Joint 48th IEEE Conference on Decision and Control and
28th Chinese Control Conference
Shanghai, P.R. China, December 16-18, 2009

Transformability from Discrete-time Periodic Non-homogeneous Systems to Time-invariant Ones

Yoshikazu Hayakawa and Tomohiko Jimbo

***Abstract*— This paper considers when a discrete-time periodic non-homogeneous system can be transformed to a time-invariant one by using regular linear mappings of state variables, inputs and outputs, respectively. The problem on a homogeneous system has been already solved as discrete-time Floquet transformation, and also the similarity classes of Floquet transformations have been characterized. Those previous results are used to derive some conditions on transformability of non-homogeneous systems.**

I. Introduction

A lot of model-based control method for periodic systems have been proposed, e.g., [2] [3] [10] [11] [13] and references therein. In order to establish model-based control designs for automotive engine, the authors proposed a model representation of V6 Spark Ignition SICE benchmark engine [9]; a continuous-time periodic nonlinear state space model is constructed first, then the model is discretized to get a discrete-time periodic nonlinear one, and finally by introducing a concept of "role state variables", the discrete-time periodic model can be transformed to a time-invariant one. In the last process, the discrete-time periodic system $x_{k+1} = f_k(x_k, u_k),\ y_k = g_k(x_k, u_k)$ has been transformed to the time-invariant one $\xi_{k+1} = f(\xi_k, \tau_k),\ \eta_k = g(\xi_k, \tau_k)$ by using periodic regular mappings $\xi_k = p_k(x_k),\ \tau_k = q_k(u_k),\ \eta_k = r_k(y_k)$. Even if the original periodic system is *nonlinear*, it is easy to find out those regular mappings because of the characteristics of automotive engine. Of course, it is very difficult to develop the above transformability for the general nonlinear systems. In order to challenge this difficult problem, as the first step, this paper aims to establish the transformability for the general *linear* systems.

It is well known [1] [4] as *theory of Floquet* that every continuous-time periodic linear homogeneous system $\dot{x}_c(t) = A_c(t)x_c(t)$ with a period T can be transformed to a linear time-invariant system $\dot{\xi}_c(t) = A_c\xi_c(t)$ by a state transformation $\xi_c(t) = P(t)x_c(t)$ with $P(t)$ being nonsingular and periodic $P(t+T) = P(t)$.

For discrete-time periodic linear homogeneous system $x_{k+1} = A_k x_k$, so-called *discrete-time version for theory of Floquet* has been derived in [12] [6], which show that all discrete-time periodic homogeneous systems can not be transformed to time-invariant ones.

Furthermore, it has been shown in [6] that if a discrete-time periodic homogeneous system has a Floquet transformation, then the system has a lot of Floquet transformations, and so three kinds of similarities in the set of Floquet transformations are introduced and it has be derived that the set of Floquet transformations splits into exactly a finite number of equivalent classes induced by the similarity.

In this paper, we consider a discrete-time periodic linear *non-homogeneous* system $x_{k+1} = A_k x_k + B_k u_k,\ y_k = C_k x_k$ and investigate when the non-homogeneous system can be transformed to a time-invariant one by using periodic linear regular mappings of x_k, u_k, y_k, respectively.

Section II formulates a problem to be considered in this paper, then reviews the previous results on discrete-time Floquet transformations and also gives an example to understand a essential difficulty in the problem. Section III is a main part of this paper, where the previous results on similarity classes of Floquet transformations are summarized first and new results on transformability are derived. Some conclusion remarks are stated in Section IV.

Notation: $\boldsymbol{R}$ is a set of all real numbers, $\boldsymbol{C}$ a set of all complex numbers, and $\boldsymbol{Z}$ a set of all integers.

For any positive integer $n \in \boldsymbol{Z}$, $\underline{n} := \{1, 2, \cdots, n\}$ and $\underline{n}^- := \{0, 1, \cdots, n-1\}$. For any $k \in \boldsymbol{Z}$, $\mathrm{mod}(k/n)$ denotes "k modulo n" .

Associated with $A, B \in \boldsymbol{C}^{n\times n}$, $A \simeq B$ means that A is similar to B, i.e., there exists a nonsingular matrix $S \in \boldsymbol{C}^{n\times n}$ such that $B = SAS^{-1}$. Note that this notation "$\simeq$" is also used for similarity of Floquet transformations, which will be defined in Section III, however, it is clear from the context whether similarity is used for matrices or Floquet transformations.

$J_m(\lambda)$ denotes an $m \times m$ Jordan block with eigenvalue λ, and for any square matrices C and D,

$$C \oplus D := \begin{bmatrix} C & 0 \\ 0 & D \end{bmatrix}.$$

II. Problem Statement

Consider a discrete-time periodic non-homogeneous system with a period N

$$\begin{aligned} x_{k+1} &= A_k x_k + B_k u_k \\ y_k &= C_k x_k \end{aligned} \qquad (1)$$

where $k \in \boldsymbol{Z}$ is time, $x_k \in \boldsymbol{R}^n$ is state, $u_k \in \boldsymbol{R}^m$ input, and $y_k \in \boldsymbol{R}^p$ output. The matrices A_k, B_k, C_k are real with appropriate sizes, N is a positive integer greater than or equal to 2, and it is assumed that

$$A_k = A_{\mathrm{mod}(k/N)},\ B_k = B_{\mathrm{mod}(k/N)},\ C_k = C_{\mathrm{mod}(k/N)}. \qquad (2)$$

Y. Hayakawa is with Department of Mechanical Science and Engineering, Nagoya University, 464-8603 Japan. He is also with R.T.C. Riken. hayakawa@nuem.nagoya-u.ac.jp
T. Jimbo is with Toyota Central R & D Labs., Inc., 480-1192 Japan. t-jmb@mosk.tytlabs.co.jp

978-1-4244-3872-3/09/$25.00 ©2009 IEEE

Suppose that there exist matrices A, B, C with appropriate sizes and also there exist *nonsingular* matrices $P_k = P_{\mathrm{mod}(k/N)} \in \boldsymbol{C}^{n\times n}$, $Q_k = Q_{\mathrm{mod}(k/N)} \in \boldsymbol{C}^{m\times m}$, $R_k = R_{\mathrm{mod}(k/N)} \in \boldsymbol{C}^{p\times p}$ such that for any $k \in \boldsymbol{Z}$,

$$P_{k+1}A_k = AP_k \ , \tag{3}$$

$$P_{k+1}B_k = BQ_k \ , \tag{4}$$

$$R_kC_k = CP_k \ . \tag{5}$$

Then the following equivalent transformations

$$\xi_k = P_kx_k,\ \tau_k = Q_ku_k,\ \eta_k = R_ky_k, \tag{6}$$

can transform the system (1) to a time-invariant system

$$\begin{array}{rcl} \xi_{k+1} & = & A\xi_k + B\tau_k \\ \eta_k & = & C\xi_k \ . \end{array} \tag{7}$$

This paper aims to make clear a necessary and sufficient condition for the system (1) to have A, B, C and $P_k = P_{\mathrm{mod}(k/N)}$, $Q_k = Q_{\mathrm{mod}(k/N)}$, $R_k = R_{\mathrm{mod}(k/N)}$ in (3)-(5). When the system (1) can be transformed to the time-invariant system (7), the system (1) is called to be *transformable to time-invariant* one (abbreviated as *TTI*).

A set of A and $P_k = P_{\mathrm{mod}(k/N)}$ satisfying (3) is denoted by $\mathcal{AP} = \{A, P_k = P_{\mathrm{mod}(k/N)} \mid k \in \boldsymbol{Z}\}$ and $\mathcal{AP}$ is called a *Floquet transformation*.

Associated with the system (1), recall that the following matrix Φ is called a monodromy matrix.

$$\Phi := A_{N-1}\cdots A_1A_0 \tag{8}$$

A necessary condition for the system (1) to have a Floquet transformation is that the monodromy matrix Φ has an N-th root matrix. In this sense, the N-th root matrices of Φ is very important [5] [6] [7] [8].

A necessary and sufficient condition for the system (1) to have a Floquet transformation has been derived in [12] and [6]. The following theorem is given in [6].

Theorem 1: [6] A discrete-time periodic system (1) with $A_k = A_{\mathrm{mod}(k/N)} \in R^{n\times n}$ and a period $N \geq 2$ has a Floquet transformation $\mathcal{AP} = \{A, P_k = P_{\mathrm{mod}(k/N)} \mid k \in \boldsymbol{Z}\}$ if and only if it holds that

$$\begin{array}{c} \mathrm{rank}A_{k-1}A_{k-2}\cdots A_{h+1}A_h = \mathrm{rank}A^{k-h} \\ \text{for } h \in \underline{N}^- \text{ and } k-h \in \underline{n} \end{array} \tag{9}$$

where $A \in \boldsymbol{C}^{n\times n}$ is any matrix similar to one of N-th roots of the monodromy Φ given by (8). □

Let the system (1) satisfy the rank condition (9) of Theorem 1, and suppose we have constructed a Floquet transformation $\mathcal{AP} = \{A, P_k = P_{\mathrm{mod}(k/N)} \mid k \in \boldsymbol{Z}\}$. Then it is easy to see [6] that the following theorem holds.

Theorem 2: [6] Suppose that the system (1) has a Floquet transformation $\mathcal{AP} = \{A, P_k = P_{\mathrm{mod}(k/N)} \mid k \in \boldsymbol{Z}\}$. Then there exist B, C and nonsingular $Q_k = Q_{\mathrm{mod}(k/N)}$, $R_k = R_{\mathrm{mod}(k/N)}$ satisfying (4),(5) if and only if it holds that for any $k \in \boldsymbol{Z}$,

$$\mathrm{Im}P_{k+1}B_k = \mathrm{Im}P_kB_{k-1} \tag{10}$$

$$\mathrm{Ker}C_kP_k^{-1} = \mathrm{Ker}C_{k-1}P_{k-1}^{-1}. \tag{11}$$

□

Theorem 2 with Theorem 1 seems to give a necessary and sufficient condition for the system (1) to be *TTI*, but you have to understand Theorem 2 very carefully. See the following example.

Example 1: Consider the following system with a period $N = 3$

$$\left\{ \begin{array}{rcl} x_{k+1} & = & A_kx_k + B_ku_k \\ y_k & = & C_kx_k \end{array} \right. \tag{12}$$

where

$$A_0 = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}, A_1 = \begin{bmatrix} 0.8 & 0 \\ 0 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 0 \\ 5 & 0 \end{bmatrix}$$

$$B_0 = \begin{bmatrix} 2 \\ 1 \end{bmatrix},\quad B_1 = \begin{bmatrix} 1.2 \\ 1 \end{bmatrix},\quad B_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$C_0 = \begin{bmatrix} 1 & 2 \end{bmatrix}, C_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}, C_2 = \begin{bmatrix} 5 & 3 \end{bmatrix}.$$

The monodromy matrix Φ is obtained as $\Phi = A_2A_1A_0 = \mathrm{diag}(0, 8)$. Choose $A = \mathrm{diag}(2, 0)$ as a cube root matrix of Φ and then it is easy to see that the rank condition (9) of Theorem 1 holds.

According to the way of constructing Floquet transformations (See [6] in detail), we can get a Floquet transformation $\mathcal{AP}_{11} = \{A, P_k = P_{\mathrm{mod}(k/3)} \mid k \in \boldsymbol{Z}\}$ where

$$P_0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, P_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, P_2 = \begin{bmatrix} 2.5 & 0 \\ 0 & 1 \end{bmatrix}.$$

Now let us check whether the condition (10) of Theorem 2 holds or not. In fact, we get

$$P_1B_0 = \begin{bmatrix} 2 \\ 1 \end{bmatrix},\quad P_2B_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix},\quad P_0B_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and therefore we conclude that the condition (10) does not hold and the system (12) is not *TTI* at least under the Floquet transformation $\mathcal{AP}_{11}$.

Note that the Floquet transformation is not unique for the system (12). The system has another Floquet transformation $\mathcal{AP}_{12} = \{A, P'_k = P'_{\mathrm{mod}(k/3)} \mid k \in \boldsymbol{Z}\}$ where

$$P'_0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, P'_1 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, P'_2 = \begin{bmatrix} 2.5 & 0 \\ 0 & 3 \end{bmatrix}.$$

In this case, it is easy to see that the conditions (10) and (11) of Theorem 2 hold. In fact, it follows that

$$P'_1B_0 = \begin{bmatrix} 2 \\ 2 \end{bmatrix},\quad P'_2B_1 = \begin{bmatrix} 3 \\ 3 \end{bmatrix},\quad P'_0B_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$C_0(P'_0)^{-1} = \begin{bmatrix} 2 & 1 \end{bmatrix},\quad C_1(P'_1)^{-1} = \begin{bmatrix} 1 & 0.5 \end{bmatrix}$$

$$C_2(P'_2)^{-1} = \begin{bmatrix} 2 & 1 \end{bmatrix}.$$

Therefore, by using $Q_0 = 0.5, Q_1 = 1/3, Q_2 = 1, Q_k = Q_{\mathrm{mod}(k/3)}$ and $R_0 = 1, R_1 = 2, R_2 = 1, R_k = R_{\mathrm{mod}(k/3)}$, we can get a time-invariant system (A, B, C) with

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 2 & 1 \end{bmatrix},$$

and so we can conclude that the system (12) is *TTI*. □

III. Main Results

Associated with the discrete-time periodic non-homogeneous system (1), if we can find a Floquet transformation by which the conditions (10) and (11) of Theorem 2 hold, then we can conclude that the system (1) is *TTI*. On the other hand, it is very difficult to conclude from Theorem 2 that the system (1) is not *TTI* because, as shown in the previous example and also in [7], the system (1) has an infinite number of Floquet transformations if it has.

Therefore, in order to handle the problem on *TTI*, it is first needed to understand a structure of the set of all Floquet transformations and also to characterize all the Floquet transformations.

Suppose the discrete-time periodic non-homogeneous system (1) has Floquet transformations, and let a set of all the Floquet transformations be denoted by $\mathcal{FT}$, i.e., every element of $\mathcal{FT}$ is a Floquet transformation $\mathcal{AP} := \{A, P_k = P_{\mathrm{mod}(k/N)} \mid k \in \boldsymbol{Z}\}$ with A,P_k's satisfying (3). Then we consider the following similarities between two Floquet transformations.

Definition 1: [7] Suppose $\mathcal{AP}, \mathcal{AP}' \in \mathcal{FT}$ with $\mathcal{AP} = \{A, P_k = P_{\mathrm{mod}(k/N)} \mid k \in \boldsymbol{Z}\}$ and $\mathcal{AP}' = \{A', P'_k = P'_{\mathrm{mod}(k/N)} \mid k \in \boldsymbol{Z}\}$.

1) $\mathcal{AP}'$ is *strongly similar* to $\mathcal{AP}$, denoted by $\mathcal{AP}' \simeq^s \mathcal{AP}$, when $A' \simeq A$ and there exist a nonsingular constant matrix $V \in \boldsymbol{C}^{n\times n}$ such that
$$P'_k = VP_k \tag{13}$$
2) $\mathcal{AP}'$ is *similar* to $\mathcal{AP}$, denoted by $\mathcal{AP}' \simeq \mathcal{AP}$, when $A' \simeq A$ and there exist nonsingular matrices $V_k = V_{\mathrm{mod}(k/N)} \in \boldsymbol{C}^{n\times n}$ such that
$$P'_k = V_kP_k \tag{14}$$
3) $\mathcal{AP}'$ is *weakly similar* to $\mathcal{AP}$, denoted by $\mathcal{AP}' \simeq^w \mathcal{AP}$, when $(A')^N \simeq A^N$ and there exist nonsingular matrices $V_k = V_{\mathrm{mod}(k/N)} \in \boldsymbol{C}^{n\times n}$ such that
$$P'_k = V_kP_k \tag{15}$$

□

Each similarity can induce an equivalence relation in $\mathcal{FT}$, and so denote its similarity class (i.e., equivalence class) respectively as follows.

$$\overline{\mathcal{AP}}^s := \{\mathcal{AP}' \in \mathcal{FT} \mid \mathcal{AP}' \simeq^s \mathcal{AP}\} \tag{16}$$
$$\overline{\mathcal{AP}} := \{\mathcal{AP}' \in \mathcal{FT} \mid \mathcal{AP}' \simeq \mathcal{AP}\} \tag{17}$$
$$\overline{\mathcal{AP}}^w := \{\mathcal{AP}' \in \mathcal{FT} \mid \mathcal{AP}' \simeq^w \mathcal{AP}\} \tag{18}$$

Under the above definitions, it is easy to see [7] that $\overline{\mathcal{AP}}^s \subset \overline{\mathcal{AP}} \subset \overline{\mathcal{AP}}^w = \mathcal{FT}$.

Define
$$\mathcal{A} := \{A \in \boldsymbol{C}^{n\times n} \mid A^N \simeq \Phi,\ A \text{ satisfies (9)}\}\ , \tag{19}$$
recalling that $\mathcal{A}$ has a finite number n_e of similarity classes $\mathcal{A}_i$'s, i.e., $\mathcal{A} = \bigcup_{i\in\underline{n_e}} \mathcal{A}_i$, $\mathcal{FT}$ also splits into n_e similarity classes as follows [7].
$$\mathcal{FT} = \bigcup_{i\in\underline{n_e}} \overline{\mathcal{AP}}_i, \quad \overline{\mathcal{AP}}_i \bigcap \overline{\mathcal{AP}}_j = \phi \text{ for } i \neq j \tag{20}$$

Figure 1 shows the structure of $\mathcal{FT}$; there exist a finite number of similarity classes $\overline{\mathcal{AP}}_1, \overline{\mathcal{AP}}_2, \cdots, \overline{\mathcal{AP}}_{n_e}$. Each similarity class $\overline{\mathcal{AP}_i}$ consists of an infinite number of strong similarity classes $\overline{\mathcal{AP}}^s_{i1}, \overline{\mathcal{AP}}^s_{i2}, \cdots$.

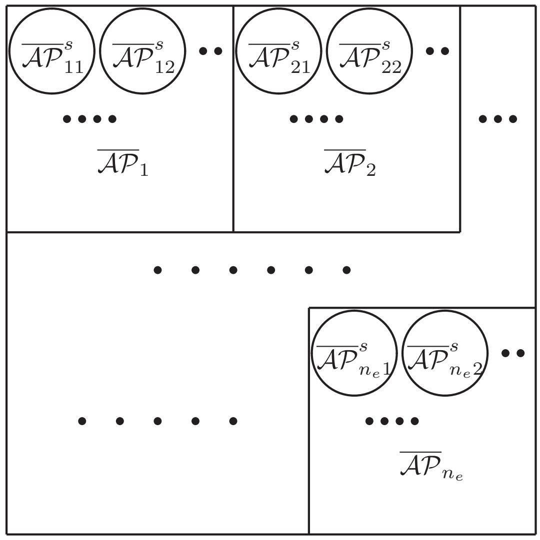

Fig. 1. The structure of $\mathcal{FT}$

Now suppose we obtain a Floquet transformation, e.g., $\mathcal{AP} = \{A, P_k = P_{\mathrm{mod}(k/N)} \mid k \in \boldsymbol{Z}\} \in \overline{\mathcal{AP}}^s_{11}$. Beginning with $\mathcal{AP}$, a way of constructing all Floquet transformations of $\overline{\mathcal{AP}}^s_{11}$, $\overline{\mathcal{AP}}_1$, and all the other $\overline{\mathcal{AP}}_i$'s is given as follows. Before showing the way, we need some notations.

Associated with $A, A' \in \mathcal{A}$, a set of all nonsingular and commutative matrices S's with A and A' is denoted as $\mathcal{C}(A, A')$, i.e.,
$$\mathcal{C}(A, A') := \left\{ S \in \boldsymbol{C}^{n\times n} : \text{nonsingular} \;\middle|\; SA = A'S \right\}. \tag{21}$$
It is trivial that $\mathcal{C}(A, A')$ is not empty if and only if $A \simeq A'$. When $A = A'$, we use a simple notation $\mathcal{C}(A)$ for $\mathcal{C}(A, A)$.

A set of all N *periodic sequences* $\mathcal{S}$'s *of commutative matrices* with $A, A' \in \mathcal{A}$ is denoted as $\mathcal{SC}(A, A', N)$, i.e.,
$$\mathcal{S} := \{S_k = S_{\mathrm{mod}(k/N)} \in \boldsymbol{C}^{n\times n} \mid k \in \boldsymbol{Z}\} \in \mathcal{SC}(A, A', N)$$
where every S_k is nonsingular and satisfies
$$S_{k+1}A = A'S_k\ . \tag{22}$$
It is easy to verify that $\mathcal{SC}(A, A', N)$ is not empty if and only if $A^N \simeq (A')^N$ because (22) implies $S_kA^N = (A')^NS_k$. When $A = A'$, we also use a simple notation $\mathcal{SC}(A, N)$ for $\mathcal{SC}(A, A, N)$. Therefore, $\mathcal{W} \in \mathcal{SC}(A, N)$ means that $\mathcal{W}$ is a set of nonsingular matrices $W_k = W_{\mathrm{mod}(k/N)}$ with
$$W_{k+1}A = AW_k. \tag{23}$$

Note that $\mathcal{C}(A,A') = \mathcal{SC}(A,A',1)$. For any $W \in \mathcal{C}(A)$, define $\mathcal{W} = \{W_k = W \mid k \in \boldsymbol{Z}\}$, then it is easy to see that $\mathcal{W} \in \mathcal{SC}(A,N)$ for any positive $N \in \boldsymbol{Z}$. In this sense, we could say that $\mathcal{C}(A) \subset \mathcal{SC}(A,N)$. This fact holds for $\mathcal{C}(A,A')$ and $\mathcal{SC}(A,A',N)$, and so we could also say that $\mathcal{C}(A,A') \subset \mathcal{SC}(A,A',N)$.

The next theorem characterizes each similarity class by using $\mathcal{C}(A)$, $\mathcal{SC}(A,N)$, $\mathcal{C}(A,A')$ and $\mathcal{SC}(A,A',N)$.

Theorem 3: Suppose $\mathcal{AP} = \{A, P_k = P_{\mathrm{mod}(k/N)} \mid k \in \boldsymbol{Z}\} \in \mathcal{FT}$.

1) $\mathcal{AP}' = \{A', P'_k = P'_{\mathrm{mod}(k/N)} \mid k \in \boldsymbol{Z}\} \in \overline{\mathcal{AP}}^s$ if and only if there exists an $S \in \mathcal{C}(A,A')$ such that
$$P'_k = SP_k\ . \tag{24}$$
2) $\mathcal{AP}' = \{A', P'_k = P'_{\mathrm{mod}(k/N)} \mid k \in \boldsymbol{Z}\} \in \overline{\mathcal{AP}}$ if and only if there exist $S \in \mathcal{C}(A,A')$ and $\mathcal{W} = \{W_k = W_{\mathrm{mod}(k/N)} \mid k \in \boldsymbol{Z}\} \in \mathcal{SC}(A,N)$ such that
$$P'_k = SW_kP_k\ . \tag{25}$$
3) $\mathcal{AP}' = \{A', P'_k = P'_{\mathrm{mod}(k/N)} \mid k \in \boldsymbol{Z}\} \in \overline{\mathcal{AP}}^w$ if and only if there exist an $\mathcal{S} = \{S_k = S_{\mathrm{mod}(k/N)} \mid k \in \boldsymbol{Z}\} \in \mathcal{SC}(A,A',N)$ such that
$$P'_k = S_kP_k\ . \tag{26}$$

□

(Proof)
1) Instead of (24), the item 1) has been given as $P'_k = SWP_k$ with $S \in \mathcal{C}(A,A')$ and $W \in \mathcal{C}(A)$ in [7]. Note that $SW \in \mathcal{C}(A,A')$ because it holds that $(SW)A = SWA = SAW = A'SW = A'(SW)$.
2) This item is just the same as given in [7].
3) **Necessity part:** Note that $P_{k+1}A_k = AP_k$ and $P'_{k+1}A_k = A'P'_k$. Therefore it follows that $(P'_{k+1})^{-1}A'P'_k = A_k = P^{-1}_{k+1}AP_k$. Define $S_k := P'_kP_k^{-1}$, it is trivial that (26) holds and also we obtain $A' = S_{k+1}AS_k^{-1}$, which means $\mathcal{S} := \{S_k = S_{mod(k/N)} \mid k \in \boldsymbol{Z}\} \in \mathcal{SC}(A,A',N)$.
Sufficient part: Existence of $\mathcal{S} \in \mathcal{SC}(A,A',N)$ implies that $(A')^N \simeq A^N$ because $(A')^N = (S_NAS^{-1}_{N-1})\cdots(S_2AS_1^{-1})(S_1AS_0^{-1})$ with $S_N = S_0$. Therefore this and (26) mean $\mathcal{AP}' \in \overline{\mathcal{AP}}^w$.

(Q.E.D.)

The above theorem means that $\mathcal{C}(A,A')$ and $\mathcal{SC}(A,A',N)$ are very important to construct all Floquet transformations of $\overline{\mathcal{AP}}^s_{11}$, $\overline{\mathcal{AP}}_1$, and all the other $\overline{\mathcal{AP}}_i$'s.

The set $\mathcal{C}(A,A')$ is well known, so here only the set $\mathcal{SC}(A,A',N)$ is summarized below. Note that $\mathcal{SC}(A,N) = \mathcal{SC}(A,A,N)$.

The following lemma is trivial.

Lemma 1: Suppose that J_A and $J_{A'}$ be the Jordan forms of A and A' respectively and also suppose that $S \in \mathcal{C}(J_A,A)$ and $S' \in \mathcal{C}(J_{A'},A')$.

1) If $\mathcal{W} = \{W_k = W_{\mathrm{mod}(k/N)} \mid k \in \boldsymbol{Z}\} \in \mathcal{SC}(J_A,N)$, then $\mathcal{W}' = \{SW_kS^{-1} \mid k \in \boldsymbol{Z}\} \in \mathcal{SC}(A,N)$.
2) If $\mathcal{S} = \{S_k = S_{\mathrm{mod}(k/N)} \mid k \in \boldsymbol{Z}\} \in \mathcal{SC}(J_A,J_{A'},N)$, then $\mathcal{S}' = \{S'S_kS^{-1} \mid k \in \boldsymbol{Z}\} \in \mathcal{SC}(A,A',N)$.

□

Therefore, in order to characterize $\mathcal{SC}(A,N)$ and $\mathcal{SC}(A,A',N)$, without loss of generality, it can be assumed that A and A' are in the Jordan forms.

Associated with $A, A' \in \mathcal{A}$, suppose that
$$J_A = J_{A\sigma} \oplus J_\nu \quad J_{A'} = J_{A'\sigma} \oplus J_\nu \tag{27}$$
where $J_{A\sigma}, J_{A'\sigma} \in \boldsymbol{C}^{\bar{n}\times\bar{n}}$ are nonsingular, and J_ν is nilpotent. Recall Theorem 1, which claims that $A^N \simeq (A')^N \simeq \Phi$, i.e., $J_A^N \simeq J_{A'}^N \simeq \Phi$. And also notice that J_A and $J_{A'}$ satisfy the rank condition (9) of Theorem 1, which is the reason why we could assume that J_A and $J_{A'}$ have the same nilpotent J_ν in (27).

Lemma 2: Suppose that J_A and $J_{A'}$ are given by (27).

1) Every $\mathcal{S} = \{S_k = S_{\mathrm{mod}(k/N)} \mid k \in \boldsymbol{Z}\} \in \mathcal{SC}(J_A,J_{A'},N)$ is given by
$$S_k = S_{\sigma,k} \oplus W_{\nu,k} \tag{28}$$
where $\mathcal{S}_\sigma := \{S_{\sigma,k} = S_{\sigma,\mathrm{mod}(k/N)} \mid k \in \boldsymbol{Z}\} \in \mathcal{SC}(J_{A_\sigma},J_{A'_\sigma},N)$ and $\mathcal{W}_\nu := \{W_{\nu,k} = W_{\nu,\mathrm{mod}(k/N)} \mid k \in \boldsymbol{Z}\} \in \mathcal{SC}(J_\nu,N)$.
2) If $\mathcal{S} := \{S_k = S_{\mathrm{mod}(k/N)} \mid k \in \boldsymbol{Z}\} \in \mathcal{SC}(J_{A_\sigma},J_{A'_\sigma},N)$, then each S_k is given by
$$S_k = J_{A'_\sigma}^{\mathrm{mod}(k/N)}S_0J_{A_\sigma}^{-\mathrm{mod}(k/N)} \tag{29}$$
with any $S_0 \in \mathcal{C}(J^N_{A_\sigma}, J^N_{A'_\sigma})$.
3) Suppose that J_ν is given as $J_\nu = J_{m_1}(0)\oplus\cdots\oplus J_{m_q}(0)$ with $m_1 \geq \cdots \geq m_q$. Then every $\mathcal{W} := \{W_k = W_{\mathrm{mod}(k/N)} \mid k \in \boldsymbol{Z}\} \in \mathcal{SC}(J_\nu,N)$ is given by
$$W_k = \begin{bmatrix} W_{1,1,k} & W_{1,2,k} & \cdots & W_{1,q,k} \\ W_{2,1,k} & W_{2,2,k} & \cdots & W_{2,q,k} \\ \vdots & \vdots & \ddots & \vdots \\ W_{q,1,k} & W_{q,2,k} & \cdots & W_{q,q,k} \end{bmatrix} \tag{30}$$
where $W_{i,j,k} \in \boldsymbol{C}^{m_i\times m_j}$ is given as
$$W_{i,j,k} = \begin{cases} \begin{bmatrix} E_{m_j,k} \\ 0_{(m_i-m_j)\times m_j} \end{bmatrix} & \text{for } i<j \\ E_{m_i,k} & \text{for } i=j \\ \begin{bmatrix} 0_{m_i\times(m_j-m_i)} & E_{m_i,k} \end{bmatrix} & \text{for } i>j \end{cases}$$
$$E_{m,k} = \begin{bmatrix} e_{1,k} & e_{2,k} & e_{3,k} & \cdots & e_{m,k} \\ 0 & e_{1,k+1} & e_{2,k+1} & \cdots & e_{m-1,k+1} \\ 0 & 0 & e_{1,k+2} & \cdots & e_{m-2,k+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & e_{1,k+m-1} \end{bmatrix}$$
$$e_{j,k} = e_{j,\mathrm{mod}(k/N)} \quad \text{for } k \in \boldsymbol{Z},\ j \in \underline{m}$$
and also the parameters $e_{j,k}$'s must be chosen such that W_k is nonsingular.

□

The proof of this lemma can be given as almost same ways of the proof for Lemmas 5 and 6 in [7], and so it is omitted here.

The following theorem is derived easily from Theorem 2 and Theorem 3.

Theorem 4: The discrete-time periodic non-homogeneous system (1) is *TTI* if and only if

1) There exists a Floquet transformation $\mathcal{AP} = \{A, P_k = P_{\mathrm{mod}(k/N)} \mid k \in \boldsymbol{Z}\}$.
2) There exists $\mathcal{S} = \{S_k = S_{\mathrm{mod}(k/N)} \mid k \in \boldsymbol{Z}\} \in \bigcup_{A' \in \mathcal{A}} \mathcal{SC}(A, A', N)$ such that

$$\begin{aligned} \mathrm{Im} S_{k+1} P_{k+1} B_k &= \mathrm{Im} S_k P_k B_{k-1} & (31)\\ \mathrm{Ker} C_k P_k^{-1} S_k^{-1} &= \mathrm{Ker} C_{k-1} P_{k-1}^{-1} S_{k-1}^{-1} & (32) \end{aligned}$$

□

Note that the Floquet transformations $\mathcal{AP}_{11}$ and $\mathcal{AP}_{12}$ in the example 1 are similar in the sense of Definition 1, i.e., $\mathcal{AP}_{11}, \mathcal{AP}_{12} \in \overline{\mathcal{AP}}_1$. Therefore the example 1 shows that a *similarity* class $\overline{\mathcal{AP}}$ could include the Floquet transformation by which either (10) or (11) of Theorem 2 does not hold even if the system (1) is *TTI*.

Beside the above observations, the following theorem is very interesting.

Theorem 5: Suppose that the system (1) has a Floquet transformation $\mathcal{AP}$ by which Theorem 2 holds. Then the conditions (10) and (11) of Theorem 2 hold under all Floquet transformations in the *strong similarity* class $\overline{\mathcal{AP}}^s$. □

(Proof) Suppose that $\mathcal{AP} = \{A, P_k = P_{\mathrm{mod}(k/N)} \mid k \in \boldsymbol{Z}\}$ satisfies (10) and (11), i.e.,

$$\begin{aligned} \mathrm{Im} P_{k+1} B_k &= \mathrm{Im} P_k B_{k-1}\\ \mathrm{Ker} C_k P_k^{-1} &= \mathrm{Ker} C_{k-1} P_{k-1}^{-1}. \end{aligned}$$

Let $\mathcal{AP}' = \{A', P'_k = P'_{\mathrm{mod}(k/N)} \mid k \in \boldsymbol{Z}\}$ be strongly similar to $\mathcal{AP}$. From Theorem 3, there exists an $S \in \mathcal{C}(A, A')$ such that $P'_k = S P_k$. Therefore it follows that

$$\begin{aligned} &\mathrm{Im} P'_{k+1} B_k = \mathrm{Im} S P_{k+1} B_k = S \mathrm{Im} P_{k+1} B_k\\ &\quad = S \mathrm{Im} P_k B_{k-1} = \mathrm{Im} S P_k B_{k-1} = \mathrm{Im} P'_k B_{k-1}\\ &\mathrm{Ker} C_k (P'_k)^{-1} = \mathrm{Ker} C_k (S P_k)^{-1} = \mathrm{Ker} C_k P_k^{-1} S^{-1}\\ &\quad = S \mathrm{Ker} C_k P_k^{-1} = S \mathrm{Ker} C_{k-1} P_{k-1}^{-1}\\ &\quad = \mathrm{Ker} C_{k-1} P_{k-1}^{-1} S^{-1} = \mathrm{Ker} C_{k-1} (S P_{k-1})^{-1}\\ &\quad = \mathrm{Ker} C_{k-1} (P'_{k-1})^{-1} \end{aligned}$$

which means that the conditions (10) and (11) hold for $\mathcal{AP}'$.
(Q.E.D.)

Example 2: Consider the system (12) again. We can construct another new Floquet transformation $\mathcal{AP}_{21} = \{A'', P''_k = P''_{\mathrm{mod}(k/N)} \mid k \in \boldsymbol{Z}\}$, which is not similar to $\mathcal{AP}_{11}$ and $\mathcal{AP}_{12}$ given in the example 1, as follows.

$$A'' = \begin{bmatrix} 2e^{j2\pi/3} & 0 \\ 0 & 0 \end{bmatrix}, \quad P''_0 = \begin{bmatrix} 0 & 1 \\ 0.5 & 0 \end{bmatrix},$$
$$P''_1 = \begin{bmatrix} e^{j2\pi/3} & 0 \\ 0 & e^{j2\pi/3} \end{bmatrix}, P''_2 = \begin{bmatrix} 2.5e^{j4\pi/3} & 0 \\ 0 & 1.5e^{j4\pi/3} \end{bmatrix}$$

It is easy to see that $\mathcal{AP}_{21}$ satisfies Theorem 2 as well as $\mathcal{AP}_{12}$. In fact,

$$P''_1 B_0 = e^{j2\pi/3} \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad P''_2 B_1 = e^{j4\pi/3} \begin{bmatrix} 3 \\ 1.5 \end{bmatrix},$$
$$P''_0 B_2 = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix},$$
$$C_0 (P''_0)^{-1} = \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad C_1 (P''_1)^{-1} = e^{-j2\pi/3} \begin{bmatrix} 1 & 1 \end{bmatrix}$$
$$C_2 (P''_2)^{-1} = e^{-j4\pi/3} \begin{bmatrix} 2 & 2 \end{bmatrix}.$$

Let the similarity classes to $\mathcal{AP}_{12}$ and $\mathcal{AP}_{21}$ be denoted as $\overline{\mathcal{AP}}_1$ and $\overline{\mathcal{AP}}_2$, respectively. Then we can construct the third Floquet transformation $\mathcal{AP}_{31} = \{A''', P'''_k = P'''_{\mathrm{mod}(k/N)} \mid k \in \boldsymbol{Z}\}$, which is not included in either $\overline{\mathcal{AP}}_1$ or $\overline{\mathcal{AP}}_2$, as follows.

$$A''' = \begin{bmatrix} 2e^{j4\pi/3} & 0 \\ 0 & 0 \end{bmatrix}, \quad P'''_0 = \begin{bmatrix} 0 & 1 \\ 0.5e^{j4\pi/3} & 0 \end{bmatrix},$$
$$P'''_1 = \begin{bmatrix} e^{j4\pi/3} & 0 \\ 0 & 1 \end{bmatrix}, P'''_2 = \begin{bmatrix} 2.5e^{j2\pi/3} & 0 \\ 0 & 1.5e^{j4\pi/3} \end{bmatrix}$$

And it is easy to see that $\mathcal{AP}_{31}$ satisfies Theorem 2 as well.
□

From the examples 1 and 2, we can see that the set of Floquet transformations for the system (12) splits three similarity classes $\overline{\mathcal{AP}}_1, \overline{\mathcal{AP}}_2$ and $\overline{\mathcal{AP}}_3$ and each similarity class includes at least one Floquet transformation by which Theorem 2 holds.

The above observation would give us the following conjecture; when a discrete-time periodic system (1) is *TTI*, every *similarity class* of Floquet transformations has at least a Floquet transformation by which the conditions (10) and (11) of Theorem 2 hold.

But this conjecture is not true, which will be shown later.

As a case study, consider a system (1) with the following assumptions.

A1) All A_k's are nonsingular.
A2) The monodromy matrix Φ has n eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$, which are mutually different.
A3) Denote N-th roots of λ_i by $\gamma_{i1}, \gamma_{i2}, \cdots, \gamma_{iN}$. Then all γ_{im} $(i \in \underline{n},\ m \in \underline{N})$ are mutually different.

The following theorem claims that the structure of $\mathcal{FT}$ is very simple under the above assumptions.

Theorem 6: Consider a system (1) with the assumptions A1)-A3). Then the system has Floquet transformations and the set $\mathcal{FT}$ consists of N^n similarity classes $\overline{\mathcal{AP}}_i$, $i \in \underline{N^n}$.

Furthermore, every similarity class $\overline{\mathcal{AP}}_i$ consists of only one strong similarity class $\overline{\mathcal{AP}}^s_{i1}$, i.e., $\overline{\mathcal{AP}}_i = \overline{\mathcal{AP}}^s_{i1}$. □

(Proof) Under the assumptions A1) -A3), it is easy to see that Φ has N^n similarity classes of N-th roots and the rank condition (9) of Theorem 1 holds. Therefore the system (1) has Floquet transformations.

The set $\mathcal{A}$ defined in (19) has N^n similarity classes whose representatives are given by

$$J_{m_1, m_2, \cdots, m_n} = \mathrm{diag}(\gamma_{1m_1}, \gamma_{2m_2}, \cdots, \gamma_{nm_n}) \quad (33)$$

where $m_1, m_2, \cdots, m_n \in \underline{N}$.

It is easy to verify from the assumption A3) that $\mathcal{SC}(J_{m_1,m_2,\cdots,m_n}, N) = \mathcal{C}(J_{m_1,m_2,\cdots,m_n})$. This fact means $\overline{\mathcal{AP}} = \overline{\mathcal{AP}}^s$. (Q.E.D.)

From Theorem 5 and Theorem 6, we can see that in order to decide whether the system (1) with A1)-A3) is *TTI* or not, we have to check the conditions (10) and (11) at most for N^n Floquet transformations.

As we mentioned before, it would be conjectured from the above examples 1 and 2 that when a discrete-time periodic system (1) is *TTI*, every *similarity class* of Floquet transformations has at least a Floquet transformation by which the conditions (10) and (11) of Theorem 2 hold. But, understanding Theorems 5 and 6, the following example shows that the conjecture is not true.

Example 3: Consider the following system with a period $N = 3$

$$\begin{cases} x_{k+1} &= A_k x_k + B_k u_k \\ y_k &= C_k x_k \end{cases} \tag{34}$$

where

$$A_0 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, A_1 = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} -4 & 0 \\ 4 & 1 \end{bmatrix}$$

$$B_0 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 9 \\ -14 \end{bmatrix}, \quad B_2 = \begin{bmatrix} -13 \\ 6 \end{bmatrix}$$

$$C_0 = \begin{bmatrix} 7 & 0 \end{bmatrix}, C_1 = \begin{bmatrix} 6 & -14 \end{bmatrix}, C_2 = \begin{bmatrix} -14 & 4 \end{bmatrix}.$$

It is easy to see that the system satisfies the assumptions A1)-A3).

The monodromy matrix is given as

$$\Phi = A_2 A_1 A_0 = \begin{bmatrix} 0 & -8 \\ 1 & 9 \end{bmatrix}$$

and the rank condition (9) of Theorem 1 holds. Therefore we can get a Floquet transformation $\mathcal{AP} = \{A, P_k = P_{\mathrm{mod}(k/3)} \mid k \in \boldsymbol{Z}\} \in \overline{\mathcal{AP}}_1$ where

$$P_0 = \begin{bmatrix} 1 & 1 \\ 1 & 8 \end{bmatrix}, P_1 = \begin{bmatrix} 1 & 0 \\ 2 & 14 \end{bmatrix}, P_2 = \begin{bmatrix} 0 & 1 \\ 14 & 4 \end{bmatrix}.$$

By this $\mathcal{AP}$ the conditions (10) and (11) of Theorem 2 hold.

Consider another similarity class $\overline{\mathcal{AP}}_2$ which has a Floquet transformation $\mathcal{AP}' = \{A', P'_k = P'_{\mathrm{mod}(k/3)} \mid k \in \boldsymbol{Z}\}$ where $A' = \mathrm{diag}(e^{j2\pi/3}, 2e^{j4\pi/3})$ and

$$P'_0 = \begin{bmatrix} 1 & 1 \\ 1 & 8 \end{bmatrix}, \quad P'_1 = \begin{bmatrix} e^{j2\pi/3} & 0 \\ 2e^{j4\pi/3} & 14e^{j4\pi/3} \end{bmatrix}$$

$$P'_2 = \begin{bmatrix} 0 & e^{j4\pi/3} \\ 14e^{j2\pi/3} & 4e^{j2\pi/3} \end{bmatrix}.$$

In this case we get

$$P'_1 B_0 = 2 \begin{bmatrix} e^{j2\pi/3} \\ -5e^{j4\pi/3} \end{bmatrix}, \quad P'_2 B_1 = -14 \begin{bmatrix} e^{j4\pi/3} \\ -5e^{j2\pi/3} \end{bmatrix}$$

$$P'_0 B_2 = -7 \begin{bmatrix} 1 \\ -5 \end{bmatrix}$$

and therefore by this $\mathcal{AP}'$ the condition (10) does not hold.

Therefore, from Theorems 5 and 6, it follows that the similarity class $\overline{\mathcal{AP}}_2$ has no Floquet transformation by which Theorem 2 does not hold even though Theorem 2 holds under the similarity class $\overline{\mathcal{AP}}_1$, and we can see that the above conjecture is not true. □

IV. CONCLUSION

This paper has considered a condition for discrete-time periodic *non-homogeneous* linear systems to be transformed to time-invariant ones. Because the condition depends on Floquet transformations, before using the condition, you have to know all Floquet transformations and its structure, where a concept of similarity classes and some special sets of commutative matrices play important roles.

When the condition is used, it has been shown that every Floquet transformation in a strong similarity class gives the same conclusion whereas not in a similarity class.

In the case that the monodromy matrix is nonsingular with diagonalizable N-th roots, the structure of all Floquet transformations is very simple; similarity is reduced to strong similarity. Therefore, in order to get a conclusion, it is enough to check the condition for a finite number of Floquet transformations.

The future researches are to consider the same problems from the view point of *impulse responses* of the system and to extend all the results to the case of *nonlinear* systems.

REFERENCES

[1] R.W. Brockett, "Finite Dimensional Linear System", *John Wiley and Sons,inc.*, p46-48, 1970.

[2] J.Chauvin, N.Petit, P.Rouchon and G.Corde, "Periodic Input Observer Design: Application for Imbalance Diagnosis", *2006 SAE World Congress*, 2006-01-0181, 2006.

[3] J. Chauvin, G. Corde, N. Petit and P.Rouchon, "Periodic input estimation for linear periodic systems: Automotive engine applications", *Automatica*, V0l.43, pp971-980, 2007.

[4] C.T. Chen, "Linear System Theory and Design", *Holt, Rinehart and Winston*, p153-154, 1984.

[5] F.R. Gantmacher, "The Theory of Matrices", Vol.1, *Chelsea*, New York, Chapter VIII, 1959.

[6] Y. Hayakawa and T. Jimbo, "Floquet Transformations for Discrete-time Systems: Equivalence between periodic systems and time-invariant ones", *the 47th IEEE CDC*, pp5140-5145, 2008.

[7] Y. Hayakawa and T. Jimbo, "On Similarity Classes of Discrete-time Floquet Transformations", *2009 American Control Conference*, pp3757-3763, 2009.

[8] G.A. Horn and C.R. Johnson, "Topics in Matrix Analysis", *Cambridge University Press*, Chapter 6, 1991.

[9] T. Jimbo and Y. Hayakawa, "Physical-Model-Based Control of Engine Cold Start via Role State Variables", *17th World Congress of IFAC*(Seoul,Korea), pp1024-1029, 2008.

[10] P. Montagnier and R.J. Spiteri, "A Gramian-Based Controller for Linear Periodic Systems", *IEEE Trans. Automat. Contr.*, Vol.49, No.8, pp1380-1385, 2004.

[11] D.Peaucelle, Y. Ebihara and D. Arzelier, "Robust H_2 performance of discrete-time periodic systems: LMIs with reduced dimension", *17th World Congress of IFAC*(Seoul,Korea), pp1348-1353, 2008.

[12] P. Van Dooren and J. Sreedhar, "When is a Periodic Discrete-Time System Equivalent to a Time-Invariant One?", *Linear Algebra and its Applications*, 212/213, pp131-151, 1994.

[13] J.J. Yamé and R.Hanus, "On Stabilization and Spectrum Assignment in Periodically Time-Varying Continuous-Time Systems", *IEEE Trans. Automat. Contr.*, Vol.46, No.6, pp979-983, 2001.