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Rigidity of the Weyl chamber flow, and vanishing theorems of Matsushima and Weil

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Abstract. The aim of the present paper is to reveal an unforeseen link between the classical vanishing theorems of Matsushima and Weil, on the one hand, and rigidity of the Weyl chamber flow, a dynamical system arising from a higher-rank non-compact Lie group, on the other. The connection is established via 'transverse extension theorems': roughly speaking, they claim that a tangential 1-form of the orbit foliation of the Weyl chamber flow that is tangentially closed (and satisfies a certain mild additional condition) can be extended to a closed 1-form on the whole space in a canonical manner. In particular, infinitesimal rigidity of the orbit foliation of the Weyl chamber flow is proved as an application.

1. *Introduction*

The old vanishing theorems, such as those of Weil or of Matsushima, have emerged in a series of attempts to grasp rigidity phenomena observed in group homomorphisms into *finite-dimensional* Lie groups: for instance, the vanishing theorem of Matsushima says that there is no homomorphism of a lattice Γ of a higher-rank Lie group into the additive group $\mathbb{R}$ other than the trivial one, while Weil's vanishing is a key step in the proof of his local rigidity of the inclusion map of a lattice Γ into the ambient Lie group G. In contrast, the Weyl chamber flow is a smooth group action: it is a continuous homomorphisms into a diffeomorphism group, which is a typical example of *'infinite-dimensional'* Lie groups. The infinite-dimensionality causes serious difficulties which seem inevitable in the understanding of actions of non-compact groups (cf. [**Ka, KS2, KS3**]). A huge gap must lie between finite- and infinite-dimensional realms. However, the gap is bridged, surprisingly without much effort, by our 'transverse extension theorems', which have their origin in the recent work of Matsumoto and Mitsumatsu [**MM**]. Before embarking on the extension theorems, we should quickly review the old vanishing theorems we are interested in, and the definition of the Weyl chamber flow which plays literally the central role in the present paper.

1.1. *Classical vanishing theorems.* Let G be a connected semisimple Lie group with finite center. Assume that the Lie algebra $\mathfrak{g}$ of G is of real rank greater than or equal to two, and has neither compact factor nor simple factor isomorphic to $\mathfrak{so}(k, 1)$, $\mathfrak{su}(k, 1)$. Furthermore, let Γ be a torsion-free irreducible cocompact lattice of G. Then the following vanishing theorems hold:

$$H^1(\Gamma; \mathbb{R}) = 0; \qquad \text{(Matsushima [**M, KN**])}$$

$$H^1(\Gamma; \mathfrak{g}) = 0. \qquad \text{(Weil [**W1, W2**])}$$

In Weil's vanishing, Γ acts on $\mathfrak{g}$ through the adjoint representation of G on $\mathfrak{g}$. These are the classical vanishing theorems with which we concern ourselves.

1.2. *The Weyl chamber flow.* Meanwhile, a dynamical system that intrigues us is constructed in the following manner. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a Cartan decomposition of $\mathfrak{g}$ with $\mathfrak{k}$ being compact. Take a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$, which gives rise to a restricted root space decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\lambda \in \Lambda} \mathfrak{g}_\lambda \quad \text{with } \mathfrak{g}_0 = \mathfrak{m} \oplus \mathfrak{a},$$

where $\mathfrak{m}$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$. Denote by M and A the connected Lie subgroups of G with Lie algebras $\mathfrak{m}$ and $\mathfrak{a}$, respectively. The group A is a free abelian group of rank equal to the real rank of $\mathfrak{g}$. Since the actions of A and M on the coset space $\Gamma \setminus G$ from the right commute, the former descends to the double coset space $W = \Gamma \setminus G/M$. The resulting action of A on W, to which we restrict our attention, is referred to as the *Weyl chamber flow*. Since the Weyl chamber flow is a locally free action, it gives rise to the foliation $\mathcal{B}$ of W by the orbits.

1.3. *Parameter rigidity of the Weyl chamber flow.* The Weyl chamber flow exhibits hyperbolicity in the direction transverse to the orbits. It serves as a typical example of what are called the Anosov actions of higher-rank abelian groups, which have been investigated extensively in the past years. Among such works are those of Katok and Spatzier. To describe their result, denote by $H^*(W, \mathcal{B}; \mathbb{R})$ the tangential (or leafwise) de Rham cohomology of the foliated manifold $(W, \mathcal{B})$. It is the cohomology of the tangential de Rham complex $\{\Omega^*(W, \mathcal{B}; \mathbb{R}), d_{\mathcal{B}}\}$ of the foliated manifold $(W, \mathcal{B})$: an element of $\Omega^p(W, \mathcal{B}; \mathbb{R})$ is an *$\mathbb{R}$-valued tangential p-form*, which is by definition a C^∞ section of the vector bundle $\bigwedge^p T^*\mathcal{B}$ over W with $T^*\mathcal{B}$ the cotangent bundle of $\mathcal{B}$, while the coboundary operator is the *tangential exterior derivative* $d_{\mathcal{B}}$ that is defined in the same manner as the ordinary exterior derivative except that the differential is performed only in the direction tangent to $\mathcal{B}$; that is, for $\omega \in \Omega^p(W, \mathcal{B}; \mathbb{R})$, its tangential exterior derivative $d_{\mathcal{B}}\omega \in \Omega^{p+1}(W, \mathcal{B}; \mathbb{R})$ is defined by

$$\begin{aligned}(d_{\mathcal{B}}\omega)(X_0, \ldots, X_p) = {} & \sum_q (-1)^q X_q \cdot \omega(X_0, \ldots \hat{X}_q \ldots, X_p) \\ & + \sum_{q<r} (-1)^{q+r} \omega([X_q, X_r], X_0, \ldots \hat{X}_q \cdots \hat{X}_r \ldots, X_p), \end{aligned} \tag{1.1}$$

where $X_0, \ldots, X_p$ are vector fields on W that are tangent to the foliation $\mathcal{B}$. One of the results established by Katok and Spatzier is the following theorem:

$$H^1(W, \mathcal{B}; \mathbb{R}) \cong \mathfrak{a}^*. \qquad \text{(Katok and Spatzier [KS2])}$$

Since the Weyl chamber flow is a locally free action of A on W, the tangent space $T_w\mathcal{B}$ of the orbit foliation $\mathcal{B}$ at each point $w \in W$ is naturally identified with the Lie algebra $\mathfrak{a}$ of A. The dual space $\mathfrak{a}^*$ is therefore embedded into $\Omega^1(W, \mathcal{B}; \mathbb{R})$ in such a manner that each element of $\mathfrak{a}^*$ is regarded as a 'constant' tangential 1-form. The theorem of Katok and Spatzier says that the embedding $\mathfrak{a}^* \subset \Omega^1(W, \mathcal{B}; \mathbb{R})$ induces an isomorphism between the tangential de Rham cohomology $H^1(W, \mathcal{B}; \mathbb{R})$ and $\mathfrak{a}^*$. In other words, a tangential 1-form $\alpha \in \Omega^1(W, \mathcal{B}; \mathbb{R})$ is of the form $\alpha = d_{\mathcal{B}}u$ with u a C^∞ function on W whenever it satisfies $d_{\mathcal{B}}\alpha = 0$ and $\int_W \alpha(H) = 0$ for all $H \in \mathfrak{a}$, where the integration is taken with respect to the standard volume form on W.

The theorem of Katok and Spatzier readily implies the parameter rigidity of the Weyl chamber flow [**KS2**]: *any smooth action of A on W that is smoothly orbit-equivalent to the Weyl chamber flow has to be smoothly conjugate to the Weyl chamber flow up to a (continuous) automorphism of A.*

1.4. *Parameter rigidity from Matsushima's vanishing and vice versa.* A cochain homomorphism of the ordinary de Rham complex $\Omega^*(W; \mathbb{R})$ into the tangential one $\Omega^*(W, \mathcal{B}; \mathbb{R})$ is given by ignoring the direction transverse to the foliation $\mathcal{B}$. The first result of the present article provides a kind of inverse procedure; namely, a transverse extension of the tangential 1-form.

THEOREM 1.1. *For any tangential* 1*-form $\alpha \in \Omega^1(W, \mathcal{B}; \mathbb{R})$ such that* (i) $d_{\mathcal{B}}\alpha = 0$ *and that* (ii) $\int_W \alpha(H) = 0$ *for all $H \in \mathfrak{a}$, there exists a unique closed* 1*-form $\theta \in \Omega^1(W; \mathbb{R})$ that extends α.*

We refer to the work of Kononenko [**K1**, Theorem 6.1] which is closely related to ours. Indeed, one can derive Theorem 1.1 from that of Kononenko, and *vice versa*, without much effort. There is also an unpublished paper of A. Katok and S. Ferleger prepared in 1997, in which they proved Theorem 1.1.

Theorem 1.1 does yield

$$(\textbf{Matsushima}) \quad \Longleftrightarrow \quad (\textbf{Katok and Spatzier});$$

Matsushima's vanishing and the theorem of Katok–Spatzier are equivalent to each other. The proof of the implication '$\Longleftarrow$' goes as follows. Notice that we have an isomorphism $H^1(\Gamma; \mathbb{R}) \cong H^1(W; \mathbb{R})$, to start with. Take a closed 1-form $\theta \in \Omega^1(W, \mathbb{R})$. To show that the restriction $\alpha = \theta|_{\mathcal{B}}$ of θ to $\mathcal{B}$ satisfies the second condition (ii) in Theorem 1.1, note first that $\mathfrak{a}$ is generated by the elements of the form $[X_\lambda, X_{-\lambda}]$, where $X_{-\lambda}$ is the image of $X_\lambda \in \mathfrak{g}_\lambda$ $(\lambda \in \Lambda)$ under the Cartan involution attached to the Cartan decomposition chosen earlier. Since $X_{\pm\lambda}$ which are thought of as vector fields on W preserve the canonical volume form on W, we have

$$\int_W X_{\pm\lambda} \cdot \theta(X_{\mp\lambda}) = 0 \quad \text{and} \quad 0 = \int_W (d\theta)(X_{-\lambda}, X_\lambda) = -\int_W \theta([X_{-\lambda}, X_\lambda]).$$

Thus the condition (ii) has been verified for $\alpha = \theta|_{\mathcal{B}}$. Due to the theorem of Katok and Spatzier we can find a smooth function u of W such that $d_{\mathcal{B}}u = \alpha = \theta|_{\mathcal{B}}$. Since both du and θ are extensions of $\theta|_{\mathcal{B}}$ to closed 1-forms of W, the uniqueness part of Theorem 1.1 implies that those two have to coincide. This proves '$\Longleftarrow$'. Meanwhile, the proof of the converse is straightforward.

However, what has been shown could not be regarded as a new proof of the theorem of Katok and Spatzier, for *we do need the theorem of Katok and Spatzier in the proof of Theorem 1.1* (cf. §2).

1.5. *Infinitesimal rigidity vs Weil's vanishing.* The parameter rigidity is a rigidity in the direction tangent to the orbits. As to rigidity in the transverse direction, Katok and Spatzier [**KS3**] proved the local rigidity of the foliation $\mathcal{B}$: *a smooth foliation of W that is of the same dimension as $\mathcal{B}$ and has the tangent bundle close to that of $\mathcal{B}$ in an appropriate topology is smoothly conjugate to $\mathcal{B}$ (i.e. there is a diffeomorphism of W that sends the original foliation $\mathcal{B}$ to the perturbed one).* Another transverse rigidity is the infinitesimal rigidity of the foliation $\mathcal{B}$, which could be interpreted as a linearized version of the local one, and is formulated in terms of a variant of the tangential de Rham cohomology; namely, the cohomology $H^*(W, \mathcal{B}; N\mathcal{B})$ of the cochain complex $\{\Omega^*(W, \mathcal{B}; N\mathcal{B}), d^D_{\mathcal{B}}\}$. The cochains $\omega \in \Omega^p(W, \mathcal{B}; N\mathcal{B})$ are the tangential p-forms *taking values in the normal bundle $N\mathcal{B}$* of the foliation $\mathcal{B}$, i.e. the C^∞ sections of $\bigwedge^p T^*\mathcal{B} \otimes N\mathcal{B}$. In the meantime, the tangential exterior derivative $d^D_{\mathcal{B}}$ is defined in terms of the linear holonomy D of $\mathcal{B}$: in the definition (1.1) of the tangential exterior derivative of $\mathbb{R}$-valued tangential differential forms, the directional derivative $X_q\cdot$ appearing in the first term of the right-hand side is to be replaced by the covariant derivative $D_{X_q}\cdot$ with respect to the linear holonomy D. The foliation $\mathcal{B}$ is said to be *infinitesimally rigid* if the following vanishing holds (cf. [**Z2**]):

$$H^1(W, \mathcal{B}; N\mathcal{B}) = 0. \qquad \text{(Infinitesimal rigidity)}$$

The infinitesimal rigidity of the foliation $\mathcal{B}$ is also tied to a classical vanishing theorem: indeed, we will show that

$$(\textbf{Weil}) \quad \Longrightarrow \quad (\textbf{Infinitesimal rigidity});$$

that is, the infinitesimal rigidity of the foliation $\mathcal{B}$ follows from the vanishing theorem of Weil. This is also established by means of some transverse extension theorem. In order to talk about an extension of an $N\mathcal{B}$-valued tangential differential form of $(W, \mathcal{B})$, we need a vector bundle over W that contains $N\mathcal{B}$ as a subbundle, and a *flat* connection of that bundle in terms of which the covariant exterior derivatives of differential forms taking values in that bundle are defined. Unfortunately, we are not able to find a pair of such a bundle and a connection (as far as we seek 'natural' ones). To avoid this trouble, we work with $V = \Gamma \setminus G$ instead of $W = \Gamma \setminus G/M$, and the right action of A on V. The foliation of V by the orbits of the action would be denoted by $\mathcal{A}$. On the other hand, denote by $\Omega^*(V; \mathfrak{g})$ the space of $\mathfrak{g}$-valued C^∞ differential forms on V. A twisted exterior derivative acting on $\mathfrak{g}$-valued differential forms is defined by $d^\nabla = d + \mathrm{ad}$ with d the ordinary exterior derivative and ad defined by

$$(\mathrm{ad}\,\omega)(X_0, \ldots, X_p) = \sum_q (-1)^q \mathrm{ad}(X_q) \cdot \omega(X_0, \ldots \widehat{X}_q \ldots, X_p)$$

for $\omega \in \Omega^p(V; \mathfrak{g})$ and $X_0, \ldots, X_p \in TV$. The cohomology of the cochain complex $\{\Omega^*(V; \mathfrak{g}), d^\nabla\}$ is denoted by $H^*(V; \mathfrak{g})$. Note that $H^1(V; \mathfrak{g}) \cong H^1(\Gamma; \mathfrak{g})$. Similarly, we are able to introduce the tangential de Rham cohomology of $(V, \mathcal{A})$ with non-trivial coefficients. In particular, since the linear subspace $\bigoplus_{\lambda \in \Lambda} \mathfrak{g}_\lambda$ of $\mathfrak{g}$ is invariant under the adjoint action of A, we are provided with the cochain complex $\{\Omega^*(V, \mathcal{A}; \bigoplus \mathfrak{g}_\lambda), d_{\mathcal{A}}^\nabla\}$ consisting of the $(\bigoplus_{\lambda \in \Lambda} \mathfrak{g}_\lambda)$-valued C^∞ tangential differential forms of $(V, \mathcal{A})$ and of the twisted tangential exterior derivative $d_{\mathcal{A}}^\nabla = d_{\mathcal{A}} + \mathrm{ad}|_{\mathfrak{a}}$ defined in the same manner as d^∇, and its cohomology $H^*(V, \mathcal{A}; \bigoplus \mathfrak{g}_\lambda)$, as well. Here is another extension theorem which works for differential forms with coefficients in $\bigoplus \mathfrak{g}_\lambda$.

THEOREM 1.2. *For $\alpha \in \Omega^1(V, \mathcal{A}; \bigoplus \mathfrak{g}_\lambda)$ with $d_{\mathcal{A}}^\nabla \alpha = 0$, there exists $\theta \in \Omega^1(V; \mathfrak{g})$ that is an extension of α and satisfies $d^\nabla \theta = 0$. Furthermore, $\theta' \in \Omega^1(V, \mathfrak{g})$ has the same properties if and only if it is of the form $\theta' = \theta + \mathrm{ad}(Z)$ for some $Z \in \mathfrak{g}_0$.*

Notice that, although the values of the original tangential 1-form α are assumed to have no $\mathfrak{g}_0$-component, the values of the resulting 1-form θ are allowed to have a non-trivial $\mathfrak{g}_0$-component.

As we are going to see later in §2, the theorem yields the following.

COROLLARY 1.3. *The orbit foliation $\mathcal{B}$ of the Weyl chamber flow is infinitesimally rigid.*

Kononenko proved an essentially equivalent claim [**K2**, Theorem 11.1] under some extra assumptions, which include the assumptions that the real rank is at least three, and that G is split.

Combining Corollary 1.3 with the parameter rigidity of Katok and Spatzier gives rise to the infinitesimal rigidity of the Weyl chamber flow. It is also possible to derive the infinitesimal rigidity of the action of the lattice Γ on 'boundaries' of G for which local rigidity has already been established by Katok and Spatzier [**KS3**] (see also [**Ka**] for a partial result). See [**LZ, Z2, L, H, P, Q, B**] for other dynamical systems for which infinitesimal rigidity holds.

1.6. *In the rank-one case.* In [**MM**], Matsumoto and Mitsumatsu established a transverse extension theorem for $\mathbb{R}$-valued tangential 1-forms of the unit tangent bundle of a closed hyperbolic surface equipped with the Anosov foliation, namely, the weak (un)stable foliation of the geodesic flow. The present work is an outcome of an attempt to see to what extent the extension theorem of Matsumoto and Mitsumatsu remains valid for other foliated spaces. As a direct generalization of their theorem into the higher-dimensional rank-one case, we obtain the following result, which unfortunately has no direct application to rigidity problems.

Take a closed locally symmetric Riemannian manifold of strictly negative curvature: its universal cover is isometric to either the real, complex, quaternion or Cayley hyperbolic space (up to multiplying the metric by a positive constant). The geodesic flow ψ^t defined on the unit tangent bundle W of the locally symmetric space is a rank-one counterpart of the Weyl chamber flow. However, rigidity phenomena such as the parameter rigidity of Katok and Spatzier fail for the geodesic flow: this is the big difference between the rank-one and the higher-rank cases. We still have a transverse extension theorem for a foliation attached to the geodesic flow. Since the geodesic flow, which is defined as a non-singular flow on the unit tangent bundle W, is known to be Anosov, we are able to speak of the weak unstable foliation $\mathcal{F}$ of the geodesic flow. Since the underlying manifold is locally symmetric, the foliation $\mathcal{F}$ is C^∞. Let $\dot{\psi}$ be the geodesic spray, namely, the vector field on W that generates the geodesic flow ψ^t. We then have the following.

THEOREM 1.4. *For $\alpha \in \Omega^1(W, \mathcal{F}; \mathbb{R})$ with $d_{\mathcal{F}}\alpha = 0$ and with $\int_W \alpha(\dot{\psi}) = 0$, there always exists a unique closed 1-form $\theta \in \Omega^1(W; \mathbb{R})$ that extends α.*

In the theorem, integration is taken with respect to the Liouville measure, a measure on W represented by a smooth volume form invariant under ψ^t. The case where the dimension of the underlying locally symmetric space is two is due to Matsumoto and Mitsumatsu [**MM**], as mentioned earlier.

As an immediate consequence, we are provided with

$$H^1(W, \mathcal{F}; \mathbb{R}) \cong H^1(W, \mathbb{R}) \oplus \mathbb{R}.$$

Recall in addition that the first cohomology $H^1(W, \mathbb{R})$ of W is known to be trivial in the case when the underlining locally symmetric space is covered by either the quaternion or Cayley hyperbolic space.

1.7. *Notation and setup.* Theorems 1.1, 1.2 and Corollary 1.3 are to be proved in §2, and Theorem 1.4 in §3. Before embarking on the proofs of the theorems, let us clarify the notation and setup that are common in the rest of the paper. Denote by G a connected semisimple Lie group with finite center, and by $\mathfrak{g}$ its Lie algebra: assume that none of the simple factors of $\mathfrak{g}$ are compact. Also, take a torsion-free irreducible cocompact lattice Γ of G, and form the quotient $V = \Gamma \setminus G$ on which G acts from the right. The action induces a natural injective homomorphism of the Lie algebra $\mathfrak{g}$ into that of smooth vector fields on V. By abuse of notation, an element of $\mathfrak{g}$ and the corresponding vector field on V are denoted by the same symbol.

Furthermore, take a Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ with $\mathfrak{k}$ compact subalgebra. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$, which gives rise to the restricted root space decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\lambda \in \Lambda} \mathfrak{g}_\lambda \quad \text{with } \mathfrak{g}_0 = \mathfrak{a} + \mathfrak{m},$$

where $\mathfrak{m}$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$, and Λ is the set of non-zero restricted roots. For each $\lambda \in \Lambda$, $X_\lambda, Y_\lambda, \ldots$ represent the elements of the restricted root space $\mathfrak{g}_\lambda$, while $X_0, Y_0, \ldots$ represent those of $\mathfrak{m}$. The elements of $\mathfrak{a}$ are denoted by $H, H_0, H_1, \ldots$.

The connected Lie subgroups of G corresponding to the Lie subalgebras $\mathfrak{a}$ and $\mathfrak{m}$ are respectively denoted by A and M. The former is just a real vector space of dimension equal to the real rank of G. The double coset space $\Gamma \setminus G/M$ is denoted by W. The action of A on W (from the right) is called the Weyl chamber flow in the case where the real rank is greater than one. Meanwhile, in the case where the real rank is equal to one, W is the unit tangent bundle of the locally symmetric Riemannian manifold $\Gamma \setminus G/K$ with K being the connected compact subgroup of G with Lie algebra $\mathfrak{k}$, and the right action of $A \cong \mathbb{R}$ on W is exactly the geodesic flow of $\Gamma \setminus G/K$. Although our main concern is as to the action of A on W, which is the Weyl chamber flow in the case when the real rank is greater than one and is the geodesic flow in the case when the real rank is equal to one, we would mainly handle the right action of A on V, instead of that on W; this is partially because there is a natural global framing on V. The foliation of V (respectively of W) by the orbits of the action of A on V (respectively on W) is denoted by $\mathcal{A}$ (respectively by $\mathcal{B}$). We are going to employ the following fact repeatedly, which follows from Moore's ergodicity theorem (cf. [**Z1**]).

LEMMA 1.5. *The flow of V generated by a non-zero element of $\mathfrak{a}$ is always ergodic with respect to the standard measure of V. In particular, every A-invariant L^2-function on V is constant almost everywhere.*

2. *Proofs of Theorems 1.1, 1.2 and Corollary 1.3*

Throughout the present section, it is assumed in addition that $\mathfrak{g}$ is of real rank greater than one, and has no simple factor isomorphic to $\mathfrak{so}(k, 1)$ or $\mathfrak{su}(k, 1)$.

2.1. *The theorem of Katok and Spatzier, revisited.*

The proofs of our Theorems 1.1 and 1.2 heavily rely on the existence and the smoothness of a solution of the differential equation

$$(d_{\mathcal{A}} - \lambda)u = f \tag{2.1}$$

with $\lambda \in \Lambda \cup \{0\}$, where the given f is an $\mathbb{R}$-valued C^∞ tangential 1-form of $(V, \mathcal{A})$, while the unknown u is an $\mathbb{R}$-valued function on V: equation (2.1) is shorthand for $H \cdot u - \lambda(H)u = f(H)$ $(H \in \mathfrak{a})$. The case $\lambda = 0$ is covered by the aforementioned theorem of Katok and Spatzier. Let us quickly review their theorem, as the remaining case $\lambda \in \Lambda$ as well follows from their theorem.

THEOREM 2.1. (Katok and Spatzier [**KS2**]) *Let f be an $\mathbb{R}$-valued C^∞ tangential 1-form of $(V, \mathcal{A})$ such that* (i) *$d_{\mathcal{A}} f = 0$, and that* (ii) *$\int_V f(H) = 0$ for all $H \in \mathfrak{a}$. Then there always exists a C^∞ function u of V satisfying the differential equation*

$$d_{\mathcal{A}} u = f. \tag{2.2}$$

Moreover, the solution u is unique up to adding a constant.

The proof of the theorem goes as follows. Fix a regular element $H_0 \in \mathfrak{a}$, to start with. Denote by $\phi^t_{H_0}$ the flow on V generated by the vector field H_0. The differential equation $H_0 \cdot u = f(H_0)$ along the orbits of $\phi^t_{H_0}$ has 'formal' solutions $u = u^+$ and u^- with

$$u^+ = -\int_0^\infty f(H_0) \circ \phi^t_{H_0}\, dt, \quad u^- = \int_{-\infty}^0 f(H_0) \circ \phi^t_{H_0}\, dt.$$

Due to the exponential decay of the matrix coefficients, those formal solutions turn out to be well defined as distributions. It was readily seen that both u^+ and u^- satisfy equation (2.2) in the distribution sense. Moreover, partial regularity of those distributions follows from the hyperbolicity of the Weyl chamber flow: u^+ (respectively u^-) is differentiable repeatedly in the directions X_μ $(\mu \in \Lambda)$ with $\mu(H_0) > 0$ (respectively $\mu(H_0) < 0$).

The crucial point of the proof is the equality $u^+ = u^-$: it would imply the differentiability of $u = u^+ = u^-$ in any direction X_μ with $\mu \in \Lambda$, and eventually the regularity or the smoothness of u on the whole space, which is a consequence of a subelliptic regularity theorem in [**KS1**]. The equality $u^+ = u^-$ follows from the matrix coefficient estimate and the higher-rank condition employed in a smart manner. This is a digest of the proof of Theorem 2.1 given in [**KS2**].

We now return to the case (2.1). Let $\lambda \in \Lambda$ be a non-zero restricted root, and f an $\mathbb{R}$-valued C^∞ tangential 1-form of $(V, \mathcal{A})$. In order for equation (2.1) to possess a smooth solution, the in-homogeneous term f has to satisfy the integrability condition

$$d_{\mathcal{A}} f = \lambda \wedge f. \tag{2.3}$$

As to the existence and the smoothness of a solution of (2.1), we have the following.

THEOREM 2.2. *Let $\lambda \in \Lambda$. Suppose that f is an $\mathbb{R}$-valued C^∞ tangential 1-form of $(V, \mathcal{A})$ satisfying the condition (2.3).*

(1) *The differential equation (2.1) has a unique continuous solution u. The solution u is differentiable in the direction X_μ repeatedly with $\mu \in \Lambda \cup \{0\}$ being zero or a restricted root that differs from a multiple of λ by a negative constant.*

(2) *Moreover, u is C^∞ whenever the following conditions are satisfied:*
   (i) *$d_{\mathcal{A}}(X_{-\lambda} \cdot f) = 0$ for all $X_{-\lambda} \in \mathfrak{g}_{-\lambda}$, where $X_{-\lambda} \cdot f$ is a tangential 1-form of $(V, \mathcal{A})$ defined by $(X_{-\lambda} \cdot f)(H) = X_{-\lambda} \cdot f(H)$ $(H \in \mathfrak{a})$;*
   (ii) *in the case when $\mu = \lambda/2 \in \Lambda$, $d_{\mathcal{A}}(Y_{-\mu} X_{-\mu} \cdot f) = 0$ for all $X_{-\mu}, Y_{-\mu} \in \mathfrak{g}_{-\mu}$.*

The conditions (i) and (ii) in Theorem 2.2 are also necessary ones for the regularity of the solution u.

Proof of Theorem 2.2. (1) Take a regular element $H_0 \in \mathfrak{a}$. Equation (2.1) gives

$$(H_0 - \lambda(H_0)) \cdot u = f(H_0),$$

which could be thought of as a linear ordinary differential equation on the orbits of the flow $\phi^t_{H_0}$ generated by the vector field H_0. There is a unique continuous solution u of the equation, whose explicit form is

$$u = -\int_0^\infty e^{-\lambda(H_0)t}\, f(H_0) \circ \phi^t_{H_0}\, dt$$

in the case when $\lambda(H_0) > 0$, and $-\int_0^\infty$ in the formula is to be replaced by $\int_{-\infty}^0$ in the case when $\lambda(H_0) < 0$. In addition, since

$$(d\phi^t_{H_0})(X_\mu) = e^{-\mu(H_0)t} X_\mu$$

$(\mu \in \Lambda \cup \{0\})$, u is differentiable in the direction X_μ provided $\lambda(H_0)$ and $\lambda(H_0) + \mu(H_0)$ have the same sign. In fact, for instance, in the case when $\lambda(H_0) > 0$, the differential is given by

$$X_\mu \cdot u = -\int_0^\infty e^{-(\lambda+\mu)(H_0)t}\, (X_\mu \cdot f(H)) \circ \phi^t_{H_0}\, dt.$$

In particular, it turns out that u is differentiable in the direction $\mathfrak{a}$ repeatedly.

Next, we want to prove that the function u fulfills equation (2.1). For this purpose, put $u_0 = u$. Take another regular element $H_1 \in \mathfrak{a}$, and let u_1 be the continuous solution of $(H_1 - \lambda(H_1)) \cdot u_1 = f(H_1)$. Then we have

$$(H_0 - \lambda(H_0))(H_1 - \lambda(H_1)) \cdot (u_1 - u_0) = (d_{\mathcal{A}} f - \lambda \wedge f)(H_0, H_1) = 0$$

by (2.3). In consequence, it turns out that $u_0 = u_1$. Since the regular elements are dense in $\mathfrak{a}$, $u = u_0$ turns out to be a solution of (2.1).

Unless $\mu \in \Lambda \cup \{0\}$ is a multiple of λ by a negative constant, we can find a regular element $H \in \mathfrak{a}$ with $\lambda(H), \mu(H) > 0$. Thus the integral representation of u along the orbits of the flow generated by H implies that u is differentiable in the directions $X_\mu \in \mathfrak{g}_\mu$ for those μ.

(2) What is still missing is the differentiability of u in the directions X_μ with μ being negative multiples of λ. Notice that the restricted roots that are negative multiples of λ are $-\lambda$ and possibly $-\lambda/2, -2\lambda$.

First, we deal with the differentiability in the direction $X_{-\lambda}$. Again take a regular element $H \in \mathfrak{a}$ with $\lambda(H) > 0$. Recall that a C^∞ solution v of the equation $d_{\mathcal{A}} v = X_{-\lambda} \cdot f$ exists, and is given by

$$v = -\int_0^\infty (X_{-\lambda} \cdot f(H)) \circ \phi_H^t \, dt.$$

This is due to Theorem 2.1 and the assumption (i) in Theorem 2.2 (2). On the other hand, applying formally $X_{-\lambda}$ to the integral representation of u gives rise to

$$X_{-\lambda} \cdot u = -\int_0^\infty (X_{-\lambda} \cdot f(H)) \circ \phi_H^t \, dt.$$

However, the right-hand side is equal to v, which is known to be a C^∞ function. This shows that u is differentiable in the direction $X_{-\lambda}$ as many times as desired.

Similarly, we can handle the case $-\mu = -\lambda/2 \in \Lambda$. From the integral representation of u, it is readily seen that $X_{-\mu} \cdot u$ exists as a continuous function. In addition, it follows from the assumption (ii) in the theorem that $Y_{-\mu} X_{-\mu} \cdot u$ exists and is C^∞. Finally, consider the case where $-2\lambda \in \Lambda$. The differentiability in the direction $X_{-2\lambda}$ follows from that in the direction $X_{-\lambda}$ and the fact that $\mathfrak{g}$ is generated as a Lie algebra by the linear subspace $\bigoplus_{\mu \in \Lambda \setminus \{-2\lambda, 2\lambda\}} \mathfrak{g}_\mu$. The smoothness of u now follows from the subelliptic regularity theorem of Katok and Spatzier [**KS1**]. □

2.2. *The case of the trivial coefficients.* To prove Theorem 1.1, we show the corresponding result on $(V, \mathcal{A})$.

THEOREM 2.3. *For any $\mathbb{R}$-valued C^∞ tangential 1-form $\alpha \in \Omega^1(V, \mathcal{A}; \mathbb{R})$ such that (i) $d_{\mathcal{A}} \alpha = 0$, and that (ii) $\int_V \alpha(Z) = 0$ for all $Z \in \mathfrak{a}$, there exists a unique closed 1-form $\theta \in \Omega^1(V; \mathbb{R})$ that extends α.*

Proof. To determine $\theta(X_\lambda)$ ($\lambda \in \Lambda \cup \{0\}$), take a look at the condition $(d\theta)(H, X_\lambda) = 0$, $H \in \mathfrak{a}$. Since

$$(d\theta)(H, X_\lambda) = (H - \lambda(H)) \cdot \theta(X_\lambda) - X_\lambda \cdot \alpha(H),$$

the equation

$$(d_{\mathcal{A}} - \lambda) \cdot \theta(X_\lambda) = f(X_\lambda) \tag{2.4}$$

has to be fulfilled with $f(X_\lambda) \in \Omega^1(V, \mathcal{A}; \mathbb{R})$ being defined by

$$f(X_\lambda)(H) = X_\lambda \cdot \alpha(H), \quad H \in \mathfrak{a}.$$

The tangential 1-form $f(X_\lambda)$ satisfies the integrability condition (2.3). In addition, in the case when $\lambda \neq 0$ it also fulfills the conditions in Theorem 2.2 that guarantee the smoothness of the solution $\theta(X_\lambda)$. In consequence, the existence of a C^∞ solution $\theta(X_\lambda)$ of (2.4) follows from Theorems 2.1 and 2.2. Moreover, $\theta(X_\lambda)$ is uniquely determined provided $\lambda \neq 0$. Meanwhile, in the case when $\lambda = 0$, we force $\theta(X_0)$ to satisfy the normalization condition

$$\int_V \theta(X_0) = 0, \tag{2.5}$$

whose necessity will be clarified later.

To insure that θ is closed, notice that

$$0 = (d^2\theta)(H, X_\lambda, X_\mu) = (H - (\lambda + \mu)(H)) \cdot (d\theta)(X_\lambda, X_\mu).$$

Thus, in the case when $\lambda + \mu \neq 0$, we immediately have $(d\theta)(X_\lambda, X_\mu) = 0$. On the other hand, in the case when $\lambda + \mu = 0$, we obtain $(d\theta)(X_\lambda, X_{-\lambda}) = \text{const}$. Besides, the condition (ii) imposed on $\alpha = \theta|_{\mathcal{A}}$ in the theorem and the normalization condition (2.5) give us

$$\int_V (d\theta)(X_\lambda, X_{-\lambda}) = -\int_V \theta([X_\lambda, X_{-\lambda}]) = 0.$$

Thus, we are led to $(d\theta)(X_\lambda, X_{-\lambda}) = 0$, as well. □

2.3. *The adjoint representation as the coefficient.* We now go on to the proof of Theorem 1.2. Define an affine connection ∇ of the manifold V by

$$\nabla_X \xi = L_X \xi \quad \text{for } X \in \mathfrak{g} \text{ and a vector field } \xi \text{ on } V,$$

where L denotes the Lie derivative. Since the tangent bundle of V carries a trivialization $TV = V \times \mathfrak{g}$ that arises from the right action of G on V, a vector field ξ of V is simultaneously thought of as a $\mathfrak{g}$-valued function on V. In terms of this identification, the covariant derivative turns into the form

$$\nabla_X \xi = X \cdot \xi + \mathrm{ad}(X)\xi, \quad X \in \mathfrak{g}.$$

It is readily seen that ∇ is flat. For a $\mathfrak{g}$-valued (or equivalently, TV-valued) p-form $\omega \in \Omega^p(V; \mathfrak{g})$ on V, its exterior derivative in terms of ∇ is defined by

$$\begin{aligned}(d^\nabla \omega)(X_0, \ldots, X_p) = &\sum_q (-1)^q \nabla_{X_q} \cdot \omega(X_0, \ldots \hat{X}_q \ldots, X_p) \\ &+ \sum_{q<r} (-1)^{q+r} \omega([X_q, X_r], X_0, \ldots \hat{X}_q \cdots \hat{X}_r \ldots, X_p)\end{aligned}$$

for $X_0, \ldots, X_p \in \mathfrak{g}$. The cohomology of the cochain complex $\{\Omega^*(V; \mathfrak{g}), d^\nabla\}$ is denoted by $H^*(V; \mathfrak{g})$. It is known that $H^1(V; \mathfrak{g}) \cong H^1(\Gamma; \mathfrak{g})$.

Meanwhile, since $\mathfrak{g}_\lambda$ are closed under the adjoint action of A, we are also provided with the cochain complex $\{\Omega^*(V, \mathcal{A}; \bigoplus_{\lambda\in\Lambda} \mathfrak{g}_\lambda), d^\nabla\}$ consisting of the $\bigoplus_{\lambda\in\Lambda} \mathfrak{g}_\lambda$-valued C^∞ tangential differential forms of $(V, \mathcal{A})$ and the tangential covariant exterior derivative d^∇. Its cohomology will be denoted by $H^*(V, \mathcal{A}; \bigoplus \mathfrak{g}_\lambda)$.

Proof of Theorem 1.2. First, we determine $\theta(X_\lambda)$ $(\lambda \in \Lambda \cup \{0\})$ so that

$$(d^\nabla\theta)(H, X_\lambda) = 0, \qquad H \in \mathfrak{a},$$

holds. Taking the $\mathfrak{g}_\mu$-component $(\mu \in \Lambda \cup \{0\})$ of both sides gives

$$(d_{\mathcal{A}} - (\lambda - \mu)) \cdot \theta_\mu(X_\lambda) = f_\mu(X_\lambda), \tag{2.6}$$

where $\theta_\mu(X_\lambda)$ denotes the $\mathfrak{g}_\mu$-component of $\theta(X_\lambda)$, and $f_\mu(X_\lambda)$ the $\mathfrak{g}_\mu$-valued tangential 1-form of $(V, \mathcal{A})$ defined by

$$f_\mu(X_\lambda)(H) = (\nabla_{X_\lambda} \cdot \alpha(H))_\mu, \qquad H \in \mathfrak{a},$$

the $\mathfrak{g}_\mu$-component of $\nabla_{X_\lambda} \cdot \alpha(H)$. It is straightforward to see that the inhomogeneous term $f_\mu(X_\lambda)$ satisfies the integrability condition (2.3).

In addition, in the case when $\mu \neq \lambda$, $f_\mu(X_\lambda)$ satisfies the conditions in Theorem 2.2 as well so that Theorem 2.2 guarantees the unique existence of a C^∞ solution $\theta_\mu(X_\lambda)$ of equation (2.6). Due to the uniqueness, $\theta_\mu(X_\lambda)$ is readily seen to be linear in $X_\lambda \in \mathfrak{g}_\lambda$.

Meanwhile, the case $\lambda = \mu$ is covered by Theorem 2.1. Since the values of α have no $\mathfrak{g}_0$-components, we get $f_\lambda(X_\lambda)(H) = (\nabla_{X_\lambda} \cdot \alpha(H))_\lambda = X_\lambda \cdot \alpha_\lambda(H)$. In particular, $\int_V f_\lambda(X_\lambda)(H) = 0$ $(H \in \mathfrak{a})$ is satisfied, and the existence of a C^∞ solution $\theta_\lambda(X_\lambda)$ of equation (2.6) follows. Unlike the case of $\theta_\mu(X_\lambda)$ with $\lambda \neq \mu$, $\theta_\lambda(X_\lambda)$ is not uniquely determined by (2.6): it determines $\theta_\lambda(X_\lambda)$ just up to adding constant. Meanwhile, $\theta_\lambda(X_\lambda)$ has to be linear in X_λ. This is realized if and only if the following condition is satisfied by

$$I_\lambda(X_\lambda) = \int_V \theta_\lambda(X_\lambda)$$

(where the integration is normalized so that the total volume of V is equal to 1):

$$I_\lambda(X_\lambda) \text{ is linear in } X_\lambda. \tag{2.7}$$

Note that the condition is fulfilled, for instance, by $I_\lambda \equiv 0$.

We now go on to the proof of the identity

$$(d^\nabla\theta)(X_\lambda, X_\mu) = 0 \tag{2.8}$$

for $\lambda, \mu \in \Lambda \cup \{0\}$. First, note that

$$0 = (d^{\nabla 2}\theta)(H, X_\lambda, X_\mu) = (\nabla_H - (\lambda + \mu)(H)) \cdot (d^\nabla\theta)(X_\lambda, X_\mu).$$

Taking the $\mathfrak{g}_\nu$-components $(\nu \in \Lambda \cup \{0\})$ of both sides and denoting the $\mathfrak{g}_\nu$-component of $(d^\nabla\theta)(X_\lambda, X_\mu)$ by $(d^\nabla\theta)_\nu(X_\lambda, X_\mu)$, we obtain

$$(H - (\lambda + \mu - \nu)(H)) \cdot (d^\nabla\theta)_\nu(X_\lambda, X_\mu) = 0,$$

which implies

$$(d^\nabla\theta)_\nu(X_\lambda, X_\mu) = 0 \quad \text{provided that} \nu \neq \lambda + \mu.$$

Meanwhile, in the case when $\nu = \lambda + \mu$, we have

$$H \cdot (d^\nabla\theta)_{\lambda+\mu}(X_\lambda, X_\mu) = 0$$

for all $H \in \mathfrak{a}$; that is, $(d^\nabla\theta)_{\lambda+\mu}(X_\lambda, X_\mu)$ is constant on V. In addition, we have

$$\begin{aligned}\int (d^\nabla\theta)_{\lambda+\mu}(X_\lambda, X_\mu) &= \int \{X_\lambda \cdot \theta_{\lambda+\mu}(X_\mu) + \mathrm{ad}(X_\lambda)\cdot\theta_\mu(X_\mu) \\ &\quad - X_\mu \cdot \theta_{\lambda+\mu}(X_\lambda) - \mathrm{ad}(X_\mu)\cdot\theta_\lambda(X_\lambda) - \theta_{\lambda+\mu}([X_\lambda, X_\mu])\} \\ &= \mathrm{ad}(X_\lambda)I_\mu(X_\mu) - \mathrm{ad}(X_\mu)I_\lambda(X_\lambda) - I_{\lambda+\mu}([X_\lambda, X_\mu]).\end{aligned}$$

Thus, in order to get $(d^\nabla\theta)_{\lambda+\mu}(X_\lambda, X_\mu) = 0$, we need to assume that

$$I_{\lambda+\mu}([X_\lambda, X_\mu]) = \mathrm{ad}(X_\lambda)I_\mu(X_\mu) - \mathrm{ad}(X_\mu)I_\lambda(X_\lambda). \tag{2.9}$$

Again, this is fulfilled by $I_\lambda \equiv 0$. We are thus led to (2.8), and the proof of the existence of a solution θ to the extension problem has been completed.

Finally, we address ourselves to the uniqueness. Recall that the conditions imposed on $\theta_\lambda(X_\lambda)$ are the differential equation (2.6) with $\mu = \lambda$, (2.7) and (2.9). Let $\theta_\lambda^{(0)}(X_\lambda)$ be the solution of (2.6) with $\mu = \lambda$ that corresponds to $I_\lambda \equiv 0$, and $\theta_\lambda(X_\lambda)$ another solution with an arbitrary I_λ satisfying (2.7) and (2.9). Since both $\theta_\lambda^{(0)}(X_\lambda)$ and $\theta_\lambda(X_\lambda)$ satisfy the equation (2.6), they differ from each other just by a constant. Meanwhile, we are able to define a linear endomorphism I of $\mathfrak{g}$ by setting $I|_{\mathfrak{g}_\lambda} = I_\lambda$ $(\lambda \in \Lambda \cup \{0\})$. According to (2.9), I is a derivation of $\mathfrak{g}$. Since $\mathfrak{g}$ is semisimple, I has to be realized as the adjoint action of a certain element $Z \in \mathfrak{g}$: $I = \mathrm{ad}(Z)$. Moreover, Z has to be an element of $\mathfrak{g}_0$, since $I = \mathrm{ad}(Z)$ preserves the restricted root space decomposition. It thus follows that $\theta_\lambda = \theta_\lambda^{(0)} + \mathrm{ad}(Z)$ for some $Z \in \mathfrak{g}_0$. Conversely, θ_λ defined by the last identity clearly satisfies those conditions imposed on it. □

We now go on to the following.

Proof of Corollary 1.3. Take an $N\mathcal{B}$-valued C^∞ tangential 1-form of $(W, \mathcal{B})$ that is tangentially closed with respect to $d_{\mathcal{B}}^D$. Lifting it onto $(V, \mathcal{A})$, we are provided with a $d_{\mathcal{A}}^\nabla$-closed element α of $\Omega^1(V, \mathcal{A}; \bigoplus \mathfrak{g}_\lambda)$ which is M-equivariant in the sense that $R_m^*\alpha = \mathrm{Ad}(m^{-1})\alpha$ for $m \in M$, where R_m denotes the right action of m on V. To show the corollary, it is sufficient to find an M-equivariant $\mathfrak{g}$-valued C^∞ function u on V such that $d_{\mathcal{A}}^\nabla u = \alpha$.

Applying Theorem 1.2 to α yields its extension $\theta \in \Omega^1(V; \mathfrak{g})$ with $d^\nabla\theta = 0$. According to the vanishing theorem of Weil, there must be a $\mathfrak{g}$-valued C^∞ function $\bar{u}$ of V such that $d^\nabla\bar{u} = \theta$. It obviously satisfies $d_{\mathcal{A}}^\nabla\bar{u} = \alpha$. Decompose $\bar{u}$ into $\mathfrak{g}_\lambda$-valued functions $\bar{u}_\lambda$ $(\lambda \in \Lambda \cup \{0\})$, each of which satisfies the differential equation $(d_{\mathcal{A}} - \lambda)\cdot\bar{u}_\lambda = \alpha_\lambda$, where α_λ is the $\mathfrak{g}_\lambda$-component of α. For $\lambda \neq 0$, $\bar{u}_\lambda$ is completely determined by the equation. In the meantime, $\bar{u}_0$ is determined just up to a constant. We now define a new $\mathfrak{g}$-valued function u of V by $u = \bar{u} - u_0$, which also satisfies $d_{\mathcal{A}}^\nabla u = \alpha$, as α has no $\mathfrak{g}_0$-component. It is the unique solution to the last equation with no $\mathfrak{g}_0$-component. Thus it has to be M-equivariant. This completes the proof of the corollary. □

3. *Proof of Theorem 1.4*

In what follows, G is assumed to be of real rank one. By identifying $\mathfrak{a}^*$ with $\mathbb{R}$, the restricted root space decomposition of the Lie algebra $\mathfrak{g}$ of G turns into the form

$$\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{+1} \oplus \mathfrak{g}_{+2} \quad \text{with } \mathfrak{g}_0 = \mathfrak{a} + \mathfrak{m};$$

we adopt the convention that $\mathfrak{g}_{\pm 2} = 0$ when the corresponding symmetric space G/K is the *real* hyperbolic space. Take the element H of $\mathfrak{a}$ that generates the geodesic flow ψ^t defined on the unit tangent bundle $W = \Gamma \setminus G/M$. Each root space $\mathfrak{g}_\lambda$ ($\lambda = 0, \pm 1, \pm 2$) is then characterized by $\mathrm{ad}(H)|_{\mathfrak{g}_\lambda} = \lambda$. Denote by ϕ^t the corresponding flow on V. The foliation of V that integrates the subbundle of TV arising from the Borel subalgebra $\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0$ of $\mathfrak{g}$ is denoted by $\mathcal{E}$. Theorem 1.4 is an immediate consequence of the following.

THEOREM 3.1. *Any $\mathbb{R}$-valued C^∞ tangential 1-form α of the foliated manifold $(V, \mathcal{E})$ such that $d_{\mathcal{E}}\alpha = 0$ and that $\int_V \alpha(X) = 0$ for all $X \in \mathfrak{g}_0$ can be uniquely extended to an $\mathbb{R}$-valued C^∞ closed 1-form θ on V.*

To prove the theorem, we need to work with the following differential equation for a function u of V, where f is a given C^∞ function on V:

$$(H - 1) \cdot u = f. \tag{3.1}$$

The equation has a unique solution

$$u = -\int_0^\infty e^{-t}(f \circ \phi^t)\, dt \tag{3.2}$$

that is differentiable on each orbit of the flow ϕ^t, and is continuous on the whole space V. Since

$$(d\phi^t)(X_\mu) = e^{-\mu t} X_\mu \quad \text{for } \mu = 0, \pm 1, \pm 2, \tag{3.3}$$

u is repeatedly continuously differentiable in the direction $E_0 \oplus E_{+1} \oplus E_{+2}$, where E_μ ($\mu = 0, \pm 1, \pm 2$) denotes the subbundle of TV that corresponds to the root space $\mathfrak{g}_\mu$. As to the differentiability in the remaining directions, we have the following.

LEMMA 3.2. *If there is a C^∞ section k of E^*_{-1} such that*

$$Y_{-1} \cdot f = H \cdot k(Y_{-1}) \quad \textit{for all } Y_{-1} \in \mathfrak{g}_{-1},$$

the solution (3.2) of equation (3.1) is C^∞.

Proof. To start with, we show the L^2-integrability of $Y_{-1} \cdot u$ ($Y_{-1} \in \mathfrak{g}_{-1}$). Note that $Y_{-1} \cdot (f \circ \phi^t) = e^t(Y_{-1} \cdot f) \circ \phi^t = e^t(H \cdot k(Y_{-1})) \circ \phi^t$, by (3.3) and the assumption made in the lemma. Take an arbitrary C^∞ function τ on V as a test function. Then, for the coupling of u and $Y_{-1} \cdot \tau$ with the L^2 inner product $\langle \cdot , \cdot \rangle_{L^2}$, we have

$$\langle u,\ Y_{-1}\cdot\tau\rangle_{L^2} \underset{\text{as } T\to\infty}{\longleftarrow} -\int_0^T e^{-t}\langle f\circ\phi^t,\ Y_{-1}\cdot\tau\rangle_{L^2}\,dt$$
$$= \int_0^T e^{-t}\langle Y_{-1}\cdot(f\circ\phi^t),\ \tau\rangle_{L^2}\,dt$$
$$= \int_0^T \langle (H\cdot k(Y_{-1}))\circ\phi^t,\ \tau\rangle_{L^2}\,dt$$
$$= \langle k(Y_{-1})\circ\phi^T - k(Y_{-1}),\ \tau\rangle_{L^2}.$$

Thus, for the directional derivative $Y_{-1}\cdot u$ in the distribution sense, we obtain

$$|\langle Y_{-1}\cdot u,\ \tau\rangle_{L^2}| \le 2\|k(Y_{-1})\|_{L^2}\,\|\tau\|_{L^2}.$$

Namely, it turns out that $Y_{-1}\cdot u \in L^2$.

Next we show that $Y_{-1}\cdot u$ is C^∞. Again in the distribution sense, we have

$$HY_{-1}\cdot u = Y_{-1}(H-1)\cdot u = Y_{-1}\cdot f = H\cdot k(Y_{-1});$$

that is, $H\cdot(Y_{-1}\cdot u - k(Y_{-1})) = 0$. Thus, the L^2 function $Y_{-1}\cdot u - k(Y_{-1})$ is invariant under the flow ϕ^t, and has to be constant almost everywhere due to the ergodicity of ϕ^t (Lemma 1.5). In particular, $Y_{-1}\cdot u = k(Y_{-1}) + \text{const}$ is C^∞.

It has been shown that u is differentiable except in the direction E_{-2}. However, the apparently missing direction E_{-2} could be recovered by the subelliptic regularity theorem [**KS1**], as $[\mathfrak{g}_{-1},\mathfrak{g}_{-1}] = \mathfrak{g}_{-2}$. □

The existence of k with the property described in the lemma is also a necessary condition for the smoothness of u, although we do not need that fact.

Proof of Theorem 3.1. To determine $\theta(X_{+\lambda})$ for $\lambda = 1, 2$, we appeal to the condition $(d\theta)(H, X_{+\lambda}) = 0$. It readily leads us to the differential equation

$$(H-\lambda)\cdot\theta(X_{+\lambda}) = f(X_{+\lambda}) \tag{3.4}$$

for $\theta(X_{+\lambda})$ with

$$f(X_{+\lambda}) = X_{+\lambda}\cdot\alpha(H),$$

which always has a unique continuous solution. In particular, the linearity of $\theta(X_{+\lambda})$ in $X_{+\lambda}$ follows from the uniqueness.

To ensure the smoothness of $\theta(X_{+1})$ by means of Lemma 3.2, take a look at

$$Y_{-1}\cdot f(X_{+1}) = Y_{-1}X_{+1}\cdot\alpha(H) = X_{+1}Y_{-1}\cdot\alpha(H) - [X_{+1}, Y_{-1}]\cdot\alpha(H).$$

Since α is tangentially closed, the first term on the right-hand side of the last equality is equal to $X_{+1}(H\cdot\alpha(Y_{-1}) + \alpha(Y_{-1})) = HX_{+1}\cdot\alpha(Y_{-1})$, while the second is equal to $H\cdot\alpha([X_{+1}, Y_{-1}])$. It follows that

$$Y_{-1}\cdot f(X_{+1}) = H\cdot\{X_{+1}\cdot\alpha(Y_{-1}) - \alpha([X_{+1}, Y_{-1}])\},$$

as desired. Thus, $\theta(X_{+1})$ turns out to be C^∞.

To show the differentiability of $\theta(X_{+2})$ for arbitrary $X_{+2} \in \mathfrak{g}_{+2}$, it is sufficient to deal with the case $X_{+2} = [Y_{+1}, Z_{+1}]$ for some $Y_{+1}, Z_{+1} \in \mathfrak{g}_{+1}$, as $[\mathfrak{g}_{+1}, \mathfrak{g}_{+1}] = \mathfrak{g}_{+2}$. The condition $(d\theta)(Y_{+1}, Z_{+1}) = 0$ is then equivalent to the identity

$$\theta(X_{+2}) = Y_{+1} \cdot \theta(Z_{+1}) - Z_{+1} \cdot \theta(Y_{+1}), \tag{3.5}$$

which especially implies the smoothness of $\theta(X_{+2})$. To prove (3.5), note that the right-hand side of the identity (3.5) is also a solution of (3.4) with $\lambda = 2$: thus uniqueness guarantees (3.5). This proves the smoothness of $\theta(X_{+2})$, and the identity $(d\theta)(Y_{+1}, Z_{+1}) = 0$, as well.

In the meantime, the identity $(d\theta)(X_{+2}, Y_{+\mu}) = 0$ $(\mu = 1, 2)$ follows from the following computation:

$$0 = (d^2\theta)(H, X_{+2}, Y_{+\mu}) = (H - (2 + \mu)) \cdot (d\theta)(X_{+2}, Y_{+\mu}).$$

What is remaining to be verified is the condition

$$(d\theta)(X_{+\lambda}, Y_{-\mu}) = 0 \quad \text{with } \lambda = 1, 2, \mu = 0, 1, 2.$$

For this purpose, note that

$$0 = (d^2\theta)(H, X_{+\lambda}, Y_{-\mu}) = (H - (\lambda - \mu)) \cdot (d\theta)(X_{+\lambda}, Y_{-\mu}).$$

Thus in the case when $\lambda \neq \mu$, we immediately have $(d\theta)(X_{+\lambda}, Y_{-\mu}) = 0$. In the meantime, in the case when $\lambda = \mu$, the ergodicity of ϕ^t implies $(d\theta)(X_{+\lambda}, Y_{-\lambda}) = \text{const}$. Integrating

$$\text{const} = (d\theta)(X_{+\lambda}, Y_{-\lambda}) = X_{+\lambda} \cdot \alpha(Y_{-\lambda}) - Y_{-\lambda} \cdot \theta(X_{+\lambda}) - \theta([X_{+\lambda}, Y_{-\lambda}])$$

over V gives $(d\theta)(X_{+\lambda}, Y_{-\lambda}) = 0$, because of the assumption $\int_V \alpha(Z) = 0$ for all $Z \in \mathfrak{g}_0$. This completes the proof of Theorem 3.1. □

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