Optimal Control Designs for Systems with Input Saturations and Rate Limiters

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***Abstract*—Most systems in practice have some input saturations due to physical restrictions. But taking input saturations into account directly is not easy in ordinary control design methods. In this report, we show that the method recently developed by one of the authors is useful to approximate the exact solutions of Hamilton-Jacobi equations for systems including input saturations. First, we propose a method to design nonlinear optimal controllers for systems with input saturations by solving Hamilton-Jacobi equations. Next, it will be extend to a design method of nonlinear optimal controllers for systems with rate limited actuators. Two numerical examples are illustrated for the input saturation and input rate limited problems.**

I. Introduction

Most systems in practice have some input saturations due to physical restrictions. But taking input saturations into account directly is not easy in ordinary control design methods. The effect of input saturations sometimes causes undesired phenomenon such as windup etc. Rate limiter is one sort of input saturation known as a main reason of oscillation phenomenon such as aircraft PIO(Pilot-Induced Oscillation)[5]. Anti-windup controller and MPC (Model Predictive Control) are well-known as general way to solve these problems. But they are not practical because of the difficulty of calculation or the online calculation load[2][3]. An alternative approach is to solve Hamilton-Jacobi equations derived from an optimal control problem including input saturations. A method using neural network is proposed in reference[4]. But it is less accurate and little theoretical rigor. Recently a solution method to Hamilton-Jacobi equations for systems with state nonlinearity is proposed[1]. In this report, we show that this method is also useful to approximate the exact solutions of Hamilton-Jacobi equations for systems including input saturations. First, we propose a method to design nonlinear optimal controllers for systems with input saturations by solving Hamilton-Jacobi equations. Next, it will be extend to a design method of nonlinear optimal controllers for systems with rate limited actuators. Two numerical examples are illustrated for the input saturation and input rate limited problems.

II. Optimal Control for Systems with Input Saturations

A. Problem Definition

We consider nonlinear optimal regulation problems to minimize quadratic form cost functions for nonlinear time-invariant systems with state nonlinearities and input saturations. The state equation Σ and the cost function J are given as (1) with R being defined as a diagonal matrix.

$$\begin{cases} \Sigma : \dot{x} = f(x) + g(x) \cdot \mathrm{sat}(u) \ , \ x(0) = x_0 \\ J = \displaystyle\int_0^\infty \left(x^T Q x + u^T R u \right) dt \end{cases} , \tag{1}$$

where

$$Q \geq 0 \ , \ R > 0 \ , \ x \in \mathbb{R}^n \ , \ u \in \mathbb{R}^m \ ,$$
$$f(\cdot) : \mathbb{R}^n \to \mathbb{R}^n \ , \ g(\cdot) : \mathbb{R}^n \to \mathbb{R}^{n \times m},$$
$$\mathrm{sat}(u) = \left[\mathrm{sat}_1(u_1) \ , \cdots , \ \mathrm{sat}_m(u_m) \right]^T$$

and

$$\mathrm{sat}_i(u_i) = \begin{cases} \overline{u}_i & (\overline{u}_i \leq u_i) \\ u_i & (\underline{u}_i < u_i < \overline{u}_i) \\ \underline{u}_i & (u_i \leq \underline{u}_i) \end{cases} ,$$
$$\underline{u}_i \leq 0 \ , \ \overline{u}_i \geq 0 \ , \ (i = 1, 2, \cdots, m).$$

B. Calculation Algorithm

In this subsection, we derive a Hamilton-Jacobi equations and solve it by the method in [1]. First the dynamic programming is applied with the Hamiltonian

$$H(x,p,u) = p^T (f(x) + g(x) \cdot \mathrm{sat}(u)) + x^T Q x + u^T R u,$$

where p is the co-state. To minimize $H(x,p,u)$ with u, we have to minimize the terms of u,

$$u^T R u + p^T g(x) \cdot \mathrm{sat}(u) = \sum_{i=1}^{m} \left\{ r_i {u_i}^2 + p^T g_i(x) \cdot \mathrm{sat}_i(u_i) \right\} ,$$

where $g(x) = \begin{bmatrix} g_1(x) & g_2(x) & \dots & g_n(x) \end{bmatrix}^T$. For each i ($i = 1, 2, \cdots, n$),

$$\begin{aligned} & r_i {u_i}^2 + p^T g_i(x) \cdot \mathrm{sat}_i(u_i) \\ & \quad = \begin{cases} r_i {u_i}^2 + p^T g_i(x) \overline{u}_i & (\overline{u}_i \leq u_i) \\ r_i {u_i}^2 + p^T g_i(x) u_i & (\underline{u}_i < u_i < \overline{u}_i) \\ r_i {u_i}^2 + p^T g_i(x) \underline{u}_i & (u_i \leq \underline{u}_i) \end{cases} . \end{aligned} \tag{2}$$

Defining $u_i^*(x,p)$ as u_i minimizing (2), $u_i^*(x,p)$ is calculated as

$$\begin{aligned} u_i^*(x,p) &= \begin{cases} \overline{u}_i & (\overline{u}_i \leq \hat{u}_i(x,p)) \\ \hat{u}_i(x,p) & (\underline{u}_i < \hat{u}_i(x,p) < \overline{u}_i) \\ \underline{u}_i & (\hat{u}_i(x,p) \leq \underline{u}_i) \end{cases} \\ \Leftrightarrow u_i^*(x,p) &= \mathrm{sat}_i\left(\hat{u}_i(x,p)\right), \end{aligned}$$

where $\hat{u}_i(x,p) = -\frac{1}{2} R^{-1} g_i(x)^T p$. Thus the optimal input $u^*(x,p)$ is obtained as

$$u^*(x,p) = \mathrm{sat}\left(\hat{u}(x,p)\right), \tag{3}$$

where $\hat{u}(x,p) = -\frac{1}{2} R^{-1} g(x)^T p$.

Then the Hamilto-Jacobi equation to be solved is

$$\begin{aligned} &\left(\frac{\partial V}{\partial x}\right)^T \left\{ f(x) + g(x) \cdot \mathrm{sat}\left(\hat{u}\left(x, \left(\frac{\partial V}{\partial x}\right)\right)\right) \right\} \\ &\quad + x^T Q x + \mathrm{sat}\left(\hat{u}\left(x, \left(\frac{\partial V}{\partial x}\right)\right)\right)^T \\ &\qquad \cdot R \cdot \mathrm{sat}\left(\hat{u}\left(x, \left(\frac{\partial V}{\partial x}\right)\right)\right) = 0. \end{aligned}$$

If a solution $V(x)$ of this equation can be obtained, the optimal feedback $u^*(x)$ can be constructed by substituting $p(x) = (\partial V / \partial x)(x)$ in (3).

Since the soltions of Hamilton-Jacobi equations and the solutions of Hamilton's canonical equations are equivalent, we consider the Hamilton's canonical equations. Defining a new function $\widetilde{\mathrm{sat}}(u) = \mathrm{sat}(u) - u$ to bring out the linear terms of the canonical equation, the new Hamiltonian H' is

$$\begin{aligned} H'(x,p) &= p^T f(x) + x^T Q x \\ &\quad - \frac{1}{4} (g(x)^T p)^T R^{-1} g(x)^T p \\ &\quad + \widetilde{\mathrm{sat}}\left(\hat{u}(x,p)\right)^T R \cdot \widetilde{\mathrm{sat}}\left(\hat{u}(x,p)\right). \end{aligned}$$

Then the canonical equation is derived by

$$\begin{cases} \dot{x} = \dfrac{\partial H'(x,p)}{\partial p} \\ \dot{p} = -\dfrac{\partial H'(x,p)}{\partial x} \end{cases} . \tag{4}$$

The solutions $x(t), p(t)$ of this equation can be calculated by the iterative calculation of the stable manifold theorem[1]. The co-state $p(x)$ in the optimal feedback (3) can be calculated by deleting the parameter t from the solutions $x(t), p(t)$ of the canonical equations. The saturation functions are approximated by C^2 class functions so that the right sides of the canonical equations are C^1 class functions. In this approximation, the smoothed parts are small enough not to affect the whole calculations.

C. A Numerical Example

The proposed method is applied to a numerical example. The state equation is given as

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \mathrm{sat}_1(u_1) \\ \mathrm{sat}_2(u_2) \end{bmatrix}.$$

The maximum and minimum values of saturations are $\underline{u}_1 = -1, \overline{u}_1 = 1, \underline{u}_2 = -100, \overline{u}_2 = 100$ and the weighting matrices are $Q = I_{2\times2}, R = I_{2\times2}$. After applying the stable manfold algorithm 25 times, the co-state $p(x)$ is approximated by 3-order polynomials. The simulation is calculated, where the initial state is $x(0) = [-64 \ -341]^T$. The system and the input responses are shown in Fig. 1. As a comparison, the system and the input responses by the linear optimal controller are shown in Fig. 2. In Fig. 1, it is observed that when actuator u_1 is saturated, actuator u_2 in the nonlinear controller takes larger value than that of the linear controller to compensate the saturation of u_1. Comparing the system responses, one observes that the states of Fig. 1 converge more rapidly. The cost function value J of the nonlinear controller is $J = 5.202$, while the value of the linear controller is $J = 5.412$.

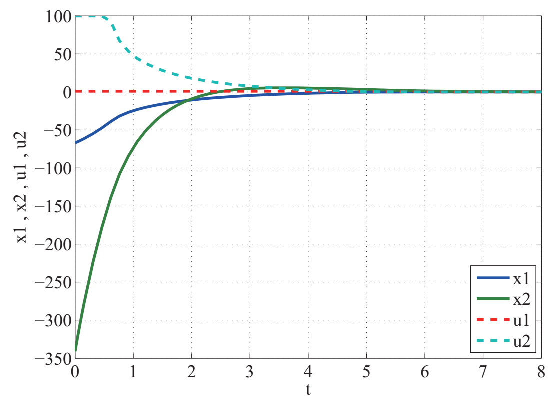

Fig. 1. Response by the nonlinear controller

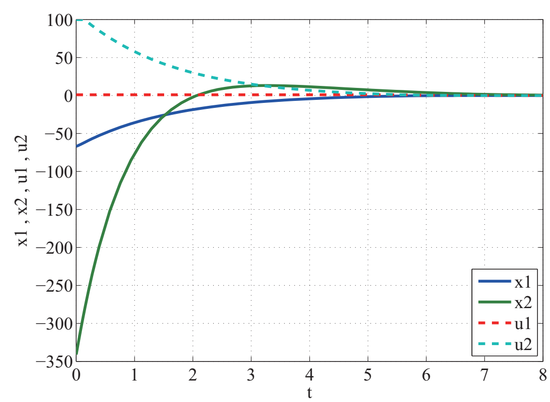

Fig. 2. Response by the linear controller

III. Optimal Control for Systems with Rate Limited Actuator

A. Problem Definition and Approximation of Rate Limiter

We consider nonlinear optimal regulation problems to minimize quadratic form cost functions for nonlinear time-invariant systems with state nonlinearities and rate limiters. The state equation Σ and the cost function J are given as (5) with R being a diagonal matrix. The parameters and the functions are defined as well as (1). The rate limiter functions are described as $\mathrm{RL}(u)$.

$$\begin{cases} \Sigma : \dot{x} = f(x) + g(x) \cdot \mathrm{RL}(u) \ , \ x(0) = x_0 \\ J = \displaystyle\int_0^\infty L(x(t), u(t))\, dt \end{cases} , \tag{5}$$

where

$$L(x, u) = x^T Q x + u^T R u \ , \ Q > 0 \ , \ R > 0.$$

In general, rate limiter is difficult to handle analytically and various approximation methods are proposed[5][6]. The most common way of approximating rate limiters is to use the integrater-feedback block in Fig. 3 (see, eg., [2]). This

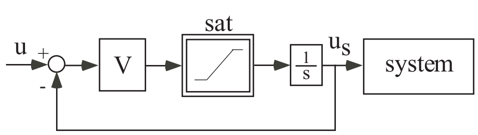

Fig. 3. Rate Limiter Approximation Block

block is written as

$$\dot{u}_s = \mathrm{sat}(V(u - u_s)). \tag{6}$$

By going through the block, the rate of inputs u by a controller is limited, yielding actual inputs u_s to the system. The maximum and minimum values of the rate limiter are $\overline{u}, \underline{u}$. $V = \mathrm{diag}[v_1 \cdots v_m]$ is a constant matrix and the bigger these values are, the better the approximation accuracy is.

B. Calculation Algorithm

We first apply the dynamic programming for the augmented system (5) and (6) by rewriting (6) as

$$\dot{u}_s = -V u_s + \overline{\mathrm{sat}}(V u_s, V u). \tag{7}$$

The function $\overline{\mathrm{sat}}(x, u)$ means a saturation function which maximum and minimum limits are $\overline{u}+x, \underline{u}+x$. The augmented system is written as

$$\begin{bmatrix} \dot{x_1} \\ \dot{u_s} \end{bmatrix} = \begin{bmatrix} f(x_1) + g(x_1) u_s \\ -V u_s \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \overline{\mathrm{sat}}(V u_s, V u). \tag{8}$$

For this state equation, the cost function J is redefined as

$$J = \int_0^\infty (x'(t)^T Q_a x'(t) + u(t)^T R u(t))\, dt,$$

$$x' = [x_1 \ u_s]^T \ , \ Q_a = \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix}.$$

Now, the same caltulation procedure as in Sec. II, except for the two variable function $\overline{\mathrm{sat}}(x, u)$, can be applied. The optimal input $u^*(x)$ is given as

$$u^*(x') = V^{-1} \overline{\mathrm{sat}}(V u_s, \hat{u}(x', p(x'))), \tag{9}$$

where $\hat{u}(x', p) = -\frac{1}{2} V^{-1} R^{-1} B^T p$, $B = [0 \ 1]^T$ and p is the co-state. The feedback thus constructed uses the augmented state u_s. In actual systems with rate limited actuators, such as aircraft longitudinal dynamics, the measurement of the saturated input (the elavator angle in the aircraft model) is available.

C. A Numerical Example

The proposed method is applied to a numerical example. The state equation is given as

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -16 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mathrm{RL}(u).$$

The maximum and minimum values of the rate limiter are $\underline{u} = -2, \overline{u} = 2$. Constants V, Q, R are $V = 20, R = I_{2\times 2}, Q = 0.01$. After applying the iterative stable manifold algorithm 30 times, the co-state $p(x)$ is approximated by 3-order polynomials. The simulation is calculated, where the initial state is $x(0) = [0.03 \ -0.257]^T, u_s(0) = 0$. The system and the input responses by the nonlinear controller are shown in Fig. 4. As a comparison, the system and input responses by linear optimal controllers for the augmented system (8) and the original system (5) are shown in Figs. 5 and 6, respectively. As a comparative controller, the response and the input by the linear optimal controller for (8) are shown in Fig. 5, the response and the input by the linear optimal controller for (5) are shown in Fig. 6.

In all figures, one sees that rates of inputs are restricted by 2. Particularly, in Fig. 6, the gap between controller input and actual input to the system is more evident. In general, controllers based on the augmented system have better performance (see, Figs. 4 and 5). This difference may be the effect of the additional u_s feedbacks. If the measurement of u_s is available, it is expected that the controllers designed for the augmented systems improve the responses. Comparing Fig. 4 and Fig. 5, the nonlinear controller achieves faster stabilization with less oscillation and smaller input. Since rate limiters induce phase lag as are well-known, the nonlinear controller, considering the saturation in the rate limiter, switches the input toward the negative direction before the linear controller does. This lag between the nonlinear and linear controller inputs may yield the difference in performance. In these simulations, cost function values are obtained as $J = 2.4977$ for the nonlinear controller, $J = 3.8277$ for the linear controller with augmented state and $J = 13.677$ for the linear controller for the original system.

IV. Conclusions

In this report, first, we proposed an optimal control design method for systems with input saturations and then, extended it to systems with rate limited actuator. The designs are

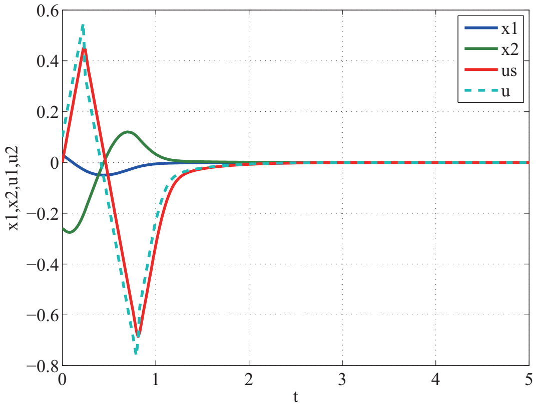

Fig. 4. Response by the nonlinear controller

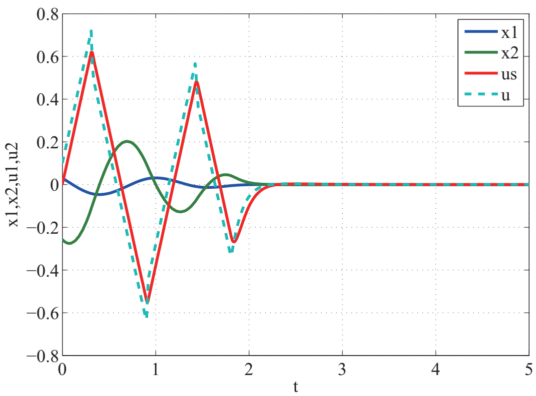

Fig. 5. Response by the linear controller for the augmented system

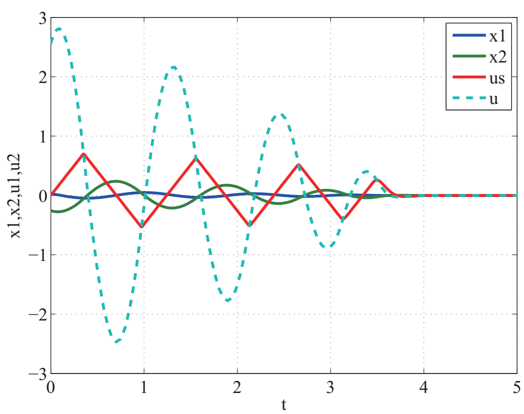

Fig. 6. Response by the linear controller for the original system

based on the stable manifold approach to approximate the solution of Hamilton-Jacobi equations by iterative calculations. The effectiveness of the proposed methods is shown by two numerical examples.

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