Optical Orthogonal Signature Pattern Codes with Maximum Collision Parameter 2 and Weight 4

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***Abstract*—An optical orthogonal signature pattern code (OOSPC) finds application in transmitting 2-dimensional images through multicore fiber in code-division multiple-access (CDMA) communication systems. Observing a one-to-one correspondence between an OOSPC and a certain combinatorial subject, called a packing design, we present a construction of optimal OOSPCs with weight 4 and maximum collision parameter 2, which generalizes a well known Köhler's construction of optimal optical orthogonal codes (OOC) with weight 4 and maximum collision parameter 2. Using this new construction enables one to obtain infinitely many optimal OOSPCs, whose existence was previously unknown. We prove that for a multiple n of 4, there exists no optimal OOSPC of size $6 \times n$ with weight 4 and maximum collision parameter 2, together with a report which shows a gap between optimal OOCs and optimal OOSPCs when 6 and n are not coprime. We also present a recursive construction of OOSPCs which are asymptotically optimal with respect to the Johnson bound. As a by-product, we obtain an asymptotically optimal $(m, n, 4, 2)$-OOSPC for all positive integers m and n.**

***Index Terms*—automorphism group, H-design, packing design, optical orthogonal code, optical orthogonal signature pattern code, space code division multiple access.**

I. Introduction

KITAYAMA [1] proposed a novel type of optical code-division multiple-access (CDMA), called space CDMA, for parallel transmission of 2-dimensional images through multicore fiber. The use of multicore fiber or spatial optical CDMA networks was proposed independently by Hassan et al. [2], Park et al. [3] and Yang et al. [4]. In space CDMA each pixel in a 2-dimensional image is encoded into a signature address, called an optical orthogonal signature pattern (OOSP). All the encoded images are multiplexed and broadcast to all receivers. Then each receiver regenerates the intended data from the multiplexed signals using its own OOSP. In order to achieve this, one requires that an OOSP is distinguishable under any space-shift of itself and any space-shift of other OOSP. Thus OOSPs are formuralized as follows.

Let m, n, k, λ be positive integers with $mn > k \geq \lambda$. An optical orthogonal signature pattern code is a family $\mathcal{C}$ of $(0, 1)$ matrices of size $m \times n$ which satisfies the following correlation properties:

(i) (Autocorrelation property)

$$\sum_{i=0}^{m-1} \sum_{j=0}^{n-1} x_{i,j} x_{i \oplus p, j \hat{\oplus} q} \begin{cases} = k & \text{if } (p,q) = (0,0) \\ \leq \lambda & \text{otherwise} \end{cases} \tag{1}$$

for any $(x_{i,j}) \in \mathcal{C}$;

(ii) (Crosscorrelation property)

$$\sum_{i=0}^{m-1} \sum_{j=0}^{n-1} x_{i,j} y_{i \oplus p, j \hat{\oplus} q} \leq \lambda \tag{2}$$

for any distinct $(x_{i,j}), (y_{i,j}) \in \mathcal{C}$,

where the additions $\oplus, \hat{\oplus}$ are respectively reduced modulo m and n. This is denoted by (m, n, k, λ)-OOSPC. Values k, λ are respectively called weight and maximum collision parameter. An OOSPC is a particular type of 2-dimensional optical orthogonal codes (2-D OOC); for example, see [5] for other variations of 2-D OOCs.

The set of all ordered pairs of subscripts (i, j) in an OOSP $(x_{i,j})$, that is, $\{(i, j) \mid i = 0, \ldots, m-1,\ j = 0, \ldots, n-1\}$, can be regarded as the group $\mathbb{Z}_m \times \mathbb{Z}_n$, where $\mathbb{Z}_l$ denotes the set of residue classes modulo l. Thus, an (m, n, k, λ)-OOSPC is a family $\mathcal{C}'$ of k-subsets of $Z_m \times \mathbb{Z}_n$ in which each k-subset X corresponds to a signature pattern $(x_{i,j})$ such that $(i, j) \in X$ if and only if $x_{i,j} = 1$, where the correlation properties are given as

(i) (Autocorrelation property)

$$|X \cap (X + (i,j))| \begin{cases} = k & \text{if } (i,j) = (0,0) \\ \leq \lambda & \text{otherwise} \end{cases} \tag{3}$$

for any $X \in \mathcal{C}'$;

(ii) (Crosscorrelation property)

$$|X \cap (Y + (i,j))| \leq \lambda \tag{4}$$

for any distinct $X, Y \in \mathcal{C}'$.

In Section II, the importance of the set-theoretic perspective of OOSPCs will be clear in connection with a certain combinatorial object, called a packing design.

Example I.1. A trivial example of OOSPC with weight 4 and maximum collision parameter 2 arises when $m = 2$ and $n = 4$:

$$\mathcal{C} = \left\{ \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix} \right\},$$

or equivalently,

$$\mathcal{C}' = \{\{(0,0),\ (0,1),\ (1,0),\ (1,2)\}\}.$$

For given m, n, k, λ, the usual objective is to determine the largest possible size, say $\phi(m, n, k, \lambda)$, of an (m, n, k, λ)-OOSPC. An (m, n, k, λ)-OOSPC is optimal if it attains $\phi(m, n, k, \lambda)$. It is well known [6] that $\phi(m, n, k, \lambda)$ is upper-bounded as

$$\phi(m,n,k,\lambda) \leq \left\lfloor \frac{1}{k} \left\lfloor \frac{mn-1}{k-1} \left\lfloor \cdots \left\lfloor \frac{mn-\lambda}{k-\lambda} \right\rfloor \cdots \right\rfloor \right\rfloor \right\rfloor =: J(mn, k, \lambda). \tag{5}$$

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This research is supported in part by the research fellow of Japan Society for the Promotion of Science for Young Scientists.

This paper appears in IEEE Transactions on Information Theory, Vol. 56, No. 7.

When m and n are coprime, an optimal (m, n, k, λ)-OOSPC is equivalent to an optimal (mn, k, λ) optical orthogonal code (OOC) [4, Construction I]. Thus in this case, using rich results on existence of optimal OOCs with maximum collision parameter one (for example, see [7], [8], [9], [10], [11], [12]) or optimal OOCs with maximum collision parameter more than one (for example, see [9], [13], [14], [15]) gives a number of optimal OOSPCs. Whereas, when m and n are not coprime, the problem of finding OOSPCs cannot be reduced to that of finding OOCs. In this case, some constructions of optimal $(m, n, k, 1)$-OOSPCs are already known for very specific values of m and n [4]. However, as far as the author knows, little is known on construction of optimal (m, n, k, λ)-OOSPC for $\lambda \geq 2$ as well as for m and n in general.

This paper is organized as follows. In Section II, it is shown that an optimal (m, n, k, λ)-OOSPC is equivalent to an optimal strictly $\mathbb{Z}_m \times \mathbb{Z}_n$-invariant $(\lambda+1)$-$(mn, k, 1)$ packing design. In Section III, a graph-theoretic construction of optimal strictly A-invariant 3-$(v, 4, 1)$ packing designs is proposed for an abelian group A of order v which contains the unique element of order 2. This is an extension of a well known Köhler's construction of optimal strictly cyclic 3-$(v, 4, 1)$ packing designs to an abelian group with the unique element of order 2. Using the basic construction yields infinitely many optimal $(m, n, 4, 2)$-OOSPCs whose existence was previously in doubt. In Section IV, it is proved that if n is a multiple of 4, then there exists no $(6, n, 4, 2)$-OOSPC attaining (5). An interesting phenomenon which shows a gap between optimal OOCs and optimal OOSPCs when m and n are not coprime is also reported. In Section V, we present a recursive construction of OOSPCs. Though the resulting OOSPCs are not optimal, there are some merits to this construction. First, it can deal with the case where $\mathbb{Z}_m \times \mathbb{Z}_n$ contains more than one elements of order 2 or a subgroup isomorphic to $\mathbb{Z}_7$, though in these cases, the graph-theoretic construction mentioned in Section III does not work. By using this construction, we also obtain an asymptotically optimal $(m, n, 4, 2)$-OOSPC for all positive integers m and n.

II. Optical Orthogonal Signature Pattern Code And Packing Design

In this section we will show a close relationship between OOSPCs and combinatorial packing designs.

First we prepare some terminologies from design theory [16]. For positive integers t, k, v such that $t < k < v$, a t-$(v, k, 1)$ packing design is an ordered pair $\mathcal{D} = (V, \mathcal{B})$ with v points V and those k-subsets $\mathcal{B}$, called blocks, such that every t-subset of V occurs in at most one block. For given t, k and v, $\mathcal{D}$ is optimal if it has the largest possible number $A(v, k, t)$ of blocks. It is well known [17] that

$$A(v,k,t) \leq \left\lfloor \frac{v}{k} \left\lfloor \frac{v-1}{k-1} \left\lfloor \cdots \left\lfloor \frac{v-(t-1)}{k-(t-1)} \right\rfloor \cdots \right\rfloor \right\rfloor \right\rfloor. \quad (6)$$

An automorphism of $\mathcal{D}$ is a permutation ξ on V such that $B^\xi \in \mathcal{B}$ for each $B \in \mathcal{B}$, where B^ξ is the set of images of elements in B under ξ, that is, $B^\xi = \{x^\xi \mid x \in B\}$. The collection of all automorphisms of $\mathcal{D}$ forms a group, called the full automorphism group and a subgroup G of the full automorphism group is an automorphism group of $\mathcal{D}$. Let $B \in \mathcal{B}$. The stabilizer of B under G is the subgroup of G consisting of all elements $\xi \in G$ such that $B^\xi = B$. The G-orbit of B is the set $\mathrm{Orb}_G(B)$ of all distinct images of B, that is, $\mathrm{Orb}_G(B) = \{B^\xi \mid \xi \in G\}$. $\mathcal{D}$ is G-invariant if it admits G as a point-regular automorphism group, that is, G is an automorphism group of $\mathcal{D}$ such that for any $x, y \in V$, there exists exactly one element $\xi \in G$ with $x^\xi = y$. In particular a $\mathbb{Z}_v$-invariant packing design is cyclic. Moreover $\mathcal{D}$ is strictly G-invariant if it is G-invariant and the stabilizer of any block in $\mathcal{D}$ under G equals the identity.

Now, take a strictly $\mathbb{Z}_m \times \mathbb{Z}_n$-invariant t-$(mn, k, 1)$ packing design. Denote by $\hat{A}(mn, k, t)$ the largest possible number of orbits of blocks under $\mathbb{Z}_m \times \mathbb{Z}_n$ in this design. In view of the Johnson bound [18] it is then easily shown that

$$\hat{A}(mn, k, t) \leq J(mn, k, t-1); \quad (7)$$

we also refer to [11] for its proof.

Theorem II.1. *An optimal (m, n, k, λ)-OOSPC is equivalent to an optimal strictly $\mathbb{Z}_m \times \mathbb{Z}_n$-invariant $(\lambda+1)$-$(mn, k, 1)$ packing design.*

Proof: Straightforward. ■

By Theorem II.1, in order to find optimal (m, n, k, λ)-OOSPCs, it suffices to find optimal $\mathbb{Z}_m \times \mathbb{Z}_n$-invariant $(\lambda+1)$-$(mn, k, 1)$ packing designs.

Remark II.2. ([4, Construction I]). When m and n are coprime, an optimal (mn, k, λ)-OOC can be regarded as an optimal (m, n, k, λ)-OOSPC from Chinese reminder theorem.

Example II.3. It is known [14] that for mn up to 44 except $mn \in \{24,\ 36,\ 42\}$ there exists an optimal $(mn, 4, 2)$-OOC. For $mn \in \{24,\ 36,\ 42\}$, an optimal $(mn, 4, 2)$-OOC has been recently constructed in [19].

Particularly when $\lambda \geq 2$ and m, n are not coprime, the problem of finding optimal OOSPCs is quite difficult one in design theory, and very little is known about the existence and construction of such OOSPCs. In the following section, we will propose a construction of optimal $(m, n, 4, 2)$-OOSPCs, which works for positive integers m and n in general.

III. Graph Theoretic Construction

The main purpose of this section is to present a graph-theoretic construction of optimal strictly A-invariant 3-$(v, 4, 1)$ packing designs for an abelian group A of order v with the unique element of order 2. In [25], the author and Munemasa showed some general results related to the basic construction. Their paper is long and requires very involved arguments from algebraic combinatorics. Thus another purpose of this section is to briefly review the generalized results by the author and Munemasa which enables us to construct infinitely many optimal $(m, n, 4, 2)$-OOSPCs, in particular for readers of information science.

Throughout this section let A be an abelian group of order $v \equiv 2$ or $4 \pmod 6$ which contains the unique element h of order 2. Group A is regarded as a permutation group consisting of all translates σ_a defined by $x^{\sigma_a} = x + a$. Let σ be the automorphism of A defined by $x^\sigma = -x$. Then

$$\hat{A} = \{\sigma_a, \sigma_a\sigma \mid a \in A\}$$

is a permutation group on A, where $\sigma_a\sigma$ is an automorphism of A which maps x to $-x+a$. For distinct $a_1, \ldots, a_t \in A\backslash\{0\}$, we abbreviate $\mathrm{Orb}_{\hat{A}}(\{0, a_1, \ldots, a_t\})$ as $[a_1, \ldots, a_t]$.

Set

$$\begin{aligned} \mathcal{T}_1(A) &= \{[a, -a] \mid a \in A \setminus \langle h \rangle\}, \\ \mathcal{T}_2(A) &= \{[a, h] \mid a \in A \setminus \langle h \rangle\}, \\ \mathcal{B}_1 &= \{[a, -a, h] \mid a \in A \setminus \langle h \rangle\}. \end{aligned}$$

Example III.1. When $A \simeq \mathbb{Z}_{10}$, $\mathcal{T}_1(A), \mathcal{T}_2(A), \mathcal{B}_1$ are given as

$$\begin{aligned} \mathcal{T}_1(A) &= \{[1, -1],\ [2, -2],\ [3, -3],\ [4, -4]\}, \\ \mathcal{T}_2(A) &= \{[1, 5],\ [2, 5]\}, \\ \mathcal{B}_1 &= \{[1, -1, 5],\ [2, -2, 5]\}. \end{aligned}$$

Under a natural one-to-one correspondence between $x \in A$ and $e^{\frac{x\pi i}{5}} \in \mathbb{C}$, the elements of $\mathcal{T}_1(A)$ and $\mathcal{T}_2(A)$ can be viewed as isoceles and rectangles respectively. Similarly, the elements of $\mathcal{B}_1$ can be viewed as kite quadruples; see [16] for these geometric terminologies.

Set

$$\begin{aligned} \mathcal{P} =& \{T \in \binom{A}{3} \mid \mathrm{Orb}_{\hat{A}}(T) \in \mathcal{T}_1(A) \cup \mathcal{T}_2(A)\}; \\ \mathcal{Q} =& \{B \in \binom{A}{4} \mid \mathrm{Orb}_{\hat{A}}(B) \in \mathcal{B}_1\}, \end{aligned}$$

where $\binom{A}{k}$ denotes the set of all k-subsets of A.

If $\{0, a, b\} \in \binom{A}{3}$, then

$$\begin{aligned} &\{T \in \binom{A}{2} \mid \{0\} \cup T \in [a, b]\} \\ &= \{\{a, b\},\ \{-a, b - a\},\ \{-b, a - b\}, \\ &\qquad \{-a, -b\},\ \{a, a - b\},\ \{b, b - a\}\}. \end{aligned} \tag{8}$$

Similarly, if $\{0, a, b, a + b\} \in \binom{A}{4}$, then

$$\begin{aligned} &\{B \in \binom{A}{3} \mid \{0\} \cup B \in [a, b, a + b]\} \\ &= \{\{a, b, a + b\},\ \{-a, b - a, b\}, \\ &\qquad \{-a, -b, -a - b\}, \{a, a - b, -b\}\}. \end{aligned} \tag{9}$$

Lemma III.2. *If $T \subset B \in \mathcal{Q}$ with $T \in \binom{A}{3}$, then $\mathrm{Orb}_{\hat{A}}(T) \in \mathcal{T}_1(A) \cup \mathcal{T}_2(A)$. Conversely, if $\mathrm{Orb}_{\hat{A}}(T) \in \mathcal{T}_1(A) \cup \mathcal{T}_2(A)$, then there exists a unique $B \in \mathcal{Q}$ such that $T \subset B$.*

Proof: Evidently, $T \subset B \in \mathcal{Q}$ with $T \in \binom{A}{3}$ implies that $\mathrm{Orb}_{\hat{A}}(T) \in \mathcal{T}_1(A) \cup \mathcal{T}_2(A)$. Conversely, if $\mathrm{Orb}_{\hat{A}}(T) \in \mathcal{T}_1(A) \cup \mathcal{T}_2(A)$, then there exists at least one $B \in \mathcal{Q}$ such that $T \subset B$. After a standard argument as done in Appendix B, we have

$$|\mathcal{P}| = 4|\mathcal{Q}|. \tag{10}$$

This implies the uniqueness of B containing T. ∎

Set

$$\mathcal{T}(A) = \{[a, b] \mid a \neq \pm b,\ 2a \notin \{0, b, 2b\},\ 2b \notin \{0, a, 2a\}\}. \tag{11}$$

Lemma III.3. (i) *$\mathcal{T}(A) \cup \mathcal{T}_1(A) \cup \mathcal{T}_2(A)$ is the set of all orbits of triples of A under $\hat{A}$.*
(ii) $(\mathcal{T}_1(A) \cup \mathcal{T}_2(A)) \cap \mathcal{T}(A) = \emptyset$.

Proof: The results follow from (8) and (11). ∎

A quadruple B of A is *symmetric* if it satisfies that $\mathrm{Orb}_{\hat{A}}(B) = \mathrm{Orb}_A(B)$, or equivalently, $B = -B + x$ for some $x \in A$ [20]. It is easily shown that if B is a symmetric block, then $\mathrm{Orb}_{\hat{A}}(B) \in \mathcal{Q} \cup \mathcal{Q}'$, where

$$\mathcal{Q}' = \{B' \in [a, b, a + b] \mid \{0, a, b, a + b\} \in \binom{A}{4}\}.$$

Set

$$\begin{aligned} \mathcal{E}(A) = \{&[a, b, a + b] \mid a, b \in A, \\ &0 \notin \{2a, 2b\},\ \{a, 2a\} \cap \{\pm b, \pm 2b\} = \emptyset\}. \end{aligned} \tag{12}$$

Proposition III.4. Assume that $[a, b, a + b] \in \mathcal{E}(A)$. Then $[a, b, a + b]$ is incident with exactly two distinct members

$$[a, b], [a, a + b]$$

of $\mathcal{T}(A)$.

Proof: Let $B = \{0, a, b, a + b\} \in \binom{A}{4}$ such that $\mathrm{Orb}_{\hat{A}}(B) \in \mathcal{E}(A)$. Then, by (12), we have

$$\{[a, b], [a, a + b]\} \subset \mathcal{T}(A).$$

By (8), it is shown that $[a, b] = [a, a+b]$ if and only if $b = 2a$ and $4a = 0$, or $a = 2b$ and $4b = 0$, or $2a = 0$, or $2b = 0$. Hence by (12), we have $[a, b] \neq [a, a + b]$.

The set of elements of $\mathcal{T}(A)$ which are incident with $\mathrm{Orb}_{\hat{A}}(B)$ is

$$\begin{aligned} &\{\mathrm{Orb}_{\hat{A}}(T) \in \mathcal{T}(A) \mid T \in \binom{B}{3}\} \\ &= \{[a, b],\ [a, a + b],\ [b, a + b],\ \mathrm{Orb}_{\hat{A}}(\{a, b, a + b\})\} \\ &= \{[a, b],\ [a, a + b]\}, \end{aligned}$$

since

$$\begin{aligned} \{0, a, b\} &= -\{a, b, a + b\} + (a + b), \\ \{0, a, a + b\} &= -\{0, b, a + b\} + (a + b). \end{aligned}$$

Thus the proof is complete. ∎

Remark III.5. Proposition III.4 implies incidence structure $(\mathcal{T}(A), \mathcal{E}(A))$ defines an ordinary graph in the sense that each edge is incident with exactly two distinct vertices. In particular, the concept of graph $(\mathcal{T}(\mathbb{Z}_v), \mathcal{E}(\mathbb{Z}_v))$ is equivalent to the first Köhler graph of order v introduced by Köhler [22]. Though the original Köhler's idea already appeared in a paper by Fitting [21], this attractive graph was so named for a Köhler's work [22]. The concept of graph $(\mathcal{T}(A), \mathcal{E}(A))$ is extended to an arbitrary abelian group A in [25, Section 2]. In this paper we omit to mention the first Köhler graphs in detail. The interested reader is referred to Köhler's paper or a summary of his work by Beth et al. [16].

Lemma III.6. *Assume $T \in \binom{B}{3}$ and $\mathrm{Orb}_{\hat{A}}(B) \in \mathcal{E}(A)$. Then B is the only member of $\mathrm{Orb}_{\hat{A}}(B)$ containing T.*

Proof: Without loss of generality we may assume $B = \{0, a, b, a+b\}$, with $a, b \in A$, $0 \notin \{2a, 2b\}$ and $\{a, 2a\} \cap \{\pm b, \pm 2b\} = \emptyset$. Also, we may assume $T = \{0, a, b\}$. Thus the result follows from (9). ■

A 1-factor of a graph is a collection of disjoint edges into which the vertices are partitioned. The existence problem of an A-invariant 3-packing design with blocksize 4 can be reduced to that of a 1-factor of $(\mathcal{T}(A), \mathcal{E}(A))$, as the following shows.

Theorem III.7. *Let A be an abelian group of order $mn \equiv 2$ or $4 \pmod 6$ which contains the unique element of order 2. Assume $A \simeq \mathbb{Z}_m \times \mathbb{Z}_n$ and there exists a 1-factor $\mathcal{F}$ of $(\mathcal{T}(A), \mathcal{E}(A))$. Then,*

$$\begin{cases} \mathcal{F} \cup \mathcal{B}_1 & \text{if } mn \equiv 2 \pmod 4 \\ (\mathcal{F} \cup \mathcal{B}_1) \setminus \{[h/2, -h/2, h]\} & \text{if } mn \equiv 0 \pmod 4 \end{cases}$$

yields an optimal $(m, n, 4, 2)$-OOSPC.

Proof: First we claim that if $(\mathcal{T}(A), \mathcal{E}(A))$ has a 1-factor, then there exists an optimal A-invariant 3-$(mn, 4, 1)$ packing design which attains (6). Denote by $\mathcal{F}$ the set of edges of $(\mathcal{T}(A), \mathcal{E}(A))$ which forms a 1-factor. Set $\mathcal{B} = \mathcal{Q} \cup \mathcal{Q}''$, where

$$\mathcal{Q}'' = \{B \in \binom{A}{4} \mid \mathrm{Orb}_{\hat{A}}(B) \in \mathcal{F}\}.$$

By Lemma III.2 and Lemma III.3 it suffices to show that every triple T with $\mathrm{Orb}_{\hat{A}}(T) \in \mathcal{T}(A)$ is contained in a unique member of $\mathcal{Q}''$. Since $\mathcal{F}$ is a 1-factor, there exists a unique element $\mathrm{Orb}_{\hat{A}}(B) \in \mathcal{F}$ incident with $\mathrm{Orb}_{\hat{A}}(T)$. By Lemma III.6, $\mathrm{Orb}_{\hat{A}}(B)$ contains a unique member containing T.

Now, by the definition of $(\mathcal{T}(A), \mathcal{E}(A))$ and the assumption that A contains the unique element h of order 2, an A-invariant packing design constructed in the above claim contains a quadruple with nontrivial stabilizer if and only if mn is divisible by 4; the only possible orbit of quadruples of length less than mn could be $[h/2, -h/2, h]$. Since $mn \equiv 2$ or $4 \pmod 6$, it follows that

$$\begin{aligned} J(mn, 4, 2) &= \left\lfloor \frac{1}{4} \left\lfloor \frac{mn-1}{3} \left\lfloor \frac{mn-2}{2} \right\rfloor \right\rfloor \right\rfloor \\ &= \left\lfloor \frac{(mn-1)(mn-2)}{24} \right\rfloor. \end{aligned}$$

Thus the result follows from Theorem II.1. ■

Remark III.8. (i) The proof of Theorem III.7 makes clear the reason why we assume that A contains the unique element of order 2.

(ii) When $A \simeq \mathbb{Z}_v$, the designs constructed in the proof of Theorem III.7 is equivalent to a theorem by Köhler [22, Satz 1]. In this paper we omit to review Köhler's theory in detail. The interested reader is referred to [16].

Hereafter we will review some recent general results by the author and Munemasa [25] that are related to the content of this section. A 3-$(v, 4, 1)$ packing design on points A is said to be A-reversible if every block is symmetric and the set of all blocks is invariant under the action of A. An A-reversible packing design is also called an S-cyclic packing design. Obviously, the A-reversibility implies the A-invariance.

Lemma III.9. ([25, Theorem 7.1]). *Let $v \geq 8$ be a positive integer such that $v \equiv 2$ or $4 \pmod 6$. The following statements are equivalent:*

(i) *There exists an A-reversible optimal 3-$(v, 4, 1)$ packing design which attains (6) for any abelian group A of order v with the unique element of order 2.*

(ii) *There exists an S-cyclic optimal 3-$(v, 4, 1)$ packing design which attains (6).*

(iii) $v \equiv 0 \pmod 2, v \not\equiv 0 \pmod 3, v \not\equiv 0 \pmod 8$, *and there exists an S-cyclic optimal 3-$(2p, 4, 1)$ packing design which attains (6) for any odd prime divisor p of v.*

Lemma III.10. ([25, Theorem 6.5]). *Let A be an abelian group of order $v \equiv 2$ or $4 \pmod 6$ which contains the unique element of order 2. Then there exists an A-reversible optimal 3-$(v, 4, 1)$ packing design which attains (6) if and only if the graph $(\mathcal{T}(A), \mathcal{E}(A))$ has a 1-factor.*

Lemma III.9 is a generalization of the well-known Piotrowski's theorem [26]. Proofs of Lemma III.9 and Lemma III.10 will be long and require very involved mathematical arguments (see [25]), and therefore we omit them here.

Lemma III.11. ([22], [23], [24]). *If p is a prime number congruent to 53 or 77 modulo 120 with $p < 500000$, or belongs to*

$$\begin{aligned} S = \{&5, 13, 17, 25, 29, 37, 41, 53, 61, 85, 89, 97, 101, 113, 137, \\ &149, 157, 169, 173, 193, 197, 229, 233, 289, 293, 317\}, \end{aligned}$$

then there exists an S-cyclic optimal 3-$(2p, 4, 1)$ packing design which attains (6).

Summarizing Theorem III.7, Lemma III.9, Lemma III.10 and Lemma III.11, we obtain the following.

Theorem III.12. *Let $\epsilon \in \{1, 2\}$ and x, y be composite numbers of prime numbers, each being less than 500000 and congruent to 53 or 77 modulo 120 or belonging to S, where S is defined in Lemma III.11. Then there exists an optimal $(2^\epsilon x, y, 4, 2)$-OOSPC which attains (5).*

Corollary III.13. With the same assumptions as in Theorem III.12, there exists an optimal $(2^\epsilon \cdot p^n,\ p^m,\ 4,\ 2)$-OOSPC which attains (5).

IV. Optimal $(m, n, 4, 2)$-OOSPCs of small order

Search techniques can also be used to produce optimal $(m, n, 4, 2)$-OOSPCs of small order mn. In this section we investigate $\phi(m, n, 4, 2)$ for small values of m, n by using GAP [27] and cliquer [28] under the guidance of Östergård and Pottonen [29].

Example IV.1. (Order 9) When $(m, n) = (3, 3)$, the right-hand side of (7) equals 2. Set up

$$A = \langle (1\ 2\ 3)(4\ 5\ 6)(7\ 8\ 9), (1\ 4\ 7)(2\ 5\ 8)(3\ 6\ 9) \rangle \simeq \mathbb{Z}_3^2.$$

Then the following quadruples generate an optimal A-invariant 3-$(9,4,1)$ packing design;

$$\{\{1,2,4,8\},\ \{1,2,5,6\}\}.$$

It is obvious that the stabilizer of any quadruple equals the identity, and so the above two quadruples correspond to an optimal $(3,3,4,2)$-OOSPC.

Example IV.2. (Order 16) When $mn = 16$, the right-hand side of (7) equals 8. Take A to be

$$\begin{aligned}&\langle(1\ 2\ 3\ 4\ 5\ 6\ 7\ 8)(9\ 10\ 11\ 12\ 13\ 14\ 15\ 16),\\ &\quad(1\ 9)(2\ 10)(3\ 11)(4\ 12)(5\ 13)(6\ 14)(7\ 15)(8\ 16)\rangle\\ &\simeq \mathbb{Z}_2\times\mathbb{Z}_8.\end{aligned}$$

Then the following quadruples generate an optimal strictly A-invariant 3-$(16,4,1)$ packing design;

$$\begin{aligned}&\{\{1,2,3,9\},\ \{1,2,4,13\},\ \{1,2,7,12\},\ \{1,2,10,16\},\\ &\ \{1,3,5,10\},\ \{1,3,6,13\},\ \{1,3,12,15\},\ \{1,4,9,14\}\}.\end{aligned}$$

Next, take A to be

$$\begin{aligned}&\langle(1\ 2\ 3\ 4)(5\ 6\ 7\ 8)(9\ 10\ 11\ 12)(13\ 14\ 15\ 16),\\ &\quad(1\ 5\ 9\ 13)(2\ 6\ 10\ 14)(3\ 7\ 11\ 15)(4\ 8\ 12\ 16)\rangle\simeq\mathbb{Z}_4^2.\end{aligned}$$

Then the following quadruples generate an optimal strictly A-invariant 3-$(16,4,1)$ packing design;

$$\begin{aligned}&\{\{1,2,5,10\},\ \{1,2,6,12\},\ \{1,2,7,8\},\ \{1,2,9,14\},\\ &\ \{1,2,11,13\},\ \{1,3,5,13\},\ \{1,3,6,16\},\ \{1,5,12,14\}\}.\end{aligned}$$

The interested reader is referred to [30] in which the isomorphism of optimal 3-$(16,4,1)$ packing designs is extensively discussed.

Recently Feng et al. [19] proved the following theorem.

Theorem IV.3. ([19]). *Assume* $v \equiv 0 \pmod{24}$. *Then there exists no strictly cyclic* 3-$(v,4,1)$ *packing design which attains (6).*

Theorem IV.3 can be extended as follows.

Theorem IV.4. *Assume* $v \equiv 0 \pmod{24}$. *Then there exists no strictly A-invariant* 3-$(v,4,1)$ *packing design which attains (6) for any abelian group A of order v.*

Proof: It is known [31] that if $v \equiv 0 \pmod 6$, then $A(v,4,3) = v(v^2-3v-6)/24$. This fact and the point-regularity of a design imply that $\hat{A}(v,4,3) \le \lfloor(v^2-3v-6)/24\rfloor$. Hence by letting $v = 24r$, we have $\hat{A}(v,4,3) \le 24r^2-3r-1$. The result then follows from the fact that $J(v,4,2) = 24r^2-3r$. ∎

Remark IV.5. In Appendix A we summarize the values $\phi(m,n,4,2)$ for all m,n with $mn < 25$. As is seen in no. 4, no. 11, no. 18 of Table II in Appendix A, there is no $(6,n,4,2)$-OOSPC for $n = 2,3,4$ which attains (5). Applying Theorem IV.4 to the case where $A \simeq \mathbb{Z}_6\times\mathbb{Z}_4$, we can explain the reason why there exists no $(6,4,4,2)$-OOSPC which attains (5). However, the gap between $J(6n,4,2)$ and $\phi(6,n,4,2)$ for $n=2,3$ has not been explained theoretically. In particular, it is interesting to note that there exists an optimal $\mathbb{Z}_{18}$-invariant 3-$(18,4,1)$ packing design attaining $J(18,4,2)$ [14], whereas, no $\mathbb{Z}_6\times\mathbb{Z}_3$-invariant 3-$(18,4,1)$ packing design attaining $J(18,4,2)$. This observation shows a gap between OOCs and OOSPCs under the restriction that m and n are not coprime.

We conclude this section by summarizing observations we made in Remark IV.5 as Table I below and by proposing an open problem.

TABLE I
SMALL OPTIMAL $(6,n,4,2)$-OOSPC

$\mathbb{Z}_6\times\mathbb{Z}_n$	$J(6n,4,2)$	$\phi(6,n,4,2)$	Source
$n=2$	4	3	Computer search
$n=3$	11	10	Computer search
$n=4$	21	20	Computer search
$n=5$	33	33	Example II.3
$n=6$	49	??	

Problem IV.6. Does there exist an optimal $(6,n,4,2)$-OOSPC attaining (5) for a positive integer n, not being a multiple of 4 in general?

V. Asymptotically Optimal Recursive constructions

The graph-theoretic construction of optimal OOSPCs discussed in Section III is valid only for an abelian group A with the unique element of order 2. Also it does not work when A contains a subgroup isomorphic to $\mathbb{Z}_7$, since $(\mathcal{T}(\mathbb{Z}_7),\mathcal{E}(\mathbb{Z}_7))$ consists of a single vertex $\mathrm{Orb}_{\hat{\mathbb{Z}}_7}(\{0,1,3\})$ and hence has no 1-factor. A recursive technique overcomes these problems. The resulting OOSPCs are not optimal but asymptotically optimal. An (m,n,k,λ)-OOSPC is asymptotically optimal with respect to (5) if it satisfies that

$$\lim_{mn\to\infty}\frac{\phi(m,n,k,\lambda)}{J(mn,k,\lambda)} = 1;$$

see [7], [32], [33] for the details of this concept.

To explain our recursive techniques, we prepare two elementary tools. An (m,k,λ)-optical orthogonal code (OOC) is a collection of binary $(0,1)$ sequences of length m and weight k, such that the auto- and cross-correlation constraints are no more than λ respectively [10]. Just as OOSPCs are, an (m,k,λ)-OOC is identified with a collection $\mathcal{C}$ of subsets of $\mathbb{Z}_m$ which satisfies that

(i) (Autocorrelation property)

$$|X\cap(X+i)|\begin{cases}=k & \text{if } i=0,\\ \le\lambda & \text{otherwise}\end{cases}$$

for any $X\in\mathcal{C}$,

(ii) (Crosscorrelation property)

$$|X\cap(Y+i)|\le\lambda$$

for any distinct $X,Y\in\mathcal{C}$.

In the case that $m=1$ or $n=1$, the definition of OOSPCs is just equivalent to that of OOCs. Therefore the bound (5) is also vaid for OOCs.

Another preliminaly tool comes from combinatorics. An H-design, denoted by $H(s,n,k,t)$, is a triple $\mathcal{D}=(V,\mathcal{G},\mathcal{B})$, where $\mathcal{G}$ is a partition of points V into s n-subsets, called groups, $\mathcal{B}$ is a collection of k-subsets of V, called blocks, such that each block intersects any given group in at most one point and each t-subset of V from t distinct groups occurs in exactly one block. Without loss of generality we may let $V=I\times\mathbb{Z}_n$ and $\mathcal{G}=\{\{(x,y)\mid y\in\mathbb{Z}_n\}\mid x\in I\}$, where I is a set of s symbols. An automorphism group of $\mathcal{D}$ is a permutation group on V which preserves $\mathcal{G}$ and $\mathcal{B}$ respectively. $\mathcal{D}$ is semi-cyclic if it admits the automorphism τ defined by $(x,y)^\tau=(x,y+1)$. Note that the stabilizer of any block in a semi-cyclic H-design is the identity, since each block intersects in at most one group. We refer the reader to [19, P. 1529] for the definition of semi-cyclic H-designs.

Lemma V.1. ([19, Corollary 3.5]). *For any positive integer n, there exists a semi-cyclic $H(4,n,4,3)$.*

Theorem V.2. *Assume that there exists an $(m,4,2)$-OOC with ℓ sequences. Then for any positive integer n, there exists an $(m,n,4,2)$-OOSPC with ℓn^2 OOSPs.*

Proof: Let $\mathcal{C}$ be the set of sequences in an $(m,4,2)$-OOC. For each $B\in\mathcal{C}$, take a semi-cyclic $H(4,n,4,3)$ with points $B\times\mathbb{Z}_n$ and then choose a collection $\mathcal{C}_B$ of quadruples which generate the H-design. The set $\bigcup_{B\in\mathcal{C}}\mathcal{C}_B$ obviously forms an OOSPC with weight 4. Recalling the definitions of OOCs and H-designs, we see that the correlation constraint is at most 2. Since the number of blocks in $H(4,n,4,3)$ counts n^2, it follows that

$$\left|\bigcup_{B\in\mathcal{C}}\mathcal{C}_B\right|=\bigcup_{B\in\mathcal{C}}|\mathcal{C}_B|=\ell n^2.$$

■

Remark V.3. Theorem V.2 is also valid without the assumption that $k=4,\lambda=2$. The modified version of Theorem V.2 is an analogy of the Basic Construction of OOCs by Chu-Golomb [34, Theorem 4].

Lemma V.4. ([35, Theorem 3.5.2]). *For any positive integer m, there exists an $(m,4,2)$-OOSPC which is asymptotically optimal with respect to (5).*

Corollary V.5. For all positive integers m,n, there exists an $(m,n,4,2)$-OOSPC which is asymptotically optimal with respect to (5).

Proof: The result immediately follows from Theorem V.2 and Lemma V.4. ■

A recent trend in the study of recursive constructions of OOCs is to use a combinatorial object, called fan design; for example see [19]. Using fan designs, we can adjoin more OOSPs to OOSPCs constructed from Theorem V.2, but the resulting new families are still asymptotically optimal. Similarly, several known recursive constructions of OOCs with weight 4 and maximum collision parameter 2 (cf. [19, Construction 2.5], [36, Theorem 1]) can be modified to those of asymptotically optimal OOSPCs, none of which are no longer extended to the constructions of optimal OOSPCs. Thus the following seems to be a challenging problem.

Problem V.6. Present a recursive construction of optimal OOSPCs with weight 4 and maximum collision parameter 2.

VI. Conclusion

When m and n are coprime, Remark II.2 implies an optimal $(m,n,4,2)$-OOSPC is equivalent to an optimal $(mn,4,2)$-OOC. Using rich existence results of optimal $(m,n,4,2)$-OOCs gives many optimal $(m,n,4,2)$-OOSPCs; see Example II.3. Whereas, if m and n are not coprime, the existence problem of optimal $(m,n,4,2)$-OOSPCs cannot be reduced to that of optimal $(mn,4,2)$-OOCs; see also Remark IV.5. Motivated by this, we have developed a construction of optimal $(m,n,4,2)$-OOSPCs by detecting a 1-factor in a certain algebraic graph, which is a generalization of that for finding optimal strictly cyclic 3-$(mn,4,1)$ packing designs proposed by Köhler. A merit in extending Köhler's theory to the abelian group case is that our approach enables one to construct infinitely many optimal OOSPCs of size $m\times n$ from a 1-factor of $(\mathcal{T}(\mathbb{Z}_{2p}),\mathcal{E}(\mathbb{Z}_{2p}))$ for some odd prime numbers p, where mn is a product of such odd prime numbers. In the future, for an abelian group A in general, an A-invariant packing design might be required for subjects in information science other than OOCs and OOSPCs. Our construction of optimal A-invariant 3-$(v,4,1)$ packing designs will be useful for us to deal effectively with such situations.

We have also developed a recursive construction of OOSPCs with weight 4 and maximum collision parameter 2. Though the resulting OOSPCs are not optimal, there are some merits to this construction. First, it can deal with the case where $\mathbb{Z}_m\times\mathbb{Z}_n$ contains more than one element of order 2 or a subgroup isomorphic to $\mathbb{Z}_7$, though the graph-theoretic construction is not valid in these cases. Second, by using this construction, we obtain an asymptotically optimal $(m,n,4,2)$-OOSPC for all positive integers m,n. On the other hand, the author recently bacame aware, through a private communication with Kenichi Kitayama and Moriya Nakamura, that the details of developments related to the theory of recursive constructions in combinatorics had not been fully recognized by researchers in information science who study OOSPCs. I hope that this paper will play a role to inform the recursive techniques to such information scientists.

Appendix A
Optimal $(m,n,4,2)$-OOSPC with $mn<25,\ m,n>1$

Appendix B
Proof of (10)

By (8), $[a,-a]=[b,h]$ if and only if $\{b,h\}\in\{\{a,-a\},\ \{a,2a\},\ \{-a,-2a\}\}$, or equivalently,

$$b=\pm a \text{ and } h=2a. \tag{13}$$

Similarly, by using (8), we have

$$[a,-a]=[b,-b] \text{ if and only if } b=\pm a, \tag{14}$$

and

$$[a,h]=[a',h] \text{ if and only if } a'\in\{\pm a,h\pm a\}. \tag{15}$$

TABLE II

no	$J(mn,4,2)$	$\phi(mn,4,2)$	Abelian groups	Source
1	1	1	$\mathbb{Z}_4\times\mathbb{Z}_2$	Example I.1
2	2	2	$\mathbb{Z}_3^2$	Example IV.1
3	3	3	$\mathbb{Z}_5\times\mathbb{Z}_2$	Example II.3
4	4	3	$\mathbb{Z}_6\times\mathbb{Z}_2$	Computer search
5	4	3	$\mathbb{Z}_4\times\mathbb{Z}_3$	[14]
6	6	6	$\mathbb{Z}_7\times\mathbb{Z}_2$	Example II.3
7	7	7	$\mathbb{Z}_5\times\mathbb{Z}_3$	Example II.3
8	8	8	$\mathbb{Z}_8\times\mathbb{Z}_2$	Example IV.2
9		8	$\mathbb{Z}_4^2$	Example IV.2
10	11	11	$\mathbb{Z}_9\times\mathbb{Z}_2$	[14]
11	11	10	$\mathbb{Z}_6\times\mathbb{Z}_3$	Computer search
12	14	14	$\mathbb{Z}_{10}\times\mathbb{Z}_2$	Computer search
13		14	$\mathbb{Z}_5\times\mathbb{Z}_4$	Example II.3
14	15	15	$\mathbb{Z}_7\times\mathbb{Z}_3$	Example II.3
15	17	17	$\mathbb{Z}_{11}\times\mathbb{Z}_2$	Example II.3
16	21	20	$\mathbb{Z}_{12}\times\mathbb{Z}_2$	Computer search
17		20	$\mathbb{Z}_8\times\mathbb{Z}_3$	[14]
18		20	$\mathbb{Z}_6\times\mathbb{Z}_4$	Computer search

Note that, for $\{0,a,b\}\in\binom{A}{3}$, $|[a,b]|=\frac{v}{3}|\{T\in\binom{A}{2}\mid\{0\}\cup T\in[a,b]\}|$. Using this shows that

$$|[a,-a]|=v\qquad\text{for } a\in A\setminus\langle h\rangle \tag{16}$$

and

$$|[b,h]|=\begin{cases} v & \text{if } v\equiv 0 \pmod 4,\ b=\pm h/2,\\ 2v & \text{otherwise.}\end{cases} \tag{17}$$

We now compute $|\mathcal{P}|$ and $|\mathcal{Q}|$.

$$\begin{aligned}
&|\{T\mid \mathrm{Orb}_{\hat{A}}(T)\in\mathcal{T}_1(A)\setminus\mathcal{T}_2(A)\}|\\
&=\left|\bigcup_{\substack{a\in A\setminus\langle h\rangle\\ [a,-a]\notin\mathcal{T}_2(A)}}[a,-a]\right|\\
&=\left|\bigcup_{a\in A\setminus\langle h\rangle,\ 2a\neq h}[a,-a]\right| && \text{(by (13))}\\
&=\sum_{a\in A\setminus\langle h\rangle,\ 2a\neq h}\frac{|[a,-a]|}{|\{a,-a\}|} && \text{(by (14))}\\
&=\frac{v}{2}\sum_{a\in A\setminus\langle h\rangle,\ 2a\neq h}. && \text{(by (16))}
\end{aligned}$$

Moreover,

$$\begin{aligned}
&|\{T\mid \mathrm{Orb}_{\hat{A}}(T)\in\mathcal{T}_2(A)\}|\\
&=\left|\bigcup_{a\in A\setminus\langle h\rangle}[a,h]\right|\\
&=\sum_{a\in A\setminus\langle h\rangle}\frac{|[a,h]|}{|\{\pm a,h\pm a\}|} && \text{(by (15))}\\
&=\frac{v}{2}\sum_{a\in A\setminus\langle h\rangle}. && \text{(by (17))}
\end{aligned}$$

Thus, we obtain

$$|\mathcal{P}|=v\sum_{a\in A\setminus\langle h\rangle,\ 2a\neq h}+\frac{v}{2}\sum_{2a=h}. \tag{18}$$

On the other hand, note that for $a,b\in A\setminus\langle h\rangle$,

$$[a,-a,h]=[b,-b,h]\text{ if and only if } b\in\{\pm a,h\pm a\}. \tag{19}$$

By (9), it is easy to see that

$$\begin{aligned}
&|[a,-a,h]|\\
&=\frac{v}{4}|\{\{a,-a,h\},\{h,h+a,h-a\},\\
&\qquad\{a,2a,h+a\},\{-a,-2a,h-a\}\}|.
\end{aligned}$$

Therefore we have

$$\begin{aligned}
|\mathcal{Q}|&=\left|\bigcup_{a\in A\setminus\langle h\rangle}[a,-a,h]\right|\\
&=\sum_{a\in A\setminus\langle h\rangle}\frac{|[a,-a,h]|}{|\{\pm a,h\pm a\}|} && \text{(by (19))}\\
&=\frac{v}{4}\sum_{a\in A\setminus\langle h\rangle,\ 2a\neq h}+\frac{v}{8}\sum_{2a=h}. && (20)
\end{aligned}$$

We thus conclude by summarizing (18) and (20) that $|\mathcal{Q}|=4|\mathcal{P}|$. □

ACKNOWLEDGMENT

Special thanks go to Prof. Masakazu Jimbo for his continuous encouragement throughout this work. The author would like to thank Prof. Ryo Fuji-Hara, Prof. Kenichi Kitayama, Dr. Moriya Nakamura for their kind response to my contact in an attempt to inquire about earlier literature on OOSPC. The author also wishes to thank Mr. Olli Pottonen and Prof. Patric R. J. Östergård for the guidance of the use of GAP and cliquer and for the fruitful discussion we had on the existence of optimal 3-packing designs of order 16 with weight four.

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