Ergod. Th. & Dynam. Sys. (2009), **29**, 579–612
doi:10.1017/S0143385708080255

Tessellation and Lyubich–Minsky laminations associated with quadratic maps, I: pinching semiconjugacies

TOMOKI KAWAHIRA

Graduate School of Mathematics, Nagoya University, Nagoya 464-8602, Japan
(e-mail: kawahira@math.nagoya-u.ac.jp)

(*Received* 14 *May* 2007 *and accepted in revised form* 22 *February* 2008)

Abstract. We construct *tessellations* of the filled Julia sets of hyperbolic and parabolic quadratic maps. The dynamics inside the Julia sets are then organized by tiles which play the role of the external rays outside. We also construct continuous families of pinching semiconjugacies associated with hyperbolic-to-parabolic degenerations *without* using quasiconformal deformation. Instead, we achieve this via tessellation and investigation of the hyperbolic-to-parabolic degeneration of linearizing coordinates inside the Julia set.

1. *Introduction*

After the work of Douady and Hubbard, the dynamics of quadratic maps $f = f_c : z \mapsto z^2 + c$ with an attracting or a parabolic cycle have been investigated intensively; this is because such parameters c of f_c are contained in the Mandelbrot set and are very important elements that determine the topology of the Mandelbrot set. See [**DH**] or [**Mi2**] for more details.

The aim of this paper is to present a new method for describing combinatorial changes of dynamics when the parameter c moves from one hyperbolic component to another via a 'parabolic parameter' (i.e. c of f_c with a parabolic cycle). The simplest example is the motion in the Mandelbrot set along a path joining $c = 0$ to the center $c_{p/q}$ of the p/q-satellite component of the main cardioid via the root of the p/q-limb. In particular, we join these points by the two segments characterized as follows:

(s1) c of f_c which has a fixed point of multiplier $re^{2\pi i p/q}$ with $0 < r \leq 1$; and

(s2) c of f_c which has a q-periodic cycle of multiplier $1 \geq r > 0$.

Note that we avoid the hyperbolic centers (c of f_c with a superattracting cycle) because we regard these as non-generic special cases far away from parabolic bifurcations.

In the magnified box of Figure 1, segments (s1) and (s2) for $p/q = 1/3$ are drawn in the Mandelbrot set. By the Douady–Hubbard theory, the change in dynamics of $f = f_c$

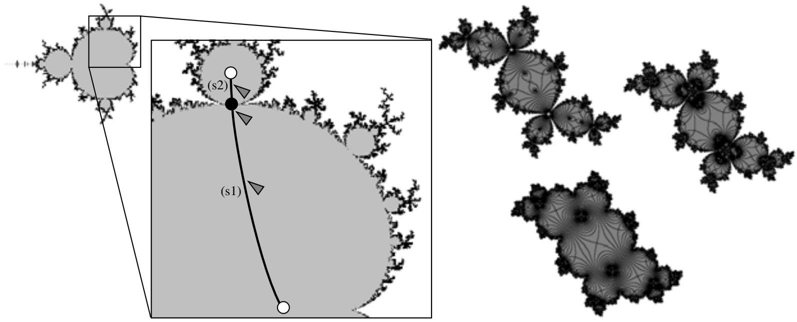

FIGURE 1. Chubby rabbits.

on and outside the Julia set is described by the external rays $R_f(\theta)$ with $\theta \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$ and their landing points $\gamma_f(\theta)$, which satisfy $f(R_f(\theta)) = R_f(2\theta)$ and $f(\gamma_f(\theta)) = \gamma_f(2\theta)$. For example, as c moves from (s1) to (s2), the map $\gamma_f : \mathbb{T} \to J_f$ loses injectivity at a dense subset Θ_f of $\mathbb{T}$ consisting of the countably many angles that eventually land on $\{1/7, 2/7, 4/7\}$ by angle doubling $\delta : \theta \mapsto 2\theta$.

On the other hand, for the dynamics inside the filled Julia set K_f, there are no particular means, such as external rays, for describing degeneration and bifurcation. However, as indicated by the pictures of filled Julia sets in Figure 1 (with equipotential curves drawn in), the interior of K_f does preserve a certain pattern along (s1) and (s2).

1.1. *Degeneration pairs and tessellation.* In this paper, we introduce *tessellation* of the interior K_f° of K_f as a means of detecting hyperbolic-to-parabolic degeneration or parabolic-to-hyperbolic bifurcation of quadratic maps.

Let X be a hyperbolic component of the Mandelbrot set. By a theorem due to Douady and Hubbard [**Mi2**, Theorem 6.5], there exists a conformal map λ_X from $\mathbb{D}$ onto X that parameterizes the multiplier of the attracting cycle of $f = f_c$ for $c \in X$. Moreover, the map λ_X has the homeomorphic extension $\lambda_X : \bar{\mathbb{D}} \to \bar{X}$ such that $\lambda_X(e^{2\pi i p/q})$ is a parabolic parameter for all $p, q \in \mathbb{N}$. A *degeneration pair* $(f \to g)$ consists of a hyperbolic $f = f_c$ and a parabolic $g = f_\sigma$, where $(c, \sigma) = (\lambda_X(re^{2\pi i p/q}), \lambda_X(e^{2\pi i p/q}))$ for some $0 < r < 1$ and coprime $p, q \in \mathbb{N}$. By letting $r \to 1$, the map f converges uniformly to g on $\bar{\mathbb{C}}$, and we have a path which generalizes segment (s1) or (s2). For a degeneration pair, we have associated tessellations which have the same combinatorics.

THEOREM 1.1. (Tessellation) *Let $(f \to g)$ be a degeneration pair. There exist families* $\mathrm{Tess}(f)$ *and* $\mathrm{Tess}(g)$ *of simply connected sets with the following properties.*

(1) *Each element of* $\mathrm{Tess}(f)$ *is called a* tile *and is identified by an* angle θ *in* $\mathbb{Q}/\mathbb{Z}$, *a* level m *in* $\mathbb{Z}$ *and a* signature $*$ *which is either* $+$ *or* $-$.
(2) *If $T_f(\theta, m, *)$ is such a tile in* $\mathrm{Tess}(f)$, *then $f(T_f(\theta, m, *)) = T_f(2\theta, m+1, *)$.*
(3) *The interiors of tiles in* $\mathrm{Tess}(f)$ *are disjoint topological disks. Tiles with the same signature are univalently mapped to each other by a branch of $f^{-i} \circ f^j$, for some $i, j > 0$.*

(4) *Let $\Pi_f(\theta, *)$ denote the union of tiles with angle θ and signature $*$. Then its interior $\Pi_f(\theta, *)^\circ$ is also a topological disk, and its boundary contains the landing point $\gamma_f(\theta)$ of $R_f(\theta)$. In particular, $f(\Pi_f(\theta, *)) = \Pi_f(2\theta, *)$.*

The above properties hold if f is replaced by g; moreover, we have the following.

(5) *There exists an f-invariant family I_f of star-like graphs such that the union of tiles in* $\mathrm{Tess}(f)$ *is $K_f^\circ - I_f$; on the other hand, the union of tiles in* $\mathrm{Tess}(g)$ *is K_g°.*

(6) *The boundaries of $T_f(\theta, m, *)$ and $T_f(\theta', m', *')$ in $K_f^\circ - I_f$ intersect if and only if the boundaries of $T_g(\theta, m, *)$ and $T_g(\theta', m', *')$ in K_g° do.*

Here the angles of tiles must be the angles of external rays which eventually land on the parabolic cycle of g. For example, if $(f \to g)$ are on (s1) or (s2) in Figure 1, the set of angles of tiles coincides with Θ_f. See §§2 and 3 for the construction of tessellations and Figure 2 for examples. The combinatorics of tessellations are found to be preserved along (s1) and (s2); this is justified in §4 in more generality. Since $f_c \in X - \{\lambda_X(0)\}$ is structurally stable, the tessellation of K_{f_c} has the same properties as $\mathrm{Tess}(f)$.

1.2. *Pinching semiconjugacy.* As an application of tessellation, we show that there exists a pinching semiconjugacy from f to g for the degeneration pair $(f \to g)$. In §§4 and 5 we will establish the following.

THEOREM 1.2. (Pinching semiconjugacy) *Let $(f \to g)$ be a degeneration pair. There exists a semiconjugacy $h : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ from f to g such that:*

(1) *h only pinches I_f to the grand orbit of the parabolic cycle of g;*

(2) *h sends all possible $T_f(\theta, m, *)$ to $T_g(\theta, m, *)$, $R_f(\theta)$ to $R_g(\theta)$, and $\gamma_f(\theta)$ to $\gamma_g(\theta)$;*

(3) *h tends to the identity as f tends to g.*

One may easily imagine the situation by looking at the figures of tessellation. As a corollary, we have convergence of the tiles when f of $(f \to g)$ tends to g (Corollary 5.2).

First, the existence of h with properties (1) and (2) will be proved in §4 (Theorem 4.1) by using combinatorial properties of tessellation. Property (3) is then proved in §5 (Theorem 5.1); this is done by means of continuity results for extended Böttcher coordinates (Theorem 5.4) on and outside the Julia sets and linearizing coordinates (i.e. Königs and Fatou coordinates) inside the Julia sets associated with $(f \to g)$ (Theorem 5.5).

In Appendix A, we present some useful results on perturbation of parabolics which are used in the proofs of the theorems.

Notes.

(1) For any $f_c \in X - \{\lambda_X(0)\}$, we have a semiconjugacy h_c which has properties similar to (1) and (2) by structural stability. From the work of Cui [**Cu**] and Haïssinsky and Tan Lei [**Ha2, HT**], it is already known that such a semiconjugacy exists. Their results are based on quasiconformal deformation theory and hold even for some geometrically infinite rational maps. On the other hand, our method is faithful to

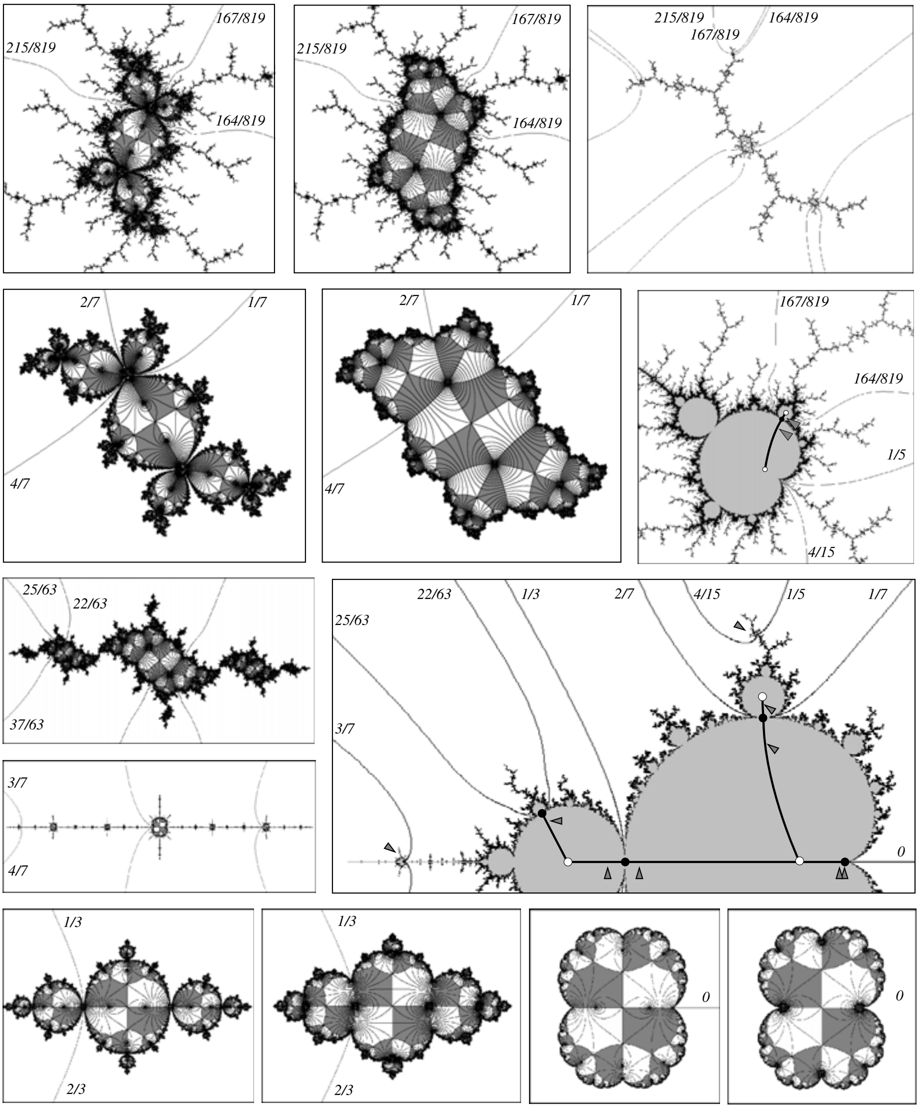

FIGURE 2. Examples of tessellation. For the two upper-left panels, parameters are taken from period-12 and period-4 hyperbolic components of the Mandelbrot set, as indicated in the picture of a small Mandelbrot set.

the quadratic dynamics, and the semiconjugacy is constructed in a more explicit way without using quasiconformal deformation. It is possible to extend our results to certain classes of higher-degree polynomials or rational maps, but such extensions are out of the scope of this paper.

(2) This paper is the first part of a project on Lyubich–Minsky laminations. In [**LM**], hyperbolic 3-laminations associated with rational maps were introduced as an analogue of the hyperbolic 3-manifolds associated with Kleinian groups. In the second part of this project [**Ka3**] we shall investigate, also by means of

tessellation and pinching semiconjugacies, combinatorial and topological changes of 3-laminations associated with hyperbolic-to-parabolic degeneration of quadratic maps.

2. *Degeneration pairs and degenerating arc systems*

Segments (s1) and (s2) in the previous section are considered as hyperbolic-to-parabolic degeneration processes of two distinct directions. Degeneration pairs generalize all of such processes in the quadratic family. The aim of this section is to give a dichotomous classification of the degeneration pairs $\{(f \to g)\}$ and to define invariant families of star-like graphs (*degenerating arc systems*) for each f of $(f \to g)$.

2.1. *Classification of degeneration pairs.* First let us define some notation to be used throughout this paper. Let p and q be relatively prime positive integers, and set $\omega := \exp(2\pi i p/q)$; we allow the case of $p = q = 1$. Take a number r from the interval $(0, 1)$, and set $\lambda := r\omega$. As in the previous section, we take a hyperbolic component X of the Mandelbrot set. Then we have a degeneration pair $(f \to g)$ that comprises a hyperbolic $f = f_c$ and a parabolic $g = f_\sigma$, where $(c, \sigma) = (\lambda_X(re^{2\pi i p/q}), \lambda_X(e^{2\pi i p/q}))$.

For the degeneration pair $(f \to g)$, let $O_f := \{\alpha_1, \ldots, \alpha_l\}$ be the attracting cycle of f with multiplier $\lambda = r\omega$ and $f(\alpha_j) = \alpha_{j+1}$ (taking subscripts modulo l). Similarly, let $O_g := \{\beta_1, \ldots, \beta_{l'}\}$ be the parabolic cycle of g with $g(\beta_{j'}) = \beta_{j'+1}$ (taking subscripts modulo l'). Let $\omega' = e^{2\pi i p'/q'}$ denote the multiplier of O_g with relatively prime positive integers p' and q' (so O_g is a parabolic cycle with q' repelling petals).

Our fundamental classification is described in the following proposition.

PROPOSITION 2.1. *Any degeneration pair $(f \to g)$ satisfies one of the following.*
Case (a): $q = q'$ *and* $l = l'$;
Case (b): $q = 1 < q'$ *and* $l = l'q'$.
In both cases, we have $lq = l'q'$.

This proposition is proved by combining the results in [**Mi2**, §§2, 4 and 6]. For example, a degeneration pair $(f \to g)$ on segment (s1) (respectively (s2)) with $q > 1$ falls into Case (a) (respectively Case(b)). Degeneration pairs $(f_c \to f_\sigma)$ with $\sigma = 1/4$ or $\sigma = -7/4$ satisfy $q = q' = 1$ and thus Case (a).

A note on terminology. According to [**Mi2**], a parabolic g with $q' = 1$ is called *primitive*. The parabolic map $g = f_\sigma$ with $\sigma = 1/4$ is also called *trivial*. For these g's, any degeneration pair $(f \to g)$ automatically belongs to Case (a) by the proposition above.

2.2. *Perturbation of O_g and degenerating arcs.* For a degeneration pair $(f \to g)$ with $r \approx 1$, the parabolic cycle O_g is approximated by an attracting or repelling cycle O'_f with the same period l' and multiplier $\lambda' \approx e^{2\pi i p'/q'}$. Let α'_1 be the point in O'_f such that $\alpha'_1 \to \beta_1$ as $r \to 1$ (cf. [**Mi2**, §4]).

In Case (a), the cycle O'_f is attracting (thus $O'_f = O_f$) and there are q' symmetrically arrayed repelling periodic points around $\alpha_1 = \alpha'_1$; we will then show that there exists an $f^{l'}$-invariant star-like graph $I(\alpha'_1)$ that joins α'_1 and the repelling periodic points by q'

arcs. In Case (b), the cycle O'_f is repelling and there are $q' = l/l'$ symmetrically arrayed attracting periodic points around α'_1; we will then show that there exists an $f^{l'}$-invariant star-like graph $I(\alpha'_1)$ that joins α'_1 and the attracting periodic points by q' arcs.

In both cases, we define the *degenerating arc system* I_f by

$$I_f := \bigcup_{n \geq 0} f^{-n}(I(\alpha'_1)).$$

The rest of this section is mainly devoted to the detailed construction of I_f, which plays the role of a parabolic cycle and its preimages. It may be helpful to look at Figure 3 first, which illustrates what we are aiming for.

2.3. *External and internal landing.* First, consider the parameter c on segment (s1) such that $f = f_c$ has an attracting fixed point $O_f = \{\alpha_1\}$ of multiplier $\lambda = r\omega$; thus $c = \lambda/2 - \lambda^2/4$. When r tends to 1, the hyperbolic map f tends to a parabolic map g which has a parabolic fixed point $O_g = \{\beta_1\}$ with multiplier $\omega = e^{2\pi i p/q}$. Note that $q = q'$ and $l = l'$ (=1), hence this situation falls into Case (a) of Proposition 2.1. It is known that the Julia set J_f of f is a quasicircle, and the dynamics on J_f is topologically the same as that of $f_0(z) = z^2$ on the unit circle. Since J_f is locally connected, for any angle $\theta \in \mathbb{R}/\mathbb{Z} = \mathbb{T}$ the external ray $R_f(\theta)$ has a unique landing point $\gamma_f(\theta)$. The same is true for $R_g(\theta)$, since J_g is also locally connected. (See [**DH**, Exposé No. X].)

2.3.1. *External landing.* By [**Mi1**, Theorem 18.11] due to Douady, there is at least one external ray with rational angle landing at β_1. Now [**GM**, Lemma 2.2] and the local dynamics of β_1 ensure the following.

LEMMA 2.2. *In the dynamics of g, there exist exactly q external rays, corresponding to angles $\theta_1, \ldots, \theta_q$ with $0 \leq \theta_1 < \cdots < \theta_q < 1$, which land on β_1. Moreover, the map g sends $R_g(\theta_j)$ onto $R_g(\theta_k)$ univalently (or equivalently, $\theta_k = 2\theta_j$ modulo 1) if and only if $k \equiv j + p \bmod q$.*

In particular, such angles are determined uniquely by the value $p/q \in \mathbb{Q}/\mathbb{Z}$. We take the subscripts of $\{\theta_j\}$ modulo q. For these angles, we call $p/q \in \mathbb{Q}/\mathbb{Z}$ the *(combinatorial) rotation number*. Note that the external rays $R_g(\theta_j)$ divide the complex plane into q open pieces, called *sectors* based at β_1.

2.3.2. *Internal landing.* On the other hand, the set of landing points $\{\gamma_f(\theta_j)\}$ of $\{R_f(\theta_j)\}$ is a repelling cycle of period q, and the corresponding rays do not divide the plane. They do, however, continuously extend and penetrate through the filled Julia set, landing at the attracting fixed point.

LEMMA 2.3. *For $\theta_1, \ldots, \theta_q$ as above, there exist q open arcs $I(\theta_1), \ldots, I(\theta_q)$ such that:*
- *for each j modulo q, the arc $I(\theta_j)$ joins α_1 and $\gamma_f(\theta_j)$;*
- *f maps $I(\theta_j)$ onto $I(\theta_k)$ univalently if and only if $k \equiv j + p \bmod q$.*

There is no canonical choice for the arcs $\{I(\theta_j)\}$, but we will make a judicious choice in the proof. An important fact is that the set $\{I(\theta_j) \cup \gamma_f(\theta_j) \cup R_f(\theta_j)\}_{j=1}^{q}$ divides the plane into q sectors based at α_1. This is topologically the same situation as for g. Indeed, as r tends to 1, we can consider the arcs $\{I(\theta_j)\}$ constructed above as degenerating to the parabolic β_1.

Sketch of the proof. (See [**Ka2**, Lemma 2.3.3] for a detailed proof.) Let $w = \phi_f(z)$ be a linearizing coordinate near α_1, where f near α_1 is viewed as $w \mapsto \lambda w$. We can extend this to $\phi_f : K_f^{\circ} \to \mathbb{C}$ and normalize it so that $\phi_f(0) = 1$ [**Mi1**, §8]. Now we pull back the qth roots of the negative real axis in the w-plane, which are q symmetrically arrayed invariant radial rays, to the original dynamics. We can then show that the pulled-back arcs land at a unique repelling cycle with external angles determined by the rotation number p/q. In particular, they are disjoint from the critical orbit. □

Definition. (Degenerating arcs) In the construction of $\{I(\theta_j)\}$, we make a particular choice of such arcs so that they are laid opposite to the critical orbit in the w-plane. We call these arcs *degenerating arcs*.

2.4. *Degenerating arc systems.* Let us now return to a general degeneration pair $(f \to g)$.

2.4.1. *Renormalization.* Let B_1 be the Fatou component containing the critical value c. We may assume that B_1 is the immediate basin of α_1 for f^l. It is known that there exists a topological disk U containing B_1 such that f^l maps U over itself properly by degree two; that is, the map $f^l : U \to f(U)$ is a quadratic-like map which is a renormalization of f. See [**Mi2**, §8] or [**Ha1**, Partie 1]. In particular, the map $f^l : U \to f(U)$ is hybrid equivalent to $f_1(z) = z^2 + c_1$ with $c_1 = \lambda/2 - \lambda^2/4$ in segment (s1), which we dealt with earlier. More precisely, the dynamics of f^l near B_1 (respectively on B_1) can be topologically (respectively conformally) identified with that of f_1 near K_{f_1} (respectively on $K_{f_1}^{\circ}$).

2.4.2. *Definition of degenerating arc systems.* In K_{f_1}, we have q degenerating arcs associated with the attracting fixed point of f_1. By pulling them back to the closure of B_1 with respect to the conformal identification above, we obtain q open arcs $\{I_j\}_{j=1}^{q}$ which are cyclic under f^l.

When $q = q'$ and $l = l'$, i.e. in Case (a), the arcs $\{I_j\}_{j=1}^{q}$ join q' repelling points (cyclic under $f^l = f^{l'}$) and $\alpha_1 = \alpha_1'$; in this case we define $I(\alpha_1')$ by the closure of the union of $\{I_j\}_{j=1}^{q}$. When $1 = q < q' = l/l'$, i.e. in Case (b), we only have I_1 that joins the repelling point α_1' (fixed under $f^{l'}$) and α_1; in this case we define $I(\alpha_1')$ by the closure of the union of $\{f^{kl'}(I_1)\}_{k=0}^{q'-1}$. In both cases, we have $I(\alpha_1')$ as desired. Now we define the *degenerating arc system* of f by

$$I_f := \bigcup_{n>0} f^{-n}(I(\alpha_1')).$$

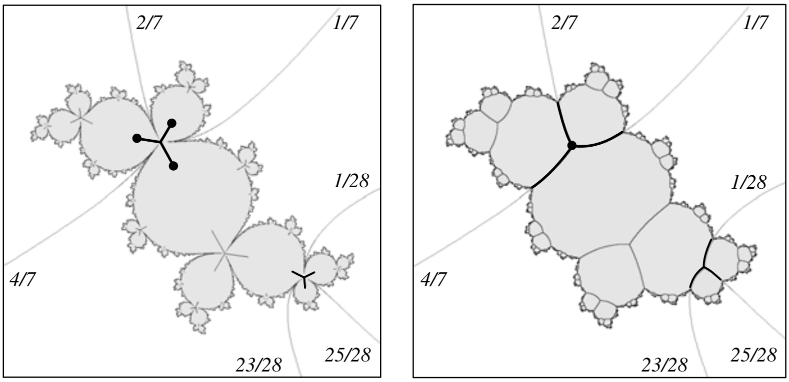

FIGURE 3. The left panel shows the Julia set of an f in segment (s2) for $p/q = 1/3$; the right panel shows the Julia set of an f in segment (s1). Their degenerating arc systems are roughly drawn in. Attracting cycles are depicted by heavy dots, and degenerating arcs of types {1/7, 2/7/4/7} and {1/28, 23/28, 25/28} are emphasized.

For $z \in I_f$, it is useful to denote the connected component of I_f containing z by $I(z)$.

For later use, we define $\alpha_f := \bigcup_{n>0} f^{-n}(\alpha_1)$, the set of all points that eventually land on the attracting cycle O_f. Note that I_f and α_f are forward and backward invariant, and are disjoint from the critical orbit. In particular, for any $z \in I_f$, the components $I(z)$ and $I(\alpha_1')$ are homeomorphic. In Case (a), the points in α_f and the connected components of I_f have a one-to-one correspondence. In Case (b), however, they have a q'-to-one correspondence. See Figure 3 and Proposition 2.5.

Similarly, for g of the degeneration pair $(f \to g)$ and one of its parabolic points $\beta_1 \in O_g$, we define

$$I_g := \bigcup_{n>0} g^{-n}(\beta_1).$$

We shall see that this naturally corresponds to I_f rather than α_f.

2.4.3. *Types.* After [**GM**], we define the *type* $\Theta(z)$ of z in J_f (or J_g) by the set of all angles of external rays which land on z. Let $\delta : \mathbb{T} \to \mathbb{T}$ be the angle-doubling map. Since J_f has no critical points, one can easily see that $\delta(\Theta(z))$ coincides with $\Theta(f(z))$. The same holds for g. By an unpublished result of Thurston, if z in the quadratic Julia set does not have a finite orbit, then $\mathrm{card}(\Theta(z)) \leq 4$. See [**Ki**] for a generalized statement and proof. In our case, (pre-)periodic points in J_f and J_g have uniformly bounded numbers of landing rays. Hence we have the following.

LEMMA 2.4. *For any point z in J_f or J_g, the set $\Theta(z)$ consists of finitely many angles.*

We shall abuse the notation $\Theta(\cdot)$: for any subset E of the filled Julia set, its *type* $\Theta(E)$ will be the set of angles of the external rays that land on points in E. (So $\Theta(E)$ is empty when E does not touch the Julia set.) For each ζ in α_f, we formally define the type of ζ by $\Theta(\zeta) := \Theta(I(\zeta))$. Then one can easily see that $\delta^n(\Theta(\zeta)) = \Theta(\alpha_1)$ for some $n > 0$. We also set $\Theta_f := \Theta(I_f)$ and $\Theta_g := \Theta(I_g)$. We will show that Θ_f is equal to Θ_g in the next proposition.

2.4.4. *Valence.* For any $\zeta \in \alpha_f$, the component $I(\zeta)$ of I_f is univalently mapped onto $I(\alpha_1)$ by iteration of f. Thus the value $\mathrm{val}(f) := \mathrm{card}(\Theta(\zeta))$ is a constant for a given f. Similarly, since a small neighborhood of ξ in I_g is sent univalently over β_1 by iteration of g, the value $\mathrm{val}(g) := \mathrm{card}(\Theta(\xi))$ is constant for g. Now we claim the following.

PROPOSITION 2.5. *For any degeneration pair* $(f \to g)$, *we have* $\Theta_f = \Theta_g$ *and* $\mathrm{val}(f) = \mathrm{val}(g)$. *Moreover:*
- *if* $q = q' = 1$ *and* $l = l' > 1$ *(i.e. Case (a) and non-trivial primitive), then* $\mathrm{val}(g) = 2$;
- *otherwise* $\mathrm{val}(g) = q'$.

We call $\mathrm{val}(f) = \mathrm{val}(g)$ the *valence* of $(f \to g)$. Note that the valence depends only on g. The proof of this proposition makes use of some facts from [**Mi2**].

Proof. The two possibilities of $\mathrm{val}(g)$ above are shown in [**Mi2**, Lemma 2.7, §6]. If we can prove that $\Theta(\alpha'_1) = \Theta(\beta_1)$, then $\Theta_f = \Theta_g$ and $\mathrm{val}(f) = \mathrm{val}(g)$ follow automatically.

Case (a): $q = q'$. (Recall that in this case we have $l = l'$ and $\alpha_1 = \alpha'_1$.) First, consider $q = q' = 1$. By the argument of [**Mi2**, Theorem 4.1], there exists a repelling cycle $\{\gamma_1, \ldots, \gamma_{l'}\}$ of f which satisfies $\gamma_{j'} \to \beta_{j'}$ as $f \to g$ and $\Theta(\gamma_{j'}) = \Theta(\beta_{j'})$ for $j' = 1, \ldots, l'$. Take the degenerating arc $\{I_1\}$ in the construction of I_f. Then I_1 joins $\alpha_1 (= \alpha'_1)$ and γ_1, and hence $\Theta(\alpha'_1) = \Theta(I(\alpha_1)) = \Theta(\gamma_1) = \Theta(\beta_1)$.

Next, consider $q = q' > 1$. When f is in segment (s1), the identity $\Theta(\alpha'_1) = \Theta(\beta_1)$ is clear by Lemma 2.3. In the general case, we use renormalization.

Let us take a path η in parameter space that joins c to σ according to the motion as $r \to 1$. By [**Ha1**, Théorème 1], there is an analytic family of quadratic-like maps $\{f^l_{c'} : U_{c'} \to f^l_{c'}(U_{c'})\}$ over a neighborhood of η such that the straightening maps are continuous and they give one-to-one correspondence between η and (s1).

Let $\alpha_1 \in O_f$ and $\beta_1 \in O_g$ be the attracting and parabolic fixed points of $f^l = f^l_c : U_c \to f^l_c(U_c)$ and $g^l = f^l_\sigma : U_\sigma \to f^l_\sigma(U_\sigma)$, respectively, satisfying $\alpha_1 \to \beta_1$ as $f \to g$. By Lemma 2.2, we can find q external rays landing at β_1 in the original dynamics of g, which is cyclic under g^l. In particular, there can be no more rays landing at β_1, because any such rays must be cyclic of period q under g^l, and this would contradict [**Mi2**, Lemma 2.7]. Similarly, in the dynamics of f, by Lemma 2.3 and continuity of the straightening, there are exactly q external rays of angles in $\Theta(\beta_1)$ landing at q ends of $I(\alpha_1) = I(\alpha'_1)$. In fact, if there is another ray of angle $t \notin \Theta(\beta_1)$ landing on such an end, then $R_g(t)$ must land on β_1 by orbit forcing [**Mi2**, Lemma 7.1]; but this is a contradiction. Thus $\Theta(\alpha_1) = \Theta(\alpha'_1) = \Theta(\beta_1)$.

Case (b): $q = 1 < q'$. By the argument of [**Mi2**, Theorem 4.1], the repelling points $O'_f = \{\alpha'_1, \ldots, \alpha'_{l'}\}$ must satisfy $\Theta(\alpha'_{j'}) = \Theta(\beta_{j'})$ for $j' = 1, \ldots, l'$. □

In both Case (a) and Case (b), it is convenient to assume that the $\alpha_{j'}, \alpha_{j'+l'}, \ldots, \alpha_{j'+(q'-1)l'}$ have the same types as $\beta_{j'}$, for each $j' = 1, \ldots, l'$. Equivalently, we assume throughout this paper that

$$I(\alpha_{j'}) = I(\alpha_{j'+l'}) = \cdots = I(\alpha_{j'+(q'-1)l'}).$$

2.5. *Critical sectors.* For ξ in I_g, the external rays of angles in $\Theta(\xi)$ cut the plane up into $\mathrm{val}(g)$ open regions, called *sectors* based at ξ. Similarly, for ζ in α_f, the union of the external rays of angles in $\Theta(\zeta)$ and $I(\zeta)$ cut the plane up into $\mathrm{val}(f) = \mathrm{val}(g)$ open regions. We abuse the term *sectors based at* $I(\zeta)$ for these regions.

Let B_0 be the Fatou component of g that contains the critical point $z = 0$. We may assume that $\beta_0 = \beta_{l'}$ is on the boundary of B_0. Now one of the sectors based at β_0 contains the critical point 0; this is called the *critical sector*. For later use, let $\theta_0^+,\ \theta_0^- \in \mathbb{R}/\mathbb{Z}$ denote the angles of external rays bounding the critical sector, such that if we take representatives $\theta_0^+ < \theta_0^- \le \theta_0^+ + 1$, the external ray of angle θ with $\theta_0^+ < \theta < \theta_0^-$ is contained in the critical sector. For example, we defined $\theta_0^+ := 4/7$ and $\theta_0^- := 1/7$ for Figure 3, while for Figure 8 we took $\theta_0^+ := 5/7$ and $\theta_0^- := 2/7$. We also define the *critical sector* based at $I(\alpha_0)$ to be one of the sectors bounded by $I(\alpha_0)$ and $R_f(\theta_0^\pm)$.

3. *Tessellation*

In this section, we develop (and make more concise) the method in [**Ka2**], and construct a *tessellation* of the interior of the filled Julia sets for a degeneration pair $(f \to g)$.

For each $\theta \in \Theta_f = \Theta_g$ and some $m \in \mathbb{Z}$ (with a condition depending on θ), we will define the tiles $T_f(\theta, m, \pm)$ and $T_g(\theta, m, \pm)$ with the properties listed in Theorem 1.1. As one can see from Figure 2, the idea of tessellation is simple, but we need to do the construction precisely in order to figure out the combinatorial structure in detail.

3.1. *Fundamental model of tessellation.* Take $R \in (0, 1)$, and consider the affine maps $F(W) = RW + 1$ and $G(W) = W + 1$ on $\mathbb{C}$ as the W-plane. The map F has a fixed point $a = 1/(1 - R)$ and one can see the action by the relation $F(W) - a = R(W - a)$.

3.1.1. *Tiles for* F. Set $I := [a, \infty)$, a half line invariant under F. For each $\mu \in \mathbb{Z}$, we define 'tiles' of level μ for F by

$$A_\mu(+) := \{W \in \mathbb{C} - I : R^{\mu+1}a \le |W - a| \le R^\mu a,\ \mathrm{Im}\, W \ge 0\},$$
$$A_\mu(-) := \{W \in \mathbb{C} - I : R^{\mu+1}a \le |W - a| \le R^\mu a,\ \mathrm{Im}\, W \le 0\}.$$

One can check that $F(A_\mu(*)) = A_{\mu+1}(*)$, where $* \in \{+, -\}$. For the boundary of each $A_\mu(*)$, we define:

- the *circular edges* by the intersection with $A_{\mu\pm1}(*)$;
- the *degenerating edge* by $\overline{A_\mu(*)} \cap I$; and
- the *critical edge* by the intersection with $(-\infty, a)$.

Note that $A_\mu(*) \subset \mathbb{C} - I$, so the degenerating edge is *not* contained in $A_\mu(*)$.

3.1.2. *Tiles for* G. Analogously, for each $\mu \in \mathbb{Z}$, we define 'tiles' of level μ for G by

$$C_\mu(+) := \{W \in \mathbb{C} : \mu \le \mathrm{Re}\, W \le \mu + 1,\ \mathrm{Im}\, W \ge 0\},$$
$$C_\mu(-) := \{W \in \mathbb{C} : \mu \le \mathrm{Re}\, W \le \mu + 1,\ \mathrm{Im}\, W \le 0\}.$$

Then one can check that $G(C_\mu(*)) = C_{\mu+1}(*)$. For the boundary of each $C_\mu(*)$, we define:

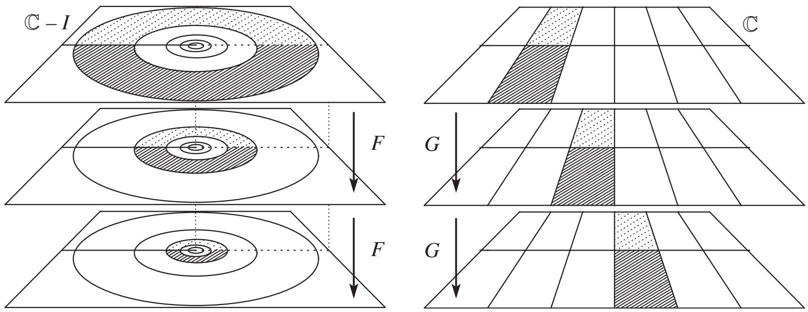

FIGURE 4. The fundamental model.

- the *circular edges* by the intersection with $C_{\mu\pm1}(*)$, which are vertical half lines;
- the *critical edge* by the intersection with $(-\infty, \infty)$.

Note that there is no degenerating edge for $C_\mu(*)$. One can consider $C_\mu(*)$ the limit of $A_\mu(*)$ as $R \to 1$.

3.2. *Tessellation for f and g.* First we reduce the dynamics of $f|_{K_f^\circ}$ and $g|_{K_g^\circ}$ to the dynamics of F and G on $\mathbb{C}$.

3.2.1. *From f to F.* Let B_0 be the Fatou component of f containing 0. We may assume that $\alpha_0 = \alpha_l \in B_0$. There exists a unique extended linearizing coordinate $\phi_f : B_0 \to \mathbb{C}$ such that $\phi_f(\alpha_0) = \phi_f(0) - 1 = 0$ and $\phi_f(f^l(z)) = \lambda\phi_f(z)$ [**Mi1**, §8]. Set $w := \phi_f(z)$ and $R := \lambda^q = r^q$. Then $f^{lq}|_{B_0}$ is semiconjugate to $w \mapsto Rw$. To reduce this situation to our fundamental model, we first take a branched covering $W = w^q$. Then $f^l|_{B_0}$ and $f^{lq}|_{B_0}$ are semiconjugate to $W \mapsto RW$ and $W \mapsto R^q W$, respectively. Next, we take an affine conjugation $W \mapsto a(1 - W)$. Then $f^l|_{B_0}$ and $f^{lq}|_{B_0}$ are finally semiconjugate to F and F^q in the fundamental model, respectively. Let Φ_f denote this final semiconjugation. Now we have $\Phi_f(0) = 0$ and $\Phi_f(B_0 \cap I_f) = I$ (the second equality comes from the construction of the degenerating arcs in Lemma 2.3). In particular, the map Φ_f branches at $z \in B_0$ if and only if either $f^{ln}(z) = 0$ for some $n \geq 0$, or $q > 1$ and $f^{ln}(z) = \alpha_0$ for some $n \geq 0$.

3.2.2. *From g to G.* Let B_0' be the Fatou component of g containing 0. We may assume that $\beta_0 = \beta_{l'} \in \partial B_0'$. There exists a unique extended Fatou coordinate $\phi_g : B_0' \to \mathbb{C}$ such that $\phi_g(0) = 0$ and $\phi_g(g^{lq}(z)) = \phi_g(z) + 1$ [**Mi1**, §10]. Set $w := \phi_g(z)$; then $g^{lq}|_{B_0}$ is semiconjugate to $w \mapsto w + 1$. To adjust the situation to that of f, we take an additional conjugacy $w \mapsto W = qw$. Then $g^{lq}|_{B_0}$ is semiconjugate to $G^q(W) = W + q$. We denote this semiconjugation $z \mapsto w \mapsto W$ by Φ_g. Note that $\Phi_g(0) = 0$ and Φ_g branches at $z \in B_0'$ if and only if $g^{lqn}(z) = 0$ for some $n \geq 0$.

Let us summarize these reduction steps. Now $\Phi_f : B_0 \to \mathbb{C}$ semiconjugates the action of $f^{lq} : B_0 - I_f \to B_0 - I_f$ to that of $F^q : \mathbb{C} - I \to \mathbb{C} - I$. Similarly, the

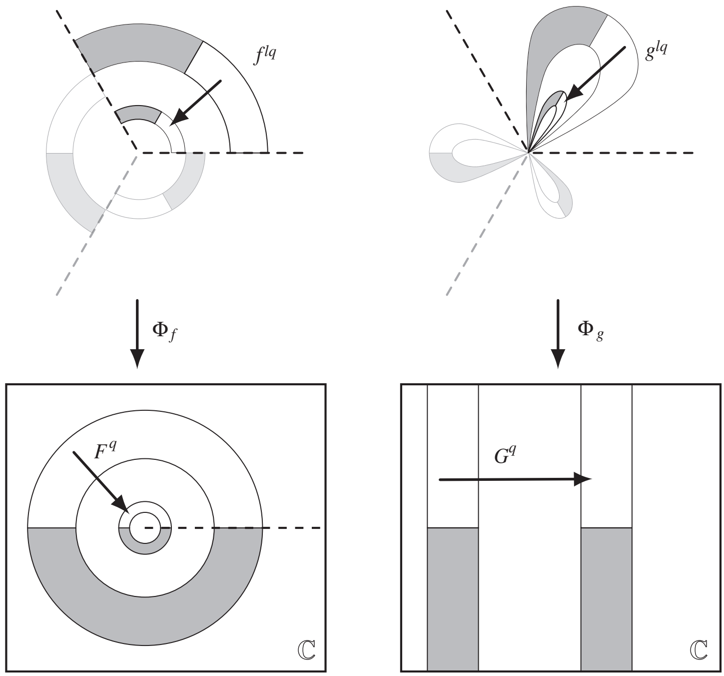

FIGURE 5. f^{lq} and g^{lq} are semiconjugate to F^q and G^q.

map $\Phi_g : B_0' \to \mathbb{C}$ semiconjugates the action of $g^{lq} : B_0' \to B_0'$ to that of $G^q : \mathbb{C} \to \mathbb{C}$ (see Figure 5). In addition, we have one important property as follows.

PROPOSITION 3.1. *The branched linearization Φ_f does not ramify over $\mathbb{C} - (-\infty, 0]$ or $\mathbb{C} - (-\infty, 0] \cup \{a\}$ according to $q = 1$ or $q > 1$. Similarly, the branched linearization Φ_g does not ramify over $\mathbb{C} - (-\infty, 0]$. In particular, both Φ_f and Φ_g do not ramify over tiles of level $\mu > 0$.*

See Theorem 5.5 for another important property of Φ_f and Φ_g.

3.2.3. *Definition of tiles and their addresses.* A subset $T \subset K_f^\circ$ is a *tile* for f if there exist $n \in \mathbb{N}$ and $\mu \in \mathbb{Z}$ such that $f^n(T)$ is contained in B_0 and $\Phi_f \circ f^n$ maps T homeomorphically onto $A_\mu(+)$ or $A_\mu(-)$. We define *circular, degenerating* and *critical edges* for T by their corresponding edges of $A_\mu(\pm)$. We call the collection of such tiles the *tessellation* of $K_f^\circ - I_f$, and denote it by $\mathrm{Tess}(f)$. In fact, one can easily check that

$$K_f^\circ - I_f = \bigcup_{T \in \mathrm{Tess}(f)} T.$$

Each $z \in K_f^\circ - I_f$ is in the interior of a unique $T \in \mathrm{Tess}(f)$, on a vertex shared by four or eight tiles in $\mathrm{Tess}(f)$ if $f^m(z) = f^n(0)$ for some $n, m > 0$, or on an edge shared by two tiles in $\mathrm{Tess}(f)$.

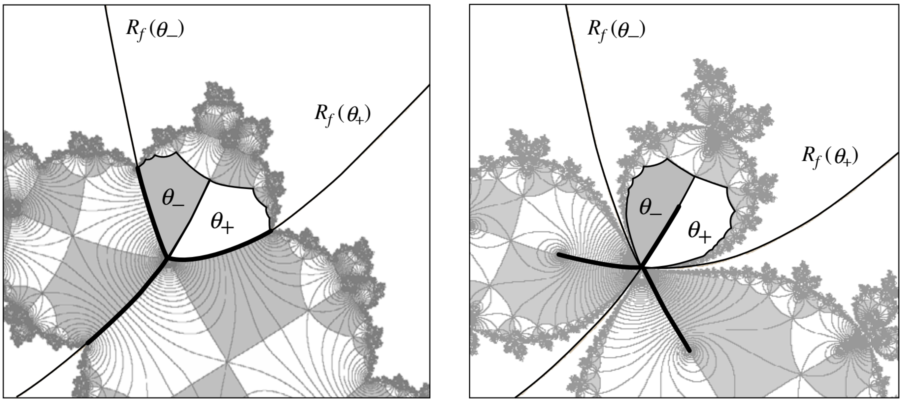

FIGURE 6. Angles of tiles in Case (a) (left) and Case (b) (right) with $q' = 3$. The thick curves represent degenerating arcs.

Tiles for g and the tessellation of $K_g^\circ - I_g = K_g^\circ$ are defined by replacing f, B_0 and $A_\mu(\pm)$ by g, B_0' and $C_\mu(\pm)$, respectively.

Each tile is identified by an *address*, which consists of an *angle*, a *level* and a *signature*. These are defined as follows.

Level and signature. For $T \in \mathrm{Tess}(f)$, i.e. $f^n(T) \subset B_0$ and $\Phi_f \circ f^n(T) = A_\mu(*)$ with $*$ being $+$ or $-$, we say that T has *level* $m = \mu l - n$ and *signature* $*$. Then the critical point $z = 0$ is a vertex of eight tiles of level 0 and $-l$.

For a tile $T' \in \mathrm{Tess}(g)$, its level and signature are defined in the same way.

Angle. For $T \in \mathrm{Tess}(f)$, there exists ζ in α_f such that $I(\zeta)$ contains the degenerating edge of T. There are then $\mathrm{val}(f) = v \geq 1$ rays landing on $I(\zeta)$, and these rays and $I(\zeta)$ divide the plane into v sectors. (In the case of $v = 1$, or equivalently $g(z) = z^2 + 1/4$, we consider the sector as the plane with a slit.) Take two angles $\theta_+ < \theta_- (\leq \theta_+ + 1)$ of external rays bounding the sector that contains T (so any external ray of angle θ with $\theta_+ < \theta < \theta_-$ would be contained in the sector). We define the *angle* of T by θ_*, where $*$ is the signature of T; see Figure 6.

For a tile $T' \in \mathrm{Tess}(g)$, one can check that there exists a unique point $\beta' \in I_g \cap \partial T'$. Since there are v rays landing on β' and they divide the plane into v sectors as in the case of $T \in \mathrm{Tess}(f)$, we can define the angle of T' in the same way as above.

We denote such tiles by $T = T_f(\theta_*, m, *)$ and $T' = T_g(\theta_*, m, *)$, and we call the triple $(\theta_*, m, *)$ the *address* of the tiles. For example, Figure 7 shows the structure of addresses for the two tessellations on the lower left of Figure 2.

Now one can verify the desired property

$$f(T_f(\theta, m, *)) = T_f(2\theta, m + 1, *).$$

The same holds if we replace f by g. One can also check properties (1) to (5) of Theorem 1.1 easily.

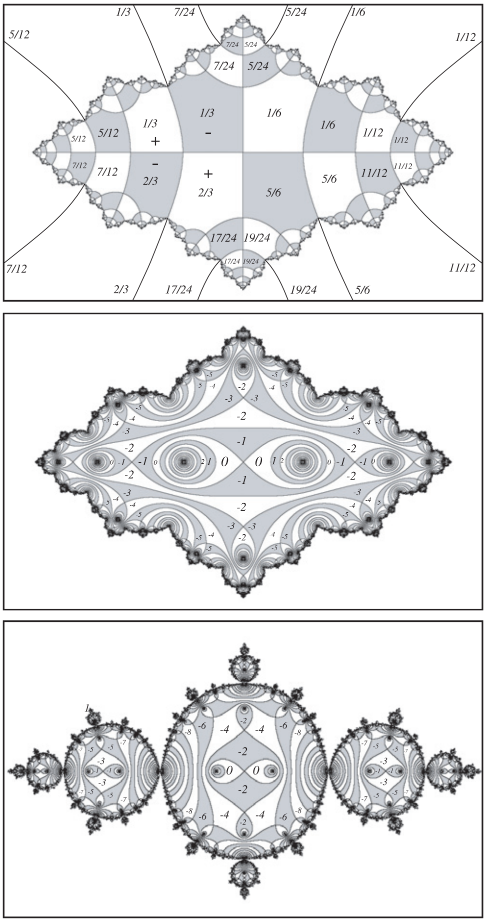

FIGURE 7. 'Checkerboard' and 'Zebras', showing the structure of the addresses of tiles. Checkerboard (top panel), with some external rays drawn in, illustrates the relation between the external angles and the angles of tiles. The invariant regions colored in white and gray correspond to tiles of signature $+$ and $-$, respectively. Zebras (middle and bottom panel) show the levels of tiles for f_c with $-1 < c < 0$; the levels get higher near the preimages of the attracting periodic points.

Remarks on angles and levels.

- We make an exception for non-trivial primitives ($q = q'$ and $l = l' > 1$). If $(f \to g)$ is non-trivial primitive, then $v = 2$ and only tiles of addresses $(\theta_\pm, m, \pm)$ are defined. However, we can formally define tiles of addresses $(\theta_\pm, m, \mp)$ by tiles of addresses $(\theta_\mp, m, \mp)$ respectively; see Figure 8.)
- For a degeneration pair $(f \to g)$, the space of possible addresses of tiles is not equal to $\Theta_f \times \mathbb{Z} \times \{+, -\}$ in general. For both f and g, all possible addresses are realized when $l = 1$. But when $l > 1$, the address $(\theta, m, \pm)$ is realized if and only if $m + n \equiv 0 \bmod l$ for some $n > 0$ with $2^n\theta = \theta_0^\pm$. In any case, note that $T_f(\theta, m, *)$ *exists if and only if* $T_g(\theta, m, *)$ *exists.*

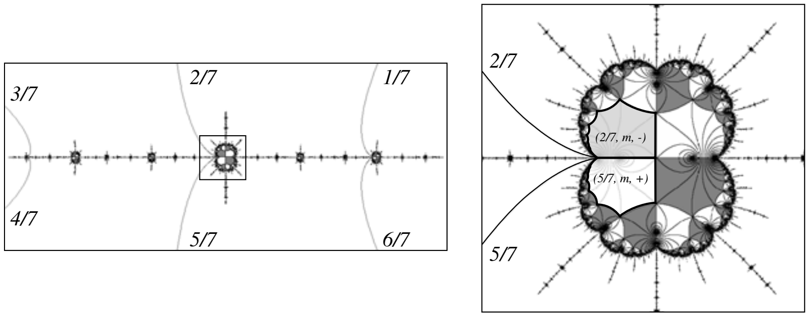

FIGURE 8. A non-trivial primitive $(f \to g)$ with $g(z) = z^2 - 7/4$. We define, for example, $T_f(2/7, m, +)$ by $T_f(5/7, m, +)$.

3.3. *Edge sharing.* Let us investigate the combinatorics of tiles in $\mathrm{Tess}(f)$ and $\mathrm{Tess}(g)$. We will prove the next proposition, which is a detailed version of Theorem 1.1(6).

PROPOSITION 3.2. *For $\theta \in \Theta_f = \Theta_g$ and $* \in \{+, -\}$, let us take an $m \in \mathbb{Z}$ such that $T = T_f(\theta, m, *)$ and $S = T_g(\theta, m, *)$ exist. Then the following hold.*

(1) *The circular edges of T and S are shared by $T_f(\theta, m \pm l, *)$ and $T_g(\theta, m \pm l, *)$, respectively.*

(2) *The degenerating edge of T is contained in $I(\zeta)$ with $\zeta \in \alpha_f$ of type $\Theta(\zeta)$ if and only if S attaches at $\xi \in I_g$ of type $\Theta(\xi) = \Theta(\zeta)$. Moreover, the degenerating edge of T is shared with $T_f(\theta, m, \bar{*})$, where $\bar{*}$ is the opposite signature of $*$.*

(3) *T shares its critical edge with $T_f(\theta', m', *')$ if and only if S does the same with $T_g(\theta', m', *')$. In this case, we have $m' = m$ and $*' = \bar{*}$.*

Thus, the combinatorics of $\mathrm{Tess}(f)$ and $\mathrm{Tess}(g)$ are the same.

Proof. (1) *Circular edges:* By Proposition 3.1, for any $n \geq 0$, the inverse map $f^{-n} \circ \Phi_f^{-1}$ over $\mathbb{C} - (-\infty, a]$ is a multivalued function with univalent branches. It follows that the property '$A_\mu(*)$ shares its circular edges with $A_{\mu\pm 1}(*)$' is translated to '$T(\theta, m, *)$ shares its circular edges with $T(\theta, m \pm l, *)$' by one of these univalent branches. The same argument works for $\Phi_g : B_0' \to \mathbb{C}$, which does not ramify over $\mathbb{C} - (-\infty, 0]$.

(2) *Degenerating edges:* The statement follows from the definitions of tiles and addresses.

(3) *Critical edges:* The combinatorics of tiles are essentially determined by the connections of critical edges, which are organized as follows.

In the fundamental model, we consider the families of curves

$$\overline{A_\mu(*)} \cap \{|W - a| = R^{\mu+1/2}\},$$
$$C_\mu(*) \cap \{\mathrm{Re}\, W = \mu R + 1/2\}$$

for $\mu \in \mathbb{Z}$, and we call these the *essential curves* of $A_\mu(*)$ and $C_\mu(*)$. Since $\Phi_f \circ f^n$ and $\Phi_g \circ g^n$ do not ramify over these essential curves, their pulled-back images in the original dynamics form 'equipotential curves' in K_f° and K_g°. The *essential curve* of a tile is the intersection with such equipotential curves.

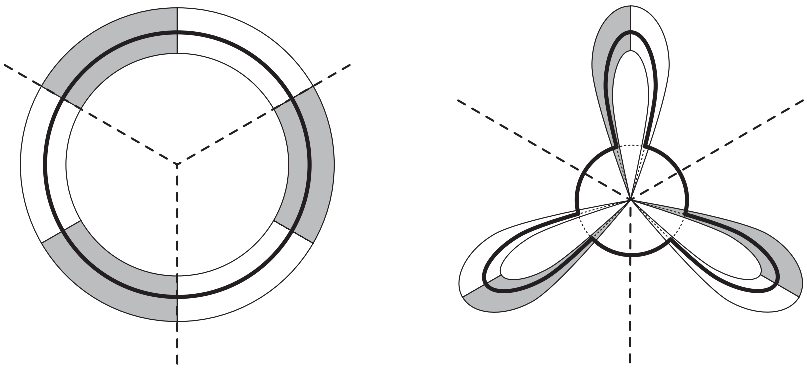

FIGURE 9. The thick curves show η_0 and η'_0 in Case (a) with $q = q' = 3$. The dashed lines indicate the degenerating arcs or external rays.

Let us consider a general tile $T \in \mathrm{Tess}(f)$ as in the statement. By taking a suitable $n \gg 0$, we may assume that $f^n(T)$ is a tile in B_0, with angle t in $\{\theta_0^+, \theta_0^-\} \subset \Theta(\alpha_0)$ and level μl for some $\mu \geq 0$. In particular, we may assume that $f^n(T)$ is in the critical sector based at $I(\alpha_0)$. Then, for S in the statement, we can take the same n and μ as those for T, so that $g^n(S)$ is a tile in B'_0 with angle t in $\Theta(\beta_0) = \Theta(\alpha_0)$ and level μl.

Case (a): $q = q'$. Let η_0 be the union of essential curves of tiles of the form $T_f(t, \mu l, *)$ with t in $\Theta(\alpha_0)$. Then η_0 forms an equipotential curve around α_0, since $\Phi_f|_{\eta_0}$ is a q-fold covering over the circle $\{|W - a| = R^{1/2+\mu}\}$. For $n > 0$, set $\eta_{-n} = f^{-n}(\eta_0)$. Then η_{-n} is a disjoint union of simple closed curves passing through tiles of level $\mu l - n$ and with angles in $\delta^{-n}(\Theta(\alpha_0))$. In particular, each curve crosses degenerating edges and critical edges alternately. More precisely, let η be a connected component of η_{-n}. Then the degree of $f^n : \eta \to \eta_0$ varies according to how many curves in $\{f^k(\eta)\}_{k=1}^n$ enclose the critical point $z = 0$. One can check the degree by counting the number of points of $f^{-n}(\alpha_0)$ inside η. Let $\zeta_1, \ldots, \zeta_N$ be these points; then η crosses each $I(\zeta_i)$, and thus η crosses the tiles of level $-n$ with angles in $\Theta(\zeta_1) \cup \cdots \cup \Theta(\zeta_N) \subset \mathbb{T}$ in cyclic order, and with signatures switching as the edges of tiles are crossed. This observation provides a description of how critical and degenerating edges are shared among tiles along η.

Now we can take η passing through T. From the above observation we deduce that: if T shares its critical edge with $T_f(\theta', m', *')$, then $m' = m$ and $*' = \bar{*}$; if T shares its degenerating edge with $T_f(\theta', m', *')$, then $\theta' = \theta$, $m' = m$ and $*' = \bar{*}$.

For S, consider a circle around β_0 which is so small that the circle and the essential curves of tiles with angle $\theta \in \Theta(\beta_0)$ and level μl bound a flower-like disk; see Figure 9. Let us denote the boundary of the disk by η'_0, which works as η_0. Since the combinatorics of pulled-back sectors based at β_0 and $I(\alpha_0)$ are the same, the observation of $g^{-n}(\eta'_0) = \eta'_{-n}$ must be the same as that of η_{-n}. This concludes the proof in Case (a).

Case (b): $q = 1 < q'$. (Recall that in this case, O_g is perturbed into the repelling cycle $O'_f = \{\alpha'_1, \ldots, \alpha'_{l'} = \alpha'_0\}$, with $\alpha'_0 \to \beta_0$ as $f \to g$.) The same argument as above works if we take η_0 and η'_0 as follows: first, in the fundamental model, take $\epsilon \ll 1$ and two radial half-lines from a with arguments $\pm\epsilon$. Then there are univalently pulled-back arcs of two

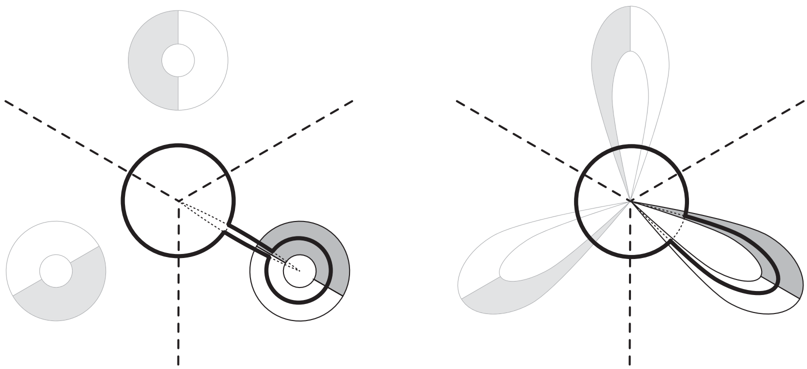

FIGURE 10. η_0 and η_0' in Case (b) with $q' = 3$.

lines in the critical sector that joins α_0 and α_0'. Next, we take simple closed curves around α_0 and α_0'. For α_0, we take the essential curves along tiles of address $(\theta_0^{\pm}, \mu l, \pm)$. For α_0', we take just a small circle around α_0'. Then the two arcs and two simple closed curves bound a dumbbell-like topological disk. We define η_0 to be its boundary curve.

Similarly, for g, the essential curves of tiles of address $(\theta_0^{\pm}, \mu l, \pm)$ and a small circle around β_0 bound a topological disk. We take η_0' as its boundary; see Figure 10. □

3.4. *Tiles and panels with small diameters.* We now show that the diameters of tiles are controlled by their angles. For θ in $\Theta_f = \Theta_g$ and $* \in \{+, -\}$, let $\Pi_f(\theta, *)$ and $\Pi_g(\theta, *)$ be the union of tiles with angle θ and signature $*$ in $\mathrm{Tess}(f)$ and $\mathrm{Tess}(g)$, respectively; we call these *panels* of angle θ and signature $*$. (For later convenience, we shall denote $\Pi_f(\theta, +) \cup \Pi_f(\theta, -)$ by $\Pi_f(\theta)$.) The *depth* of angle θ is the minimal $n \geq 0$ such that $2^n\theta = \theta_0^+$, where $\theta_0^+ \in \Theta(\alpha_0) = \Theta(\beta_0)$ is as defined in §2.5. (Note that $\Pi_f(\theta_0^+) = \Pi_f(\theta_0^-)$ when $(f \to g)$ is non-trivial primitive.) We denote such an n by $\mathrm{depth}(\theta)$.

PROPOSITION 3.3. *For any fixed degeneration pair $(f \to g)$ and any $\epsilon > 0$, there exists $N = N(\epsilon, f, g)$ such that*

$$\operatorname{diam} \Pi_f(\theta, *) < \epsilon \quad \textit{and} \quad \operatorname{diam} \Pi_g(\theta, *) < \epsilon$$

for any signature $*$ and any $\theta \in \Theta_f$ with $\mathrm{depth}(\theta) \geq N$.

Proof. We first work with f and signature $+$. One can easily check that the interior Π of $\Pi_f(\theta_0^+, +)$ is a topological disk. For any $\theta \in \Theta_f$, the panel $\Pi_f(\theta, +)^\circ$ is sent univalently onto Π by f^n with $n = \mathrm{depth}(\theta)$. Let F_θ be the univalent branch of f^{-n} which sends Π to $\Pi_f(\theta, +)^\circ$. Since the family $\{F_\theta : \theta \in \Theta_f\}$ on Π avoids the values outside the Julia set, it is normal.

Now we claim that $\operatorname{diam} \Pi_f(\theta, +) \to 0$ as $\mathrm{depth}(\theta) \to \infty$. Suppose otherwise; then one can find a sequence $\{\theta_k\}_{k>0}$ with depth $n_k \to \infty$ and a $\delta > 0$ such that $\operatorname{diam} \Pi_f(\theta_k, +) > \delta$ for any k. By passing to a subsequence, we may assume that

$F_k := F_{\theta_k}$ has a non-constant limit ϕ. Fix any point $z \in \Pi$, and set $\zeta := \phi(z) = \lim F_k(z)$. Since ϕ is holomorphic and is therefore an open map, there exists a neighborhood V of ζ such that $V \subset \phi(\Pi)$ and $V \subset F_k(\Pi)$ for all $k \gg 0$. Since $f^{n_k}(V) \subset \Pi \subset K_f^\circ$, any point in V is attracted to the cycle O_f. However, by univalence of F_k, there exists a neighborhood W of z with $W \subset F_k^{-1}(V) = f^{n_k}(V)$ for all $k \gg 0$. This is a contradiction; thus the claim is verified.

Finally, we arrange the angles of Θ_f in a sequence $\{\theta_i\}_{i>0}$ such that $\mathrm{depth}(\theta_n)$ is non-decreasing. Note that for any integer n, there are only finitely many angles with depth n. Thus there exists an integer $N = N(\epsilon, f, +)$ such that $\Pi_f(\theta, +)$ has diameter less than ϵ if $\mathrm{depth}(\theta) \geq N$.

This argument also works if we switch the map (from f to g) or the signature (from $+$ to $-$). Thus we obtain four distinct N's, and we can take $N(\epsilon, f, g)$ to be their maximum. □

Indeed, as depth tends to infinity, we have uniformly small panels for $f \approx g$ (see Proposition 5.6).

4. *Pinching semiconjugacy*

In this section we construct a semiconjugacy $h : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ associated with $(f \to g)$ by gluing tile-to-tile homeomorphisms inside the Julia sets and using the topological conjugacy induced by the Böttcher coordinates outside the Julia sets.

THEOREM 4.1. *For a degeneration pair $(f \to g)$, there exists a semiconjugacy $h : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ from f to g such that:*

(1) *h maps $\bar{\mathbb{C}} - I_f$ to $\bar{\mathbb{C}} - I_g$ homeomorphically and is a topological conjugacy between $f|_{\bar{\mathbb{C}} - I_f}$ and $g|_{\bar{\mathbb{C}} - I_g}$;*

(2) *for each $\zeta \in \alpha_f$ with type $\Theta(\zeta)$, h maps $I(\zeta)$ onto a point $\xi \in I_g$ with type $\Theta(\xi) = \Theta(\zeta)$;*

(3) *h sends all possible $T_f(\theta, m, *)$ to $T_g(\theta, m, *)$, $R_f(\theta)$ to $R_g(\theta)$, and $\gamma_f(\theta)$ to $\gamma_g(\theta)$.*

This theorem emphasizes the combinatorial properties of h. In the next section we will show that $h \to \mathrm{id}$ as f tends to g uniformly.

4.1. *Trans-component partial conjugacy and subdivision of tessellations.*

Let $(f_1 \to g_1)$ and $(f_2 \to g_2)$ be distinct satellite degeneration pairs with $g_1 = g_2$. More precisely, we consider $(f_1 \to g_1)$ and $(f_2 \to g_2)$ to be tuned copies of degeneration pairs in segments (s1) and (s2), with $q > 1$ by the same tuning operator. By composing homeomorphic parts of the conjugacies associated with $(f_1 \to g_1)$ and $(f_2 \to g_2)$, we deduce the following.

COROLLARY 4.2. *There exists a topological conjugacy $\kappa = \kappa_{f_1, f_2} : \bar{\mathbb{C}} - I_{f_1} \to \bar{\mathbb{C}} - I_{f_2}$ from f_1 to f_2.*

For example, the panel $\Pi_{f_1}(\theta, *)$ is mapped to the panel $\Pi_{f_2}(\theta, *)$. Now we can compare $\mathrm{Tess}(f_1)$ and $\mathrm{Tess}(f_2)$ via $\mathrm{Tess}(g_i)$. By comparing $\mathrm{Tess}(g_1)$ and $\mathrm{Tess}(g_2)$, one can easily check that

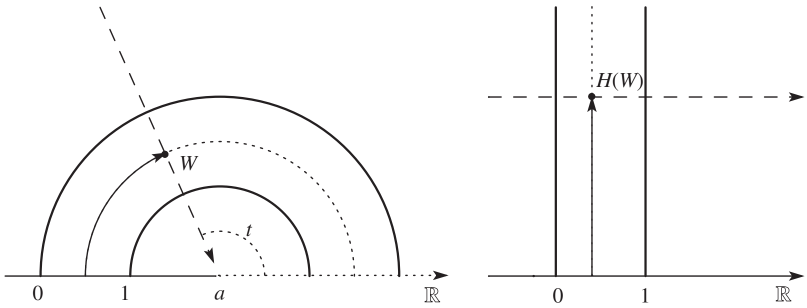

FIGURE 11. H maps $A_0(+)$ to $C_0(+)$.

$$T_{g_2}(\theta,\ \mu,\ *) = \bigcup_{j=0}^{q-1} T_{g_1}(\theta,\ \ \mu + lj,\ \ *)$$

for any $T_{g_2}(\theta, \mu, *) \in \mathrm{Tess}(g_2)$. Thus $\mathrm{Tess}(g_1)$ is just a subdivision of $\mathrm{Tess}(g_2)$.

Take a tile $T_{f_1}(\theta, m, *) \in \mathrm{Tess}(f_1)$. Then there is a homeomorphic image $T_2'(\theta, m, *) := \kappa(T_{f_1}(\theta, m, *))$ in $K_{f_2}^{\circ}$. We say that the family

$$\mathrm{Tess}'(f_2) := \{\kappa(T) : T \in \mathrm{Tess}(f_1)\}$$

is the *subdivided tessellation* of $K_{f_2}^{\circ} - I_{f_2}$. Since $\mathrm{Tess}(f_1)$ and $\mathrm{Tess}(f_2)$ have, respectively, the same combinatorics as $\mathrm{Tess}(g_1)$ and $\mathrm{Tess}(g_2)$, it follows that

$$T_{f_2}(\theta,\ \mu,\ *) = \bigcup_{j=0}^{q-1} T_{f_2}'(\theta,\ \ \mu + lj,\ \ *)$$

for any $T_{f_2}(\theta, \mu, *) \in \mathrm{Tess}(f_2)$. Now we have a natural tile-to-tile correspondence between $\mathrm{Tess}(f_1)$, $\mathrm{Tess}(g_1)$ and $\mathrm{Tess}'(f_2)$. In other words, the combinatorial property of tessellation is preserved under the degeneration from f_1 to g and the bifurcation from g to f_2.

In [**Ka3**], we will use this property to investigate the structures of the Lyubich–Minsky hyperbolic 3-laminations associated with f_1, g and f_2.

4.2. *Proof of Theorem 4.1.* The rest of this section is devoted to the proof of our main theorem. The proof can be broken down into five steps.

(1) *Conjugacy on the fundamental model.* We define a topological map $H : \mathbb{C} - I \to \mathbb{C}$ which maps $A_\mu(\pm)$ to $C_\mu(\pm)$ homeomorphically. For $W \in \mathbb{C} - I$, set $W := a + \rho e^{it}$, where $\rho > 0$ and $0 < t < 2\pi$. The map H is then given by

$$H(W) := \frac{\log \rho - \log a}{\log R} + 2ai \tan \frac{\pi - t}{2} \in \mathbb{C}.$$

One can check that H conjugates the action of F on $\mathbb{C} - I$ to that of G on $\mathbb{C}$, and H maps $A_\mu(\pm)$ homeomorphically onto $C_\mu(\pm)$.

(2) *Tile-to-tile conjugation.* Consider the critical sectors of f and g. Let Π_0 and Π'_0 denote the union of tiles of addresses $(\theta_0^{\pm}, \mu l, \pm)$ with $\mu > 0$ in $\mathrm{Tess}(f)$ and $\mathrm{Tess}(g)$, respectively.

By Proposition 3.1, the map $\Phi_f : \Pi_0 \to \mathbb{C}$ is univalent, and we can choose a univalent branch Ψ_g of Φ_g^{-1} which sends $\{W : \mathrm{Re}\, W \geq 1\}$ to Π'_0. For each point in Π_0, we define $h := \Psi_g \circ H \circ \Phi_f|_{\Pi_0}$. Then h is a conjugacy between $f^{lq}|_{\Pi_0}$ and $g^{lq}|_{\Pi'_0}$. Note that all tiles will eventually land on tiles in Π_0 or Π'_0. According to the combinatorics of tiles determined by pulling back essential curves in Π_0 and Π'_0, we can pull back h over $K_f^\circ - I_f$. Now the map $h : K_f^\circ - I_f \to K_g^\circ$ conjugates $f|_{K_f^\circ - I_f}$ and $g|_{K_g^\circ}$.

(3) *Continuous extension to the degenerating arc system.* Take $\zeta \in \alpha_f$. For any point z in $I(\zeta)$, we define $h(z)$ by the unique $\xi \in I_g$ such that $\Theta(\xi) = \Theta(\zeta)$.

We now show the continuity of the $h : K_f^\circ \cup I_f \to K_g^\circ \cup I_g$ which we have defined above. Take any z in $I(\zeta)$. We claim that any sequence $z_n \in K_f^\circ \cup I_f$ converging to z must satisfy $h(z_n) \to \xi$.

First, when z is neither ζ nor one of the endpoints of $I(\zeta)$, it is enough to consider the case of $z_n \in K_f^\circ - I_f$ for all n. Now, z is on the degenerating edges of at most four tiles; let $T = T_f(\theta, m, +)$ be one of these tiles. The subsequence z_{n_i} of z_n contained in T is mapped to $T_g(\theta, m, +)$. In the fundamental model, the sequence $h(z_{n_i})$ corresponds to a sequence whose imaginary part is getting larger. Thus $h(z_{n_i})$ converges to ξ with its type containing θ, which must coincide with $\Theta(\zeta)$. By varying the choice of T, we have $h(z_n) \to \xi$ with $\Theta(\xi) = \Theta(\zeta)$.

Next, if z is ζ or one of the endpoints of $I(\zeta)$, then it is an attracting or repelling periodic point. If z is attracting, the levels of tiles containing z_n go to $+\infty$. According to the fundamental model, we then have $h(z_n) \to \xi$.

The remaining case is that z is repelling and hence in the Julia set. We deal with this case in the next step of the proof.

(4) *Continuous extension to the Julia set.* Take any $z \in J_f$ and any sequence $z_n \in K_f^\circ \cup I_f$ converging to z. Then pick a sequence $\theta_n \in \Theta_f$ such that $z_n \in \Pi_f(\theta_n)$. After passing to a subsequence, we may assume that θ_n and $h(z_n)$ converge to some $\theta \in \mathbb{T}$ and some $w \in K_g$, respectively.

We first claim that $z = \gamma_f(\theta)$; that is, $\theta \in \Theta(z)$. If the depth of θ_n is bounded, then $\theta_n = \theta \in \Theta_f$ for all $n \gg 0$. This implies that $z_n \in \Pi_f(\theta)$ for all $n \gg 0$ and it follows that $z \in \overline{\Pi_f(\theta)} \cap J_f$. Thus $z = \gamma_f(\theta)$ by definition of $\Pi_f(\theta)$. If the depth of θ_n is unbounded, it is enough to consider subsequences with the depth of θ_n monotonically increasing. Take any $\epsilon > 0$. For $n \gg 0$, we have $|\gamma_f(\theta_n) - z_n| < \epsilon$ by Proposition 3.3, and we also have $|\gamma_f(\theta_n) - \gamma_f(\theta)| < \epsilon$ by continuity of $\gamma_f : \mathbb{T} \to J_f$. Finally, $|z - z_n| < \epsilon$ for $n \gg 0$ implies $|z - \gamma_f(\theta)| < 3\epsilon$, and we conclude the proof of the claim.

Since $h(z_n) \in \Pi_g(\theta_n)$, the same argument works for $h(z_n)$ and w. Hence we also have $w = \gamma_g(\theta) \in J_g$. It follows that for the original $z_n \to z$, the sequence $h(z_n)$ accumulates only on $\gamma_g(\theta)$, with $\theta \in \Theta(z)$.

By Theorem A.1, there exists a semiconjugacy $h_J : J_f \to J_g$ with $h_J \circ \gamma_f = \gamma_g$. Since $\gamma_f(\theta) = \gamma_f(\theta')$ for any θ, θ' in $\Theta(z)$, we have $\gamma_g(\theta) = \gamma_g(\theta')$. This implies that $h(z_n)$ accumulates on a unique point $\gamma_g(\theta)$. Thus h continuously extends to the Julia set by $h(\gamma_f(\theta)) := \gamma_g(\theta)$ for each $\theta \in \mathbb{T}$.

(5) *Global extension.* The final step is to define $h : \bar{\mathbb{C}} - K_f \to \bar{\mathbb{C}} - K_g$ via the conformal conjugacy between $f|_{\bar{\mathbb{C}}-K_f}$ and $g|_{\bar{\mathbb{C}}-K_g}$ given by the Böttcher coordinates. This conjugacy and the semiconjugacy above are continuously glued along the Julia set, thus giving a semiconjugacy on the sphere.

Properties (2) and (3) of Theorem 4.1 are clear by construction. To check property (1), we need to show that $h^{-1} : \bar{\mathbb{C}} - I_f \to \bar{\mathbb{C}} - I_g$ is continuous. Continuity in $\bar{\mathbb{C}} - K_g$ and K_g° is obvious by construction. Take any point $w \in J_g - I_g$. An argument similar to that in step 4 shows that any sequence $w_n \to w$ within $\bar{\mathbb{C}} - I_g$ is mapped to a convergent sequence $z_n \to z$ within $\bar{\mathbb{C}} - I_f$ that satisfies $\Theta(z) = \Theta(w) \subset \mathbb{T} - \Theta_g$. □

5. *Continuity of pinching semiconjugacies*

In this section we deal with continuity of the dynamics of the degeneration pair $(f \to g)$ as f tends to g. We will establish the following result.

THEOREM 5.1. *Let $h : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ be the semiconjugacy associated with a degeneration pair $(f \to g)$ as given in Theorem 4.1. Then h tends to the identity as f tends to g.*

Here are two immediate corollaries.

COROLLARY 5.2. *The closures of $T_f(\theta, m, *)$ and $\Pi_f(\theta, *)$ in* Tess(f) *converge uniformly to those of $T_g(\theta, m, *)$ and $\Pi_g(\theta, *)$ in* Tess(g) *in the Hausdorff topology.*

COROLLARY 5.3. *As $f \to g$, the diameters of the connected components of I_f tend to zero uniformly.*

Let us start with some terminology to be used in the proof. Two degeneration pair $(f_1 \to g_1)$ and $(f_2 \to g_2)$ are said to be *equivalent* if $g_1 = g_2$ and both f_1 and f_2 are in the same hyperbolic component. For a degeneration pair $(f \to g)$, by $f \approx g$ we mean that f is sufficiently close to g; in other words, the multiplier $r\omega$ of O_f is sufficiently close to ω, i.e. $r \approx 1$.

Formally, we consider a family of equivalent degeneration pairs $\{(f \to g)\}$ parameterized by $0 < r < 1$, and investigate its behavior as r tends to 1. To prove the theorem, it suffices to show the following.

(i) For any compact set K in $\bar{\mathbb{C}} - K_g$, we have $K \subset \bar{\mathbb{C}} - K_f$ for all $f \approx g$ and $h \to \mathrm{id}$ on K.
(ii) For any compact set K in K_g°, we have $K \subset K_f^\circ$ for all $f \approx g$ and $h \to \mathrm{id}$ on K.
(iii) h is equicontinuous as $f \to g$ on the sphere.

In fact, any sequence h_k associated with $f_k \to g$ has a subsequential limit h_∞ which is the identity on $\bar{\mathbb{C}} - J_g$ and continuous on $\bar{\mathbb{C}}$. Since $\bar{\mathbb{C}} - J_g$ is open and dense, the map h_∞ must be the identity on the whole sphere.

5.1. *Proof of (i).*

Let $B_f : \bar{\mathbb{C}} - \mathbb{D} \to \bar{\mathbb{C}} - K_f^\circ$ be the extended Böttcher coordinate of K_f, i.e. $B_f : \bar{\mathbb{C}} - \bar{\mathbb{D}} \to \bar{\mathbb{C}} - K_f$ is a conformal map such that $B_f(w^2) = f(B_f(w))$, $B_f(w)/w \to 1$ as $w \to \infty$, and $B_f(e^{2\pi i\theta}) := \gamma_f(\theta) \in J_f$. Now (i) follows immediately from this stronger claim below.

THEOREM 5.4. (Böttcher convergence) *As* $f \to g$, *we have uniform convergence* $B_f \to B_g$ *on* $\bar{\mathbb{C}} - \mathbb{D}$.

Note that establishing uniform convergence on compact sets in $\bar{\mathbb{C}} - \bar{\mathbb{D}}$ is not difficult. Our proof is a mild generalization of the proof of [**Po**, Theorem 2.11].

Proof. By Corollary A.2, one can easily check that $\bar{\mathbb{C}} - K_f$ converges to $\bar{\mathbb{C}} - K_g$ in the sense of Carathéodory kernel convergence with respect to ∞. Thus pointwise convergence $B_f \to B_g$ on each $z \in \bar{\mathbb{C}} - \bar{\mathbb{D}}$ is given by [**Po**, Theorem 1.8] and $B_f'(\infty) = B_g'(\infty) = 1$. To prove the theorem, it is enough to show that K_f is uniformly locally connected as $f \to g$, by [**Po**, Corollary 2.4]. In other words, it is sufficient to show that for any $\epsilon > 0$, there exists a $\delta > 0$ such that for any $f \approx g$ and any $a, b \in K_f$ with $|a - b| < \delta$, there exists a continuum $E \subset K_f$ such that $a, b \in E$ and diam $E < \epsilon$.

For contradiction, suppose we have a sequence of equivalent degenerating pairs $(f_n \to g)$ which satisfies: $f_n \to g$ uniformly; for each f_n, there exist a_n and a_n' in J_{f_n}, with $|a_n - a_n'| \to 0$, which cannot be contained in the same continuum in K_{f_n} of diameter less than $\epsilon_0 > 0$. We may set $a_n = \gamma_{f_n}(\theta_n)$ and $a_n' = \gamma_{f_n}(\theta_n')$ for some $\theta_n, \theta_n' \in \mathbb{T}$, since γ_{f_n} maps $\mathbb{T}$ onto J_{f_n}. By passing to a subsequence, we may also assume that $\theta_n \to \theta$ and $\theta_n' \to \theta'$. Since $\gamma_{f_n} \to \gamma_g$ uniformly by Corollary A.3, the assumption $|a_n - a_n'| \to 0$ implies that we have $\gamma_g(\theta) = \gamma_g(\theta') =: w \in J_g$. Now we consider the following cases.

Case 1: $\theta = \theta'$. We may assume that $\theta_n \le \theta_n'$ and that both sequences tend to θ. Set $E_n := \{\gamma_{f_n}(t) : t \in [\theta_n, \theta_n']\}$, which is a continuum containing a_n and a_n'. Then, for any $t \in [\theta_n, \theta_n']$, we have $|\gamma_{f_n}(t) - w| \le |\gamma_{f_n}(t) - \gamma_g(t)| + |\gamma_g(t) - \gamma_g(\theta)| \to 0$, since $\gamma_{f_n} \to \gamma_g$ uniformly and γ_g is continuous. This implies that diam $E_n \to 0$, which is a contradiction.

Case 2-1: $\theta \ne \theta'$ and $w \notin I_g$. First we show that $\gamma_{f_n}(\theta) = \gamma_{f_n}(\theta')$. Let $h_n : J_{f_n} \to J_g$ be the semiconjugacy given by Theorem A.1. Since $h_n \circ \gamma_{f_n} = \gamma_g$, we have

$$w = h_n \circ \gamma_{f_n}(\theta) = h_n \circ \gamma_{f_n}(\theta') \notin I_g.$$

By property (1) of Theorem A.1, this implies that $\gamma_{f_n}(\theta) = \gamma_{f_n}(\theta')$. Now set

$$E_n := \{\gamma_{f_n}(t) : |t - \theta| \le |\theta_n - \theta| \text{ or } |t - \theta'| \le |\theta_n' - \theta'|\},$$

which is a continuum containing a_n and a_n'. Again one can easily check that $|\gamma_{f_n}(t) - w| \to 0$ uniformly for any $\gamma_{f_n}(t) \in E_n$, and thus diam $E_n \to 0$, a contradiction.

Case 2-2: $\theta \ne \theta'$ and $w \in I_g$. There exists an $m \ge 0$ such that $g^m(w) = \beta_0$. Since $h_n \circ \gamma_{f_n} = \gamma_g$, we have $\gamma_{f_n}(\theta), \gamma_{f_n}(\theta') \in h_n^{-1}(w) \subset J_{f_n} \cap I_{f_n}$. If $q = 1$, then h_n is homeomorphism by Theorem A.1. Thus $\gamma_{f_n}(\theta) = \gamma_{f_n}(\theta')$, and a contradiction follows by the same argument as before.

Suppose $q > 1$. Then we are in Case (a) ($q = q'$ and $l = l'$) of Proposition 2.1. In particular, we have $w_n \in \alpha_{f_n}$ such that $w_n \to w$ and $f_n^m(w_n)$ is an attracting periodic point $\alpha_{0,n} \in O_{f_n}$ which tends to β_0. Let $\lambda_n = r_n e^{2\pi i p/q}$ be the multiplier of O_{f_n} with $r_n \nearrow 1$. In a fixed small neighborhood of w, we have

$$\begin{aligned} f^{-m} \circ f^{lq} \circ f^m(z) &= r_n^q z\,(1 + z^q + O(z^{2q})) \\ \longrightarrow\ g^{-m} \circ g^{lq} \circ g^m(z) &= z\,(1 + z^q + O(z^{2q})) \end{aligned}$$

by using suitable local coordinates as in Appendix A.2. (For simplicity, we abbreviate conjugations by the local coordinates.)

By Lemma A.7, we can find a small continuum $E'_n \subset K_{f_n}$ which joins w_n and pre-periodic points $\gamma_{f_n}(\theta)$, $\gamma_{f_n}(\theta')$. Set E_n as in Case 2-1. Now $E'_n \cup E_n$ is a continuum containing a_n and a'_n. Since diam $(E'_n \cup E_n) \to 0$, we again obtain a contradiction. □

5.2. *Proof of (ii).* We start with the following theorem.

THEOREM 5.5. (Linearization convergence) *Let K be any compact set in K_g°. Then $K \subset K_f^\circ$ for $f \approx g$, and $\Phi_f \to \Phi_g$ uniformly on K.*

Proof. One can easily check that, by Corollary A.2, $K \subset K_f^\circ$ if $f \approx g$. Let $\beta_0 \in O_g \cap \partial B'_0$. By taking a suitable $N \gg 0$, we may assume that $K' = g^N(K)$ is sufficiently close to β_0 and contained in B'_0. Then K' is attracted to β_0 along the attracting direction associated with B'_0 by iteration of $g^{l'q'}$. For simplicity, set $\bar{l} := lq = l'q'$.

Recall that Φ_f and Φ_g semiconjugate $f^{\bar{l}}$ and $g^{\bar{l}}$ to F^q and G^q, respectively, in the fundamental model. We will construct further semiconjugacies $\tilde{\Phi}_f$ and $\tilde{\Phi}_g$ which have the same property as Φ_f and Φ_g, and are such that $\tilde{\Phi}_f \to \tilde{\Phi}_g$ on compact subsets of a small attracting petal in B'_0. Then we will show that they coincide.

By Appendix A.2, there exist local coordinates $\zeta = \psi_f(z)$ and $\zeta = \psi_g(z)$, with $\psi_f \to \psi_g$ near β_0, such that we can view $f^{\bar{l}} \to g^{\bar{l}}$ as

$$\begin{aligned} f^{\bar{l}}(\zeta) &= \Lambda\zeta\ (1 + \zeta^{q'} + O(\zeta^{2q'})) \\ \longrightarrow\ g^{\bar{l}}(\zeta) &= \zeta\ (1 + \zeta^{q'} + O(\zeta^{2q'})) \end{aligned}$$

where $\Lambda \to 1$. (To simplify notation, we abbreviate conjugations by these local coordinates. For example, we write $f^{\bar{l}}(\zeta)$ to mean $\psi_f \circ f^{\bar{l}} \circ \psi_f^{-1}(\zeta)$.) There are two cases for Λ.

- In Case (a) ($q = q'$ and $l = l'$), the fixed point $\zeta = 0$ is attracting, and $\Lambda = \lambda^q = r^q = R < 1$.
- In Case (b) ($q = 1 < q' = l/l'$), the fixed point $\zeta = 0$ is repelling, and $|\Lambda| > 1$.

By taking branched coordinate changes $w = \Psi_f(\zeta) = -\Lambda^{q'}/(q'\zeta^{q'})$ and $w = \Psi_g(\zeta) = -1/(q'\zeta^{q'})$, respectively, we can view $f^{\bar{l}} \to g^{\bar{l}}$ as

$$\begin{aligned} f^{\bar{l}}(w) &= \Lambda^{-q'} w + 1 + O(1/w) \\ \longrightarrow\ g^{\bar{l}}(w) &= w + 1 + O(1/w). \end{aligned}$$

Case (a). Set $\tau = \Lambda^{-q'} = R^{-q} > 1$. By simultaneous linearization as described in Appendix A.3, we have convergent coordinate changes $W = u_f(w) \to u_g(w)$ on compact sets of $P_\rho := \{\mathrm{Re}\, w > \rho \gg 0\}$ such that $f^{\bar{l}} \to g^{\bar{l}}$ can be viewed as

$$\begin{aligned} \tilde{F}(W) &:= f^{\bar{l}}(W) = \tau W + 1 \\ \longrightarrow\ \tilde{G}(W) &:= g^{\bar{l}}(W) = W + 1. \end{aligned}$$

Let us adjust $\tilde{F} \to \tilde{G}$ to $F^q \to G^q$ in the fundamental model. Recall that the map $F(W) = RW + 1$ has an attracting fixed point at $a = 1/(1 - R)$. On the other hand, the map $\tilde{F}$ has the repelling fixed point $\tilde{a} = 1/(1 - R^{-q})$ instead. Set $T_f(W) := aW/(W - \tilde{a})$. Then $T_f(W) = qW(1 + O(W/\tilde{a})) \to T_g(W) = qW$ on any compact sets on the W-plane as $R \to 1$. By taking conjugations with T_f and T_g, we can view $\tilde{F} \to \tilde{G}$ as $F^q \to G^q$ on any compact sets of the domain of $\tilde{G}$.

Case (b). By Rouché's theorem, there exists a fixed point b of $f^{\bar{l}}(w) = \Lambda^{-q'}w + 1 + O(1/w)$ that has the form $b = 1/(1 - \Lambda^{-q'}) + O(1)$. Indeed, this b belongs to the image of the attracting cycle O_f, hence its multiplier is $r < 1$. Set $S_f(w) := bw/(b - w)$. Then $S_f(w) = w(1 + O(w/b)) \to S_g(w) = w$ on any compact sets of the w-plane as $r \to 1$. By taking conjugations by S_f and S_g, we can view $f^{\bar{l}}(w) \to g^{\bar{l}}(w)$ as

$$\begin{aligned} f^{\bar{l}}(w) &= \tau w + 1 + O(1/w) \\ \longrightarrow \ g^{\bar{l}}(w) &= w + 1 + O(1/w), \end{aligned}$$

where $\tau = 1/r > 1$. By simultaneous linearization, we have convergent coordinate changes $W = u_f(w) \to u_g(w)$ on compact sets of P_ρ, such that $f^{\bar{l}} \to g^{\bar{l}}$ is again viewed as

$$\begin{aligned} \tilde{F}(W) := f^{\bar{l}}(W) &= \tau W + 1 \\ \longrightarrow \ \tilde{G}(W) := g^{\bar{l}}(W) &= W + 1. \end{aligned}$$

Since $q = 1$, we adjust $\tilde{F} \to \tilde{G}$ to $F \to G$ in the fundamental model. Set $\tilde{b} := 1/(1 - \tau)$ and $T_f(W) := \tilde{b}W/(\tilde{b} - W)$. Then $T_f(W) = W(1 + O(W/\tilde{b})) \to T_g(W) = W$ on any compact sets on the W-plane as $r \to 1$. By taking conjugations by T_f and T_g, we can view $\tilde{F} \to \tilde{G}$ as $F \to G$ on any compact sets of the domain of $\tilde{G}$.

Adjusting critical orbits. Now we denote these final local coordinates conjugating $f^{\bar{l}} \to g^{\bar{l}}$ to $F^q \to G^q$ by $\hat{\Phi}_f \to \hat{\Phi}_g$, where the convergence holds on compact subsets of a small attracting petal P' in B_0' corresponding to P_ρ in the w-plane.

We need to compare the images of the critical orbits by $\hat{\Phi}_f \to \hat{\Phi}_g$ on the W-plane with those by Φ_f and Φ_g, and adjust their positions. We may assume that $g^{n\bar{l}}(0) \in P'$ for fixed $n \gg 0$. Then $f^{n\bar{l}}(0) \in P'$ for all $f \approx g$. Set $s := \hat{\Phi}_f(f^{n\bar{l}}(0))$ and $s' := \hat{\Phi}_g(g^{n\bar{l}}(0))$. Then $s \to s'$ as $f \to g$. On the other hand, we have

$$\Phi_f(f^{n\bar{l}}(0)) = F^{nq}(\Phi_f(0)) = F^{nq}(0) = R^{nq-1} + \cdots + 1 =: R_n$$

and $\Phi_g(g^{n\bar{l}}(0)) = nq$. Set $U_f(W) := k(W - a) + a$ and $U_g(W) := W + nq - s'$, where $k = (R_n - a)/(s - a)$. Then one can check that $U_f \to U_g$ on any compact sets in the W-plane as $f \to g$, and U_f and U_g commute with F and G, respectively. By defining $\tilde{\Phi}_f$ and $\tilde{\Phi}_g$ by $U_f \circ \hat{\Phi}_f$ and $U_g \circ \hat{\Phi}_g$, respectively, we have $\tilde{\Phi}_f \to \tilde{\Phi}_g$ on compact sets of P', with $\tilde{\Phi}_f(f^{n\bar{l}}(0)) = R_n$ and $\tilde{\Phi}_g(g^{n\bar{l}}(0)) = nq$.

Finally, we need to check that $\tilde{\Phi}_f = \Phi_f$ and $\tilde{\Phi}_g = \Phi_g$. The latter equality is clear by uniqueness of the Fatou coordinate [**Mi1**, §8]. For the former, recall that $W = \Phi_f(z)$ is given by

$$z \ \mapsto \ \phi_f(z) = w \ \mapsto \ w^q = W \ \mapsto \ a(1 - W) =: \Phi_f(z),$$

and ϕ_f is uniquely determined under the condition of $\phi_f(0) = 1$ [**Mi1**, §10]. Let us consider the local coordinate $\tilde{\phi}_f$ on a compact set of P' given by

$$z \mapsto \tilde{\Phi}_f(z) = W \mapsto \left(1 - \frac{W}{a}\right)^{1/q} =: \tilde{\phi}_f(z) = w,$$

where we take a suitable branch of the qth root so that $\tilde{\phi}_f(f^{n\bar{l}}(0)) = \lambda^{nq}$ on the w-plane. Then $\tilde{\phi}_f(f(z)) = \lambda\tilde{\phi}_f(z)$. Since $\phi_f(0) = 1$ is equivalent to $\phi_f(f^{n\bar{l}}(0)) = \lambda^{nq}$, the map $\tilde{\phi}_f$ coincides with ϕ_f. This implies the equality $\tilde{\Phi}_f = \Phi_f$.

Now we may assume that $K' = g^N(K) \subset D \Subset P'$ for some open set D. If $f \approx g$, then $f^N(K) \subset D$ and we have uniform convergence $\Phi_g \to \Phi_f$ on D. Uniform convergence on K is then obtained by $\Phi_f(z) = F^{-N}(\Phi_f(f^N(z))) \to G^{-N}(\Phi_g(g^N(z)) = \Phi_g(z)$ for $z \in K$. □

Proof of (ii). We begin by working with the fundamental model. Suppose that $\epsilon \searrow 0$, and set $R = 1 - \epsilon$. Then $F(W) = RW + 1$ fixes $a_\epsilon = 1/(1 - R) = \epsilon^{-1}$. For a fixed γ with $1/2 < \gamma < 1$, we define a compact set $Q_\epsilon \subset \mathbb{C}$ by

$$Q_\epsilon := \{W = a_\epsilon + \rho e^{(\pi - t)i} \in \mathbb{C} : |t| \leq \epsilon^\gamma,\ |\rho - a_\epsilon| \leq a_\epsilon \sin \epsilon^\gamma\}.$$

Let D be any bounded set in $\mathbb{C}$. For all $\epsilon \ll 1$, the compact set Q_ϵ contains D. Let $H : \mathbb{C} - [a_\epsilon, \infty) \to \mathbb{C}$ be the conjugacy between F and $G(W) = W + 1$ as described in §4. Then one can easily check that $|\mathrm{Re}\, W - \mathrm{Re}\, H(W)| = O(\epsilon^{2\gamma-1})$ and $|\mathrm{Im}\, W - \mathrm{Im}\, H(W)| = O(\epsilon^{2\gamma-1})$ on Q_ϵ. Thus $H \to \mathrm{id}$ uniformly on D.

Let K be any compact set in K_g°, and let D be the 1/10-neighborhood of $\Phi_g(K)$. For all $f \approx g$, we have $K \subset K_f^\circ$ and $\Phi_f(K) \subset D$ by Theorem 5.5. By the argument above for the fundamental model, the restriction $h|_K$ is a branch of $\Phi_g^{-1} \circ H \circ \Phi_f$ that converges to the identity. (The branch is determined by the tile-to-tile correspondence given by h.) □

5.3. *Proof of (iii).* To show (iii), we need two propositions on the properties of panels as $f \to g$. The first one is a refinement of Proposition 3.3, and the second is on the convergence of panels with a fixed angle.

PROPOSITION 5.6. (Uniformly small panels) *For any $\epsilon > 0$, there exists $N = N(\epsilon)$ such that for all $f \approx g$, $* = \pm$ and $\theta \in \Theta_g$ with* $\mathrm{depth}(\theta) \geq N$,

$$\mathrm{diam}\, \Pi_f(\theta, *) < \epsilon \quad \textit{and} \quad \mathrm{diam}\, \Pi_g(\theta, *) < \epsilon.$$

PROPOSITION 5.7. (Hausdorff convergence to a panel) *For a fixed angle $\theta \in \Theta_g$ and signature $* \in \{+, -\}$, we have $\overline{\Pi_f(\theta, *)} \to \overline{\Pi_g(\theta, *)}$ as $f \to g$ in the Hausdorff topology.*

Let us prove (iii) first by assuming these propositions hold.

By (i), we have equicontinuity near ∞. Assume that there exist degeneration pairs $(f_k \to g)$ with semiconjugacies h_k as given in Theorem 4.1, points $a_k, a_k' \in \mathbb{C}$ with $|a_k - a_k'| \to 0$, and $b_k = h_k(a_k)$, $b_k' = h_k(a_k')$ with $|b_k - b_k'| \geq \epsilon_0 > 0$.

Suppose that $a_k, a_k' \in \mathbb{C} - K_f^\circ$ and hence $b_k, b_k' \in \mathbb{C} - K_g^\circ$. Then there exist w_k, $w_k' \in \mathbb{C} - \mathbb{D}$ such that $a_k = B_{f_k}(w_k)$, $a_k' = B_{f_k}(w_k')$ and $b_k = B_g(w_k)$, $b_k' = B_g(w_k')$.

By Theorem 5.4, we have $B_{f_k} \to B_g$. Thus $|a_k - a'_k| \to 0$ implies $|b_k - b'_k| \to 0$, which is a contradiction.

Now it suffices to prove the case where $a_k, a'_k \in K_{f_k}$ and hence $b_k, b'_k \in K_g$. By taking subsequences, we may assume that $a_k \to a$, $a'_k \to a$, $b_k \to b$ and $b'_k \to b'$, with $|b - b'| \geq \epsilon_0/2 > 0$. Since $K_{f_k} \to K_g$ in the Hausdorff topology, a, b and b' are all in K_g.

First let us consider the case where a is bounded away from J_g. Then we have a compact neighborhood E of a such that $h_k|_E \to \mathrm{id}|_E$ and $a_k, a'_k \in E$ for all $k \gg 0$. This implies that $|b_k - b'_k| \to 0$, a contradiction.

Next, consider the case where $a \in J_g$. For $a_k \to a$ and $b_k \to b$, we claim that $a = b$. Then, by the same argument, we obtain $a = b'$ which is a contradiction.

For $a_k \in K_{f_k}$, we take any $\theta_k \in \mathbb{T}$ such that $a_k = \gamma_{f_k}(\theta_k)$ if $a_k \in J_{f_k}$, and otherwise a_k is contained in the closure of $\Pi_{f_k}(\theta_k)$. (Hence $b_k = \gamma_g(\theta_k)$, or b_k is in the closure of $\Pi_g(\theta_k)$.) By passing to a subsequence, we may assume that $\theta_k \to \theta$ for some $\theta \in \mathbb{T}$.

If $\theta_k \notin \Theta_g$, we define its depth by ∞. Then there are two more cases according to whether $\limsup \mathrm{depth}(\theta_k) = \infty$ or not.

If $\limsup \mathrm{depth}(\theta_k) = \infty$, we take a subsequence again and assume that $\mathrm{depth}(\theta_k)$ is strictly increasing. Then, by Proposition 5.6, we have $|a_k - \gamma_{f_k}(\theta_k)| \to 0$. Since $\theta_k \to \theta$ and $\gamma_{f_k} \to \gamma_g$ uniformly (Corollary A.3), we have $|a_k - \gamma_g(\theta)| \to 0$, so $a = \gamma_g(\theta)$. Similarly, we conclude that $b = \gamma_g(\theta)$, and this gives a contradiction.

If $\limsup \mathrm{depth}(\theta_k) < \infty$, we take a subsequence again and assume that $\theta_k = \theta \in \Theta_g$ for all $k \gg 0$. By Proposition 5.7, the $a_k \in \overline{\Pi_{f_k}(\theta)}$ are approximated by some $c_k \in \Pi_g(\theta)$ such that $|a_k - c_k| \to 0$; thus $c_k \to a \in J_g$. Since $\overline{\Pi_g(\theta)} \cap J_g = \{\gamma_g(\theta)\}$, we have $a = \gamma_g(\theta)$. On the other hand, if $b_k \in \Pi_g(\theta)$ is bounded away from J_g, there exists a compact neighborhood $E' \subset K_g^\circ$ of b where $h_k|_{E'} \to \mathrm{id}|_{E'}$, and this leads to a contradiction. Therefore $b \in J_g$ and must be $\gamma_g(\theta)$. Now we obtain $a = b$ as claimed. □

To complete the proof of Theorem 5.1, we need to finish the proofs of the propositions.

Proof of Proposition 5.6. We modify the argument of Proposition 3.3. Suppose that there exist $f_k \to g$ which determine equivalent degeneration pairs $(f_k \to g)$ and θ_k with $n_k = \mathrm{depth}(\theta_k) \nearrow \infty$ such that $\mathrm{diam}\, \Pi_{f_k}(\theta_k, +) \geq \epsilon_0 > 0$ for all k. Then we can take a branch F_k of $f_k^{-n_k}$ such that F_k maps $\Pi_{f_k}(\theta_0^+, +)^\circ$ onto $\Pi_{f_k}(\theta_k, +)^\circ$ univalently.

Take a small ball $B \Subset T_g(\theta_0^+, 0, +)$ and fix a point $z \in B$. By (ii), we may assume that $B \Subset T_{f_k}(\theta_0^+, 0, +)$ for all $k \gg 0$. Since the $F_k|_B$ avoid values near ∞, they form a normal family. By passing to a subsequence, we may also assume that there exists $\phi = \lim F_k|_B$ which is non-constant by assumption. Now we have a small open set $V \Subset \phi(B)$ with $V \subset F_k(B)$ for all $k \gg 0$, thus $f_k^{n_k}(V) \subset B \subset K_{f_k}^\circ$. This implies that $V \subset K_{f_k}^\circ$ for all $k \gg 0$; hence, by Corollary A.2, we have $V \subset K_g^\circ$ too. Since V is open, there exist a tile $T = T_g(\theta, m, +)$ and a small ball B' such that $B' \Subset (T \cap V)^\circ$. From $B' \subset T$ and (ii) again, we deduce that $B' \subset T_k := T_{f_k}(\theta, m, +)$ for all $k \gg 0$. Moreover, since $B' \subset V$, we have $f_k^{n_k}(B') \subset B \subset T_{f_k}(\theta_0^+, 0, +)$. Thus $f_k^{n_k}(T_k)$ must be $T_{f_k}(\theta_0^+, 0, +)$; however, $f_k^{n_k}(T_k)$ has level $m + n_k \to \infty$. This is a contradiction.

The proof is completed by following the same argument as for Proposition 3.3. □

Proof of Proposition 5.7. It is enough to consider the case of $\theta = \theta_0^+$ and $* = +$. Recall that the attracting cycle O_f has multiplier $re^{2\pi i p/q}$. We introduce a parameter $\epsilon \in [0, 1)$

of $f \to g$ such that $r^q = R = 1 - \epsilon$. Set $\Pi_\epsilon := \Pi_f(\theta_0^+, +)$ and $\Pi_0 := \Pi_g(\theta_0^+, +)$. Then the semiconjugacy $h = h_\epsilon$ sends Π_ϵ to Π_0. To conclude the proof it suffices to show the following statement: for any $\delta > 0$, we have $\Pi_0 \subset N_\delta(\Pi_\epsilon)$ and $\Pi_\epsilon \subset N_\delta(\Pi_0)$ for all $\epsilon \ll 1$, where $N_\delta(\cdot)$ denotes the δ-neighborhood.

It is easy to check that $\Pi_0 \subset N_\delta(\Pi_\epsilon)$. We can take a compact set K such that $K \subset \Pi_0^\circ \Subset N_\delta(K)$. Since $h_\epsilon \to \mathrm{id}$ on K, we have $K \subset \Pi_\epsilon^\circ$ for all $\epsilon \ll 1$. Thus $\Pi_0 \subset N_\delta(K) \subset N_\delta(\Pi_\epsilon)$.

The proof of $\Pi_\epsilon \subset N_\delta(\Pi_0)$ is more technical. Here let us assume that $q = q'$, i.e. Case (a). The proof for Case (b) ($q = 1 < q'$) is analogous.

Local coordinates. Set $B := B(\beta_0, \delta)$. For fixed δ that is sufficiently small, there exists a convergent family of local coordinates $\zeta = \psi_\epsilon(z) \to \psi_0(z)$ on B, with the following properties for all $0 \le \epsilon \ll 1$.

- There exists a $\delta' > 0$ independent of $\epsilon \ll 1$ such that $\Delta := B(0, \delta') \Subset \psi_\epsilon(B)$.
- Let $f_\epsilon := f^{lq}$, $f_0 := g^{lq}$ and $R_\epsilon = 1 - \epsilon$; then $f_\epsilon(\zeta) = R_\epsilon \zeta\,(1 + \zeta^q + O(\zeta^{2q}))$ on Δ. (See Appendix A.2.)
- ψ_ϵ maps $\Pi_\epsilon \cap \psi_\epsilon^{-1}(\Delta)$ into $\Delta' := \{\zeta \in \Delta : -\pi/2q < \arg \zeta < 3\pi/2q\}$. (This is just a technical assumption.)
- Let $E_\epsilon := \{\zeta \in \Delta' : |\arg \zeta^q| \le \pi/3,\ |\zeta^q| \ge \epsilon/2\}$; then $f_0^{-1}(\overline{E_0}) \subset E_0 \cup \{0\}$ and $f_\epsilon^{-1}(\overline{E_\epsilon}) \subset E_\epsilon$ for all $0 < \epsilon \ll 1$. (See the argument of Lemma A.7.)

Let us interpret the setting of Theorem 5.5 by using $\epsilon \in [0, 1)$. For $0 < \epsilon < 1$, we denote Φ_f, Ψ_f, u_f, T_f and U_f by Φ_ϵ, Ψ_ϵ, u_ϵ, T_ϵ and U_ϵ, respectively; for $\epsilon = 0$, these denote Φ_g, Ψ_g, etc. In particular, we consider Ψ_ϵ only on Δ'. For later use, we define $W = \chi_\epsilon(\zeta)$ for each $\zeta \in \psi_\epsilon(K_{f_\epsilon}^\circ \cap B)$ by $\chi_\epsilon := \Phi_\epsilon \circ \psi_\epsilon^{-1}$.

Now, through $w = \Psi_\epsilon(\zeta)$, we can view f_ϵ on Δ' as $f_\epsilon(w) = \tau_\epsilon w + 1 + O(1/w)$ where $\tau_\epsilon := R_\epsilon^{-q}$. On this w-plane, take $P = P_\rho = \{\mathrm{Re}\, w \ge \rho \gg 0\}$ such that for all $0 \le \epsilon \ll 1$, the set $\hat{P} := \Psi_\epsilon^{-1}(P)$ is contained in Δ' and u_ϵ is defined on P. Note that for all $0 \le \epsilon \ll 1$ we have $f_\epsilon(P) \subset P$ and $u_0(w) = w(1 + o(1))$ by Lemma A.6. One can also check that $\chi_\epsilon \circ \Psi_\epsilon^{-1}(w) = U_\epsilon \circ T_\epsilon \circ u_\epsilon(w)$ on P, and that

$$U_\epsilon \circ T_\epsilon \circ u_\epsilon(w) = U_0 \circ T_0 \circ u_0(w) + o(1) = qw(1 + o(1))$$

on compact sets of P.

Rectangles. For fixed positive integers M and N, we define the following compact sets in the W-plane:

$$C_0 := \{W \in \mathbb{C} : (N-1)q \le \mathrm{Re}\, W \le Nq,\ 0 \le \mathrm{Im}\, W \le Nq\},$$

$$Q_0 := \bigcup_{k=0}^{M} G^{-kq}(C_0) \quad \text{and} \quad C_0' := G^{-Mq}(C_0),$$

where $G(W) = W + 1$.

By taking sufficiently large N and M, we may assume the following.

(1) Let $\tilde{Q}_0 := \Pi_0 \cap \Phi_0^{-1}(Q_0)$ in the z-coordinate; then $\Pi_0 - \tilde{Q}_0 \Subset \psi_0^{-1}(\Delta)$.

(2) In the w-coordinate, we have $\chi_0^{-1}(C_0) \subset \hat{P}$ and $\chi_0^{-1}(C_0') \subset E_0$.

See Figure 12. In fact, for any compact set K with $\Pi_0 - K \Subset \psi_0^{-1}(\Delta)$, the set $\Phi_0(K)$ is compact in $\overline{\mathbb{H}}_W := \{\mathrm{Im}\, W \ge 0\}$ and covered by Q_0 if we choose sufficiently large N

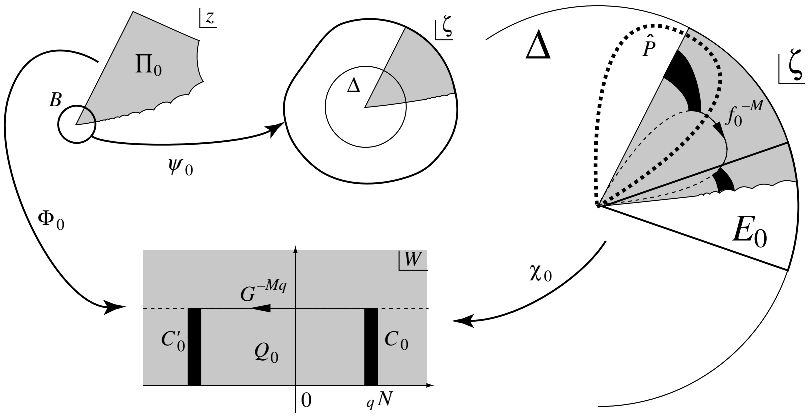

FIGURE 12. Choosing M and N.

and M. Thus we have (1). If $N \gg 0$, the set C_0 must be contained in $\chi_0(P)$. Since $\Pi_0 \cap \Phi_0^{-1}(C_0)$ is compact, it is uniformly attracted to the repelling direction by iteration of $(g|_{\Pi_0})^{-lq}$. Thus, by taking M much larger, we have (2) .

Perturbation. Fix integers N and M as described above. Now we consider perturbations of fixed rectangles C_0, C'_0 and Q_0 having properties (1) and (2). By using the conjugacy $H = H_\epsilon : \mathbb{C} - [a, \infty) \to \mathbb{C}$ between $F = F_\epsilon$ and $G = F_0$, we define C_ϵ, C'_ϵ and Q_ϵ by their homeomorphic images by H_ϵ^{-1}. Since $H_\epsilon \to \mathrm{id}$ as $\epsilon \to 0$ on any compact sets (see the proof of (ii)), we have $C_\epsilon \to C_0$, $C'_\epsilon \to C'_0$ and $Q_\epsilon \to Q_0$ in the Hausdorff topology. Moreover, the following properties hold for all $\epsilon \ll 1$.

(1′) Let $\tilde{Q}_\epsilon := \Pi_\epsilon \cap \Phi_\epsilon^{-1}(Q_\epsilon)$ in the z-coordinate; then $\tilde{Q}_\epsilon \subset N_{\delta/2}(\tilde{Q}_0)$.

(2′) In the ζ-coordinate, we have $\chi_\epsilon^{-1}(C_\epsilon) \subset \hat{P}$ and $\chi_\epsilon^{-1}(C'_\epsilon) \subset E_\epsilon$.

In fact, since $\tilde{Q}_0 = h_\epsilon(\tilde{Q}_\epsilon)$ and is compact, property (1′) follows from $\Phi_\epsilon \to \Phi_0$ as $\epsilon \to 0$. Property (2′) holds because $\chi_\epsilon \to \chi_0$ on compact sets in $\hat{P}$ and $f^{lqM} \to g^{lqM}$.

Now it is enough to show that $\Pi_\epsilon - \tilde{Q}_\epsilon \Subset \psi_\epsilon^{-1}(\Delta) \subset B$, which is equivalent to $\chi_\epsilon^{-1}(\overline{\mathbb{H}}_W - Q_\epsilon) \Subset \Delta$ in the ζ-coordinate. We consider the following three sets in $\overline{\mathbb{H}}_W$:

$$X_0 := \{W \in \overline{\mathbb{H}}_W : \operatorname{Re} W \le (N - M - 1)q,\ \operatorname{Im} W \le Nq\},$$
$$Y_0 := \{W \in \overline{\mathbb{H}}_W : \operatorname{Re} W \ge Nq,\ \operatorname{Im} W \le Nq\},$$
$$Z_0 := \{W \in \overline{\mathbb{H}}_W : \operatorname{Im} W \ge Nq\}.$$

Let X_ϵ, Y_ϵ, and Z_ϵ be their homeomorphic images by H_ϵ^{-1}. Then $X_\epsilon \cup Y_\epsilon \cup Z_\epsilon = \overline{\mathbb{H}}_W - Q_\epsilon^\circ$.

Note that $X_\epsilon = \bigcup_{k \ge 1} F_\epsilon^{-kq}(C'_\epsilon)$ and $Y_\epsilon = \bigcup_{k \ge 1} F_\epsilon^{kq}(C_\epsilon)$. Since $f_\epsilon^{-1}(E_\epsilon) \subset E_\epsilon$ and $f_\epsilon(\hat{P}) \subset \hat{P}$ in the ζ-coordinate, (2′) implies $\chi_\epsilon^{-1}(X_\epsilon) \subset E_\epsilon$ and $\chi_\epsilon^{-1}(Y_\epsilon) \subset \hat{P}$, hence we have $\chi_\epsilon^{-1}(X_\epsilon \cup Y_\epsilon) \subset \Delta$.

The proof is completed by showing that $\chi_\epsilon^{-1}(Z_\epsilon) \subset \Delta$. It suffices to prove that $\chi_\epsilon^{-1}(\partial Z_\epsilon) \subset \Delta$. Note that ∂Z_ϵ consists of two half lines: one is the interval $I_\epsilon := [a_\epsilon, \infty)$ where a_ϵ is the attracting fixed point of F_ϵ, and the other is $I'_\epsilon := H_\epsilon^{-1}(\partial Z_0)$ along the top edge of Q_ϵ.

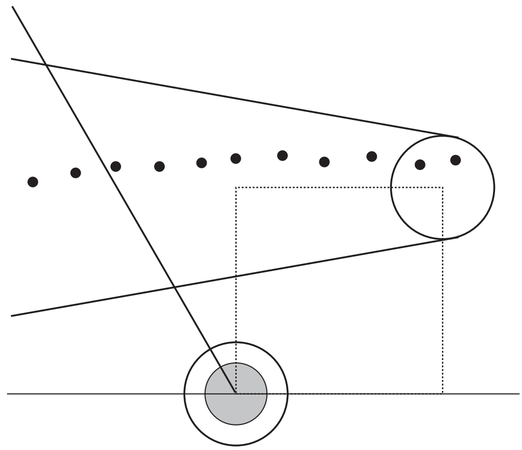

FIGURE 13. An orbit in the w-plane. The dotted square has height N.

First we show that $\chi_\epsilon^{-1}(I_\epsilon) \subset \Delta$. Recall that $\chi_\epsilon^{-1}(I_\epsilon)$ is the image of a degenerating arc in the ζ-coordinate. Let $E_0' := \{\zeta \in \Delta' : |\arg(-\zeta^q)| \le \pi/3\}$. Then one can check that $f_\epsilon(E_0') \subset E_0'$ and $f_\epsilon^{-1}(E_0) \subset E_0$ for all $\epsilon \ll 1$ as in the argument of Lemma A.7.

The real part of $g^{lqk}(0)$ in the w-coordinate increases as $k \to \infty$, thus the critical orbit of $f_0 = g^{lq}$ in Δ' is tangent to the attracting direction in the ζ-coordinate. Therefore we may assume that $g^{lqn}(0)$ in the proof of Theorem 5.5 is contained in E_0'. Hence $f^{lqn}(0) = f_\epsilon^n(0)$ in the ζ-coordinate is contained in E_0' for all $\epsilon \ll 1$. Moreover, the property $f_\epsilon(E_0') \subset E_0'$ implies that the critical orbit of $f_\epsilon = f^{lq}$ in Δ' is eventually contained in E_0'. By construction of the degenerating arcs in Lemma 2.3 and by $f_\epsilon^{-1}(E_0) \subset E_0$, the arc $\chi_\epsilon^{-1}(I_\epsilon)$ must be contained in $E_0 \subset \Delta$.

Next we show that $\chi_\epsilon^{-1}(I_\epsilon') \subset \Delta$. Let s_ϵ and ℓ_ϵ be the top edges of quadrilaterals C_ϵ and Q_ϵ that intersect I_ϵ'. Then $\ell_\epsilon = \bigcup_{k \ge 0}^{M} F_\epsilon^{-kq}(s_\epsilon)$. It is enough to show that $\chi_\epsilon^{-1}(\ell_\epsilon)$ is contained in Δ, since $\chi_\epsilon^{-1}(X_\epsilon \cup Y_\epsilon) \subset \Delta$.

Take any point w_0 in $\Psi_\epsilon \circ \chi_\epsilon^{-1}(s_\epsilon) = (U_\epsilon \circ T_\epsilon \circ u_\epsilon)^{-1}(s_\epsilon)$ in the w-plane. We may assume that N is sufficiently large and $w_0 \in B(N + Ni, N/4)$ for all $\epsilon \ll 1$, because $U_\epsilon \circ T_\epsilon \circ u_\epsilon(w) = qw(1 + o(1))$ on compact sets of P. Moreover, we may assume that $\Psi_\epsilon(\partial\Delta) \subset B(0, N/4)$.

Recall that $f_\epsilon(w) = \tau_\epsilon w + 1 + O(1/w)$ and so $f_\epsilon^{-1}(w) = \tau_\epsilon^{-1}(w-1) + O(1/w)$. Take any w with $N/4 \le |w| \le 4N$; then we have $|f_\epsilon^{-1}(w) - (w-1)| = O(\epsilon N) + O(1/N)$. Thus, for any fixed $\kappa \ll 1$, by taking $N \gg 0$ we have $|f_\epsilon^{-1}(w) - (w-1)| \le \kappa$ for all $\epsilon \ll 1$. This implies that $|\arg(f_\epsilon^{-1}(w) - w)| \le \arcsin \kappa$.

By (2′), the orbit $w_k = f_\epsilon^{-k}(w_0)$ of w_0 lands on $\Psi_\epsilon(E_\epsilon)$ by at most M iterations of f_ϵ^{-1} (so $\arg w_M > 2\pi/3$). For small enough κ, the point w_k satisfies $N/4 \le |w_k| \le 4N$ and $|\arg(w_k - w_0)| \le \arcsin \kappa$ for all $k = 0, \ldots, M$; see Figure 13. This implies that $\Psi_\epsilon \circ \chi_\epsilon^{-1}(\ell_\epsilon)$ never crosses over $\Psi_\epsilon(\partial\Delta)$, and thus we have $\chi_\epsilon^{-1}(\ell_\epsilon) \subset \Delta$. □

Acknowledgements. I would like to thank M. Lyubich for giving me opportunities to visit SUNY at Stony Brook, the University of Toronto and the Fields Institute, where part of this paper was written. I am also grateful to the referee for helpful comments. This research was partially supported by JSPS Research Fellowships for Young Scientists, JSPS Grant-in-Aid for Young Scientists, Nagoya University, the Circle for the Promotion of Science and Engineering, and the Inamori Foundation.

A. *Appendix*

In this section we present some results on the perturbation of a parabolic cycle corresponding to the degeneration pair $(f \to g)$.

A.1. *Pinching semiconjugacy on the Julia sets.*

Let $(f \to g)$ be a general degeneration pair. Recall that the attracting cycle $O_f = \{\alpha_1, \ldots, \alpha_l = a_0\}$ has multiplier $\lambda = r\omega = r\exp(2\pi i p/q)$ with $0 < r < 1$, and the parabolic cycle $O_g = \{\beta_1, \ldots, \beta_{l'} = \beta_0\}$ has multiplier $\omega' = \exp(2\pi p'/q')$.

By applying [**Ka1**, Theorem 1.1] to $(f \to g)$, we obtain the following.

THEOREM A.1. *If $f \approx g$, then there exists a unique semiconjugacy $h_J : J_f \to J_g$ with the following properties.*

(1) *If* $\mathrm{card}\, h_J^{-1}(w) \geq 2$ *for some* $w \in J_g$, *then* $w \in I_g$ *and* $\mathrm{card}\, h_J^{-1}(w) = q$ *(thus $q = q' \geq 2$).*

(2) *h_J is a homeomorphism if and only if $(f \to g)$ is of type $q = 1$.*

(3) $\sup_{z \in J_f} |z - h_J(z)| \to 0$ *as* $f \to g$.

See also Proposition 2.1. The proof of [**Ka1**, Theorem 1.1] is based on a pull-back argument and does not use quasiconformal maps. Here is a useful corollary which follows easily from property (3).

COROLLARY A.2. *As $f \to g$, the Julia set J_f converges to J_g in the Hausdorff topology.*

Since $h_J \circ \gamma_f$ and γ_g determines the same ray equivalence, we have $h_J \circ \gamma_f = \gamma_g$. For $\theta \in \mathbb{T}$, put $\gamma_f(\theta)$ into z in property (3) of Theorem A.1. Then we obtain the following corollary.

COROLLARY A.3. *As $f \to g$, the map $\gamma_f : \mathbb{T} \to J_f$ converges uniformly to $\gamma_g : \mathbb{T} \to J_g$.*

A.2. *Normalized form of a local perturbation.*

For a degeneration pair $(f \to g)$, the parabolic cycle O_g is approximated by an attracting or a repelling cycle O'_f with the same period l' and multiplier $\lambda' \approx \omega' = e^{2\pi i p'/q'}$ (see §2). Let $\alpha'_0 \in O'_f$ with $\alpha'_0 \to \beta_0$. Then, by looking through the local coordinates $\psi_f(z) = z - \alpha'_0$ and $\psi_g(z) = z - \beta_0$ near β_0, one can view the convergence $f^{l'} \to g^{l'}$ as

$$\begin{aligned} &\psi_f \circ f^{l'} \circ \psi_f^{-1}(z) = \lambda' z + O(z^2) \\ &\longrightarrow \psi_g \circ g^{l'} \circ \psi_g^{-1}(z) = \omega' z + O(z^2). \end{aligned}$$

Here we claim that replacing $\psi_f \to \psi_g$ with better local coordinates yields a normalized form of convergence

$$\begin{aligned} \psi_f \circ f^{l'} \circ \psi_f^{-1}(z) &= \lambda' z + z^{q'+1} + O(z^{2q'+1}) \\ \longrightarrow \ \psi_g \circ g^{l'} \circ \psi_g^{-1}(z) &= \omega' z + z^{q'+1} + O(z^{2q'+1}). \end{aligned}$$

More generally, we have the following.

PROPOSITION A.4. *For $\epsilon \in [0, 1]$, let $\{f_\epsilon\}$ be a family of holomorphic maps on a neighborhood of* 0 *such that as $\epsilon \to 0$,*

$$f_\epsilon(z) = \lambda_\epsilon z + O(z^2) \ \longrightarrow \ f_0(z) = \lambda_0 z + O(z^2)$$

where λ_0 is a primitive qth root of unity. Then there exists a family of holomorphic maps $\{\phi_\epsilon\}$ such that

$$\phi_\epsilon \circ f_\epsilon \circ \phi_\epsilon^{-1}(z) = \lambda_\epsilon z + z^{q+1} + O(z^{2q+1})$$

and $\phi_\epsilon \to \phi_0$ near $z = 0$.

Proof. First suppose that $f_\epsilon(z) = \lambda_\epsilon z + A_\epsilon z^n + O(z^{n+1})$, where $2 \le n \le q$. Let us consider a coordinate change $z \mapsto z - B_\epsilon z^n$ with $B_\epsilon = A_\epsilon/(\lambda_\epsilon^{n+1} - \lambda_\epsilon)$. Note that $\lambda_\epsilon^{n+1} - \lambda_\epsilon$ is bounded away from 0 when $\epsilon \ll 1$, because λ_ϵ converges to a primitive qth root of unity. In particular, the coordinate change $z \mapsto z - B_\epsilon z^n$ also converges to $z \mapsto z - B_0 z^n$ near 0. By applying these coordinate changes, we can view the family $\{f_\epsilon\}$ as

$$f_\epsilon(z) = \lambda_\epsilon z + O(z^{n+1}).$$

By repeating this process until $n = q$, we have the family $\{f_\epsilon\}$ taking the form

$$f_\epsilon(z) = \lambda_\epsilon z + C_\epsilon z^{q+1} + A'_\epsilon z^n + O(z^{n+1}),$$

where $q + 2 \le n \le 2q$. Next, for each ϵ, take a linear coordinate change $z \mapsto C_\epsilon^{1/q} z$ which effectively normalizes C_ϵ to 1. Using another coordinate change of the form $z \mapsto \zeta = z - B'_\epsilon z^n$ with $B'_\epsilon = A'_\epsilon/(\lambda_\epsilon^{n+1} - \lambda_\epsilon)$, we have

$$f_\epsilon(z) = \lambda_\epsilon z + z^{q+1} + O(z^{n+1}).$$

Repeating this process until $n = 2q$ gives the desired form of convergence. □

For this new family $\{f_\epsilon(z) = \lambda_\epsilon z + z^{q+1} + O(z^{2q+1})\}$ and $n \ge 0$, one can easily check that

$$f_\epsilon^n(z) = \lambda_\epsilon^n z + C_{\epsilon,n} z^{q+1} + O(z^{2q+1}),$$

where $C_{\epsilon,n}$ is given by the recursive formula $C_{\epsilon,n+1} = \lambda_\epsilon^{q+1} C_{\epsilon,n} + \lambda_\epsilon^n$. Let $n = q$ and set $\Lambda_\epsilon := \lambda_\epsilon^q$ ($\to 1$ as $\epsilon \to 0$). By taking linear coordinate changes $z \mapsto (C_{\epsilon,q}/\Lambda_\epsilon)^{1/q} z$, we obtain convergence of the form

$$\begin{aligned} f_\epsilon^q(z) &= \Lambda_\epsilon z\,(1 + z^q + O(z^{2q})) \\ \longrightarrow \ f_0^q(z) &= z\,(1 + z^q + O(z^{2q})). \end{aligned}$$

The application of further coordinate changes with $w = \Psi_\epsilon(z) = -\Lambda_\epsilon^q/(qz^q)$ yields

$$\begin{aligned} \Psi_\epsilon \circ f_\epsilon^q \circ \Psi_\epsilon^{-1}(w) &= \Lambda_\epsilon^{-q} w + 1 + O(1/w) \\ \longrightarrow \ \Psi_0 \circ f_0^q \circ \Psi_0^{-1}(w) &= w + 1 + O(1/w) \end{aligned}$$

on a neighborhood of infinity. Note that we have a similar representation for $f^{l'q'} \to g^{l'q'}$.

A.3. *Simultaneous linearization.* Ueda [**Ue**] recently proved a simultaneous linearization theorem which explains hyperbolic-to-parabolic degenerations of linearizing coordinates. Here we state a simpler version of the theorem which suffices for our purposes. For $R \geq 0$, let E_R denote the region $\{z \in \mathbb{C} : \operatorname{Re} z \geq R\}$.

THEOREM A.5. (Ueda) *For $\epsilon \in [0, 1]$, let $\{f_\epsilon\}$ be a family of holomorphic maps on $\{|z| \geq R > 0\}$ such that*

$$\begin{aligned} f_\epsilon(z) &= \tau_\epsilon z + 1 + O(1/z) \\ \longrightarrow \ f_0(z) &= z + 1 + O(1/z) \end{aligned}$$

uniformly as $\epsilon \to 0$, where $\tau_\epsilon = 1 + \epsilon$. If $R \gg 0$, then for any $\epsilon \in [0, 1]$ there exists a holomorphic map $u_\epsilon : E_R \to \bar{\mathbb{C}}$ such that

$$u_\epsilon(f_\epsilon(z)) = \tau_\epsilon u_\epsilon(z) + 1$$

and $u_\epsilon \to u_0$ uniformly on compact sets of E_R.

Indeed, Ueda's original theorem in [**Ue**] claims that a similar statement holds for any radial convergence $\tau_\epsilon \to 1$ outside the unit disk. In [**Ka4**] an alternative proof is given, and the error term $O(1/z)$ is refined to be $O(z^{-1/n})$ for any $n \geq 1$.

LEMMA A.6. *$u_0(z) = z(1 + o(1))$ as* $\operatorname{Re} z \to \infty$.

It is well-known that if $f_0(z) = z + 1 + a_0/z + \cdots$, then the Fatou coordinate is of the form $u_0(z) = z - a_0 \log z + O(1)$; see [**Sh**], for instance.

A.4. *Small invariant paths joining perturbed periodic points.* For a degeneration pair $(f \to g)$ in Case (a) $(q = q')$, we may consider that the parabolic cycle O_g is perturbed into the attracting cycle O_f with the same period $l = l'$ (see Proposition 2.1). In this case, the convergence $f^{lq} \to g^{lq}$ is viewed as

$$f^{lq}(z) = r^q z + z^q + O(z^{2q}) \ \longrightarrow \ g^{lq}(z) = z + z^q + O(z^{2q}),$$

with $r^q \nearrow 1$ through suitable local coordinates near $\beta_0 \in O_g$, as in §A.2.

By an additional linear coordinate change $z \mapsto z/r$, we consider, instead, a family of holomorphic maps $\{f_\epsilon\}$ of the form

$$f_\epsilon(z) = \lambda_\epsilon z(1 + z^q + O(z^{2q})),$$

where we set $r^q = \lambda_\epsilon = 1 - \epsilon \nearrow 1$. The local solution of $f_\epsilon(z) = z$ is then $z = 0$ or $z^q = \epsilon + O(\epsilon^2)$. The latter means that q symmetrically arrayed repelling fixed points are generated by the perturbation of a parabolic point with multiplicity $q + 1$. We claim that the following holds.

LEMMA A.7. *For $\epsilon \ll 1$, there exist q f_ϵ-invariant paths of diameter $O(\epsilon^{1/q})$ joining the central attracting point $z = 0$ to each of the symmetrically arrayed repelling fixed points.*

Proof. First,we show that $D := \{z : |z|^q \leq \epsilon/2\}$ satisfies $f_\epsilon(D) \subset D^\circ$. By checking the real part of $\log f_\epsilon(z)$, we find

$$|f_\epsilon(z)| = \lambda_\epsilon |z|(1 + \mathrm{Re}\, z^q + O(z^{2q})).$$

Since $\mathrm{Re}\, z^q \leq \epsilon/2$ on D, we have $|f_\epsilon(z)| = |z|(1 - \epsilon/2 + O(\epsilon^2)) < |z|$.

Next, we set

$$E := \left\{ z : \frac{\epsilon}{2} \leq |z^q| \leq 4\epsilon \text{ and } |\arg z^q| \leq \frac{\pi}{3} \right\}.$$

Note that E has q connected components around the repelling fixed points. Now we claim that E satisfies $f_\epsilon^{-1}(E) \subset E^\circ$. Since f_ϵ^{-1} is univalent near 0, it is enough to show that $f_\epsilon^{-1}(\partial E) \subset E^\circ$. Set

$$\begin{aligned}
e_1 &:= \left\{ z : |z^q| = \frac{\epsilon}{2} \text{ and } |\arg z^q| \leq \frac{\pi}{3} \right\}, \\
e_2 &:= \left\{ z : |z^q| = 4\epsilon \text{ and } |\arg z^q| \leq \frac{\pi}{3} \right\}, \\
e_3^\pm &:= \left\{ z : \frac{\epsilon}{2} \leq |z^q| \leq 4\epsilon \text{ and } \arg z^q = \pm\frac{\pi}{3} \right\}.
\end{aligned}$$

By checking $\log f_\epsilon^{-1}(z)$, we have

$$\begin{aligned}
|f_\epsilon^{-1}(z)| &= \lambda_\epsilon^{-1}|z|(1 - \lambda_\epsilon^{-q}\, \mathrm{Re}\, z^q + O(z^{2q})), \\
\arg f_\epsilon^{-1}(z) &= \arg z - \lambda_\epsilon^{-q}\, \mathrm{Im}\, z^q + O(z^{2q}).
\end{aligned}$$

If $z \in e_1$, then $\mathrm{Re}\, z^q \leq \epsilon/2$ and hence $|f_\epsilon(z)| \geq |z|(1 + \epsilon/2 + O(\epsilon^2)) > |z|$. If $z \in e_2$, then $\mathrm{Re}\, z^q \geq 2\epsilon$ and hence $|f_\epsilon(z)| \leq |z|(1 - \epsilon + O(\epsilon^2)) < |z|$. For $z \in e_3^\pm$, set $|z^q| = \rho$ with $\epsilon/2 \leq \rho \leq 4\epsilon$. Then $\arg f_\epsilon^{-1}(z) = \arg z \mp (\sqrt{3}/2)\rho(1 + O(\rho))$. Thus, overall we have $f_\epsilon^{-1}(\partial E) \subset E^\circ$.

Take any q points $\{z_1, \ldots, z_q\}$ from each connected component of e_1. Let η_j be the segment joining z_j and $f_\epsilon(z_j)$. Then the path $\bigcup_{k\in\mathbb{Z}} f_\epsilon^k(\eta_j)$ has the desired property. □

Remark. In Case (b) ($q = 1 < q'$), the cycle O'_f in §A.2 is repelling. By taking $f^{-l'q'} \to g^{-l'q'}$ near O_g, we have a form of convergence

$$f_\epsilon(z) = \lambda_\epsilon z(1 + z^{q'} + O(z^{2q'}))$$

which is similar to that in the $q = q'$ case, where $\lambda_\epsilon = 1 - \epsilon + O(\epsilon^2) \in \mathbb{C}^*$. This λ_ϵ comes from the fact that any non-zero solution of $f_\epsilon(z) = z$ has derivative $0 < r < 1$ (since it is actually a point in O_f in different coordinates). One can easily check that the argument used in Lemma A.7 works for this f_ϵ as well, and that the statement is also true when q is replaced by q'.

REFERENCES

[Cu] G. Cui. Geometrically finite rational maps with given combinatorics. *Preprint*, 1997.

[DH] A. Douady and J. H. Hubbard. *Etude dynamique des polynômes complexes I & II*. Publ. Math. Orsay, Université de Paris-Sud, Département de Mathématiques, Orsay, 1984/85, 84-2, 85-4.

[GM] L. R. Goldberg and J. Milnor. Fixed points of polynomial maps, Part II: Fixed point portraits. *Ann. Sci. École Norm. Sup.* **26** (1993), 51–98.

[Ha1] P. Haïssinsky. Modulation dans l'ensemble de Mandelbrot. *The Mandelbrot Set, Theme and Variations. (Lond. Math. Soc. Lec. Note Ser., 274)*. Ed. L. Tan. Cambridge University Press, Cambridge, 2000, pp. 37–65.

[Ha2] P. Haïssinsky. Pincement de polynômes. *Comment. Math. Helv.* **77** (2002), 1–23.

[HT] P. Haïssinsky and T. Lei. Convergence of pinching deformations and matings of geometrically finite polynomials. *Fund. Math.* **181** (2004), 143–188.

[Ka1] T. Kawahira. Semiconjugacies between the Julia sets of geometrically finite rational maps. *Ergod. Th. & Dynam. Sys.* **23** (2003), 1125–1152.

[Ka2] T. Kawahira. Semiconjugacies in complex dynamics with parabolic cycles. *Thesis*, University of Tokyo, 2003. Available at http://www.math.sunysb.edu/dynamics/theses/index.html.

[Ka3] T. Kawahira. Tessellations and Lyubich–Minsky laminations associated with quadratic maps II: Topological structures of 3-laminations. *Conform. Geom. Dyn.* to appear (arXiv:math.DS/0609836).

[Ka4] T. Kawahira. A proof of simultaneous linearization with a polylog estimate. *Bull. Pol. Acad. Sci. Math.* **55** (2007), 43–52.

[Ki] J. Kiwi. Wandering orbit portrait. *Trans. Amer. Math. Soc.* **354** (2001), 1437–1485.

[LM] M. Lyubich and Y. Minsky. Laminations in holomorphic dynamics. *J. Differential Geom.* **47** (1997), 17–94.

[Mi1] J. Milnor. *Dynamics in One Complex Variable*, 3rd edn *(Annals of Math Studies, 160)*. Princeton University Press, Princeton, NJ, 2006.

[Mi2] J. Milnor. Periodic orbits, external rays, and the Mandelbrot set: an expository account. *Géométrie complexe et systèmes dynamiques*. Eds. M. Flexor *et al. Astérisque* **261** (2000), 277–333.

[Po] Ch. Pommerenke. *Boundary Behaviour of Conformal Maps*. Springer, Berlin, 1992.

[Sh] M. Shishikura. The Hausdorff dimension of the boundary of the Mandelbrot set and Julia sets. *Ann. of Math.* (2) **147** (1998), 225–267.

[T] T. Lei. On pinching deformation of rational maps. *Ann. Sci. École Norm. Sup.* **35** (2002), 353–370.

[Ue] T. Ueda. Simultaneous linearization of holomorphic maps with hyperbolic and parabolic fixed points. *Publ. Res. Inst. Math. Sci.* **44**(1) (2008), 91–105.