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A Floquet-Like Factorization for Linear Periodic Systems

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***Abstract*— In this note, the novel representation is proposed for a linear periodic continuous-time system with T-periodic real-valued coefficients. We prove that a T-periodic real-valued factor and two real-valued matrix exponential functions can be extracted from a state transition matrix, while, in the well-known Floquet representation theorem, a $2T$-periodic real-valued factor and a real-valued matrix exponential function are extracted from the state transition matrix. Then we also proved that any T-periodic system can be transformed to a system with T-periodic real-valued trigonometric coefficients using a T-periodic real-valued coordinate transformation, while, in the well-known Lyapunov reducibility theorem, a $2T$-periodic real-valued coordinate transformation is utilized to transform the given periodic system into a time-invariant system with real coefficients. This new information can be useful for designing a T-periodic control law.**

NOTATIONS

$\mathbb{R}$	the set of all real numbers
$\mathbb{C}$	the set of all complex numbers
$\mathbb{R}^{n\times m}$	the set of all real matrices with n rows and m columns
$\mathbb{C}^{n\times m}$	the set of all complex matrices with n rows and m columns
0	the zero number or the zero matrix
I	the identity matrix
$\det X$	the determinant of a matrix X
X^{-1}	the inverse of a matrix X
e^X	the matrix exponential of a matrix X
$:=$	$X := Y$ denotes that X is defined by Y

I. INTRODUCTION

We consider the linear periodic system

$$\dot{x} = A(t)x, \quad \dot{x} := \frac{dx}{dt} \tag{1}$$

where $t \in \mathbb{R}$ is a time, $x(t) \in \mathbb{R}^n$ is a state vector, $A(t) \in \mathbb{R}^{n\times n}$ is supposed to be real-valued, continuous and periodic with a period $T > 0$, *i.e.*

$$A(t+T) = A(t) \quad \forall t \in \mathbb{R},$$

which is said to be T-*periodic* for simplicity.

Let Φ denotes the state transition matrix of (1), i.e. Φ is the unique solution of

$$\frac{\partial}{\partial s}\Phi(s,t) = A(s)\Phi(s,t) \tag{2}$$

$$\Phi(t,t) = I. \tag{3}$$

It is well known that Φ exists and satisfies the following properties (see e.g. [3]):

$$\det \Phi(s,t) > 0 \tag{4}$$

$$\Phi(s,t) = \Phi(t,s)^{-1} \tag{5}$$

$$\Phi(s,\tau)\Phi(\tau,t) = \Phi(s,t) \tag{6}$$

$$\Phi(s+T,t+T) = \Phi(s,t) \quad \forall s,t,\tau \in \mathbb{R}. \tag{7}$$

Floquet theorem plays a fundamental role in the analysis and control of linear periodic continuous-times systems. The theorem consists of two main parts: the Floquet representation theorem [2] and the Lyapunov reducibility theorem [4].

The Floquet representation theorem provides a representation of the state transition matrix Φ in terms of a periodic matrix function and a matrix exponential function. This requires matrix logarithms, and we refer the theory as follows:

Lemma 1: [1] (i) Let $M_c \in \mathbb{C}^{n\times n}$. Then there exists a complex matrix $L_c \in \mathbb{C}^{n\times n}$ such that

$$M_c = e^{L_c}, \tag{8}$$

if and only if $\det M_c \neq 0$.
(ii) Let $M_r \in \mathbb{R}^{n\times n}$. Then there exists a real matrix $L_r \in \mathbb{R}^{n\times n}$ such that

$$M_r = e^{L_r} \tag{9}$$

if and only if $\det M_r \neq 0$ and, for every negative eigenvalue λ of M_r and for every integer k, the Jordan block of M_r has an even number of $k \times k$ blocks associated with λ.

Let $M_c = \Phi(T,0)$ and apply Lemma 1, it follows that Φ always has a *T-periodic complex Floquet factorization* (see (10)). Let $M_r = \Phi(T,0)$ and apply Lemma 1, it follows that Φ always has a *$2T$-periodic real Floquet factorization* (see (11)). However, since $\Phi(T,0)$ do not always have a real logarithm, Φ do not always have a *T-periodic real Floquet factorization* (see (12)).

Corollary 1: [5] Let Φ denotes the state transition matrix of (1). Then, (i) there exists a T-periodic complex-valued matrix function $P_c(t) \in \mathbb{C}^{n\times n}$ with $\det P_c(t) \neq 0, \forall t \in \mathbb{R}$ and a complex matrix $F_c \in \mathbb{C}^{n\times n}$ such that Φ is factored by

$$\Phi(t,0) = P_c(t)e^{F_c t}, \tag{10}$$

(ii) there exists a $2T$-periodic real-valued matrix function $P_d(t) \in \mathbb{R}^{n\times n}$ with $\det P_d(t) \neq 0, \forall t \in \mathbb{R}$ and a real matrix $F_d \in \mathbb{R}^{n\times n}$ such that Φ is factored by

$$\Phi(t,0) = P_d(t)e^{F_d t}, \tag{11}$$

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(iii) there exists a T-periodic real-valued matrix function $P_r(t) \in \mathbb{R}^{n\times n}$ with $\det P_r(t) \neq 0, \forall t \in \mathbb{R}$ and a real matrix $F_r \in \mathbb{R}^{n\times n}$ such that Φ is factored by

$$\Phi(t,0) = P_r(t)e^{F_r t} \tag{12}$$

if and only if $\Phi(T,0)$ has a real logarithm, i.e. there exists a real matrix $\tilde{F}_r \in \mathbb{R}^{n\times n}$ satisfying $\Phi(T,0) = e^{\tilde{F}_r}$.

The representation of the form (10) or (11) does not give a parameterization of a set of all state transition matrices generated by T-periodic systems with real-valued coefficients. Namely, $\Phi(t,0)$ defined by (10) (or (11)) might be a fundamental solution of the equation of the form (1) with some complex-valued (or $2T$-periodic) $A(t)$ (see an example in section IV which does not have a T-periodic real Floquet factorization).

In this regard, we refer a parameterization of a set of all state transition matrices generated by periodic systems with real-valued coefficients. This requires a generalization of a real matrix logarithm as follows (see [5] for a constructive proof):

Lemma 2: [5] Suppose that a real matrix $M \in \mathbb{R}^{n\times n}$ is invertible. Then there exist real matrices $L_y, X_y \in \mathbb{R}^{n\times n}$ such that

$$M = X_y e^{L_y} = e^{L_y} X_y \tag{13}$$
$$X_y L_y = L_y X_y \tag{14}$$
$$X_y^2 = I. \tag{15}$$

Corollary 2: [7] Let Φ denotes the state transition matrix of (1) with a real-valued and T-periodic $A(t)$ which is integrable and piecewise-continuous in any finite interval. Then there exists a real-valued matrix function $P_y(t) \in \mathbb{R}^{n\times n}$ having an integrable piecewise-continuous derivative with $\det P_y(t) \neq 0, \forall t \in \mathbb{R}$ and real matrices $F_y, Y_y \in \mathbb{R}^{n\times n}$ such that Φ is factored as

$$\Phi(t,0) = P_y(t)e^{F_y t} \tag{16}$$

and $P_y(t), Y_y, F_y$ satisfy

$$P_y(t+T) = P_y(t)Y_y \tag{17}$$
$$Y_y F_y = F_y Y_y \tag{18}$$
$$Y_y^2 = I. \tag{19}$$

Conversely, let $P_y(t) \in \mathbb{R}^{n\times n}$ be arbitrary real-valued matrix function with $\det P_y(t) \neq 0, \forall t \in \mathbb{R}$ and $F_y, Y_y \in \mathbb{R}^{n\times n}$ be any real matrices satisfying conditions (17)–(19), and let $P_y(t)$ have an integrable piecewise continuous derivative. Then $\Phi(t,0)$ defined by (16) is a fundamental matrix for some equation of the form (1) with a T-periodic real-valued matrix function $A(t) \in \mathbb{R}^{n\times n}$ which is integrable and piecewise-continuous in any finite interval.

It follows from (19) that the periodic factor $P_y(t)$ is $2T$-periodic, but not necessarily T-periodic, therefore (16) is regarded as a special case of $2T$-periodic real Floquet factorization. It was pointed out in [5] that conditions (17)–(19) are sufficient, but not necessary, for the periodic factor $P_y(t)$ to be $2T$-periodic.

We also refer an alternative parameterization of a set of all state transition matrices generated by periodic systems with real-valued coefficients.

Corollary 3: [5] Let Φ denotes the state transition matrix of (1). Let real matrices $Y_m, F_m \in \mathbb{R}^{n\times n}$ satisfy

$$Y_m \Phi(T,0) = e^{F_m T}$$

and define a real-valued function $P_m(t) \in \mathbb{R}^{n\times n}$ by

$$\Phi(t,0) = P_m(t)e^{F_m t}. \tag{20}$$

Then $P_m(t)$ has a continuous derivative and satisfy

$$P_m(t+T) = P_m(t)e^{-F_m t}P_m(T)e^{-F_m t}. \tag{21}$$

The choice of Y_m affects $P_m(t)$ as follows:

$P_m(T) = Y_m^{-1}$,
$P_m(t+T) = P_m(t), \forall t \in \mathbb{R} \Leftrightarrow Y_m = I$,
$P_m(t+T) = -P_m(t), \forall t \in \mathbb{R} \Leftrightarrow Y_m = -I$,
$P_m(t+2T) = P_m(t), \forall t \in \mathbb{R} \Leftrightarrow (\Phi(T,0))^2 = (Y_m\Phi(T,0))^2$.

Conversely, let $P_m(t) \in \mathbb{R}^{n\times n}$ be arbitrary real-valued matrix function satisfying (21) with $\det P_m(t) \neq 0, \forall t \in \mathbb{R}$, and let $P_m(t)$ has a continuous derivative. Then $\Phi(t,0)$ defined by (16) is a fundamental matrix for some equation of the form (1) with a T-periodic real-valued matrix function $A(t) \in \mathbb{R}^{n\times n}$.

The factor $P_m(t)$ is not necessarily periodic. But if we restrict our attention to extract periodic real-valued factors, it is not always possible to construct T-periodic real Floquet factorization; therafore, (20) is also regarded as a special case of $2T$-periodic real Floquet factorization.

In this note, we shall firstly consider whether if it is always possible to extract a T-periodic real-valued factor from the state transition matrix $\Phi(t,0)$.

The *first purpose* of this note is to provide a novel representation by

$$\Phi(t,0) = P(t)e^{Gt}e^{Ft}$$

for some T-periodic real-valued matrix function $P(t) \in \mathbb{R}^{n\times n}$ and real matrices $F, G \in \mathbb{R}^{n\times n}$ satisfying

$$e^{GT}F = Fe^{GT}$$
$$e^{2GT} = I,$$

and then prove that this gives a parameterization of all state transition matrices generated by periodic systems with real-valued coefficients (see the formal statement in Theorem 1). It is clear that T-periodic factor $P(t)$ is extracted from $\Phi(t,0)$. In addition, it is easy to see that $P(t)e^{Gt}$ is $2T$-periodic real-valued, therefore T-periodic factor $P(t)$ is extracted from $2T$-periodic factor $P(t)e^{Gt}$ (see Corollary 4 for more detail).

Next we shall consider whether if it is possible to derive a kind of standard form for linear periodic continuous-continuous time systems utilizing the proposed factorization.

Let us recall that the Lyapunov reducibility theorem states that there exists a periodic coordinate transformation which transforms the given system (1) into a linear time-invariant system. Namely, (1) is transformed to a linear time-invariant system

$$\dot{\eta}_c = F_c \eta_c$$

with a complex matrix $F_c \in \mathbb{C}^{n\times n}$ via a T-periodic complex-valued coordinate transformation

$$\eta_c = P_c(t)^{-1}x. \tag{22}$$

(1) is also transformed to a linear time-invariant system

$$\dot{\eta}_d = F_d \eta_d$$

with the real matrix $F_d \in \mathbb{R}^{n\times n}$ via a $2T$-periodic real-valued coordinate transformation

$$\eta_d = P_d(t)^{-1}x. \tag{23}$$

The *second purpose* of this note is to provide a novel reducibility theorem such that the given system (1) is transformed to a T-periodic system of the form

$$\begin{aligned}\dot{\eta} &= H(t)\eta\\ H(t) &:= H_1 + \cos\left(\frac{2\pi t}{T}\right)H_2 + \sin\left(\frac{2\pi t}{T}\right)H_3\end{aligned}$$

for some real matrices $H_1, H_2, H_3 \in \mathbb{R}^{n\times n}$ utilizing a T-periodic real-valued coordinate transformation

$$\xi = P(t)^{-1}x$$

where $P(t)$ is the T-periodic real-valued factor introduced in the proposed factorization with an additional constraint (see the formal statement in Theorem 2 and its constructive proof).

Although the above transformed system is no longer time-invariant, we note that the given system (1) is simplified. We also note that any T-periodic system with input

$$\dot{x} = A(t)x + B(t)u$$

is transformed to the T-periodic system of the form

$$\dot{\eta} = H(t)\eta + B_h(t)u$$

for some T-periodic real-valued $B_h(t)$. On the contrary, B-matrix of the transformed system might be complex-value (or $2T$-periodic) for (22) (or (23)). This new information can be useful for designing a T-periodic control law which will be a sequential research topic. It is also important to note that, even if $A(t)$ is only continuous and is not differentiable, the coefficient matrix well simplified to a sum of a constant matrix and a sinusoidal with a single period T as shown in $H(t)$ (see Theorem 2 for more detail).

II. Extension of Floquet Representation Theorem

Firstly we prove that, if we restrict M in Lemma 2 to be $\det M > 0$, then we can choose X_y in Lemma 2 such that X_y has a real logarithm.

Lemma 3: A real matrix $M \in \mathbb{R}^{n\times n}$ satisfies $\det M > 0$ if and only if there exist real matrices $L, N \in \mathbb{R}^{n\times n}$ satisfying

$$M = e^N e^L = e^L e^N \tag{24}$$
$$e^N L = L e^N \tag{25}$$
$$e^{2N} = I. \tag{26}$$

Proof: Sufficiency: it follows from (24) that

$$\det M = \det e^L \det e^L > 0.$$

Necessity: Consider a real similarity transformation

$$M = V\begin{bmatrix}\Lambda_1 & \\ & \Lambda_2\end{bmatrix}V^{-1}$$

where $V \in \mathbb{R}^{n\times n}$ is invertible, each eigenvalue of $\Lambda_1 \in \mathbb{R}^{n_1\times n_1}$ is negative real number, each eigenvalue of $\Lambda_2 \in \mathbb{R}^{(n-n_1)\times(n-n_1)}$ is not negative real number. Note that, since $\det M > 0$ by assumption, n_1 is even. With this partitioning, since $-\Lambda_1$ and Λ_2 do not contain negative real eigenvalues, it follows from Lemma 1 (ii) that there exist real matrices $L_1 \in \mathbb{R}^{n_1\times n_1}$ and $L_2 \in \mathbb{R}^{(n-n_1)\times(n-n_1)}$ such that

$$e^{L_1} = -\Lambda_1, \quad e^{L_2} = \Lambda_2.$$

Choose real matrices $N, L \in \mathbb{R}^{n\times n}$ as follows:

$$N := V\begin{bmatrix}N_1 & \\ & 0\end{bmatrix}V^{-1},\ L := V\begin{bmatrix}L_1 & \\ & L_2\end{bmatrix}V^{-1}$$
$$N_1 := \begin{bmatrix}\begin{bmatrix}0 & \pi\\ -\pi & 0\end{bmatrix} & & \\ & \ddots & \\ & & \begin{bmatrix}0 & \pi\\ -\pi & 0\end{bmatrix}\end{bmatrix} \in \mathbb{R}^{n_1\times n_1}$$

Then their matrix exponentials are computed as

$$e^N = V\begin{bmatrix}-I & \\ & I\end{bmatrix}V^{-1},\ e^L = V\begin{bmatrix}-\Lambda_1 & \\ & \Lambda_2\end{bmatrix}V^{-1},$$

and (24)–(26) are readily verified. □

It was proved in [1] that "A real matrix $M \in \mathbb{R}^{n\times n}$ satisfies $\det M > 0$ if and only if there exist real matrices $L, N \in \mathbb{R}^{n\times n}$ satisfying (24)", and we obtained the additional conditions (25) and (26) in the necessity part of Lemma 3. Indeed we proved Lemma 3 in the same way as in [1], but those additional conditions (25) and (26) are crucial to prove the following key lemma:

Lemma 4: Let Φ denotes the state transition matrix of (1). Then, (i) there exist real matrices $F, G \in \mathbb{R}^{n\times n}$ such that the real-valued matrix function

$$P(t) := \Phi(t,0)e^{-Ft}e^{-Gt} \tag{27}$$

are T-periodic if and only if there exist real matrices $F, G \in \mathbb{R}^{n\times n}$ satisfying

$$\Phi(T,0) = e^{GT}e^{FT} \tag{28}$$
$$e^{GT}F = Fe^{GT}. \tag{29}$$

(ii) there exist real matrices $F, G \in \mathbb{R}^{n\times n}$ such that the real-valued matrix function $P(t)$ in (27) is T-periodic and

$$P_g(t) := P(t)e^{Gt} \tag{30}$$

are $2T$-periodic if and only if there exist real matrices $F, G \in \mathbb{R}^{n\times n}$ satisfying (28), (29) and

$$e^{2GT} = I. \tag{31}$$

Proof: (i) Necessity: Substituting $t = 0$ into (27), we have

$$P(T) = P(0) = I, \tag{32}$$

which proves (28). Using (6), (7), (27) and (32), it follows that $P(t)$ is T-periodic. Since $P(t)$ is T-periodic and invertible for all $t \in \mathbb{R}$, it follows that

$$e^{GT}e^{Ft} = e^{Ft}e^{GT}.$$

Differentiating both sides with respect to t

$$e^{GT}Fe^{Ft} = e^{Ft}Fe^{GT}.$$

and substituting $t = 0$, we have (29).
Sufficiency: It follows from (29) that

$$e^{GT}F^k = Fe^{GT}F^{k-1} = \cdots = F^k e^{GT} \quad \forall k \in \mathbb{N},$$

therefore

$$e^{GT}e^{Ft} = e^{Ft}e^{GT} \quad \forall t \in \mathbb{R}. \tag{33}$$

It follows from (28) that

$$P(T) = \Phi(T,0)e^{-FT}e^{-GT} = I. \tag{34}$$

Using (6), (7), (27), (33) and (34), it follows that $P(t)$ is T-periodic.
(ii) Necessity: Similar to necessity part of (i), since $P(t)$ is T-periodic, we have (28), (29). By the definition of $P_g(t)$, we have

$$\begin{aligned} P_g(t+2T) &= P(t+2T)e^{G(t+2T)} \\ &= P(t)e^{Gt}e^{2GT} \\ &= P_g(t)e^{2GT}. \end{aligned}$$

Since $P_g(t)$ is $2T$-periodic and invertible for all $t \in \mathbb{R}$, we have (31).
Sufficiency: Similar to sufficiency part of (i), (28) and (29) prove the T-periodicity of $P(t)$. Similar to necessity part of (ii), (31) proves the $2T$-periodicity of $P_g(t)$. □

Now we prove that any state transition matrix of a T-periodic system is factored by a T-periodic matrix and two matrix exponentials. In addition, we prove that this factorization gives a parameterization of all set of state transition matrix generated by T-periodic systems with real-valued coefficients.

Theorem 1: Let Φ denotes the state transition matrix of (1). Then there exist a T-periodic real-valued matrix function $P(t) \in \mathbb{R}^{n\times n}$ with $\det P(t) \neq 0, \forall t \in \mathbb{R}$ and real matrices $F, G \in \mathbb{R}^{n\times n}$ such that Φ is factored as

$$\Phi(t,0) = P(t)e^{Gt}e^{Ft}, \tag{35}$$

and F and G satisfy (28), (29) and (31).

Conversely, let $P(t) \in \mathbb{R}^{n\times n}$ be any continuously differentiable, T-periodic and real-valued matrix function with $\det P(t) \neq 0, \forall t \in \mathbb{R}$ and let $F, G \in \mathbb{R}^{n\times n}$ be any matrices satisfying (29). Then

$$\Phi(s,t) := P(s)e^{Gs}e^{F(s-t)}e^{-Gt}P(t)^{-1} \tag{36}$$

is a state transition matrix for some equation of the form (1) with a T-periodic continuous real-valued function $A(t) \in \mathbb{R}^{n\times n}$.

Moreover, let $P(t) \in \mathbb{R}^{n\times n}$ be any continuously differentiable, T-periodic and real-valued matrix function with $\det P(t) \neq 0, \forall t$ and let $F, G \in \mathbb{R}^{n\times n}$ be any matrices satisfying (29) and (31). Then (36) is a state transition matrix for some equation of the form (1) with a T-periodic continuous real-valued function $A(t) \in \mathbb{R}^{n\times n}$ and $P(t)e^{Gt}$ is $2T$-periodic.

Proof: It follows from (4) that $\det \Phi(T,0) > 0$. Let $M := \Phi(T,0)$ and apply Lemma 3, then there exist real matrices $L, N \in \mathbb{R}^{n\times n}$ satisfying (24)–(26). Define real matrices $F := \frac{1}{T}L$ and $G := \frac{1}{T}N$. Since (24)–(26) are equivalent to (28), (29) and (31), the sufficiency part of Lemma 4 (ii) proves the existence of T-periodic invertible real-valued matrix function $P(t) \in \mathbb{R}^{n\times n}$ satisfying (35).

Next consider a continuously differentiable, T-periodic and invertible real-valued matrix function $P(t) \in \mathbb{R}^{n\times n}$ and real matrices $F, G \in \mathbb{R}^{n\times n}$ satisfying (29). Since $P(t)$ is invertible for all $t \in \mathbb{R}$, Φ defined in (36) is well defined. Substituting $s = t$ into (36), it follows that Φ satisfies (3). Define a real-valued matrix function by

$$A(t) := (\dot{P}(t)e^{Gt} + P(t)Ge^{Gt} + P(t)e^{Gt}F)e^{-Gt}P(t)^{-1}$$

By the definition of Φ, it follows that

$$A(s) = \frac{\partial \Phi(s,t)}{\partial s}\Phi(s,t)^{-1}$$

which proves that Φ satisfies an equation of the form (2). Using (29) and T-periodicity of $P(t)$ (and therefore T-periodicity of $\dot{P}(t)$), $A(s)$ is shown to be T-periodic. Since $P(t)$ is continuously differentiable and real-valued and F and G are real, $A(t)$ is continuous and real-valued. Since Φ is the solution of (2) and (3) with a T-periodic continuous function $A(t)$, Φ is the state transition matrix of the form (1) with a T-periodic continuous function $A(t)$.

Moreover, it follows from the sufficiency part of Lemma 4 (ii) that the additional condition (31) implies that $P(t)e^{Gt}$ is $2T$-periodic. □

As proved in the above theorem, given a T-periodic real-valued $P(t)$ and real matrices F, G satisfying conditions (29), (31), $\Phi(t,0)$ defined by (36) is factored as

$$\Phi(t,0) = P_g(t)e^{Ft}$$

with $2T$-periodic real-valued matrix function $P_g(t) := P(t)e^{Gt}$. Conversely, one may think whether if it is possible to factor a $2T$-periodic real-valued matrix function constituting a $2T$-periodic real Floquet factorization by some T-periodic real-valued function and some real-valued matrix exponential. This is not possible in general, but it is interesting to see that this is possible for $2T$-periodic factors $P_y(t)$ discussed in Corollary 2.

Corollary 4: For any $2T$-periodic real-valued matrix function $P_y(t) \in \mathbb{R}^{n\times n}$ with $\det P_y(t) \neq 0$, $\forall t \in \mathbb{R}$ and any real matrices $F_y, Y_y \in \mathbb{R}^{n\times n}$ satisfying (17)–(19), choose $F = F_y$, then there exists a T-periodic real-valued matrix function $P(t) \in \mathbb{R}^{n\times n}$ and real matrix $G \in \mathbb{R}^{n\times n}$ such that satisfying (28), (29), (31) and

$$P_y(t)e^{F_y t} = P(t)e^{Gt}e^{Ft} \quad \forall t \in \mathbb{R}. \tag{37}$$

Proof: Consider the real Jordan factorization of $Y_y \in \mathbb{R}^{n\times n}$ by

$$Y_y = V^{-1}\begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_m \end{bmatrix} V$$

where $V \in \mathbb{R}^{n\times n}$ is invertible, $J_i (i = 1, \cdots, m)$ are real Jordan matrices with appropriate dimensions. It follows from (19) that

$$Y_y^2 = V^{-1}\begin{bmatrix} J_1^2 & & \\ & \ddots & \\ & & J_m^2 \end{bmatrix} V = I,$$

therefore

$$J_i^2 = I$$

for each $i = 1, \cdots, m$. Then it can be shown that $J_i = 1$ or $J_i = I \in \mathbb{R}^{2\times 2}$ or $J_i = -I \in \mathbb{R}^{2\times 2}$ for each $i = 1, \cdots, m$. By reordering subscripts of J_i, Y_y is written as

$$Y_y = V^{-1}\begin{bmatrix} -I & \\ & I \end{bmatrix} V$$

for some invertible real matrix $V \in \mathbb{R}^{n\times n}$. Then a real logarithm of Y_y is given by

$$L_y := V^{-1}\begin{bmatrix} \begin{bmatrix} \begin{bmatrix} 0 & \pi \\ -\pi & 0 \end{bmatrix} & & \\ & \ddots & \\ & & \begin{bmatrix} 0 & \pi \\ -\pi & 0 \end{bmatrix} \end{bmatrix} & \\ & 0 \end{bmatrix} V.$$

Choose $G := \frac{1}{T}L_y \in \mathbb{R}^{n\times n}$, then it follows that $Y_y = e^{GT}$. Choose $P(t) := P_y(t)e^{-Gt}$, then it follows that $P(t)$ is T-periodic. It is immediate to prove that (37) is satisfied and that (17)–(19) are equivalent to (28), (29) and (31) respectively. □

Remark: The statement of Corollary 4 is not satisfied for arbitrary $2T$-periodic real-valued matrix function without assuming (17)–(19). Consider a $2T$-periodic real-valued matrix function

$$P_y(t) := \begin{bmatrix} \cos\left(\frac{\pi t}{T}\right) & \sin\left(\frac{\pi t}{T}\right) + \frac{1}{2} - \frac{1}{2}\cos\left(\frac{\pi t}{T}\right) \\ -\sin\left(\frac{\pi t}{T}\right) & \cos\left(\frac{\pi t}{T}\right) \end{bmatrix}$$

with $\det P_y(t) \neq 0, \forall t \in \mathbb{R}$, then $P_y(t)^{-1}P_y(t+T)$ is not constant, therefore there is no constant matrix $Y_y \in \mathbb{R}^{n\times n}$ satisfying (17). Suppose that there exist a T-periodic real-valued matrix function $P(t) \in \mathbb{R}^{2\times 2}$ and a real matrix $G \in \mathbb{R}^{2\times 2}$ satisfying (37). Then, using the T-periodicity of $P(t)$ and (37), we have

$$\begin{aligned} P_y(t+T) &= P(t+T)e^{G(t+T)} \\ &= P(t)e^{G(t+T)} \\ &= P_y(t)e^{-Gt}e^{G(t+T)} \\ &= P_y(t)e^{GT}. \end{aligned}$$

Substituting $t = 0$, we have

$$\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} = e^{GT}.$$

However, it follows from Lemma 1 (ii) that there is no real matrix $G \in \mathbb{R}^{2\times 2}$ satisfying the above equation, therefore we have a contradiction. □

III. Extension of Lyapunov Reducibility Theorem

In the previous section, we have shown that any state transition matrix Φ of a T-periodic system can be factored as (35) with a T-periodic real-valued matrix function $P(t)$ and real matrices F, G satisfying (28), (29), (31). In this section, we impose an additional limitation on the choice of a real matrix G to derive a novel standard form of linear-periodic continuous-time systems. We shall prove that any T-periodic system can be transformed to a T-periodic system with T-periodic real-valued trigonometric coefficients using a T-periodic real-valued coordinate transformation.

Theorem 2: Let Φ denotes the state transition matrix of (1). Then there exists a continuously differentiable, T-periodic and real-valued matrix function $P(t)$ with $\det P(t) \neq 0, \forall t \in \mathbb{R}$ such that a T-periodic coordinate transformation

$$\eta = P(t)^{-1}x$$

transforms (1) to a T-periodic system of the form

$$\dot{\xi} = H(t)\xi \tag{38}$$

$$H(t) := H_1 + \cos\left(\tfrac{2\pi t}{T}\right)H_2 + \sin\left(\tfrac{2\pi t}{T}\right)H_3 \tag{39}$$

for some constant $H_1, H_2, H_3 \in \mathbb{R}^{n\times n}$.

Proof: As shown in the proof Lemma 3, there exist real matrices $F, G \in \mathbb{R}^{n\times n}$ which satisfy (28), (29), (31). Moreover, it is possible to choose G of the form

$$G := V^{-1}\begin{bmatrix} \begin{bmatrix} \begin{bmatrix} 0 & \frac{\pi}{T} \\ -\frac{\pi}{T} & 0 \end{bmatrix} & & \\ & \ddots & \\ & & \begin{bmatrix} 0 & \frac{\pi}{T} \\ -\frac{\pi}{T} & 0 \end{bmatrix} \end{bmatrix} & \\ & 0 \end{bmatrix} V$$

for some $V \in \mathbb{R}^{n\times n}$. Define a real-valued matrix function $P(t) \in \mathbb{R}^{n\times n}$ by

$$P(t) = \Phi(t,0)e^{-Ft}e^{-Gt}.$$

Then it follows from Lemma 4 (i) that $P(t)$ is T-periodic. Consider a T-periodic coordinate transformation $\xi = P(t)x$, then $H(t)$ in (38) is given by

$$H(t) = \frac{d(e^{Gt}e^{Ft})}{dt}(e^{Gt}e^{Ft})^{-1} = G + e^{Gt}Fe^{-Gt}.$$

Since each elements of e^{Gt} is consisted of a constant and $2T$-periodic functions $\cos\left(\frac{\pi t}{T}\right)$ and $\sin\left(\frac{\pi t}{T}\right)$, each elements of $H(t)$ is consisted of a constant, $2T$-periodic functions $\cos\left(\frac{\pi t}{T}\right)$ and $\sin\left(\frac{\pi t}{T}\right)$ and T-periodic functions $\cos\left(\frac{2\pi t}{T}\right)$ and $\sin\left(\frac{2\pi t}{T}\right)$. $H(t)$ is T-periodic, and therefore $H(t)$ do not contain $2T$-periodic terms. Hence $H(t)$ is represented by the form (39) with

$$H_1 := G + \frac{1}{T}\int_0^T e^{Gt}Fe^{-Gt}dt \tag{40}$$

$$H_2 := F - \frac{1}{T}\int_0^T e^{Gt}Fe^{-Gt}dt \tag{41}$$

$$H_3 := e^{\frac{GT}{4}}Fe^{-\frac{GT}{4}} - \frac{1}{T}\int_0^T e^{Gt}Fe^{-Gt}dt. \tag{42}$$

Taking the average of $H(t)$ over $[0,T]$, we have (40). Substituting $t = 0$ into (39), we have (41). Substituting $t = \frac{T}{4}$ into (39), we have (42). □

Although the transformed system (38) is no longer time-invariant, we note that the given system (1) is well simplified in the sense that the solution of the above transformed system can be explicitly computed. Indeed we have derived the equation (38) from its solution $e^{Gt}e^{Ft}$.

Remark: With the same choice of $P(t), F, G$ in Theorem 2, $e^{-Gt}P(t)^{-1}$ becomes $2T$-periodic real-valued. Hence (1) is transformed to a time-invariant system $\dot{\eta} = F\eta$ with real coefficients using a $2T$-periodic real-valued coordinate transformation $\eta = e^{-Gt}P(t)^{-1}x$. □

IV. Example

Let $T = 1$ and consider a 1-periodic system (1) with

$$A(t) := \begin{bmatrix} -1 - \frac{\sin(2\pi t)}{2} + \frac{\dot{p}(t)}{p(t)} & -\frac{1}{2} + \pi - \frac{\cos(2\pi t)}{2} \\ \frac{1}{2} - \pi - \frac{\cos(2\pi t)}{2} & -1 + \frac{\sin(2\pi t)}{2} + \frac{\dot{p}(t)}{p(t)} \end{bmatrix}$$

where $p(t) \in \mathbb{R}$ is supposed to be 1-periodic, continuously differentiable, real-valued and positive for all $t \in \mathbb{R}$.

Then the state transition matrix is given by

$$\Phi(t,0) = e^{-t}p(t)\begin{bmatrix} \cos(\pi t) & -t\cos(\pi t) + \sin(\pi t) \\ -\sin(\pi t) & t\sin(\pi t) + \cos(\pi t) \end{bmatrix}.$$

Since the monodromy matrix

$$\Phi(1,0) = \frac{p(1)}{e}\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$$

has one real Jordan block belonging to a negative eigenvalue -1, it follows from Lemma 1 that there is no 1-periodic real Floquet factorization of the form (12).

2-periodic real Floquet factorization of the form (11) is computed by choosing 2-periodic real-valued matrix function $P_d(t)$ and a real matrix F_d as follows:

$$P_d(t) := p(t)\begin{bmatrix} \cos(\pi t) & \sin(\pi t) \\ -\sin(\pi t) & \cos(\pi t) \end{bmatrix},$$

$$F_d := \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix}.$$

On the contrary, as shown in Theorem 1, $\Phi(t,0)$ is factored as (35) by choosing T-periodic real-valued matrix function $P(t)$ and a real matrix F, G as follows:

$$P(t) := p(t)I,\ G := \begin{bmatrix} 0 & \pi \\ -\pi & 0 \end{bmatrix},\ F := \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix}.$$

In addition, as shown in Theorem 2, the given system is transformed to

$$\dot{\xi} = H(t)\xi$$

$$H(t) := \begin{bmatrix} -1 - \frac{\sin(2\pi t)}{2} & -\frac{1}{2} + \pi - \frac{\cos(2\pi t)}{2} \\ \frac{1}{2} - \pi - \frac{\cos(2\pi t)}{2} & -1 + \frac{\sin(2\pi t)}{2} \end{bmatrix}$$

by 1-periodic real-valued coordinate transformation $\xi = P(t)^{-1}x$. Note that $H(t)$ consists of 1-periodic real-valued trigonometric functions.

V. Conclusion

In this note, we firstly focus on the problem of a parameterization of state transition matrices generated by T-periodic systems with real-valued coefficients. To this end, we proposed a novel factorization of state transition matrices in terms of a T-periodic factor and two matrix exponentials. The proposed factorization has several characteristics. This gives a parameterization as expected. We can always extract a T-periodic factors from a $2T$-periodic factors in [7], therefore this achieves a special class of $2T$-periodic real Floquet factorization. We derived a kind of standard form for periodic systems in terms of T-periodic real-valued trigonometric coefficients utilizing T-periodic coordinate transformation. This new information can be useful for designing a T-periodic control law.

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