cdots(s+n-1)+s(s+2)\cdots(s+n-1)+ \\ &\quad +\cdots+s(s+1)(s+2)\cdots(s+n-2)\} \quad (n\geq 2).\end{aligned}$$

When $n=0$, since $\zeta'(0)=-(1/2)\log 2\pi$, we have

$$\frac{d}{ds}Y^A(0;s)\bigg|_{s=0} = -\frac{1}{2}(\log 2\pi)\zeta_{B,r-1}(0;\alpha,(w_1,\ldots,w_{r-1})). \tag{5.10}$$

When $n=1$, noting the Laurent expansions

$$\zeta(s+1)=\frac{1}{s}+\gamma+\cdots, \qquad \zeta'(s+1)=-\frac{1}{s^2}+\cdots,$$

we have

$$\frac{d}{ds}Y^A(1;s)\bigg|_{s=0} = -\gamma\zeta_{B,r-1}(-1;\alpha,(w_1,\ldots,w_{r-1})). \tag{5.11}$$

When $n\geq 2$, since

$$\frac{d}{ds}\binom{-s}{n}\bigg|_{s=0} = \frac{(-1)^n}{n}, \qquad \binom{-s}{n}\bigg|_{s=0} = 0,$$

we have

$$\frac{d}{ds}Y^A(n;s)\Big|_{s=0} = \frac{(-1)^n}{n}\zeta_{B,r-1}(-n;\alpha,(w_1,\ldots,w_{r-1}))\zeta(n) \quad (n\geq 2). \tag{5.12}$$

Next consider the integral term on the right-hand side of (5.5). We see that

$$\begin{aligned}
&\frac{d}{ds}\left\{\frac{\Gamma(s+z)}{\Gamma(s)}\zeta_{B,r-1}(s+z;\alpha,(w_1,\ldots,w_{r-1}))\right\}\\
&=\frac{\Gamma'(s+z)}{\Gamma(s)}\zeta_{B,r-1}(s+z;\alpha,(w_1,\ldots,w_{r-1}))\\
&\quad+\frac{\Gamma(s+z)}{\Gamma(s)}\zeta'_{B,r-1}(s+z;\alpha,(w_1,\ldots,w_{r-1}))\\
&\quad-\Gamma(s+z)\frac{\Gamma'(s)}{\Gamma(s)^2}\zeta_{B,r-1}(s+z;\alpha,(w_1,\ldots,w_{r-1})).
\end{aligned}$$

The first and the second terms on the right-hand side vanish at $s=0$, while the third term at $s=0$ is

$$\Gamma(z)\zeta_{B,r-1}(z;\alpha,(w_1,\ldots,w_{r-1}))$$

because

$$\frac{\Gamma'(s)}{\Gamma(s)^2}\Big|_{s=0}=-1$$

(see p.248 of [10]). Hence the integral term on the right-hand side of (5.5) is

$$\frac{1}{2\pi i}\int_{(-N+\varepsilon)}\Gamma(z)\zeta_{B,r-1}(z;\alpha,(w_1,\ldots,w_{r-1}))\Gamma(-z)\zeta(-z)w_r^z dz, \tag{5.13}$$

which is $O(|w_r|^{-N+\varepsilon})$. Substituting this estimate and (5.6), (5.7), (5.8), (5.10), (5.11), (5.12) into (5.5), we obtain

$$\begin{aligned}
&\zeta'_{B,r}(0;\alpha,(w_1,\ldots,w_r))\\
&=\zeta'_{B,r-1}(0;\alpha,(w_1,\ldots,w_{r-1}))\\
&\quad+(-1)^{r-1}\sum_{k=1}^{r-1}\frac{1}{k!}S_{r-1,1}^{(k+1)}(\alpha)\\
&\qquad\times\left\{\left(\left(1+\frac{1}{2}+\cdots+\frac{1}{k}\right)\zeta(-k)+\zeta'(-k)\right)w_r^k-\zeta(-k)w_r^k\log w_r\right\}\\
&\quad+\frac{1}{2}\zeta_{B,r-1}(0;\alpha,(w_1,\ldots,w_{r-1}))(\log w_r-\log 2\pi)\\
&\quad+\zeta_{B,r-1}(-1;\alpha,(w_1,\ldots,w_{r-1}))(w_r^{-1}\log w_r-\gamma w_r^{-1})\\
&\quad+\sum_{n=2}^{N-1}\frac{(-1)^n}{n}\zeta_{B,r-1}(-n;\alpha,(w_1,\ldots,w_{r-1}))\zeta(n)w_r^{-n}\\
&\quad+O(|w_r|^{-N+\varepsilon}).
\end{aligned} \tag{5.14}$$

Now we consider the limit $\alpha \to 0$ of the right-hand side. From (4.4) and (5.3) we have

$$\begin{aligned}\lim_{\alpha\to 0}\zeta_{B,r-1}(-n;\alpha,(w_1,\dots,w_{r-1})) &= \frac{(-1)^{r-1}}{n+1}S'_{r-1,n+1}(0)\\ &= (-1)^{r-1}B_{r-1,n+1}(w_1,\dots,w_{r-1}). \qquad (5.15)\end{aligned}$$

Also, the error estimate on the right-hand side of (5.14) is uniform in α. In fact, by (5.13) this claim is reduced to the uniformity of the factor $\zeta_{B,r-1}(z;\alpha,(w_1,\dots,w_{r-1}))$ in α, and, by formula (4.4) of [13] (for $r-1$ instead of r), this is further reduced to the uniformity of the Hurwitz zeta-function $\zeta(z,\alpha)$ in α for $z=-k$ ($k=0,1,\dots,M-1$) and for $\Re z = -M+\varepsilon$, where M is a positive integer. The latter claim is verified by the formula $\zeta(0,\alpha)=(1/2)-\alpha$ for $z=0$, and by Lemma 2 of [12] for all other z. Hence the error estimate on the right-hand side of (5.14) is still valid when $\alpha\to 0$.

Therefore, adding $\log\alpha$ to the both sides of (5.14) and taking the limit $\alpha\to 0$, we obtain

$$\begin{aligned}&-\log\rho_r(w_1,\dots,w_r)\\ &= -\log\rho_{r-1}(w_1,\dots,w_{r-1})\\ &+(-1)^{r-1}\sum_{k=1}^{r-1}\frac{1}{k!}S^{(k+1)}_{r-1,1}(0)\\ &\quad\times\left\{\left(\left(1+\frac{1}{2}+\cdots+\frac{1}{k}\right)\zeta(-k)+\zeta'(-k)\right)w_r^k-\zeta(-k)w_r^k\log w_r\right\}\\ &+\frac{1}{2}(-1)^{r-1}B_{r-1,1}(w_1,\dots,w_{r-1})(\log w_r-\log 2\pi)\\ &+(-1)^{r-1}B_{r-1,2}(w_1,\dots,w_{r-1})(w_r^{-1}\log w_r-\gamma w_r^{-1})\\ &+\sum_{n=2}^{N-1}\frac{(-1)^n}{n}(-1)^{r-1}B_{r-1,n+1}(w_1,\dots,w_{r-1}))\zeta(n)w_r^{-n}\\ &+O(|w_r|^{-N+\varepsilon}). \qquad (5.16)\end{aligned}$$

From (5.14) and (5.16), we obtain the form of formula (5.4), with a slightly weaker error estimate $O(|w_r|^{-N+\varepsilon})$. However, considering the same form of formula with $N+1$ instead of N, we see that the error term can be replaced by $O(|w_r|^{-N})$. This completes the proof of Theorem 4.

Remark It is also possible to show an asymptotic expansion for $\log\Gamma_r$ when $|w_r|$ is small, but Theorem 3 of [13] is not suitable as the starting point, because if k is positive, then $\zeta_{B,r-1}(k;\alpha,(w_1,\dots,w_{r-1}))$ is not uniformly bounded in α when $\alpha\to 0$. We have to introduce

$$\zeta^*_{B,r-1}(k;\alpha,(w_1,\dots,w_{r-1}))=\zeta_{B,r-1}(k;\alpha,(w_1,\dots,w_{r-1}))-\alpha^{-s}$$

and state the expansion formula in terms of $\zeta^*_{B,r-1}$, as in the case of (1.7) of [10].

6. Connections with Shintani's results on Hecke L-functions of totally real number fields

We conclude this paper with a discussion on certain connections between our theory and Shintani's results on Hecke L-functions.

In his important series of papers [17] [18] [19] [20], Shintani discovered the relationship between Barnes multiple gamma-functions and Hecke L-functions of totally real algebraic number fields. Let F be a totally real algebraic number field, and $\mathfrak{f}$ an integral ideal of F. Let χ be a character of the group of narrow ideal classes modulo $\mathfrak{f}$ of F, and $L_F(s,\chi)$ the associated Hecke L-function. Shintani (Theorem 1 of [19]) proved that, for a certain type of χ, the value $L_F(1,\chi)$ can be written as a linear combination of $\log\Gamma_r$. Therefore, combining with our Theorem 4, we can deduce a certain asymptotic expansion formula for $L_F(1,\chi)$. We do not state the result here, because it would require further notations and pages. But it should be a generalization of the expansion formula for $L_F(1,\chi)$ when F is a real quadratic field, which has been proved in the Corrigendum and Addendum of [9].

In order to prove his results, Shintani introduced the following type of multiple zeta-functions. Let $\alpha_1,\ldots,\alpha_r$ be non-negative real numbers, $(\alpha_1,\ldots,\alpha_r)\neq(0,\ldots,0)$, and

$$L_j(X_1,\ldots,X_r)=w_{j1}X_1+\cdots+w_{jr}X_r \qquad (1\leq j\leq r)$$

be linear forms with positive coefficients. Shintani's zeta-functions are defined by

$$\sum_{m_1=0}^{\infty}\cdots\sum_{m_r=0}^{\infty}\prod_{j=1}^{r}L_j(\alpha_1+m_1,\ldots,\alpha_r+m_r)^{-s} \quad . \tag{6.1}$$

Later Hida [5] introduced the multi-variable version, that is

$$\begin{aligned}
&\zeta_{SH,r}((s_1,\ldots,s_r);(\alpha_1,\ldots,\alpha_r),W_r)\\
&=\sum_{m_1=0}^{\infty}\cdots\sum_{m_r=0}^{\infty}\prod_{j=1}^{r}L_j(\alpha_1+m_1,\ldots,\alpha_r+m_r)^{-s_j}\\
&=\sum_{m_1=0}^{\infty}\cdots\sum_{m_r=0}^{\infty}\prod_{j=1}^{r}(w_{j1}(\alpha_1+m_1)+\cdots+w_{jr}(\alpha_r+m_r))^{-s_j},
\end{aligned} \tag{6.2}$$

where W_r is the matrix $(w_{jh})_{1\leq j\leq r,1\leq h\leq r}$. This is convergent absolutely when $\Re s_j>1$ $(1\leq j\leq r)$. It is clear that Shintani's function (6.1) is the special case $s_1=\cdots=s_r=s$ of (6.2). Shintani discovered that $L_F(s,\chi)$ can be expressed as a linear combination of his zeta-functions (6.1), and the value at $s=0$ of the derivative of (6.1) can be expressed in terms of $\log\Gamma_r$. Therefore (6.1) plays a vital role in his theory. Hence the study of analytic properties of (6.1) and (6.2) is an important problem.

The meromorphic continuation of (6.1) to $\mathbf{C}$ was done by Shintani [17]. Hida (Theorem 1 in Section 2.4 of [5]) gives the continuation of (6.2) to $\mathbf{C}^r$, following the idea of Shintani. On the other hand, the continuation of (6.2) to $\mathbf{C}^r$ can also be shown by (a slight generalization of) Theorem 3 of [15], whose proof is based on the Mellin-Barnes integral formula. In particular, a recurive integral formula can be shown for $\zeta_{SH,r}$ (see (2.4) of [15]), which is the key of the proof. A prototype of such a kind of integral formula was already given in Section 8 of [12] for $\zeta_{SH,2}$.

However, in the case of Shintani zeta-functions, a different type of recursive integral formula, similar to (2.4) or (3.5) of the present paper, can be proved. For this purpose, we introduce the following *modified Shintani multiple zeta-functions*. Let $1 \leq k \leq r$, and define

$$\begin{aligned}
&\zeta_{SH,k,r}((s_1,\ldots,s_r);(\alpha_1,\ldots,\alpha_k),W_k)\\
&=\sum_{m_1=0}^{\infty}\cdots\sum_{m_k=0}^{\infty}\prod_{j=1}^{r}(w_{j1}(\alpha_1+m_1)+\cdots+w_{jk}(\alpha_k+m_k))^{-s_j},
\end{aligned}\tag{6.3}$$

where $W_k=(w_{jh})_{1\leq j\leq r, 1\leq h\leq k}$. This series is convergent absolutely when

$$\Re s_j > 1 \quad (1 \leq j \leq k), \qquad \Re s_j > 0 \quad (k+1 \leq j \leq r). \tag{6.4}$$

The meromorphic continuation of $\zeta_{SH,k,r}$ to $\mathbf{C}^r$ is again verified by Theorem 3 of [15].

Assume $\alpha_k > 0$, and also at least one of $\alpha_1,\ldots,\alpha_{k-1}$ is not 0. Then

$$\begin{aligned}
&\zeta_{SH,k,r}((s_1,\ldots,s_r);(\alpha_1,\ldots,\alpha_k),W_k)\\
&=\sum_{m_1=0}^{\infty}\cdots\sum_{m_k=0}^{\infty}\prod_{j=1}^{r}(w_{j1}(\alpha_1+m_1)+\cdots+w_{j,k-1}(\alpha_{k-1}+m_{k-1}))^{-s_j}\\
&\quad\times\left(1+\frac{w_{jk}(\alpha_k+m_k)}{w_{j1}(\alpha_1+m_1)+\cdots+w_{j,k-1}(\alpha_{k-1}+m_{k-1})}\right)^{-s_j}.
\end{aligned}\tag{6.5}$$

Applying the Mellin-Barnes formula ((3.1) of [13]), we obtain

$$\begin{aligned}
&\zeta_{SH,k,r}((s_1,\ldots,s_r);(\alpha_1,\ldots,\alpha_k),W_k)\\
&=\sum_{m_1=0}^{\infty}\cdots\sum_{m_k=0}^{\infty}\prod_{j=1}^{r}(w_{j1}(\alpha_1+m_1)+\cdots+w_{j,k-1}(\alpha_{k-1}+m_{k-1}))^{-s_j}\\
&\qquad\times\frac{1}{2\pi i}\int_{(c_j)}\frac{\Gamma(s_j+z_j)\Gamma(-z_j)}{\Gamma(s_j)}\\
&\qquad\times\left(\frac{w_{jk}(\alpha_k+m_k)}{w_{j1}(\alpha_1+m_1)+\cdots+w_{j,k-1}(\alpha_{k-1}+m_{k-1})}\right)^{z_j}dz_j\\
&=\frac{1}{(2\pi i)^r}\int_{(c_r)}\cdots\int_{(c_1)}\left(\prod_{j=1}^{r}\frac{\Gamma(s_j+z_j)\Gamma(-z_j)w_{jk}^{z_j}}{\Gamma(s_j)}\right)\\
&\qquad\times\sum_{m_1=0}^{\infty}\cdots\sum_{m_{k-1}=0}^{\infty}\prod_{j=1}^{r}(w_{j1}(\alpha_1+m_1)+\cdots+w_{j,k-1}(\alpha_{k-1}+m_{r-1})^{-s_j-z_j}\\
&\qquad\times\sum_{m_k=0}^{\infty}(\alpha_k+m_k)^{z_1+\cdots+z_r}dz_1\cdots dz_r,
\end{aligned}\tag{6.6}$$

where $-\Re s_j < c_j < 0$ $(1 \leq j \leq r)$. Under assumption (6.4), we can choose $c_1,\ldots,c_r$ for which

$$\begin{cases}1-\Re s_j < c_j < 0 & (1\leq j\leq k-1),\\ -\Re s_k < c_k < -1 & (j=k),\\ -\Re s_j < c_j < 0 & (k+1\leq j\leq r)\end{cases}\tag{6.7}$$

holds. Then $\Re(s_j+z_j) > 1$ $(1 \leq j \leq k-1)$, $\Re(s_j+z_j) > 0$ $(k \leq j \leq r)$, and $\Re(z_1+\cdots+z_r) < -1$. Hence both of the series on the right-hand side of (6.6) converge, and we obtain

Proposition 3. *Let $1 \leq k \leq r$. If $\alpha_k > 0$, and at least one of $\alpha_1,\ldots,\alpha_{k-1}$ is not 0, then*

$$\begin{aligned}
&\zeta_{SH,k,r}((s_1,\ldots,s_r);(\alpha_1,\ldots,\alpha_k),W_k)\\
&=\frac{1}{(2\pi i)^r}\int_{(c_r)}\cdots\int_{(c_1)}\left(\prod_{j=1}^{r}\frac{\Gamma(s_j+z_j)\Gamma(-z_j)w_{jk}^{z_j}}{\Gamma(s_j)}\right)\\
&\qquad\times\zeta_{SH,k-1,r}((s_1+z_1,\ldots,s_r+z_r);(\alpha_1,\ldots,\alpha_{k-1}),W_{k-1})\\
&\qquad\times\zeta(-z_1-\cdots-z_r,\alpha_k)dz_1\cdots dz_r
\end{aligned}\tag{6.8}$$

in the region (6.4), where $c_1,\ldots,c_r$ are constants satisfying (6.7).

Obviously $\zeta_{SH,r,r} = \zeta_{SH,r}$, while

$$\begin{aligned}&\zeta_{SH,1,r}((s_1,\ldots,s_r);\alpha_1,W_1)\\&=\sum_{m_1=0}^{\infty}\prod_{j=1}^{r}(w_{j1}(\alpha_1+m_1))^{-s_j}=\left(\prod_{j=1}^{r}w_{j1}^{-s_j}\right)\zeta(s_1+\cdots+s_r,\alpha_1).\end{aligned}\tag{6.9}$$

Therefore, when all α_j $(1 \leq j \leq r)$ are positive, the above proposition gives the recursive structure

$$\zeta_{SH,r} = \zeta_{SH,r,r} \to \zeta_{SH,r-1,r} \to \zeta_{SH,r-2,r} \to \cdots \to \zeta_{SH,1,r},$$

where the last one is essentially the Hurwitz zeta-function. In view of this recursive structure of the family of (modified) Shintani multiple zeta-functions, (6.8) may be useful as a starting point of further analytic studies of this family.

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