The discrete universality of L-functions of newforms

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In the paper a discrete universality theorem of Voronin's type for L-functions of newforms is obtained.

References: 34 titles.

1. Introduction. In the work [1] S. M. Voronin, see also [2] – [6], obtained a brilliant theorem on the universality of the Riemann zeta-function $\zeta(s)$, $s = \sigma + it$. He proved that every analytic non-vanishing on the disc $\left\{s \in \mathbb{C} : |s| < r, r < \frac{1}{4}\right\}$ function which is continuous on the boundary of this disc can be uniformly approximated with desirable accuracy by translations of the function $\zeta(s)$. This theorem was improved and generalized for other zeta-functions and Dirichlet series by many authors. We will state the last version of the Voronin theorem [7]. Let $\mathrm{meas}\{A\}$ denote the Lebesgue measure of the set A, and let, for $T > 0$,

$$\nu_T(\ldots) = \frac{1}{T}\mathrm{meas}\{\tau \in [0, T] : \ldots\},$$

where the dots mean some condition satisfied by τ. Suppose that K is a compact subset with connected complement of the strip $\{s \in \mathbb{C} : 1/2 < \sigma < 1\}$, and let a function $f(s)$ be continuous and non-vanishing on K, and analytic in the interior of K. Then, for every $\varepsilon > 0$,

$$\liminf_{T\to\infty} \nu_T(\sup_{s\in K} |\zeta(s + i\tau) - f(s)| < \varepsilon) > 0.$$

The later theorem shows that the set f shifts of the Riemann zeta-function which approximate a given analytic function is sufficiently rich: it has a positive lower density.

The universality property for other functions defined by Dirichlet series has been obtained by S. M. Gonek [8], A. Reich [9], [10], B. Bagchi [11], [12], A. Laurinčikas [7], [13] – [17], A. Laurinčikas and K. Matsumoto [18], [19], A. Laurinčikas, K. Matsumoto and J. Steuding [20], K. Matsumoto [22], J. Steuding [23], [24], H. Mishou [25], [26], H. Bauer [27], R. Garunkštis [28], A. Laurinčikas, W. Schwarz and J. Steuding [21], and by other authors. It turned out that the majority of classical zeta and L- functions are universal in the Voronin sense. The Linnik - Ibragimov conjecture says that all functions in a certain half-plane given by Dirichlet series, analytically continuable to the left of the half-plane of absolute convergence and satisfying some natural growth conditions are universal. Without any doubt the latter conjecture is complicated thouth all examples support it.

In [20] we obtained the universality of L-functions of newforms by using a method based on the method of the work [19], where the case of newforms of level 1 was considered. Let $SL(2,\mathbb{Z})$ be the full modular group, and let

$$\Gamma_0(q) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) : c \equiv 0 \, (\mathrm{mod}\, q) \right\}$$

be its Hecke subgroup. A function $F(z)$ analytic in the upper half-plane $\mathrm{Im}\, z > 0$ is called the cusp form of weight $\varkappa$ and level q if, for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(q)$,

$$F\left(\frac{az+b}{cz+d}\right) = (cz+d)^{\varkappa} F(z),$$

and if $F(z)$ is analytic and vanishing at the cusps. In this case $F(z)$ has at ∞ the following Fourier series expansion

$$F(z) = \sum_{m=1}^{\infty} c(m) e^{2\pi i m z}.$$

A form $F(z)$ is called a newform if it is not a cusp form of level less that q, and if it is an eigenfunction of all Hecke operators. This implies the inequality $c(1) \neq 0$, and we can suppose that $c(1) = 1$.

To each normalized newform we can attach the L-function

$$L(s,F) = \sum_{m=1}^{\infty} \frac{c(m)}{m^s}.$$

The later series converges absolutely for $\sigma > (\varkappa+1)/2$. Moreover, the function $L(s,F)$ is analytically continuable to the whole s-plane and is an entire function. In the half-plane $\sigma > (\varkappa+1)/2$ the function $L(s,F)$ has the Euler product over primes

$$L(s,F) = \prod_{p|q} \left(1 - \frac{c(p)}{p^s}\right)^{-1} \prod_{p\nmid q} \left(1 - \frac{c(p)}{p^s} + \frac{1}{p^{2s+1-\kappa}}\right)^{-1}.$$

Let $D = \{s \in \mathbb{C} : \kappa/2 < \sigma < (\kappa+1)/2\}$. Then in [20] the following statement has been obtained.

Theorem 1. *Suppose that $F(z)$ is a normalized newform of weight $\varkappa$ and level q. Let K be a compact subset of the strip D with connected complement,* *and let $f(s)$ be a continuous non-vanishing on K function which is analytic in the interior of K. Then, for every $\varepsilon > 0$*

$$\liminf_{T\to\infty} \nu_T(\sup_{s\in K} |L(s+i\tau, F) - f(s)| < \varepsilon) > 0.$$

The aim of this note is to obtain a discrete version of Theorem 1. Let, for positive integer N,

$$\mu_N(\ldots) = \frac{1}{N+1}\#(0 \le m \le N : \ldots),$$

where the dots mean a certain condition satisfied by m. Suppose that h is a fixed positive number such that $\exp\left\{\frac{2\pi k}{h}\right\}$ is irrational for all integers $k \neq 0$. For example, we can take $h = \pi$, since by the Lindemann theorem $\exp\{2k\}$, $k \neq 0$, is irrational.

Theorem 2. *Let $F(z)$, K and $f(s)$ satisfy the conditions of Theorem 1. Then, for every $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \mu_N(\sup_{s\in K} |L(s+imh, F) - f(s)| < \varepsilon) > 0.$$

Recently, the Schimura-Taniyama conjecture has been proved [29]. Consequently, every non-singular elliptic curve over the field of rational number

$$E : y^2 = x^3 + ax + b, \quad \Delta = -16(4a^3 + 27b^2) \neq 0,$$

is modular, therefore its L-function

$$L_E(s) = \prod_{p\mid\Delta}\left(1 - \frac{\lambda(p)}{p^s}\right)^{-1} \prod_{p\nmid\Delta}\left(1 - \frac{\lambda(p)}{p^s} + \frac{1}{p^{2s-1}}\right)^{-1}, \quad \sigma > \frac{3}{2},$$

coincides with L-function of a certain newform of weight 2 of some Hecke subgroup. Then, in view of Theorem 2 $L_E(s)$ has the discrete universality property. Here in the difinition of $L_E(s)$ $\lambda(p) = p - \nu(p)$, where $\nu(p)$ is the number of solutions of the congruence

$$y^2 \equiv a^3 + ax + b(\mathrm{mod} p).$$

Similarly to Theorem 2 the following statement can be obtained.

Theorema 3. *Let $F(z)$ and K satisfy the conditions of Theorem 2, and let the function $f(s)$ be continuous on K and analytic in the interior of K. Then, for every $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \mu_N(\sup_{s\in K} |L'(s+imh, F) - f(s)| < \varepsilon) > 0.$$

Note that in Theorem 3 the function $f(s)$ may have zeros on K.

Theorema 4. *For arbitrary σ_1, σ_2, $\kappa/2 < \sigma_1 < \sigma_2 < (\kappa+1)/2$, there exists a constant $c = c(\sigma_1, \sigma_2) > 0$ such that for sufficiently large N the function $L'(s+imh, F)$ has a zero on the disc $|s - \hat{\sigma}| \leq (\sigma_2 - \sigma_1)/2$, $\hat{\sigma} = \frac{\sigma_1+\sigma_2}{2}$, for more than cN numbers m, $0 \leq m \leq N$.*

2. A limit theorem for the function $L(s, F)$. The proof of Theorem 2 uses a discrete limit theorem in the sence of the weak convergence of probability measures in the space of analytic functions for the function $L(s, F)$. Let $H(G)$ be the space of analytic on G functions equipped with the topology of uniform convergence on compacta. Denote by $\mathcal{B}(S)$ the class of Borel sets of the space S. In this section we will study the weak convergence of the probability measure

$$P_N(A) = \mu_N(L(s+imh, F) \in A), \quad A \in \mathcal{B}(H(D_M)),$$

where $D_M = \{s \in \mathbb{C} : \kappa/2 < \sigma < (\kappa+1)/2, \, |t| < M\}$.

Let $\gamma = \{s \in \mathbb{C} : |s| = 1\}$ denote the unit circle on the complex plane, and

$$\Omega = \prod_p \gamma_p,$$

where $\gamma_p = \gamma$ for all primes p. With product topology and pointwise multiplication the infinite-dimensional torus Ω is a compact topological Abelian group. Therefore on $(\Omega, \mathcal{B}(\Omega))$ the probability Haar measure m_H exists, and this leads to a probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Denote by $\omega(p)$ the projection of element $\omega \in \Omega$ to the coordinate space γ_p, and on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$ define an $H(D_M)$-valued random element $L(s, \omega, F)$ by the formula

$$L(s, \omega, F) = \prod_{p|q} \Big(1 - \frac{c(p)\omega(p)}{p^s}\Big)^{-1} \prod_{p\nmid q} \Big(1 - \frac{c(p)\omega(p)}{p^s} + \frac{\omega^2(p)}{p^{2s+1-\kappa}}\Big)^{-1}. \qquad (1)$$

Let P_L stand for the distribution of the random element $L(s, \omega, F)$, i. e.

$$P_L(A) = m_H(\omega \in \Omega : L(s, \omega, F) \in A), \quad A \in \mathcal{B}(H(D_M)).$$

Lemma 1. *The probability measure P_N weakly converges to P_L as $N \to \infty$.*

Proof. The lemma is a consequence of a general discrete limit theorem for the Matsumoto zeta function obtained in [30].

For primes p define positive integers $g(p)$. Moreover, let $a_p^{(j)} \in \mathbb{C}$ and $f(j,p)$, $1 \le j \le g(p)$, be positive integers. Define a polynomial

$$A_p(x) = \prod_{j=1}^{g(p)} \left(1 - a_p^{(j)} x^{f(j,p)}\right)$$

of degree $f(1,p) + \ldots + f(g(p),p)$. In [31] K. Matsumoto began to study the following zeta-function

$$\varphi(s) = \prod_p A_p^{-1}(p^{-s}). \tag{2}$$

Under restrictions $g(p) \le cp^{\alpha}$ and $|a_p^{(j)}| \le p^{\beta}$ with positive constant c and non-negative constants α and β the infinite product in (2) aconverges absolutely for $\sigma > \alpha + \beta + 1$ and defines there an analytic function with no zeros.

In the work [30] a discrete limit theorem in the space of meromorphic functions for the function $\varphi(s)$ has been obtained. Suppose that the function $\varphi(s)$ is meromorphically continuable to the region $\hat{D} = \{s \in \mathbb{C} : \sigma > \rho_0\}$ where $\alpha + \beta + \frac{1}{2} \le \rho_0 \le \alpha + \beta + 1$, and that all poles in this region are included in a compact set. Moreover, let, for $\sigma > \rho_0$,

$$\varphi(\sigma + it) = O(|t|^{\delta}), \quad |t| \ge t_0,$$

with some $\delta > 0$, and

$$\int_0^T |\varphi(\sigma + it)|^2 \mathrm{d}t = O(T), \quad T \to \infty.$$

Denote by $M(\hat{D})$ the space of meromorphic on $\hat{D}$ functions with the topology of uniform convergence on compacta, and let

$$\varphi(s,\omega) = \prod_p \prod_{j=1}^{g(p)} \left(1 - \frac{\omega^{f(j,p)}(p) a_p^{(j)}}{p^{sf(j,p)}}\right)^{-1}$$

be an $H(\hat{D})$-valued random element on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Then in [30] it was proved that the probability measure

$$\mu_N(\varphi(s + imh) \in A), \quad A \in \mathcal{B}(M(\hat{D})),$$

weakly converges to the distribution of the random element $\varphi(s,\omega)$ as $N \to \infty$.

For $\sigma > (\kappa+1)/2$ the function $L(s,F)$ can be written in the form

$$L(s,F)=\prod_{p|q}\left(1-\frac{c(p)}{p^s}\right)^{-1}\prod_{p\nmid q}\left(1-\frac{a(p)}{p^s}\right)^{-1}\left(1-\frac{b(p)}{p^s}\right)^{-1},$$

where $c(p)=a(p)+b(p)$, and in view of the Deligne estimates

$$|a(p)|\le p^{\frac{\kappa-1}{2}},\quad |b(p)|\le p^{\frac{\kappa-1}{2}}.$$

Consequently, the function $L(s,F)$ is a particular case of the Matsumoto zeta-function with $\alpha=0$ and $\beta=(\kappa-1)/2$.

The function $L(s,F)$ satisfies the functional equation

$$q^{\frac{s}{2}}(2\pi)^{-s}\Gamma(s)L(s,F)=\varepsilon(-1)^{\frac{\kappa-s}{2}}(2\pi)^{s-\kappa}\Gamma(\kappa-s)L(\kappa-s,F),$$

where, as usual, $\Gamma(s)$ is the Euler gamma-function, and $\varepsilon=\pm1$ is the sign of the function equation. The latter functional equation and the Phragmén-Lindelöf principle imply, for $\sigma>\kappa/2$, the estimate

$$L(\sigma+it,F)=O(|t|^\delta),\quad |t|\ge t_0,$$

with some $\delta>0$. Moreover, in [32] it was proved that, for $\sigma>\kappa/2$,

$$\int_0^T|L(\sigma+it,F)|^2\mathrm{d}t=O(T),\quad T\to\infty.$$

Thus, we have that all hypotheses of the theorem from [30] are satisfied with $\rho_0=\frac{\kappa}{2}$, and, since the function $L(s,F)$ is entire, the probability measure

$$\mu_N(L(s+imh,F)\in A),\quad A\in\mathcal{B}(H(\hat{D})),$$

weakly converges to the distribution of the $H(\hat{D})$-valued random element defined by (1) as $N\to\infty$. Since the function $u:H(\hat{D})\to H(D_M)$ given by the coordinatewise restriction is continuous, hence and from Theorem 2.1 [33] we obtain the assertion of the lemma.

3. The support of the measure P_L. Let P be a probability measure on $(S,\mathcal{B}(S))$. We recall that a support of the measure P is the minimal closed set $S_P\subset S$ such that $P(S_P)=1$. The set S_P consists of all elements $x\in S$ such that for every neighbourhood G of element x the inequality $P(G)>0$ takes place.

Let

$$S_M=\{g\in H(D_M): g(s)\ne 0\quad\text{or}\quad g(s)\equiv 0\}.$$

Lemma 2. *The support of the measure P_L is the set S_M.*

Proof. In [20] it was proved that the probability measure

$$\nu_T(L(s+i\tau, F) \in A), \quad A \in \mathcal{B}(H(D_M)),$$

weakly converges to the measure P_L as $T \to \infty$, and also there it was obtained that the support of the measure P_L is the set S_M.

4. Proof of Theorem 2. Obviously, there exists $M > 0$ such that the compact set $K \subset D_M$.

First we suppose that the function $f(s)$ has a non-vanishing analytic continuation to the region D_M. Denote by G the set of functions $g \in H(D_M)$ such that

$$\sup_{s\in K} |g(s) - f(s)| < \varepsilon.$$

In view of Lemma 2 the set G lies in the support of the measure P_L. Therefore $P_L(G) > 0$. Moreover, the set G is open. Consequently, Theorem 2.1 from [33] and Lemma 1 yield.

$$\liminf_{N\to\infty} \mu_N(\sup_{s\in K} |L(s+imh, F) - f(s)| < \varepsilon) \geq P_L(G) > 0. \tag{3}$$

Now let $f(s)$ satisfy the hypotheses of Theorem 2. Then by the Mergelyan theorem [34] there exists a polynomial $p_n(s)$, $p_n(s) \neq 0$ on K, such that

$$\sup_{s\in K} |f(s) - p_n(s)| < \frac{\varepsilon}{4}. \tag{4}$$

The polynomial $p_n(s)$ has only finitely many of zeros. Therefore there is a region G_1 with conected complement such that $K \subset G_1$ and $p_n(s) \neq 0$ on G_1. This shows that on G_1 there exists a continuous branch $\log p_n(s)$ which is analytic in the interior of G_1. By the Margelyan theorem again there exists a polynomial $q_m(s)$ such that

$$\sup_{s\in K} |p_n(s) - \exp\{q_m(s)\}| < \frac{\varepsilon}{4}.$$

This and (4) show that

$$\sup_{s\in K} |f(s) - \exp\{q_m(s)\}| < \frac{\varepsilon}{2}. \tag{5}$$

However, $\exp\{q_m(s)\} \neq 0$. Consequently, by (3)

$$\liminf_{N\to\infty} \mu_N \left(\sup_{s\in K} |L(s+imh, F) - \exp\{q_m(s)\}| < \frac{\varepsilon}{2}\right) > 0.$$

This and (5) yield the assertion of Theorem 2.

The proof of Theorem 3 is similar to that of Theorem 2. In this case the support of the limit measure of the measure

$$\mu_N(L'(s+imh,F)\in A),\quad A\in\mathcal{B}(H(D_M)),$$

as $N\to\infty$ is the whole of $H(D_M)$. This is obtained by asing the arguments similar to those used in [17].

Proof of Theorem 4. Let

$$\hat\sigma=\frac{\sigma_1+\sigma_2}{2},\quad \sigma_0=\max\left\{\left|\sigma_1-\frac{2\kappa+1}{4}\right|,\left|\sigma_2-\frac{2\kappa+1}{4}\right|\right\},$$

and $f(s)=s-\hat\sigma$ and $0<\varepsilon<(\sigma_2-\sigma_1)/10$. Then, by Theorem 3 there exists a constant $c=c(\sigma_1,\sigma_2)>0$ such that for sufficiently large N

$$\mu_N\left(\max_{|s-\frac{2\kappa+1}{4}|\le\sigma_0}|L'(s+imh,F)-f(s)|<\varepsilon\right)>c. \tag{6}$$

The circle $|s-\hat\sigma|=(\sigma_2-\sigma_1)/2$ lies in the disc

$$\left|s-\frac{2\kappa+1}{4}\right|\le\sigma_0.$$

Therefore for m satisfying (6)

$$\max_{|s-\hat\sigma|=\frac{\sigma_2-\sigma_1}{2}}|L'(s+imh,F)-(s-\hat\sigma)|<\frac{\sigma_2-\sigma_1}{10}.$$

This shows that the functions $(s-\hat\sigma)$ and $L'(s+imh)-(s-\hat\sigma)$ on the disc $|s-\hat\sigma|\le(\sigma_2-\sigma_1)/2$ satisfy the hypotheses of Rouche's theorem. However, the function $s-\hat\sigma$ on this disc has exactly one zero. Therefore, by Rou che's theorem the function $L'(s+imh,F)$ on the disc $|s-\hat\sigma|\le(\sigma_2-\sigma_1)/2$ has one zero. Since the number of such m, $0\le m\le N$, in view of (6) is greater then cN, the theorem is proved.

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