A new method of producing functional relations among multiple zeta-functions

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Abstract

In this paper, we introduce a new method of producing functional relations among multiple zeta-functions. This method can be regarded as a kind of multiple analogue of Hardy's one of proving the functional equation for the Riemann zeta-function. Using this method, we give new functional relations for multiple zeta-functions. In particular, substituting positive integers into variables of them, we obtain known relation formulas for the multiple zeta values.

1. Introduction

Let $\mathbb{N}$ be the set of natural numbers, $\mathbb{Z}$ the ring of rational integers, $\mathbb{Q}$ the field of rational numbers, $\mathbb{R}$ the field of real numbers and $\mathbb{C}$ the field of complex numbers.

The Euler-Zagier multiple zeta-function of depth r defined by

$$\zeta_{EZ,r}(s_1, s_2, \ldots, s_r) = \sum_{m_1,\ldots,m_r=1}^{\infty} \frac{1}{m_1^{s_1}(m_1+m_2)^{s_2}\cdots(m_1+\cdots+m_r)^{s_r}} \tag{1.1}$$

can be meromorphically continued to the whole complex space $\mathbb{C}^r$ (see, for example, [1, 10, 24]). The origin of this function goes back to Euler. Indeed, Euler studied the values of double zeta-function, and gave some fascinating relation formulas for them. For example, he proved $\zeta_{EZ,2}(1, 2) = \zeta(3)$, and furthermore the sum formulas for double zeta values

$$\sum_{j=2}^{k-1} \zeta_{EZ,2}(k-j, j) = \zeta(k) \quad (k \geq 3), \tag{1.2}$$

where $\zeta(s)$ is the Riemann zeta-function (see [5]).

In early 1990's, Zagier and Hoffman studied the values of (1.1) at positive integers independently, and gave some relation formulas for them (see [7, 23]). Following their works, a lot of mathematicians have given various relation formulas for these values. For details, see [3, 4].

On the other hand, from analytic point of view, we would like to consider the following problem which has already been presented by the first author (see [12]).

Problem. *Are the known relation formulas for multiple zeta values valid only at positive integer points, or valid also at other values?*

It seems to be difficult to give the answer to this problem, except for the trivial relation

$$\zeta(s_1)\zeta(s_2) = \zeta_{EZ,2}(s_1, s_2) + \zeta_{EZ,2}(s_2, s_1) + \zeta(s_1 + s_2)$$

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which is valid for not only positive integers but also complex numbers.

In order to think about this problem, we consider the r-ple Mordell-Tornheim zeta-function defined by

$$\zeta_{MT,r}(s_1,\ldots,s_r,s_{r+1})=\sum_{m_1,\ldots,m_r=1}^{\infty}\frac{1}{m_1^{s_1}\cdots m_r^{s_r}(m_1+\cdots+m_r)^{s_{r+1}}}. \tag{1.3}$$

The values of $\zeta_{MT,2}$ at positive integers were originally studied by Tornheim and Mordell (see [14, 16]). As its generalization, $\zeta_{MT,r}$ was defined and was meromorphically continued to $\mathbb{C}^{r+1}$ by the first author (see [8, 9]). Recently, in [19, 20, 21], the second author has given some functional relations for $\zeta_{MT,2}(s_1,s_2,s_3)$. For example,

$$\zeta_{EZ,2}(1,s+1)-\zeta_{MT,2}(s,1,1)+\zeta(s+2)=0 \tag{1.4}$$

holds for $s\in\mathbb{C}$ except for singularities of three functions on the left-hand side of (1.4). Let $s=1$ in (1.4). Since we can easily check that $\zeta_{MT,2}(1,1,1)=2\zeta_{EZ,2}(1,2)$ from the decomposition into partial fractions, we see that (1.4) in the case $s=1$ gives Euler's formula $\zeta_{EZ,2}(1,2)=\zeta(3)$. In other words, (1.4) can be regarded as a continuous generalization of Euler's formula. Furthermore we can see that (1.4) in the case $s=k(\in\mathbb{N})$ gives the sum formula for double zeta values (1.2) (see Corollaries 2.5 and 2.6 below). Thus (1.4) is a kind of concrete example of the answer to the above problem. By using this method, we gave certain functional relations for Witten zeta-functions associated with $\mathfrak{sl}(4)$, and gave new relation formulas for the values of these functions at positive integers (see [13]). However this method is actually complicated. Hence it seems to be hard to apply this method to general multiple zeta-functions.

In the present paper, we introduce a new method of finding functional relations among multiple zeta-functions (see Theorem 3.1). This method can be regarded as a kind of multiple analogue of Hardy's one ([6], see also [15] Section 2.2) of proving the functional equation for the Riemann zeta-function. Using this method, we give another proof of (1.4) (see Proposition 2.4). Furthermore, as multiple analogues of (1.4), we give non-trivial functional relations for certain multiple zeta-functions (see Proposition 5.3). For example, we give a new functional relation

$$\begin{aligned}2\zeta_{EZ,3}(1,1,s+1)-\zeta_{MT,3}(s,1,1,1)+2\zeta_{MT,2}(1,2,s)&\\ +2\zeta_{MT,2}(s,2,1)-2\zeta(2)\zeta(s+1)+4\zeta(s+3)&=0\end{aligned} \tag{1.5}$$

for $s\in\mathbb{C}$ except for singularities of six functions on the left-hand side of (1.5) (see Proposition 5.4). In particular, (1.5) in the case $s=1$ gives the well-known relation $\zeta_{EZ,3}(1,1,2)=\zeta(4)$ obtained in [7].

Using our method, we will give more functional relations for various multiple zeta-functions in our next papers.

2. Functional relations for double zeta-functions

In this section, we give another proof of (1.4). The following proof is a simple example of our method. Since the following proof includes the essence of the method, we present it before proceeding to the treatment of the general case.

We make use of the notation and quote some results in [17, 18]. We fix a $\delta\in\mathbb{R}$ with $\delta>0$. For any $u\in\mathbb{R}$ with $1\le u\le 1+\delta$, we define

$$\phi(s;u)=\sum_{m=1}^{\infty}\frac{(-u)^{-m}}{m^s}\qquad (s\in\mathbb{C}). \tag{2.1}$$

If $u > 1$ then $\phi(s;u)$ is convergent for any $s \in \mathbb{C}$. In the case when $u = 1$, we have $\phi(s;1) = (2^{1-s} - 1)\zeta(s)$. Corresponding to $\phi(s;u)$, we define a set of numbers $\{\mathcal{E}_m(u)\}$ by

$$F(x;u) = \frac{(1+u)e^x}{e^x + u} = \sum_{n=0}^{\infty} \mathcal{E}_n(u)\ \frac{x^n}{n!} \quad (|x| < \pi). \tag{2.2}$$

We see that

$$\mathcal{E}_{2j}(1) = 0 \quad (j \in \mathbb{N}) \tag{2.3}$$

and that $\phi(s;u)$ can be meromorphically continued to $\mathbb{C}$ and

$$\phi(-k;u) = -\frac{1}{1+u}\mathcal{E}_k(u) \quad (k \in \mathbb{N} \cup \{0\}). \tag{2.4}$$

Further we see that for any γ with $0 < \gamma < \pi$ there exists a constant M_γ independent of n and u such that

$$\frac{|\mathcal{E}_n(u)|}{n!} \le \frac{M_\gamma}{\gamma^n} \ \ (n \in \mathbb{N} \cup \{0\}; u \in [1, 1+\delta]). \tag{2.5}$$

For $k \in \mathbb{N} \cup \{0\}$, $x \in \mathbb{R}$ and $u \in [1, 1+\delta]$, we let

$$J_1(x; 2k+1; u) = \sum_{m=1}^{\infty} \frac{(-u)^{-m}\sin(mx)}{m^{2k+1}} - \sum_{j=0}^{k} \phi(2k-2j;u)\frac{(-1)^j x^{2j+1}}{(2j+1)!}. \tag{2.6}$$

This function is continuous in u on $[1, 1+\delta]$. Indeed, if $k \in \mathbb{N}$ then $J_1(x; 2k+1; u)$ is uniformly convergent with respect to $u \in [1, 1+\delta]$. If $k = 0$ then the continuity is implied by the following lemma and Abel's theorem.

Lemma 2.1 $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n e^{inx}}{n}$ *is convergent for* $x \in (-\pi, \pi)$, *uniformly on any compact subset of* $(-\pi, \pi)$.

Proof. Let $M, N \in \mathbb{N}$ with $M < N$. For $x \in (-\pi, \pi)$, we have

$$\begin{aligned}\sum_{M<n\le N} (-1)^n e^{inx} &= \sum_{M<n\le N} e^{in\pi} \cdot e^{inx} = \sum_{M<n\le N} e^{in(\pi+x)} \\ &= e^{i(M+1)(\pi+x)} \cdot \frac{1 - e^{i(N-M)(\pi+x)}}{1 - e^{i(\pi+x)}}.\end{aligned}$$

Hence we have

$$\left| \sum_{M<n\le N} (-1)^n e^{inx} \right| \le \left| \frac{1 - e^{i(N-M)(\pi+x)}}{1 - e^{i(\pi+x)}} \right| \le \frac{2}{|1 - e^{i(\pi+x)}|} = \frac{2}{|1 + e^{ix}|}.$$

Note that $1 + e^{ix} \ne 0$ because $-\pi < x < \pi$. Hence we put $C(x) = 2/|1 + e^{ix}|$. Then $C(x)$ is independent of M and N. Using the partial summation formula, we have

$$\begin{aligned}\left| \sum_{M<n\le N} \frac{(-1)^n e^{inx}}{n} \right| &= \left| \sum_{M<n\le N} (-1)^n e^{inx} \cdot \frac{1}{N} + \int_M^N \left(\sum_{M<n\le \xi} (-1)^n e^{inx} \right) \cdot \frac{d\xi}{\xi^2} \right| \\ &\le C(x) \cdot \frac{1}{N} + \int_M^N C(x) \cdot \frac{d\xi}{\xi^2} \\ &= C(x) \cdot \left(\frac{1}{N} + \left[-\frac{1}{\xi} \right]_M^N \right) = \frac{C(x)}{M} \to 0 \ \ (M, N \to \infty).\end{aligned}$$

By Cauchy's criterion, we obtain the assertion. $\square$

Suppose $u \in (1, 1+\delta]$. For $k \in \mathbb{N} \cup \{0\}$ and $x \in (-\pi, \pi) \subset \mathbb{R}$, using (2.4), we have

$$\begin{aligned}
\sum_{m=1}^{\infty} \frac{(-u)^{-m}\sin(mx)}{m^{2k+1}} &= \sum_{m=1}^{\infty} \frac{(-u)^{-m}}{m^{2k+1}} \sum_{j=0}^{\infty} \frac{(-1)^j (mx)^{2j+1}}{(2j+1)!} \\
&= \sum_{j=0}^{\infty} \phi(2k-2j;u) \frac{(-1)^j x^{2j+1}}{(2j+1)!} \\
&= \sum_{j=0}^{k} \phi(2k-2j;u) \frac{(-1)^j x^{2j+1}}{(2j+1)!} - \frac{1}{1+u} \sum_{j=k+1}^{\infty} \mathcal{E}_{2j-2k}(u) \frac{(-1)^j x^{2j+1}}{(2j+1)!}. \qquad (2.7)
\end{aligned}$$

Hence, for $u > 1$, we have

$$J_1(x;2k+1;u) = -\frac{1}{1+u} \sum_{j=k+1}^{\infty} \mathcal{E}_{2j-2k}(u) \frac{(-1)^j x^{2j+1}}{(2j+1)!}. \qquad (2.8)$$

Suppose $x \in (-\pi, \pi)$. Then it follows from (2.5) that the right-hand side of (2.8) is uniformly convergent with respect to $u \in [1, 1+\delta]$, so is continuous on $[1, 1+\delta]$. On the other hand, we have already seen that the right-hand side of (2.6) is also continuous on $[1, 1+\delta]$. Hence we can let $u \to 1$ on both sides of (2.8) for $k \in \mathbb{N} \cup \{0\}$. Then it follows from (2.3) that

$$J_1(x;2k+1;1) = 0 \quad (k \in \mathbb{N} \cup \{0\}; x \in (-\pi, \pi)). \qquad (2.9)$$

In particular when $k = 0$, by $\phi(0;1) = -1/2$, we obtain

$$\sum_{m=1}^{\infty} \frac{(-1)^m \sin(mx)}{m} = -\frac{1}{2}x \quad (x \in (-\pi, \pi)). \qquad (2.10)$$

Using these facts, we prove the following lemma.

Lemma 2.2 *For $x \in (-\pi, \pi)$,*

$$\begin{aligned}
\sum_{m=1}^{\infty} \frac{(-1)^m \cos(mx)}{m^2} &+ \frac{1}{2} \sum_{m,n=1}^{\infty} \frac{(-1)^{m+n} \cos((m+n)x)}{mn} \\
&- \frac{1}{2} \sum_{\substack{m,n=1 \\ m \neq n}}^{\infty} \frac{(-1)^{m+n} \cos((m-n)x)}{mn} = 0. \qquad (2.11)
\end{aligned}$$

Proof. From (2.7), we have

$$\frac{\partial}{\partial x} J_1(x;3;u) = \sum_{m=1}^{\infty} \frac{(-u)^{-m} \cos(mx)}{m^2} - \phi(2;u) - \frac{x^2}{2(1+u)}, \qquad (2.12)$$

because $\phi(0;u) = -\frac{1}{1+u}$ from (2.4). Suppose $u > 1$. Then from (2.8), we have

$$\frac{\partial}{\partial x} J_1(x;3;u) = -\frac{1}{1+u} \sum_{j=2}^{\infty} \mathcal{E}_{2j-2}(u) \frac{(-1)^j x^{2j}}{(2j)!}.$$

As well as (2.9), we obtain

$$\lim_{u \to 1} \frac{\partial}{\partial x} J_1(x;3;u) = 0 \quad (x \in (-\pi, \pi)). \qquad (2.13)$$

On the other hand, by (2.7), we have

$$J_1(x;1;u) = \sum_{m=1}^{\infty} \frac{(-u)^{-m} \sin(mx)}{m} + \frac{1}{1+u} x,$$

namely

$$x=-(1+u)\left\{\sum_{m=1}^{\infty}\frac{(-u)^{-m}\sin(mx)}{m}-J_1(x;1;u)\right\}\quad(x\in(-\pi,\pi)). \tag{2.14}$$

Substitute (2.14) into (2.12) and let $u\to 1$ which is also justified by Lemma 2.1 and Abel's theorem. Then from (2.9) and (2.13), we obtain

$$\sum_{m=1}^{\infty}\frac{(-1)^m\cos(mx)}{m^2}=\phi(2;1)+\sum_{m,n=1}^{\infty}\frac{(-1)^{m+n}\sin(mx)\sin(nx)}{mn}. \tag{2.15}$$

Using the relation $\cos(\alpha-\beta)-\cos(\alpha+\beta)=2\sin\alpha\sin\beta$, we have

$$\begin{aligned}&\sum_{m,n=1}^{\infty}\frac{(-1)^{m+n}\sin(mx)\sin(nx)}{mn}\\&=\frac{1}{2}\left\{\sum_{\substack{m,n=1\\m\neq n}}^{\infty}\frac{(-1)^{m+n}\cos((m-n)x)}{mn}+\zeta(2)-\sum_{m,n=1}^{\infty}\frac{(-1)^{m+n}\cos((m+n)x)}{mn}\right\}.\end{aligned} \tag{2.16}$$

Substituting (2.16) into (2.15) and using the fact $\phi(2;1)=-\frac{1}{2}\zeta(2)$, we obtain (2.11). This completes the proof. $\square$

In (2.11), we put $t=x+\pi$ for $-\pi<x<\pi$. Using the fact that $\cos((m\pm n)\pi)=(-1)^{m+n}$, we have

$$\sum_{m=1}^{\infty}\frac{\cos(mt)}{m^2}+\frac{1}{2}\sum_{m,n=1}^{\infty}\frac{\cos((m+n)t)}{mn}-\frac{1}{2}\sum_{\substack{m,n=1\\m\neq n}}^{\infty}\frac{\cos((m-n)t)}{mn}=0 \tag{2.17}$$

for $t\in\mathbb{R}$ with $0<t<2\pi$. Hence (2.17) holds for any $t\in\mathbb{R}\setminus 2\pi\mathbb{Z}$.

Now we consider the function

$$f(t)=\sum_{m=1}^{\infty}\frac{\sin(mt)}{m^3}+\frac{1}{2}\sum_{m,n=1}^{\infty}\frac{\sin((m+n)t)}{mn(m+n)}-\frac{1}{2}\sum_{\substack{m,n=1\\m\neq n}}^{\infty}\frac{\sin((m-n)t)}{mn(m-n)} \tag{2.18}$$

for $t\in\mathbb{R}$. We see that $f(t)$ is continuous for all $t\in\mathbb{R}$ with period 2π and belongs to $C^1(\mathbb{R}\setminus 2\pi\mathbb{Z})$. By (2.17) and Lemma 2.1, we have $f'(t)=0$ for $t\in\mathbb{R}\setminus 2\pi\mathbb{Z}$. Hence $f(t)=C_n$ with a certain constant C_n for each interval $(2n\pi,2(n+1)\pi)$ $(n\in\mathbb{Z})$. But f is continuous on $\mathbb{R}$ and $f(2n\pi)=0$ for any $n\in\mathbb{Z}$. Hence each C_n is to be equal to 0, that is, $f(t)=0$ for all $t\in\mathbb{R}$. Put $l=m-n$ and $j=n-m$ according as $m>n$ and $m<n$, respectively, in (2.18). Then we have

$$\sum_{\substack{m,n=1\\m\neq n}}^{\infty}\frac{\sin((m-n)t)}{mn(m-n)}=\sum_{l,n=1}^{\infty}\frac{\sin(lt)}{ln(l+n)}+\sum_{j,m=1}^{\infty}\frac{\sin(jt)}{jm(j+m)}.$$

Hence we obtain the following.

Proposition 2.3 *For $t\in\mathbb{R}$,*

$$f(t)=\sum_{m=1}^{\infty}\frac{\sin(mt)}{m^3}+\frac{1}{2}\sum_{m,n=1}^{\infty}\frac{\sin((m+n)t)}{mn(m+n)}-\sum_{m,n=1}^{\infty}\frac{\sin(mt)}{mn(m+n)}=0. \tag{2.19}$$

Now we make use of Hardy's method of proving the functional equation for $\zeta(s)$ ([6], see also

[15] Section 2.2). For $s\in\mathbb{R}$ with $0<s<1$, from (2.19), we have

$$\begin{aligned}0&=\int_0^\infty x^{s-1}f(x)dx\\&=\int_0^\infty x^{s-1}\left\{\sum_{m=1}^\infty\frac{\sin(mx)}{m^3}+\frac{1}{2}\sum_{m,n=1}^\infty\frac{\sin((m+n)x)}{mn(m+n)}-\sum_{m,n=1}^\infty\frac{\sin(mx)}{mn(m+n)}\right\}dx.\end{aligned}\tag{2.20}$$

We need to justify term-by-term integration on the right-hand side of (2.20). Since (2.19) is absolutely convergent, in particular boundedly convergent, we have only to check the following three conditions in order to justify term-by-term integration.

$$\lim_{\lambda\to\infty}\sum_{m=1}^\infty\frac{1}{m^3}\int_\lambda^\infty\frac{\sin(mx)}{x^{1-s}}dx=0;\tag{2.21}$$

$$\lim_{\lambda\to\infty}\sum_{m,n=1}^\infty\frac{1}{mn(m+n)}\int_\lambda^\infty\frac{\sin((m+n)x)}{x^{1-s}}dx=0;\tag{2.22}$$

$$\lim_{\lambda\to\infty}\sum_{m,n=1}^\infty\frac{1}{mn(m+n)}\int_\lambda^\infty\frac{\sin(mx)}{x^{1-s}}dx=0.\tag{2.23}$$

Note that (2.21) has been already proved in [15] Section 2.1 (p.15). By the same way as the argument given there, we have

$$\begin{aligned}\int_\lambda^\infty\frac{\sin(Nx)}{x^{1-s}}dx&=\left[-\frac{\cos(Nx)}{Nx^{1-s}}\right]_\lambda^\infty+\frac{s-1}{N}\int_\lambda^\infty\frac{\cos(Nx)}{x^{2-s}}dx\\&=O\left(\frac{1}{N\lambda^{1-s}}\right)+O\left(\frac{1}{N}\int_\lambda^\infty\frac{dx}{x^{2-s}}\right)=O\left(\frac{1}{N\lambda^{1-s}}\right)\end{aligned}\tag{2.24}$$

for $N\in\mathbb{N}$. Since

$$\left|\sum_{m,n=1}^\infty\frac{1}{mn(m+n)^2}\right|<\infty,\ \left|\sum_{m,n=1}^\infty\frac{1}{m^2n(m+n)}\right|<\infty\tag{2.25}$$

(see [16]), we see that (2.22) and (2.23) hold. Thus we can justify term-by-term integration on the right-hand side of (2.20).

Using these facts, we give another proof of (1.4) as follows. We recall that $\Gamma(s)\Gamma(1-s)=\pi/\sin\pi s$ and

$$\int_0^\infty\frac{\sin bx}{x^{1-s}}dx=\frac{\pi}{2}b^{-s}\ \frac{\mathrm{cosec}(\pi(1-s)/2)}{\Gamma(1-s)}$$

for $b>0$ and $0<s<1$ (see [22] Chapter 12, p.239 and p.260). Hence we have

$$\int_0^\infty\frac{\sin bx}{x^{1-s}}dx=b^{-s}\ \sin\left(\frac{\pi s}{2}\right)\Gamma(s).\tag{2.26}$$

Carrying out term-by-term integration on the right-hand side of (2.20) and using (2.26), we obtain

$$0=\sin\left(\frac{\pi s}{2}\right)\Gamma(s)\left\{\zeta(s+3)+\frac{1}{2}\zeta_{MT,2}(1,1,s+1)-\zeta_{MT,2}(s+1,1,1)\right\}\tag{2.27}$$

for $0<s<1$. Since $\sin\left(\frac{\pi s}{2}\right)\Gamma(s)\neq0$ for $0<s<1$, we can remove this term from (2.27). Using

$$\frac{1}{mn}=\left(\frac{1}{m}+\frac{1}{n}\right)\frac{1}{m+n},\tag{2.28}$$

we have $\zeta_{MT,2}(1,1,s+1)=2\zeta_{EZ,2}(1,s+2)$. Hence we have

$$\zeta(s+3)+\zeta_{EZ,2}(1,s+2)-\zeta_{MT,2}(s+1,1,1)=0\tag{2.29}$$

for $0 < s < 1$. Since each term on the left-hand side of (2.29) is meromorphically continued to $\mathbb{C}$ (see [8, 10, 11], see also [1, 2, 24]), we see that (2.29) holds for all $s \in \mathbb{C}$ except for the singularities of three functions in (2.29). Thus we obtain the following which is (1.4).

Proposition 2.4

$$\zeta_{EZ,2}(1, s+1) - \zeta_{MT,2}(s, 1, 1) + \zeta(s+2) = 0 \tag{2.30}$$

holds for all $s \in \mathbb{C}$ except for the singularities of three functions on the left-hand side.

Letting $s = 1$, and furthermore $s = k - 2$ for $k \in \mathbb{N}$ with $k \geq 3$ in (2.30), we have the following results given by Euler (see [5]).

Corollary 2.5 $\zeta_{EZ,2}(1, 2) = \zeta(3)$.

Corollary 2.6 *For $k \in \mathbb{N}$ with $k \geq 3$,*

$$\sum_{j=2}^{k-1} \zeta_{EZ,2}(k-j, j) = \zeta(k).$$

Proof. Using (2.28) repeatedly, we have

$$\begin{aligned} \zeta_{MT,2}(1, k-2, 1) &= \zeta_{MT,2}(1, k-3, 2) + \zeta_{EZ,2}(k-2, 2) \\ &= \zeta_{MT,2}(1, k-4, 3) + \zeta_{EZ,2}(k-3, 3) + \zeta_{EZ,2}(k-2, 2) \\ &\vdots \\ &= \zeta_{MT,2}(1, 0, k-1) + \sum_{j=2}^{k-1} \zeta_{EZ,2}(k-j, j). \end{aligned} \tag{2.31}$$

Note that $\zeta_{MT,2}(1, 0, k-1) = \zeta_{EZ,2}(1, k-1)$. On the other hand, letting $s = k-2$ in (2.30), we have

$$\zeta_{MT,2}(1, k-2, 1) = \zeta_{MT,2}(k-2, 1, 1) = \zeta_{EZ,2}(1, k-1) + \zeta(k). \tag{2.32}$$

Combining (2.31) and (2.32), we have the assertion. □

Note that Corollary 2.6 is what is called the sum formula for double zeta values (see [3, 4]). Hence (2.30) contains these well-known formulas as discrete relations among their special values.

3. The general principle

In this section, we prove our general theorem, which is a generalization of Proposition 2.4. We consider a Dirichlet series

$$Z(s) = \sum_{m=1}^{\infty} \frac{a_m}{m^s}, \tag{3.1}$$

where $\{a_n\} \subset \mathbb{C}$. Let $\Re s = \rho$ ($\rho \in \mathbb{R}$) be the convergent line of $Z(s)$. This means that if $\Re s > \rho$ then $Z(s)$ is convergent and if $\Re s < \rho$ then $Z(s)$ is not convergent. We further assume that $0 \leq \rho < 1$.

Theorem 3.1 *Assume that*

$$\sum_{m=1}^{\infty} a_m \sin(mt) = 0 \tag{3.2}$$

or

$$\sum_{m=1}^{\infty} a_m \cos(mt) = 0 \tag{3.3}$$

is boundedly convergent for $t > 0$ and that, for $\rho < s < 1$,

$$\lim_{\lambda\to\infty} \sum_{m=1}^{\infty} a_m \int_{\lambda}^{\infty} t^{s-1} \sin(mt)dt = 0 \tag{3.4}$$

(if we assume (3.2)) or

$$\lim_{\lambda\to\infty} \sum_{m=1}^{\infty} a_m \int_{\lambda}^{\infty} t^{s-1} \cos(mt)dt = 0 \tag{3.5}$$

(if we assume (3.3)). Then $Z(s)$ defined by (3.1) can be continued meromorphically to $\mathbb{C}$, and actually $Z(s) = 0$ for all $s \in \mathbb{C}$.

Proof. By the same way as in the proof of Proposition 2.4, we consider

$$0 = \int_0^{\infty} \sum_{m=1}^{\infty} a_m t^{s-1} \sin(mt)dt \tag{3.6}$$

for $\rho < s < 1$. From (3.4), we can justify term-by-term integration on the right-hand side of (3.6). Hence, using (2.26), it follows from (3.6) that

$$\sin\left(\frac{\pi s}{2}\right)\Gamma(s)Z(s) = 0 \quad (\rho < s < 1), \tag{3.7}$$

which means that $Z(s)$ can be continued meromorphically to $\mathbb{C}$ and $Z(s) = 0$ for all $s \in \mathbb{C}$ except for $s \in 2\mathbb{Z}$. The points $s \in 2\mathbb{Z}$ are removable singularities by Riemann's theorem. Hence we may define $Z(s) = 0$ at these points, which implies the assertion. Furthermore, if (3.3) and (3.5) hold, then we can similarly obtain the assertion. This completes the proof. $\square$

Theorem 3.1 itself is just a general abstract principle. From this result, however, we can find a lot of functional relations among multiple zeta-functions. In the rest of this paper, we aim to give functional relations explicitly as multiple generalizations of Proposition 2.4. Furthermore we will give more functional relations among various multiple zeta-functions in our next papers.

4. A key lemma

In this section, we prepare a key lemma to give functional relations explicitly. For $k \in \mathbb{N}$, let

$$V_k = \{\sigma = (\sigma_1, \ldots, \sigma_k) \in \{\pm 1\}^k \,|\, \sigma_1 = 1\}.$$

For $\sigma = (\sigma_1, \ldots, \sigma_k) \in V_k$, let

$$\sigma(X_1, \ldots, X_k) = \sigma_1 X_1 + \cdots + \sigma_k X_k.$$

For any $s \in \mathbb{R}$ with $s > 0$, we consider the sum

$$\mathcal{X}_k(s;\sigma) = \sum_{\substack{m_1,\ldots,m_k \geq 1 \\ \sigma(m_1,\ldots,m_k)\neq 0}} \frac{1}{m_1 \cdots m_k \cdot |\sigma(m_1, \ldots, m_k)|^s}. \tag{4.1}$$

Lemma 4.1 *For any $s > 0$ and $\sigma \in V_k$, $\mathcal{X}_k(s;\sigma)$ is convergent.*

Proof. First we consider the case $\sigma = (1, \ldots, 1)$. We can easily see that

$$m_1 + m_2 + \cdots + m_k \geq (m_1 \cdot m_2 \cdots m_k)^{1/k}.$$

Hence we have

$$\mathcal{X}_k(s;\sigma) \le \left\{\zeta\left(1+\frac{s}{k}\right)\right\}^k < \infty.$$

Next, we consider the case $\sigma \neq (1,\ldots,1)$. It is sufficient to consider the case when $\sigma_1 = \cdots = \sigma_r = 1$ and $\sigma_{r+1} = \cdots \sigma_k = -1$ with $1 \le r < k$, namely

$$\mathcal{X}_k(s;\sigma) = \sum_{\substack{m_1+\cdots+m_r \\ \neq m_{r+1}+\cdots+m_k}} \frac{1}{m_1\cdots m_k \cdot |(m_1+\cdots+m_r)-(m_{r+1}+\cdots+m_k)|^s},$$

where $m_1,\ldots,m_k \ge 1$. By putting $p = m_1+\cdots+m_r$ and $q = m_{r+1}+\cdots+m_k$, and changing the order of summation, we have

$$\mathcal{X}_k(s;\sigma) = \sum_{\substack{p,q\ge 1 \\ p\neq q}} \frac{1}{|p-q|^s} \sum_{\substack{m_1,\ldots,m_r\ge 1 \\ m_1+\cdots+m_r=p}} \frac{1}{m_1\cdots m_r} \sum_{\substack{m_{r+1},\ldots,m_k\ge 1 \\ m_{r+1}+\cdots+m_k=q}} \frac{1}{m_{r+1}\cdots m_k}, \tag{4.2}$$

where the empty sum is nil. The change of the order of summation will be justified when we prove the (absolute) convergence. In order to estimate (4.2), we consider

$$\mathcal{Y}_r(p) = \sum_{\substack{m_1,\ldots,m_r\ge 1 \\ m_1+\cdots+m_r=p}} \frac{1}{m_1\cdots m_r}. \tag{4.3}$$

We claim that

$$\mathcal{Y}_r(p) \le \frac{2^{r-1}}{p}(1+\log p)^{r-1} \quad (p\in\mathbb{N}) \tag{4.4}$$

for $r\in\mathbb{N}$ with $r\ge 2$. We prove this by induction on $r \ge 2$. Suppose $r=2$. Then we have

$$\mathcal{Y}_2(p) = \sum_{m=1}^{p-1} \frac{1}{m(p-m)} = \frac{1}{p}\sum_{m=1}^{p-1}\left(\frac{1}{m}+\frac{1}{p-m}\right) = \frac{2}{p}\sum_{m=1}^{p-1}\frac{1}{m}.$$

We further obtain

$$\sum_{m=1}^{p-1}\frac{1}{m} \le 1 + \int_1^{p-1}\frac{1}{x}dx = 1+\log(p-1) \le 1+\log p.$$

Hence we have

$$\mathcal{Y}_2(p) \le \frac{2}{p}(1+\log p), \tag{4.5}$$

which implies assertion (4.4) in the case $r=2$.

Now we assume (4.4) in the case of $r-1$ for $r>2$ and consider the case of r. Using the assumption of induction and (4.5), we have

$$\begin{aligned}
\mathcal{Y}_r(p) &= \sum_{\substack{m_1,\ldots,m_r\ge 1 \\ m_1+\cdots+m_r=p}} \frac{1}{m_1\cdots m_r} = \sum_{\substack{j,m_r\ge 1 \\ j+m_r=p}} \frac{1}{m_r} \sum_{\substack{m_1,\ldots,m_{r-1}\ge 1 \\ m_1+\cdots+m_{r-1}=j}} \frac{1}{m_1\cdots m_{r-1}} \\
&\le \sum_{\substack{j,m_r\ge 1 \\ j+m_r=p}} \frac{1}{m_r}\cdot\frac{2^{r-2}}{j}(1+\log j)^{r-2} \\
&\le 2^{r-2}(1+\log p)^{r-2}\cdot\mathcal{Y}_2(p) \le \frac{2^{r-1}}{p}(1+\log p)^{r-1},
\end{aligned}$$

which implies assertion (4.4) in the case r. Thus we obtain (4.4) for all $r\ge 2$ by induction.

Using (4.4), we can obtain the proof of Lemma 4.1 as follows. Combining (4.2) and (4.4), we have

$$\begin{aligned}
|\mathcal{X}_k(s;\sigma)| &\le \sum_{\substack{p,q\ge 1\\ p\ne q}} \frac{1}{|p-q|^s}\,\mathcal{Y}_r(p)\mathcal{Y}_{k-r}(q)\\
&\le \sum_{p\ne q} \frac{1}{|p-q|^s}\frac{2^{r-1}}{p}(1+\log p)^{r-1}\frac{2^{k-r-1}}{q}(1+\log q)^{k-r-1}\\
&\le 2^{k-2}\sum_{p\ne q}\frac{1}{|p-q|^s pq}(1+\log p)^{r-1}(1+\log q)^{k-r-1}. \qquad (4.6)
\end{aligned}$$

Put $j=p-q$ and $l=q-p$ according as $p>q$ and $p<q$, respectively. Then (4.6) is less than or equal to

$$\begin{aligned}
\mathfrak{M} &= 2^{k-2}\sum_{j,q\ge 1}\frac{1}{j^s q(j+q)}(1+\log(j+q))^{r-1}(1+\log q)^{k-r-1}\\
&\quad + 2^{k-2}\sum_{p,l\ge 1}\frac{1}{l^s p(p+l)}(1+\log p)^{r-1}(1+\log(p+l))^{k-r-1}. \qquad (4.7)
\end{aligned}$$

Let $\varepsilon\in\mathbb{R}$ be a sufficiently small positive number. Since $1+\log x \ll x^{\varepsilon}$, we have

$$\begin{aligned}
\mathfrak{M} &\ll 2^{k-2}\left\{\sum_{j,q\ge 1}\frac{(j+q)^{\varepsilon(r-1)}q^{\varepsilon(k-r-1)}}{j^s q(j+q)} + \sum_{p,l\ge 1}\frac{p^{\varepsilon(r-1)}(p+l)^{\varepsilon(k-r-1)}}{l^s p(p+l)}\right\}\\
&< 2^{k-2}\left\{\sum_{j,q\ge 1}\frac{1}{j^s q^{1-\eta}(j+q)^{1-\eta}} + \sum_{p,l\ge 1}\frac{1}{l^s p^{1-\eta}(p+l)^{1-\eta}}\right\}, \qquad (4.8)
\end{aligned}$$

where $\eta=\max\{\varepsilon(r-1),\varepsilon(k-r-1)\}$. Since k,r are fixed, we can take a sufficiently small η with $0<\eta<s/4$. Using the fact

$$(x+y)^{1-\eta}=(x+y)^{1-3\eta}(x+y)^{2\eta}>x^{1-3\eta}y^{2\eta}$$

for $x,y>0$, we see that the right-hand side of (4.8) is less than

$$2^{k-1}\sum_{j,q\ge 1}\frac{1}{j^{s+1-3\eta}q^{1+\eta}}=2^{k-1}\zeta\left(1+s-3\eta\right)\zeta\left(1+\eta\right)<\infty,$$

because $0<\eta<s/4$. This completes the proof of Lemma 4.1. $\square$

5. Explicit functional relations

For $p\in\mathbb{N}\cup\{0\}$ and

$$\sigma=(\sigma_1,\ldots,\sigma_{2p+1})\in V_{2p+1}=\{\sigma=(\sigma_j)\in\{\pm 1\}^{2p+1}\,|\,\sigma_1=1\},$$

let

$$\Delta_\sigma=(-1)^p\prod_{j=1}^{2p+1}\sigma_j\in\{\pm 1\}. \qquad (5.1)$$

Lemma 5.1 *For $p\in\mathbb{N}\cup\{0\}$,*

$$\prod_{j=1}^{2p+1}\sin X_j=\frac{1}{2^{2p}}\sum_{\sigma\in V_{2p+1}}\Delta_\sigma\cdot\sin(\sigma(X_1,\ldots,X_{2p+1})), \qquad (5.2)$$

where $\sigma(X_1,\ldots,X_{2p+1})=\sigma_1X_1+\cdots+\sigma_{2p+1}X_{2p+1}$.

Proof. We prove this lemma by induction on $p\geq 0$. The case of $p=0$ is trivial. Hence we assume the case of $p-1$ for $p\geq 1$, and consider the case of p.

Using the assumption of induction and the additive properties of $\cos X$ and $\sin X$, we have

$$\begin{aligned}
\prod_{j=1}^{2p+1}\sin X_j &= \left(\prod_{j=1}^{2p-1}\sin X_j\right)\cdot\sin X_{2p}\cdot\sin X_{2p+1}\\
&=\frac{1}{2^{2p-2}}\sum_{\sigma\in V_{2p-1}}\Delta_\sigma\cdot\sin(\sigma(X_1,\ldots,X_{2p-1}))\times\frac{1}{2}\left\{\cos(X_{2p}-X_{2p+1})-\cos(X_{2p}+X_{2p+1})\right\}\\
&=\frac{1}{2^{2p-1}}\sum_{\sigma\in V_{2p-1}}\Delta_\sigma\Big\{\sin(\sigma(X_1,\ldots,X_{2p-1}))\cos(X_{2p}-X_{2p+1})\\
&\qquad\qquad -\sin(\sigma(X_1,\ldots,X_{2p-1}))\cos(X_{2p}+X_{2p+1})\Big\}\\
&=\frac{1}{2^{2p}}\sum_{\sigma\in V_{2p-1}}\Delta_\sigma\Big\{\sin(\sigma(X_1,\ldots,X_{2p-1})+X_{2p}-X_{2p+1})\\
&\qquad +\sin(\sigma(X_1,\ldots,X_{2p-1})-X_{2p}+X_{2p+1})-\sin(\sigma(X_1,\ldots,X_{2p-1})+X_{2p}+X_{2p+1})\\
&\qquad -\sin(\sigma(X_1,\ldots,X_{2p-1})-X_{2p}-X_{2p+1})\Big\}.
\end{aligned}\tag{5.3}$$

Note that $V_{2p+1}=V_{2p-1}\times\{\pm1\}^2$. Hence, for any $\sigma=(\sigma_1,\ldots,\sigma_{2p-1})\in V_{2p-1}$ and $(\sigma_{2p},\sigma_{2p+1})\in\{\pm1\}^2$, we let

$$\overline{\sigma}=(\sigma_1,\ldots,\sigma_{2p+1})\in V_{2p+1}.$$

Then we have

$$\Delta_{\overline{\sigma}}=(-1)^p\prod_{j=1}^{2p-1}\sigma_j=\Delta_\sigma\cdot(-\sigma_{2p}\cdot\sigma_{2p+1}).$$

Therefore we can write (5.3) as

$$\frac{1}{2^{2p}}\sum_{\overline{\sigma}\in V_{2p+1}}\Delta_{\overline{\sigma}}\cdot\sin(\overline{\sigma}(X_1,\ldots,X_{2p+1})),$$

which implies that (5.2) in the case of p holds. This completes the proof. □

From (2.7) and (2.9), we have

$$\sum_{m=1}^{\infty}\frac{(-1)^m\sin(mx)}{m^{2k+1}}=\sum_{j=0}^{k}\phi(2k-2j;1)\frac{(-1)^jx^{2j+1}}{(2j+1)!}\quad(-\pi<x<\pi)\tag{5.4}$$

for $k\in\mathbb{N}\cup\{0\}$. For simplicity, we denote the left-hand side of (5.4) by $S_{2k+1}(x)$ and let $\phi(s):=\phi(s;1)=\left(2^{1-s}-1\right)\zeta(s)$. Then we have

$$\begin{pmatrix}S_1(x)\\ S_3(x)\\ \vdots\\ S_{2p+1}(x)\end{pmatrix}=\begin{pmatrix}\phi(0)&0&\cdots&0\\ \phi(2)&-\frac{1}{3!}\phi(0)&\cdots&0\\ \vdots&&\ddots&\vdots\\ \phi(2p)&-\frac{1}{3!}\phi(2p-2)&\cdots&\frac{(-1)^p}{(2p+1)!}\phi(0)\end{pmatrix}\begin{pmatrix}x\\ x^3\\ \vdots\\ x^{2p+1}\end{pmatrix}\tag{5.5}$$

for $p\in\mathbb{N}\cup\{0\}$. We denote the matrix on the right-hand side of (5.5) by $\mathfrak{D}_p$. Note that $\mathfrak{D}_p\in M_{p+1}(\mathbb{Q}[\pi^2])$, the set of $(p+1,p+1)$-matrices with components in $\mathbb{Q}[\pi^2]$, because $\phi(s)=\left(2^{1-s}-1\right)\zeta(s)$.

Since $\phi(0)=-\frac{1}{2}$, we have

$$\det \mathfrak{D}_p=\phi(0)^{p+1}\prod_{j=0}^{p}\frac{(-1)^j}{(2j+1)!}\in\mathbb{Q}^{\times}.$$

Hence $\mathfrak{D}_p\in GL_{p+1}(\mathbb{Q}[\pi^2])$. We denote

$$\mathfrak{D}_p^{-1}=(\beta_{ij})_{0\leq i,j\leq p+1}\in M_{p+1}(\mathbb{Q}[\pi^2]). \tag{5.6}$$

Then we have

$$x^{2p+1}=\sum_{j=0}^{p}\beta_{pj}S_{2j+1}(x)\quad(-\pi<x<\pi). \tag{5.7}$$

On the other hand, it follows from (2.10) that

$$x=-2S_1(x)=\sum_{m=1}^{\infty}\frac{(-1)^m\sin(mx)}{m}\quad(-\pi<x<\pi).$$

Then

$$x^{2p+1}=(-2)^{2p+1}\sum_{m_1,\ldots,m_{2p+1}\geq 1}\frac{(-1)^{m_1+\cdots+m_{2p+1}}\prod_{j=1}^{2p+1}\sin(m_jx)}{m_1\cdots m_{2p+1}}.$$

The sum of $S_1(x)$ is not convergent absolutely, but this change of the order of summation is possible, because in the proof of Lemma 2.1 we have shown

$$\left|\sum_{n>M}\frac{(-1)^ne^{inx}}{n}\right|\leq\frac{C(x)}{M}.$$

By using Lemma 5.1, we have

$$x^{2p+1}=-2\sum_{\sigma\in V_{2p+1}}\Delta_\sigma\sum_{m_1,\ldots,m_{2p+1}\geq 1}\frac{(-1)^{m_1+\cdots+m_{2p+1}}\sin(\sigma(m_1,\ldots,m_{2p+1})x)}{m_1\cdots m_{2p+1}}, \tag{5.8}$$

when $-\pi<x<\pi$. Combining (5.7) and (5.8), we have

$$\begin{aligned}&-2\sum_{\sigma\in V_{2p+1}}\Delta_\sigma\sum_{m_1,\ldots,m_{2p+1}\geq 1}\frac{(-1)^{m_1+\cdots+m_{2p+1}}\sin(\sigma(m_1,\ldots,m_{2p+1})x)}{m_1\cdots m_{2p+1}}\\&=\sum_{j=0}^{p}\beta_{pj}\sum_{m=1}^{\infty}\frac{(-1)^m\sin(mx)}{m^{2j+1}}\quad(-\pi<x<\pi).\end{aligned} \tag{5.9}$$

Putting $t=x+\pi$ in (5.9) and using the relation

$$\cos(\sigma(m_1,\ldots,m_{2p+1})\pi)=(-1)^{m_1+\cdots+m_{2p+1}}$$

for any $\sigma\in V_{2p+1}$, we have the following.

Proposition 5.2 *For $t\in\mathbb{R}$ with $0<t<2\pi$,*

$$2\sum_{\sigma\in V_{2p+1}}\Delta_\sigma\sum_{m_1,\ldots,m_{2p+1}\geq 1}\frac{\sin(\sigma(m_1,\ldots,m_{2p+1})t)}{m_1\cdots m_{2p+1}}+\sum_{j=0}^{p}\beta_{pj}\sum_{m=1}^{\infty}\frac{\sin(mt)}{m^{2j+1}}=0, \tag{5.10}$$

where $\{\beta_{pj}\in\mathbb{Q}[\pi^2]\,|\,0\leq j\leq p\}$ are defined by (5.6). Furthermore (5.10) is boundedly convergent for $t>0$.

Proof. We have only to check the bounded convergency. The latter part of the left-hand side of (5.10) is boundedly convergent for $t>0$ (see [15], p. 17). We denote by I the former part of the left-hand side of (5.10). Then, by Lemma 5.1, we have

$$\begin{aligned} I &= 2^{2p+1} \sum_{m_1,\ldots,m_{2p+1}\geq 1} \frac{\prod_{j=1}^{2p+1} \sin(m_j t)}{m_1 \cdots m_{2p+1}} \\ &= 2^{2p+1} \prod_{j=1}^{2p+1} \sum_{m_j \geq 1} \frac{\sin(m_j t)}{m_j}. \end{aligned}$$

Hence I is also boundedly convergent for $t > 0$. This completes the proof. □

Now we aim to apply Theorem 3.1 to (5.10). From Proposition 5.2, the left-hand side of (5.10) is boundedly convergent for $t > 0$, which means (3.2). Hence we will check that the left-hand side of (5.10) satisfies (3.4). For $\sigma \in V_{2p+1}$, we put

$$f^\sigma(t) = \sum_{N=1}^{\infty} \left\{ \sum_{\substack{m_1,\ldots,m_{2p+1}\geq 1 \\ \sigma(m_1,\ldots,m_{2p+1})=N}} \frac{\sin(Nt)}{m_1 \cdots m_{2p+1}} - \sum_{\substack{m_1,\ldots,m_{2p+1}\geq 1 \\ \sigma(m_1,\ldots,m_{2p+1})=-N}} \frac{\sin(Nt)}{m_1 \cdots m_{2p+1}} \right\},$$

$$g_j(t) = \sum_{m=1}^{\infty} \frac{\sin(mt)}{m^{2j+1}} \qquad (0 \leq j \leq p)$$

for $t > 0$. Then $g_j(t)$ satisfies (3.4) (see [15] Section 2.1, p.15). Next we consider $f^\sigma(t)$. Let $0 < s < 1$ and $\lambda > 0$. Then

$$\begin{aligned} &\sum_{N=1}^{\infty} \Bigg\{ \sum_{\substack{m_1,\ldots,m_{2p+1}\geq 1 \\ \sigma(m_1,\ldots,m_{2p+1})=N}} \frac{1}{m_1 \cdots m_{2p+1}} \int_{\lambda}^{\infty} t^{s-1} \sin(Nt)dt \\ &\quad - \sum_{\substack{m_1,\ldots,m_{2p+1}\geq 1 \\ \sigma(m_1,\ldots,m_{2p+1})=-N}} \frac{1}{m_1 \cdots m_{2p+1}} \int_{\lambda}^{\infty} t^{s-1} \sin(Nt)dt \Bigg\} \\ &= \sum_{N=1}^{\infty} \sum_{\substack{m_1,\ldots,m_{2p+1}\geq 1 \\ \sigma(m_1,\ldots,m_{2p+1})=N}} \frac{1}{m_1 \cdots m_{2p+1}} \times \left\{ \left[-\frac{\cos(Nt)}{Nt^{1-s}} \right]_{\lambda}^{\infty} + \frac{s-1}{N} \int_{\lambda}^{\infty} \frac{\cos(Nt)}{t^{2-s}} dt \right\} \\ &\quad - \sum_{N=1}^{\infty} \sum_{\substack{m_1,\ldots,m_{2p+1}\geq 1 \\ \sigma(m_1,\ldots,m_{2p+1})=-N}} \frac{1}{m_1 \cdots m_{2p+1}} \times \left\{ \left[-\frac{\cos(Nt)}{Nt^{1-s}} \right]_{\lambda}^{\infty} + \frac{s-1}{N} \int_{\lambda}^{\infty} \frac{\cos(Nt)}{t^{2-s}} dt \right\}. \end{aligned} \tag{5.11}$$

It follows from Lemma 4.1 that $\mathcal{X}_k(1;\sigma)$ is convergent. Hence we see that the right-hand side of (5.11) has the order $O\left(\lambda^{s-1}\right)$, which tends to 0 as $\lambda \to \infty$ because $0 < s < 1$. This means (3.4) for $f^\sigma(t)$.

Now we define

$$\begin{aligned} \mathcal{Z}_{2p+1}(s) = 2 \sum_{\sigma \in V_{2p+1}} \Delta_\sigma \Bigg\{ &\sum_{\substack{m_1,\ldots,m_{2p+1}\geq 1 \\ \sigma(m_1,\ldots,m_{2p+1})>0}} \frac{1}{m_1 \cdots m_{2p+1} \sigma(m_1,\ldots,m_{2p+1})^s} \\ &- \sum_{\substack{m_1,\ldots,m_{2p+1}\geq 1 \\ \sigma(m_1,\ldots,m_{2p+1})<0}} \frac{1}{m_1 \cdots m_{2p+1} (-\sigma(m_1,\ldots,m_{2p+1}))^s} \Bigg\} + \sum_{j=0}^{p} \beta_{pj} \zeta(s+2j+1) \end{aligned} \tag{5.12}$$

for $s \in \mathbb{C}$ with $\Re s > 0$, where $\{\beta_{pj} \in \mathbb{Q}[\pi^2] \,|\, 0 \leq j \leq p\}$ are defined by (5.6). Applying Theorem 3.1 to $\mathcal{Z}_{2p+1}(s)$ and $2\sum_{\sigma \in V_{2p+1}} \Delta_\sigma f^\sigma(t) + \sum_{j=0}^{p} \beta_{pj} g_j(t)$, we immediately obtain the following.

Proposition 5.3 *For $p \in \mathbb{N}\cup\{0\}$, $\mathcal{Z}_{2p+1}(s)$ can be continued meromorphically to $\mathbb{C}$, and $\mathcal{Z}_{2p+1}(s) = 0$ for all $s \in \mathbb{C}$.*

This result gives certain functional relations for $(2p+1)$-ple zeta-functions. For example, we will calculate the case when $p = 1$ explicitly as follows.

In (5.5), we let $p = 1$. Using $\phi(0) = -\frac{1}{2}$ and $\phi(2) = -\frac{1}{2}\zeta(2)$, we have

$$\begin{pmatrix} S_1(x) \\ S_3(x) \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & 0 \\ -\frac{1}{2}\zeta(2) & \frac{1}{12} \end{pmatrix} \begin{pmatrix} x \\ x^3 \end{pmatrix},$$

namely

$$\begin{pmatrix} x \\ x^3 \end{pmatrix} = -24 \begin{pmatrix} \frac{1}{12} & 0 \\ \frac{1}{2}\zeta(2) & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} S_1(x) \\ S_3(x) \end{pmatrix}.$$

Hence $x^3 = -12\zeta(2)S_1(x) + 12S_3(x)$, namely $\beta_{10} = -12\zeta(2)$ and $\beta_{11} = 12$. Since

$$V_3 = \{(1,1,1),(1,1,-1),(1,-1,1),(1,-1,-1)\},$$

we obtain from (5.12) that

$$\begin{aligned}
\mathcal{Z}_3(s) = 2\Bigg\{ &- \sum_{m_1,m_2,m_3 \geq 1} \frac{1}{m_1m_2m_3(m_1+m_2+m_3)^s} \\
&+ \sum_{\substack{m_1,m_2,m_3 \geq 1 \\ m_1+m_2>m_3}} \frac{1}{m_1m_2m_3(m_1+m_2-m_3)^s} - \sum_{\substack{m_1,m_2,m_3 \geq 1 \\ m_1+m_2<m_3}} \frac{1}{m_1m_2m_3(m_3-m_1-m_2)^s} \\
&+ \sum_{\substack{m_1,m_2,m_3 \geq 1 \\ m_1+m_3>m_2}} \frac{1}{m_1m_2m_3(m_1-m_2+m_3)^s} - \sum_{\substack{m_1,m_2,m_3 \geq 1 \\ m_1+m_3<m_2}} \frac{1}{m_1m_2m_3(m_2-m_1-m_3)^s} \\
&- \sum_{\substack{m_1,m_2,m_3 \geq 1 \\ m_1>m_2+m_3}} \frac{1}{m_1m_2m_3(m_1-m_2-m_3)^s} + \sum_{\substack{m_1,m_2,m_3 \geq 1 \\ m_1<m_2+m_3}} \frac{1}{m_1m_2m_3(m_2+m_3-m_1)^s} \Bigg\} \\
&- 12\zeta(2)\zeta(s+1) + 12\zeta(s+3).
\end{aligned} \tag{5.13}$$

The triple series of the second term, fourth term and seventh term in the curly parentheses on the right-hand side of (5.13) are the same, hence we denote the triple series of the second term by $G_1(s)$. Similarly, the triple series of the third term, fifth term and sixth term are the same, hence we denote the third term by $G_2(s)$. From (1.3), the first term equals to $\zeta_{MT,3}(1,1,1,s)$. Suppose $\Re s > 1$. Then it follows from Lemma 4.1 that $G_1(s)$ and $G_2(s)$ are convergent absolutely.

First we consider $G_2(s)$. Putting $l = m_3 - m_1 - m_2$, we have

$$G_2(s) = \sum_{l,m_1,m_2 \geq 1} \frac{1}{l^s m_1 m_2 (l+m_1+m_2)} = \zeta_{MT,3}(s,1,1,1). \tag{5.14}$$

Secondly, in order to consider $G_1(s)$, we prepare some elementary transformations as follows. Recall (1.1) and (1.3). We have

$$\sum_{l,m,n \geq 1} \frac{1}{lm(l+m+n)^s} = \sum_{l,m,n \geq 1} \left(\frac{1}{l} + \frac{1}{m}\right) \frac{1}{(l+m)(l+m+n)^s} = 2\zeta_{EZ,3}(1,1,s). \tag{5.15}$$

Using $\frac{1}{m(l+n)} = \left(\frac{1}{m} + \frac{1}{l+n}\right)\frac{1}{l+m+n}$, we have

$$\begin{aligned}
\sum_{l,m,n \geq 1} \frac{1}{lm(l+n)(l+m+n)^s} &= \sum_{l,m,n \geq 1} \left\{ \frac{1}{lm(l+m+n)^{s+1}} + \frac{1}{l(l+n)(l+m+n)^{s+1}} \right\} \\
&= 3\zeta_{EZ,3}(1,1,s+1),
\end{aligned} \tag{5.16}$$

by (5.15). Furthermore we have

$$\sum_{l,m,n\geq 1}\frac{1}{lm(l+n)(m+n)^s}=3\zeta_{EZ,3}(1,1,s+1)+\zeta_{MT,2}(1,2,s). \tag{5.17}$$

In fact, the left-hand side equals to

$$\begin{aligned}
&\sum_{m,n\geq 1}\frac{1}{m(m+n)^s}\sum_{l\geq 1}\frac{1}{l(l+n)}=\sum_{m,n\geq 1}\frac{1}{mn(m+n)^s}\sum_{l\geq 1}\left(\frac{1}{l}-\frac{1}{l+n}\right)\\
&=\sum_{m,n\geq 1}\frac{1}{mn(m+n)^s}\sum_{l=1}^{n}\frac{1}{l}=\sum_{\substack{l,m,n\geq 1\\ l\leq n}}\frac{1}{lmn(m+n)^s}\\
&=\sum_{m,n\geq 1}\frac{1}{mn^2(m+n)^s}+\sum_{l,m,j\geq 1}\frac{1}{lm(l+j)(l+m+j)^s}.
\end{aligned}$$

By (5.16), we obtain (5.17).

Using these relations, we consider $G_1(s)$. In the definition of $G_1(s)$, we put $l=m_2-m_3$ and $j=m_3-m_2$ according as $m_2>m_3$ and $m_2<m_3$, respectively. Then we have

$$\begin{aligned}
G_1(s)=&\sum_{m_1,m_3,l\geq 1}\frac{1}{m_1m_3(m_3+l)(m_1+l)^s}+\sum_{m_1,m_2\geq 1}\frac{1}{m_1^{s+1}m_2^2}\\
&+\sum_{\substack{m_1,m_2,j\geq 1\\ m_1>j}}\frac{1}{m_1m_2(m_2+j)(m_1-j)^s}.
\end{aligned} \tag{5.18}$$

The first term on the right-hand side of (5.18) equals to $3\zeta_{EZ,3}(1,1,s+1)+\zeta_{MT,2}(1,2,s)$ by (5.17). The second term equals to $\zeta(s+1)\zeta(2)$. Furthermore the third term equals to

$$\sum_{m_2,j,k\geq 1}\frac{1}{(k+j)m_2(m_2+j)k^s}, \tag{5.19}$$

by putting $k=m_1-j$. We aim to prove that this equals to

$$\zeta_{MT,2}(s,2,1)+\frac{1}{2}\zeta_{MT,3}(s,1,1,1). \tag{5.20}$$

In fact, as in the proof of (5.17), we see that (5.19) equals to

$$\begin{aligned}
&\sum_{j,k\geq 1}\frac{1}{k^s(k+j)}\sum_{m_2\geq 1}\frac{1}{m_2(m_2+j)}=\sum_{\substack{m_2,j,k\geq 1\\ m_2\leq j}}\frac{1}{k^sjm_2(k+j)}\\
&=\zeta_{MT,2}(s,2,1)+\sum_{m_2,l,k\geq 1}\frac{1}{k^sm_2(m_2+l)(k+m_2+l)},
\end{aligned} \tag{5.21}$$

by putting $l=j-m_2$. Denote the second term on the right-hand side of (5.21) by $\mathfrak{N}$. Since $\frac{1}{m_2(m_2+l)}=\frac{1}{l}\left(\frac{1}{m_2}-\frac{1}{m_2+l}\right)$, we have

$$\begin{aligned}
\mathfrak{N}=&\sum_{m_2,l,k\geq 1}\left\{\frac{1}{k^sm_2l(k+m_2+l)}-\frac{1}{k^sl(m_2+l)(k+m_2+l)}\right\}\\
=&\,\zeta_{MT,3}(s,1,1,1)-\mathfrak{N}.
\end{aligned}$$

Hence we see that (5.19) equals to (5.20). Combining these results, we have

$$\begin{aligned} G_1(s) &= \zeta_{MT,2}(1,2,s) + 3\zeta_{EZ,3}(1,1,s+1) + \zeta(2)\zeta(s+1) \\ &\quad + \zeta_{MT,2}(s,2,1) + \frac{1}{2}\zeta_{MT,3}(s,1,1,1). \end{aligned} \tag{5.22}$$

Lastly, using (2.28) and

$$\frac{1}{lmn} = \left(\frac{1}{lm} + \frac{1}{mn} + \frac{1}{ln}\right)\frac{1}{l+m+n},$$

we have

$$\zeta_{MT,3}(1,1,1,s) = 6\zeta_{EZ,3}(1,1,s+1). \tag{5.23}$$

Substituting (5.14), (5.22) and (5.23) into (5.13), we have

$$\begin{aligned} \mathscr{Z}_3(s) =& 6\zeta_{EZ,3}(1,1,s+1) - 3\zeta_{MT,3}(s,1,1,1) + 6\zeta_{MT,2}(1,2,s) \\ &+ 6\zeta_{MT,2}(s,2,1) - 6\zeta(2)\zeta(s+1) + 12\zeta(s+3). \end{aligned} \tag{5.24}$$

Note that each function on the right-hand side of (5.24) is meromorphic on $\mathbb{C}$. Hence, from Proposition 5.3, we obtain the following.

Proposition 5.4

$$\begin{aligned} &2\zeta_{EZ,3}(1,1,s+1) - \zeta_{MT,3}(s,1,1,1) + 2\zeta_{MT,2}(1,2,s) \\ &+ 2\zeta_{MT,2}(s,2,1) - 2\zeta(2)\zeta(s+1) + 4\zeta(s+3) = 0 \end{aligned} \tag{5.25}$$

holds for all $s \in \mathbb{C}$ except for the singularities of functions on the left-hand of (5.25).

Remark Let $s = 1$ in (5.25). By (5.23), we have $\zeta_{MT,3}(1,1,1,1) = 6\zeta_{EZ,3}(1,1,2)$. Furthermore it is known that $\zeta_{MT,2}(1,2,1) = \frac{1}{2}\zeta(2)^2$ (see [16]). Thus we obtain

$$\zeta_{EZ,3}(1,1,2) = \zeta(4), \tag{5.26}$$

which is a well-known relation (see [7]). In other words, (5.25) is a non-trivial continuous relation interpolating (5.26). As well as Proposition 5.3, we can also obtain some functional relations for $2p$-ple zeta functions. Furthermore we can find other non-trivial functional relations among various multiple zeta-functions from Theorem 3.1. Hence our investigation gives a kind of partial positive answer to the problem introduced in Section 1.

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