PAPER

Index Reduction of Overlapping Strongly Sequential Systems

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SUMMARY Huet and Lévy showed that index reduction is a normalizing strategy for every orthogonal strongly sequential term rewriting system. Toyama extended this result to root balanced joinable strongly sequential systems. In this paper, we present a class including all root balanced joinable strongly sequential systems and show that index reduction is normalizing for this class. We also propose a class of left-linear (possibly overlapping) NV-sequential systems having a normalizing strategy.

key words: *term rewriting system, normalizing strategy, strong sequentiality, index reduction*

1. Introduction

Normalizing strategies, which guarantee to find the normal form of terms whenever their normal forms exist, play an important role in the implementation of functional programming languages such as Miranda [18] and Clean [2]. A normalizing strategy for a larger subclass of term rewriting systems makes realization of more flexible programming languages possible [1], [7], [10], [11], [14].

Huet and Lévy [7] formulated strong sequentiality criterion for orthogonal term rewriting systems. They showed that for every strongly sequential orthogonal term rewriting system $\mathcal{R}$, index reduction is a normalizing strategy, that is, by rewriting a redex called an index at each step, every reduction starting with a term having a normal form eventually terminates at the normal from. Here, the index is defined as a needed redex concerning an approximation of $\mathcal{R}$ which is obtained by analyzing the left-hand sides alone of the rewrite rules of term rewriting systems. Oyamaguchi [15] introduced the notion of NV-sequentiality which is a proper extension of strong sequentiality. NV-sequentiality is not only based on the analysis of the left-hand sides of the rewrite rules of term rewriting systems but also on the non-variable parts of the right-hand sides. Extensions of NV-sequentiality were proposed by Nagaya et al. [13], Comon [3] and Jacquemard [8]. The notion of strong sequentiality was extended to left-linear term rewriting systems by Toyama [16]. He showed that index reduction is a normalizing strategy for every root balanced joinable strongly sequential system.

In this paper, we show that index reduction is normalizing for the class of stable balanced joinable strongly sequential systems. A stable balanced joinable system is a left-linear term rewriting system in which every critical pair is joinable with balanced stable reduction. In stable reduction, transitive index being stable under substitutions is contracted. This class includes all root balanced joinable strongly sequential systems. In stable balanced joinable strongly sequential systems, index reduction has the balanced weakly Church-Rosser property. Thus we can show the normalizability of index reduction by using Toyama's theorem [16] concerning reduction strategy. We show that every NV-stable balanced joinable NV-sequential system has a normalizing strategy by introducing the notions of transitivity and stability for indices with respect to NV-sequentiality. In this paper, we do not consider more general sequential systems (NVNF-[13], shallow [3] or growing [8] sequential systems). The reason is that index reduction is not balanced weakly Church-Rosser even if the system is orthogonal.

2. Term Rewriting Systems

An abstract reduction system is a structure $A = \langle D, \rightarrow \rangle$ consisting of a set D and a binary relation $\rightarrow$ on D, called a reduction relation. The identity of elements of D is denoted by $\equiv$. The transitive-reflexive closure of $\rightarrow$ is denoted by $\xrightarrow{*}$. $\xrightarrow{k}$ is the k-steps reduction of $\rightarrow$, $\leftrightarrow$ is the symmetric closure of $\rightarrow$ and $=$ is the transitive-reflexive-symmetric closure of $\rightarrow$. We write $t \leftarrow s$ if $s \rightarrow t$. We say $x \in D$ is a normal form if there exists no $y \in D$ such that $x \rightarrow y$. NF is the set of normal forms. We say that x has a normal form if $x \xrightarrow{*} y$ for some normal form y.

We say that A (or $\rightarrow$) is strongly normalizing if there are no infinite reduction sequences $x_0 \rightarrow x_1 \rightarrow x_2 \rightarrow \cdots$. A (or $\rightarrow$) is Church-Rosser if $\forall x, y, z \in D$, $x \xrightarrow{*} y$ and $x \xrightarrow{*} z$ imply $y \xrightarrow{*} w$ and $z \xrightarrow{*} w$ for some $w \in D$. A has the normal form property if $\forall x \in D$, $\forall y \in \text{NF}$, $x = y$ implies $x \xrightarrow{*} y$.

Let $\mathcal{F}$ be a finite set of function symbols denoted by $f, g, h, \ldots$, and let $\mathcal{V}$ be an enumerable set of variables denoted by $x, y, z, \ldots$ where $\mathcal{F} \cap \mathcal{V} = \phi$. The set

Manuscript received October 1, 1997.

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of all terms built from $\mathcal{F}$ and $\mathcal{V}$ is denoted by $\mathcal{T}(\mathcal{F},\mathcal{V})$. The set $\mathcal{T}(\mathcal{F},\mathcal{V})$ is sometimes denoted by $\mathcal{T}$. Terms not containing variables are called ground terms. The set of all ground terms built from $\mathcal{F}$ is denoted by $\mathcal{T}(\mathcal{F})$.

A substitution θ is a mapping from $\mathcal{V}$ into $\mathcal{T}(\mathcal{F},\mathcal{V})$. Substitutions are extended into homomorphisms from $\mathcal{T}(\mathcal{F},\mathcal{V})$ into $\mathcal{T}(\mathcal{F},\mathcal{V})$. We write $t\theta$ instead of $\theta(t)$. A term s is an instance of a term t if there exists a substitution θ such that $s \equiv t\theta$.

Let $\Box$ be an extra constant. A context $C[,\ldots,]$ is a term in $\mathcal{T}(\mathcal{F}\cup\{\Box\},\mathcal{V})$. If $C[,\ldots,]$ is a context with n occurrences of $\Box$ and $t_1,\ldots,t_n \in \mathcal{T}(\mathcal{F},\mathcal{V})$ then $C[t_1,\ldots,t_n]$ is the result of replacing from left to right the occurrences of $\Box$ by $t_1,\ldots,t_n$. A context containing precisely one occurrence of $\Box$ is denoted by $C[\,]$. A term s is a subterm occurrence of a term t, written by $s \subseteqq t$, if there exists a context $C[\,]$ such that $t \equiv C[s]$. If t has an occurrence of some (function or variable) symbol e then we write $e \in t$. The variable occurrence z of $C[z]$ is fresh if $z \notin C[\,]$.

A term rewriting system is a pair $(\mathcal{F},\mathcal{R})$ consisting of a set $\mathcal{F}$ of function symbols and a finite set $\mathcal{R}$ of rewrite rules. A rewrite rule is a pair $\langle l,r\rangle$ of terms such that $l \notin \mathcal{V}$ and any variable in r also occurs in l. We write $l \rightarrow r$ for $\langle l,r\rangle$. An instance of the left-hand side of a rewrite rule is a redex. The rewrite rules of a term rewriting system $(\mathcal{F},\mathcal{R})$ define a reduction relation $\rightarrow_{\mathcal{R}}$ on $\mathcal{T}(\mathcal{F},\mathcal{V})$ as follows: $t \rightarrow_{\mathcal{R}} s$ iff there exist a rewrite rule $l \rightarrow r \in \mathcal{R}$, a substitution θ and a context $C[\,]$ such that $t \equiv C[l\theta]$ and $s \equiv C[r\theta]$. We may write $t \xrightarrow{\Delta}_{\mathcal{R}} s$ to specify the redex occurrence $\Delta \equiv l\theta$ of t in this reduction. When no confusion can arise, we omit the subscript $\mathcal{R}$.

A rewrite rule $l \rightarrow r$ is ground if l and r are ground. $l \rightarrow r$ is left-linear if every variable in l occurs only once. A term rewriting system $\mathcal{R}$ is left-linear if every $l \rightarrow r \in \mathcal{R}$ is left-linear.

Let $l \rightarrow r$ and $l' \rightarrow r'$ be two rewrite rules of $\mathcal{R}$. We assume that they are renamed to have no common variables. Assume that $s \notin \mathcal{V}$ is a subterm occurrence of l, namely $l \equiv C[s]$, such that s and l' are unifiable, i.e. $s\theta \equiv l'\theta$ for a most general unifier θ. Then we say that $l \rightarrow r$ and $l' \rightarrow r'$ are overlapping and the pair $\langle C[r']\theta,\, r\theta\rangle$ is a critical pair of $\mathcal{R}$. If $l \rightarrow r$ and $l' \rightarrow r$ are same rule, we do not consider the case $s \equiv l$. $\mathcal{R}$ is called non-overlapping if $\mathcal{R}$ has no critical pair. $\mathcal{R}$ is orthogonal if $\mathcal{R}$ is left-linear and non-overlapping.

A reduction relation $\rightarrow_s$ on $\mathcal{T}$ is a reduction strategy for $\mathcal{R}$ (or $\rightarrow_{\mathcal{R}}$) if $\rightarrow_s \subseteqq \xrightarrow{+}_{\mathcal{R}}$ and every normal form with respect to $\rightarrow_s$ is also a normal form with respect to $\rightarrow_{\mathcal{R}}$. A reduction strategy $\rightarrow_s$ is normalizing if for each t having a normal form with respect to $\rightarrow_{\mathcal{R}}$, there are no infinite sequences $t \equiv t_0 \rightarrow_s t_1 \rightarrow_s \cdots$.

In this paper we restrict ourselves to the class of left-linear term rewriting systems.

3. Normalizing Strategy for Stable Balanced Joinable Systems

We first explain the notions and the results concerning strong sequentiality [3], [7], [9], [12], [16].

Let Ω be a new constant symbol representing an unknown part of a term. The set $\mathcal{T}(\mathcal{F}\cup\{\Omega\},\mathcal{V})$ is abbreviated to $\mathcal{T}_\Omega$. Elements of $\mathcal{T}_\Omega$ are called Ω-terms. An Ω-normal form is an Ω-term without redexes, containing at least one occurrence of Ω. t_Ω denotes the Ω-term obtained from t by replacing all variables in t by Ω. The prefix ordering $\leq$ on $\mathcal{T}_\Omega$ is defined as follows: (i) $\Omega \leqq t$ for all $t \in \mathcal{T}_\Omega$, (ii) $f(s_1,\ldots,s_n) \leqq f(t_1,\ldots,t_n)$ if $s_i \leqq t_i$ $(1 \leqq i \leqq n)$, (iii) $x \leqq x$ for all $x \in \mathcal{V}$. We write $t < s$ if $t \leqq s$ and $t \not\equiv s$.

Two Ω-terms t and s are compatible, written by $t \uparrow s$, if there exists an Ω-term r such that $t \leqq r$ and $s \leqq r$; otherwise, t and s are imcompatible, which is indicated by $t \# s$. Let $S \subseteqq \mathcal{T}_\Omega$. We write $t \uparrow S$ if there exists some $s \in S$ such that $t \uparrow s$; otherwise, $t \# S$. *Red* is the set $\{\, l_\Omega \mid l \rightarrow r \in \mathcal{R}\}$. The Ω-reduction $\rightarrow_\Omega$ is defined on $\mathcal{T}_\Omega$ as $C[t] \rightarrow_\Omega C[\Omega]$ where $t \uparrow Red$ and $t \not\equiv \Omega$. The set of normal forms with respect to Ω-reduction is denoted by NF_Ω.

Lemma 3.1 ([12]): Ω-reduction is Church-Rosser and strongly normalizing.

$\omega(t)$ denotes the normal form of t with respect to Ω-reduction. $\omega(t)$ is well-defined according to the previous lemma. We write $e \in \omega(t)$ if the normal form of t with respect to Ω-reduction has an occurrence of some symbol e.

Definition 3.2: The displayed occurrence of Ω in $C[\Omega]$ is an index if $z \in \omega(C[z])$ where z is fresh. We indicate the index with $C[\Omega_I]$. Let $C[\Omega_I]$ and Δ be a redex occurrence of $C[\Delta]$. Then Δ is also called an index of $C[\Delta]$ and we write $C[\Delta_I]$.

Lemma 3.3 ([7], [9], [12]):

(i) If $C[\Omega_I]$ and $C[z] \leqq C'[z]$ where z is fresh, then $C'[\Omega_I]$.

(ii) If $C[C'[\Omega_I]]$ then $C'[\Omega_I]$.

Definition 3.4: A left-linear term rewriting system is strongly sequential if every Ω-normal form has an index.

The index reduction $\rightarrow_I$ is defined on $\mathcal{T}$ as follows: $t \rightarrow_I s$ iff $t \xrightarrow{\Delta} s$ for some index Δ. If $\mathcal{R}$ is strongly sequential then index reduction is a reduction strategy for $\mathcal{R}$ [16]. Huet and Lévy [7] showed that index reduction is a normalizing strategy for every orthogonal strongly sequential system. Toyama [16] generalized this result to the class of root balanced joinable strongly sequential systems. A term rewriting system $\mathcal{R}$ is root balanced joinable if for any critical pair $\langle p,\, q\rangle$, there exist t and $k \geqq 0$ such that $p \xrightarrow{k}_r t$ and $q \xrightarrow{k}_r t$, where the root reduction $t \rightarrow_r s$ is defined as $t \xrightarrow{\Delta} s$ and $t \equiv \Delta$.

Theorem 3.5 ([16]): If $\mathcal{R}$ is root balanced joinable and strongly sequential then $\mathcal{R}$ has the normal form property and index reduction is a normalizing strategy for $\mathcal{R}$.

The decidability of strong sequentiality for orthogonal term rewriting systems was first shown by Huet and Lévy [7] and simplified proofs were presented by Klop and Middeldorp [12]. Jouannaud and Sadfi [9] proved the decidability of strong sequentiality assuming left-linearity instead of orthogonality. Also this result was proven by Comon [3].

Theorem 3.6: Strong sequentiality of left-linear term rewriting systems is decidable.

3.1 Stable Balanced Joinability

In this section, we define stable balanced joinable term rewriting systems. For that purpose, we need the notions of transitivity, which was introduced by Toyama et al. [17], and stability for indices. Although the notion of transitivity is restricted to orthogonal term rewriting systems in [17], we define transitive indices assuming only left-linearity.

Definition 3.7: The displayed index in $C[\Omega_I]$ is transitive if $C'[C[\Omega_I]]$ for any $C'[\Omega_I]$. The transitive index is denoted by $C[\Omega_T]$.

Example 3.8: Let $Red = \{f(g(\Omega))\}$. The Ω-occurrence in $g(\Omega)$ is an index. However, this index in $g(\Omega)$ is not transitive because the Ω-occurrence in $f(g(\Omega))$ is not an index.

Lemma 3.9: If $C[\Omega_T]$ and $C[z] \leqq C'[z]$ where z is fresh, then $C'[\Omega_T]$.

Proof : Let $C''[\Omega_I]$. Then we have $C''[C[\Omega_I]]$. Clearly $C''[C[z]] \leqq C''[C'[z]]$. By Lemma 3.3 (i), $C''[C'[\Omega_I]]$. Thus $C'[\Omega_T]$. □

Definition 3.10: The displayed transitive index in $C[\Omega_T]$ is stable, which is denoted by $C[\Omega_S]$, if $C\theta[\Omega_T]$ for any θ.

The stable reduction $\rightarrow_S$ is defined as $C[l\theta] \rightarrow_S C[r\theta]$ where $C[\Omega_S]$ and $l \rightarrow r \in \mathcal{R}$.

Lemma 3.11: If $t \rightarrow_S s$ and $C[\Omega_I]$ then $C[t\theta] \rightarrow_I C[s\theta]$ for any θ.

Proof : Let $t \equiv C'[l\theta'] \rightarrow_S C'[r\theta'] \equiv s$. From $C'[\Omega_S]$, it follow that $C'\theta[\Omega_T]$ for any θ. By the definition of transitivity, we have $C[C'\theta[\Omega_I]]$. Thus $C[t\theta] \equiv C[C'\theta[l\theta'\theta]] \rightarrow_I C[C'\theta[r\theta'\theta]] \equiv C[s\theta]$. □

Definition 3.12: A critical pair $\langle p, q\rangle$ is stable balanced joinable if $p \xrightarrow{k}_S t$ and $q \xrightarrow{k}_S t$ for some t and $k \geqq 0$. A term rewriting system $\mathcal{R}$ is stable balanced joinable if every critical pair is stable balanced joinable.

Note that every root balanced joinable term rewriting system is stable balanced joinable because $\rightarrow_r \subseteqq \rightarrow_S$.

3.2 Normalizing Strategy

In this section, we show that index reduction is normalizing for every stable balanced joinable strongly sequential system. Our proof uses the theorem of Toyama [16] concerning reduction strategy. We first explain this theorem.

A reduction relation $\rightarrow$ on $\mathcal{T}$ is balanced weakly Church-Rosser if $\forall t, t', t'' \in \mathcal{T}$, $t \rightarrow t'$ and $t \rightarrow t''$ imply $t' \xrightarrow{k} s$ and $t'' \xrightarrow{k} s$ for some $s \in \mathcal{T}$ and $k \geqq 0$. We write $t \longleftrightarrow\!\!\!\!\!\rightarrow s$ if there exists a connection $t \xrightarrow{m_1} \cdot \xleftarrow{n_1} \cdot \xrightarrow{m_2} \cdot \xleftarrow{n_2} \cdots \xrightarrow{m_p} \cdot \xleftarrow{n_p} s$ with $\sum m_i > \sum n_i$. We write $t \twoheadleftarrow\!\!\!\!\rightarrow s$ if $s \longleftrightarrow\!\!\!\!\!\rightarrow t$.

Theorem 3.13 ([16]): Let $\rightarrow_s$ be a reduction strategy for $\rightarrow$ such that:

(i) $\rightarrow_s$ is balanced weakly Church-Rosser,

(ii) If $t \rightarrow s$ then $t =_s s$ or $t \longleftrightarrow\!\!\!\!\!\rightarrow_s \cdot \leftrightarrow \cdot \twoheadleftarrow\!\!\!\!\rightarrow_s s$.

Then $\rightarrow$ has the normal form property and $\rightarrow_s$ is a normalizing strategy.

Let Δ and Δ' be two redex occurrences of $t \in \mathcal{T}$. Let $\Delta \equiv C[s_1, \ldots, s_n]$ and $C[\Omega, \ldots, \Omega] \in Red$. We say that Δ and Δ' (or Δ' and Δ) are overlapping if $\Delta' \subseteqq \Delta$ and $\Delta' \not\subseteqq s_i$ for any i [16].

Lemma 3.14: Let $\mathcal{R}$ be stable balanced joinable. Let $t \xrightarrow{\Delta}_I t'$ and $t \xrightarrow{\Delta'} t''$, where $\Delta' \subseteqq \Delta$ and Δ and Δ' are overlapping. Then $t' \xrightarrow{k}_I s$ and $t'' \xrightarrow{k}_I s$ for some s and $k \geqq 0$.

Proof : Let $t \equiv C[\Delta] \equiv C[C'[\Delta']]$. Then $t' \equiv C[q\theta]$ and $t'' \equiv C[p\theta]$ for some critical pair $\langle p, q\rangle$ and θ. Since $\mathcal{R}$ is stable balanced joinable, we have $p \xrightarrow{k}_S s'$ and $q \xrightarrow{k}_S s'$ for some s'. Thus, from Lemma 3.11 and $C[\Omega_I]$, we obtain $t' \equiv C[q\theta] \xrightarrow{k}_I C[s'\theta]$ and $t'' \equiv C[p\theta] \xrightarrow{k}_I C[s'\theta]$. □

Lemma 3.15 ([16]): Let $C[\Delta_I, \Delta']$. Then $C[\Delta_I, t]$ for any t.

Lemma 3.16: Let $\mathcal{R}$ be stable balanced joinable. If $t \rightarrow_I t'$ and $t \rightarrow_I t''$ then $t' \xrightarrow{k}_I s$ and $t'' \xrightarrow{k}_I s$ for some s and $k \geqq 0$.

Proof : Let $t \xrightarrow{\Delta}_I t'$ and $t \xrightarrow{\Delta'}_I t''$. If Δ and Δ' are disjoint then from Lemma 3.15 the lemma follows. If Δ and Δ' are not disjoint, then by the definition of index, Δ and Δ' must be overlapping. Thus the lemma holds by Lemma 3.14. □

The parallel redution $t \mathrel{+\!\!\!\!\!\!\longrightarrow} s$ is defined as $t \equiv C[\Delta_1, \ldots, \Delta_n] \xrightarrow{\Delta_1} \cdots \xrightarrow{\Delta_n} s$ $(n \geqq 0)$. We write $t \mathrel{+\!\!\!\!\!\!\longrightarrow}' s$ if $t \overset{\Delta_1 \cdots \Delta_n}{\mathrel{+\!\!\!\!\!\!\longrightarrow}} s$ and $n > 0$.

Lemma 3.17: Let $\mathcal{R}$ be strongly sequential and stable balanced joinable and $t \mathrel{+\!\!\!\!\!\!\longrightarrow} s$. Then $t =_I s$ or $t \longleftrightarrow\!\!\!\!\!\rightarrow_I \cdot \mathrel{+\!\!\!\!\!\!\longrightarrow} \cdot \twoheadleftarrow\!\!\!\!\rightarrow_I s$.

Proof : Let $t \overset{\Delta_1\cdots\Delta_n}{\twoheadrightarrow\!\!\!\!\!+} s$. We prove the lemma by induction on n. The case $n = 0$ is trivial. Let $t \overset{\Delta_1\cdots\Delta_n}{\twoheadrightarrow\!\!\!\!\!+} s$ $(n > 0)$. There are two cases.

(1) Some Δ_i, say Δ_1, is an index. Let $t \overset{\Delta_1}{\to}_I t' \overset{\Delta_2\cdots\Delta_n}{\twoheadrightarrow\!\!\!\!\!+} s$. Applying induction hypothesis to $t' \overset{\Delta_2\cdots\Delta_n}{\twoheadrightarrow\!\!\!\!\!+} s$, we obtain the lemma.

(2) No Δ_i is an index. Since $\mathcal{R}$ is strongly sequential, t has an index. Let Δ be an index of t and $t \overset{\Delta}{\to}_I t''$. Furthermore, consider the following two cases.

(2-1) Δ and Δ_i are non-overlapping for any i. Using the left-linearity of $\mathcal{R}$ and Lemma 3.15, we can easily show that $t'' \twoheadrightarrow\!\!\!\!\!+ s'$ and $s \to_I s'$ for some s'. Thus we have $t \longleftrightarrow\!\!\!\!\!\!\twoheadrightarrow_I \cdot \twoheadrightarrow\!\!\!\!\!+ \cdot \twoheadleftarrow\!\!\!\longrightarrow_I s$.

(2-2) Δ and some Δ_i, say Δ_1, are overlapping. Let $t \overset{\Delta_1}{\to} t' \overset{\Delta_2\cdots\Delta_n}{\twoheadrightarrow\!\!\!\!\!+} s$. By the definition of index, we have $\Delta_1 \subseteqq \Delta$. From Lemma 3.14, it follow that $t'' \overset{k}{\to}_I s'$ and $t' \overset{k}{\to}_I s'$ for some s' and $k \geqq 0$. Thus we have $t \longleftrightarrow\!\!\!\!\!\!\twoheadrightarrow_I t'$. Applying induction hypothsis to $t' \overset{\Delta_2\cdots\Delta_n}{\twoheadrightarrow\!\!\!\!\!+} s$, we obtain the lemma. □

Theorem 3.18: Let $\mathcal{R}$ be strongly sequential and stable balanced joinable. Then $\mathcal{R}$ has the normal form property, and index reduction $\to_I$ is a normalizing strategy for $\mathcal{R}$.

Proof : It is obvious that $\to_I$ is a reduction strategy for $\twoheadrightarrow\!\!\!\!\!+'$. Take $\to_I$ as $\to_s$ and $\twoheadrightarrow\!\!\!\!\!+'$ as $\to$ in Theorem 3.13. From Lemmas 3.16 and 3.17, the conditions (i) and (ii) in Theorem 3.13 are satisfied. Thus, from $\to \subseteqq \twoheadrightarrow\!\!\!\!\!+' \subseteqq \overset{*}{\to}$, the theorem follows. □

Quasi-index reduction (or hyper-index reduction) is defined as $\overset{*}{\to} \cdot \to_I$. In Theorem 3.18 index reduction can be relaxed into quasi-index reduction.

Theorem 3.19: Let $\mathcal{R}$ be strongly sequential and stable balanced joinable. Then quasi-index reduction $\overset{*}{\to} \cdot \to_I$ is a normalizing strategy for $\mathcal{R}$.

Proof : Similar to Theorem 7.2 in [16]. □

3.3 Decidability of Stable Transitive Indices

Stable balanced joinability is an undecidable property for left-linear term rewriting systems. Because the halting problem for Turing machines is reducible to this problem by the construction of a left-linear term rewriting system which can simulate the computations of a Turing machine. (For a construction, see [11].) In this section, we show that for a given $C[\Omega]$ we can determine whether the displayed occurrence of Ω in $C[\Omega]$ is a stable transitive index. Thus, stable balanced joinability is semi-decidable.

Lemma 3.20: Let $C[t, \Omega_I]$. Then $C[x, \Omega_I]$ where x is a fresh variable.

Proof : Suppose that the displayed occurrence of Ω in $C[x, \Omega]$ is not an index. Let θ be a substitution such that $x\theta \equiv t$ and $y\theta \equiv y$ for any $y \not\equiv x$. Then $C[t, z] \equiv C[x, z]\theta$ where z is fresh. Because Ω-reduction is closed under substitutions, $C[t, z] \equiv C[x, z]\theta \overset{*}{\to}_\Omega \omega(C[x, z])\theta$. From $\omega(C[t, z]) \leq \omega(C[x, z])\theta$ and $z \notin \omega(C[x, z])$, it follows that $z \notin \omega(C[t, z])$. However, this is contradictory to $C[t, \Omega_I]$. □

Definition 3.21: The set *Red** is defined as follows:

$$Red^* = \{\, t_\Omega \mid l \equiv C[t],\ C[\Omega_I],\ l \to r \in \mathcal{R} \,\}.$$

Note that the above definition of *Red** is different from the original one by Toyama et al. [17]. In fact, our *Red** is a subset of theirs, and these two sets are equal if $\mathcal{R}$ is orthogonal.

Example 3.22: Let

$$\mathcal{R} = \begin{cases} f(a, x) \to a \\ f(b, g(x)) \to g(b) \\ b \to b. \end{cases}$$

Then $Red^* = \{\, f(a, \Omega),\ f(b, g(\Omega)),\ a,\ b \,\}$.

Lemma 3.23: Let $C[\Omega_I]$ and $C[t] \uparrow Red$. Then $t \uparrow Red^*$.

Proof : Since $C[t] \uparrow Red$, there exists a left-hand side l of $\mathcal{R}$ such that $C[t] \uparrow l_\Omega$. Because $C[\Omega_I]$, we have $l \equiv C'[s]$ for some s and $C'[\ ]$ such that $t \uparrow s_\Omega$ and $C[z] \uparrow C'_\Omega[z]$ where z is fresh. Now we show that $s_\Omega \in Red^*$. Without loss of generality, we may state that $C[z] \equiv C''[s_1, \ldots, s_n, z, \Omega, \ldots, \Omega]$ and $C'[z] \equiv C''[x_1, \ldots, x_n, z, t_1, \ldots, t_m]$ where $C''[, \ldots ,]$ does not contain variables and $\Omega < t_{i\Omega}$ for $i = 1, \ldots, m$. Repeated application of Lemma 3.20 yields $C''[x_1, \ldots, x_n, \Omega_I, \Omega, \ldots, \Omega]$. Since $C''[x_1, \ldots, x_n, z, \Omega, \ldots, \Omega] \leqq C'[z]$, it follows from Lemma 3.3 (i) that $C'[\Omega_I]$. Thus $s_\Omega \in Red^*$. □

Lemma 3.24: Let $C[\Omega] \in \mathcal{T}_\Omega$. Then $C[\Omega_T]$ iff $z \in \omega(C[z])$ and $\omega(C[z]) \# Red^*$ where z is fresh.

Proof :

($\Rightarrow$) It is clear that $z \in \omega(C[z])$. Let $C'[z] \equiv \omega(C[z])$. Suppose $C'[z] \uparrow s$ for some $s \in Red^*$. Then there exists $C''[\ ]$ such that $C''[\Omega_I]$ and $C''_\Omega[s] \in Red$. Since $C''[C'[z]] \uparrow Red$, $\omega(C''[C[z]]) \equiv \omega(C''[C'[z]]) \equiv \Omega$. But this contradicts $C[\Omega_T]$. Hence $\omega(C[z]) \# Red^*$.

($\Leftarrow$) It is clear that $C[\Omega_I]$. We will prove $C'[C[\Omega_I]]$ for any $C'[\Omega_I]$. Let $\omega(C[z]) \equiv C_1[z]$ and $\omega(C'[z]) \equiv C'_1[z]$. It suffices to show that $C'_1[C_1[z]] \in \mathrm{NF}_\Omega$. Suppose $C'_1[C_1[z]] \notin \mathrm{NF}_\Omega$. Because $C_1[z] \in \mathrm{NF}_\Omega$ and $C'_1[z] \in \mathrm{NF}_\Omega$, there exists $C''[C_1[z]] \subseteqq C'_1[C_1[z]]$ such that $C''[C_1[z]] \uparrow Red$. From $C'_1[\Omega_I]$ and Lemma 3.3 (ii), $C''[\Omega_I]$. Using Lemma 3.23 we obtain $C_1[z] \uparrow Red^*$. However, this contradicts $\omega(C[z]) \# Red^*$. □

Lemma 3.25: Let $C[\Omega] \in \mathcal{T}_\Omega$. Then $C[\Omega_S]$ iff $C_\Omega[\Omega_T]$.

Proof :

($\Rightarrow$) Let θ be a substitution such that $x\theta$ is a redex for any $x \in C[\ ]$. Note that $C\theta[\Omega_T]$ and $\omega(C\theta[z]) \equiv \omega(C_\Omega[z])$. We will show that $C'[C_\Omega[\Omega_I]]$ for any $C'[\Omega_I]$. Because $C'[C\theta[\Omega_I]]$, we have $z \in \omega(C'[C\theta[z]]) \equiv \omega(C'[\omega(C\theta[z])]) \equiv \omega(C'[\omega(C_\Omega[z])]) \equiv \omega(C'[C_\Omega[z]])$. Thus $C'[C_\Omega[\Omega_I]]$.

($\Leftarrow$) Clearly $C_\Omega[z] \leq C\theta[z]$ for any θ. From Lemma 3.9 and $C_\Omega[\Omega_T]$, it follows that $C\theta[\Omega_T]$ for any θ. Therefore we obtain $C[\Omega_S]$. □

Lemma 3.26: Let $C[\Omega] \in \mathcal{T}_\Omega$. Then $C[\Omega_S]$ iff $z \in \omega(C_\Omega[z])$ and $\omega(C_\Omega[z]) \# Red^*$ where z is fresh.

Proof : It is trivial from Lemmas 3.24 and 3.25. □

Therefore, by the previous lemma, we can decide whether $C[\Omega_S]$ for a given $C[\Omega]$.

Example 3.27: Consider $\mathcal{R}$ of Example 3.22. $\mathcal{R}$ has only one critical pair $\langle f(b, g(x)), g(b)\rangle$. Because $\omega(g(z)) \equiv g(z) \# Red^*$, we obtain $g(\Omega_S)$ using Lemma 3.26. By $f(b, g(x)) \to_S g(b) \leftarrow_S g(b)$, $\mathcal{R}$ is stable balanced joinable. Note that $\mathcal{R}$ is not root balanced joinable. $\mathcal{R}$ is strongly sequential because $\mathcal{R}$ is left-normal [16]. Thus, from Theorem 3.18, index reduction is a normalizing strategy for $\mathcal{R}$.

4. Normalizing Strategy for NV-Stable Balanced Joinable Systems

In this section, we show that NV-index reduction is a normalizing strategy for every NV-stable balanced joinable NV-sequential system. We first discuss the notion of NV-sequentiality, which is introduced by Oyamaguchi [15].

The Ω_V-reduction $\to_{\Omega_V}$ is defined on $\mathcal{T}_\Omega$ as $C[t] \to_{\Omega_V} C[r_\Omega]$ where $t \uparrow l_\Omega$ and $t \not\equiv \Omega$ for some $l \to r \in \mathcal{R}$ [15].

Definition 4.1: The displayed occurrence of Ω in $C[\Omega]$ is called an NV-index if $z \in t$ for each Ω-term t such that $C[z] \xrightarrow{*}_{\Omega_V} t$ where z is fresh. If the displayed occurrence of Ω in $C[\Omega]$ is an NV-index then we write $C[\Omega_{I_V}]$; otherwise $C[\Omega_{NI_V}]$. If $C[\Omega_{I_V}]$ then a redex occurrence Δ in $C[\Delta]$ is also called an NV-index. If Δ is an NV-index of $C[\Delta]$ then we write $C[\Delta_{I_V}]$; otherwise $C[\Delta_{NI_V}]$.

Note that $C[\Omega_I]$ implies $C[\Omega_{I_V}]$. The above definition of NV-index is different form the one in [15]. By using the following property of NV-indices, we can see that they are equivalent.

Lemma 4.2 ([15]): Let $C[z] \in \mathcal{T}_\Omega$ where z is a fresh variable. $C[\Omega_{NI_V}]$ iff there exist $C'[z] \subseteqq C[z]$ and t such that $C'[z] \xrightarrow{*}_{\Omega_V} t$, $t \uparrow Red$ and $z \in t$.

The following lemma is used later.

Lemma 4.3 ([15]):

(i) If $C[\Omega_{I_V}]$ and $C[z] \leqq C'[z]$ where z is fresh, then $C'[\Omega_{I_V}]$.

(ii) If $C[C'[\Omega_{I_V}]]$ then $C'[\Omega_{I_V}]$.

Definition 4.4: A left-linear term rewriting system is NV-sequential if every Ω-normal form has an NV-index.

Oyamaguchi [15] proved that NV-indices are computable in polynomial time for arbitrary left-linear NV-sequential systems. He also showed that NV-sequentiality of orthogonal term rewriting systems is decidable. This result was generalized to left-linear term rewriting systems by Comon [3].

Theorem 4.5: NV-sequentiality is a decidable property of left-linear term rewriting systems.

4.1 NV-Stable Balanced Joinable Systems and Normalizing Strategy

The NV-index reduction $\to_{I_V}$ is defined on $\mathcal{T}$ as follows: $t \to_{I_V} s$ iff $t \xrightarrow{\Delta} s$ for some NV-index Δ.

Definition 4.6: The displayed NV-index in $C[\Omega_{I_V}]$ is transitive if $C'[C[\Omega_{I_V}]]$ for any Ω-term $C'[\Omega_{I_V}]$. If the displayed occurrence of Ω in $C[\Omega]$ is a transitive NV-index then we write $C[\Omega_{T_V}]$; otherwise $C[\Omega_{NT_V}]$.

The following example shows that a transitive index is not always a transitive NV-index.

Example 4.7: Consider $\mathcal{R}$ of Example 3.22. We can show $g(\Omega_T)$ by using Lemma 3.24. However, we have $g(\Omega_{NT_V})$ because $f(b, g(\Omega_{NI_V}))$ for $f(b, \Omega_{I_V})$.

Definition 4.8: The displayed transitive NV-index in $C[\Omega_{T_V}]$ is stable if $C\theta[\Omega_{T_V}]$ for any θ. If the displayed occurrence of Ω in $C[\Omega]$ is a stable transitive NV-index then we write $C[\Omega_{S_V}]$; otherwise $C[\Omega_{NS_V}]$.

The NV-stable reduction $\to_{S_V}$ is defined as $C[l\theta] \to_{S_V} C[r\theta]$ where $C[\Omega_{S_V}]$ and $l \to r \in \mathcal{R}$.

Lemma 4.9: If $t \to_{S_V} s$ and $C[\Omega_{I_V}]$ then $C[t\theta] \to_{I_V} C[s\theta]$ for any θ.

Proof : Similar to Lemma 3.11. □

Definition 4.10: A critical pair $\langle p, q\rangle$ is NV-stable balanced joinable if $p \xrightarrow{k}_{S_V} t$ and $q \xrightarrow{k}_{S_V} t$ for some t and $k \geqq 0$. A term rewriting system $\mathcal{R}$ is NV-stable balanced joinable if every critical pair is NV-stable balanced joinable.

We can prove the following theorems by a argument similar to that in Sect. 3.2.

Theorem 4.11: Let $\mathcal{R}$ be NV-sequential and NV-stable balanced joinable. Then $\mathcal{R}$ has the normal form property, and NV-index reduction $\to_{I_V}$ is a normalizing strategy for $\mathcal{R}$.

Quasi-NV-index reduction (or hyper-NV-index reduction) is defined as $\xrightarrow{*} \cdot \to_{I_V}$.

Theorem 4.12: Let $\mathcal{R}$ be NV-sequential and NV-stable balanced joinable. Then quasi-NV-index reduction $\xrightarrow{*} \cdot \to_{I_V}$ is a normalizing strategy for $\mathcal{R}$.

Note that the class of NV-stable balanced joinable systems includes all root balanced joinable systems. However, this class does not include all stable balanced joinable systems. Consider $\mathcal{R}$ of Example 3.22 which is stable balanced joinable.. $\mathcal{R}$ has only one critical pair $\langle f(b, g(x)), g(b)\rangle$. Because $g(\Omega_{NT_V})$, $g(b)$ cannot be reduced by $\rightarrow_{S_V}$. Thus, $\mathcal{R}$ is not NV-stable balanced joinable. Since $\rightarrow_I \subseteq \rightarrow_{I_V}$, we obtain the following corollary. The calculating on index is much easier than NV-index.

Corollary 4.13: Let $\mathcal{R}$ be strongly sequential and NV-stable balanced joinable. Then index reduction $\rightarrow_I$ is a normalizing strategy for $\mathcal{R}$.

4.2 Decidability of Stable Transitive NV-Indices

We next show that for a given $C[\Omega]$ it is decidable whether $C[\Omega_{S_V}]$. Hence, NV-stable balanced joinability is also semi-decidable.

Definition 4.14: The set Red^*_V is defined as follows:

$$Red^*_V = \{\, t_\Omega \mid l \equiv C[t],\ C[\Omega_{I_V}],\ l \rightarrow r \in \mathcal{R} \,\}.$$

Lemma 4.15: Let $C[t, \Omega_{I_V}]$. Then $C[x, \Omega_{I_V}]$ where x is fresh.

Proof : Similar to Lemma 3.20. □

Lemma 4.16: Let $C[\Omega_{I_V}]$ and $C[t] \uparrow Red$. Then $t \uparrow Red^*_V$.

Proof : Similar to Lemma 3.23. □

Lemma 4.17: Let $C[\Omega_{I_V}]$. Then $C[\Omega_{NT_V}]$ iff there exists t such that $C[z] \xrightarrow{*}_{\Omega_V} t$ and $t \uparrow Red^*_V$ where z is fresh.

Proof :

($\Rightarrow$) Let $C'[C[\Omega_{NI_V}]]$ for $C'[\Omega_{I_V}]$. By Lemma 4.2 and $C[\Omega_{I_V}]$, there exist $C''[C[z]] \subseteq C'[C[z]]$ and s such that $C''[C[z]] \xrightarrow{*}_{\Omega_V} s$, $s \uparrow Red$ and $z \in s$. We have $s \equiv C''_1[C_1[z]]$ for some $C''_1[\,]$ and $C_1[\,]$ such that $C''[z] \xrightarrow{*}_{\Omega_V} C''_1[z]$ and $C[z] \xrightarrow{*}_{\Omega_V} C_1[z]$. By Lemma 4.3 (ii) and $C'[\Omega_{I_V}]$, $C''[\Omega_{I_V}]$ and therefore $C''_1[\Omega_{I_V}]$. From Lemma 4.16, it is follows that $C_1[z] \uparrow Red^*_V$.

($\Leftarrow$) Let s be an Ω-term such that $t \uparrow s$ and $s \in Red^*_V$. Then by the definition of Red^*_V there exists $C'[\Omega_{I_V}]$ such that $C'_\Omega[s] \in Red$. It is clear that $C'[C[z]] \xrightarrow{*}_{\Omega_V} C'[t]$ and $C'[t] \uparrow Red$. Since $C[\Omega_{I_V}]$, $z \in t$ and therefore $z \in C'[t]$. From Lemma 4.2, it follows that $C'[C[\Omega_{NI_V}]]$. Thus $C[\Omega_{NT_V}]$. □

We use tree automata techniques in our poof. The definition of tree automaton is given as follows.

Definition 4.18: Let Σ be a finite ranked alphabet. A finite tree automaton over Σ is a triple $\mathcal{A} = (Q, Q_f, R)$, where Q is a finite set of states, Q_f is a finite set of final states ($Q_f \subseteq Q$), R is a set of ground rewrite rules of the form $f(q_1, q_2, \ldots, q_n) \rightarrow q$ or $q \rightarrow q'$ where $f \in \Sigma$, $q_1, \ldots, q_n, q, q' \in Q$.

We write $\rightarrow_{\mathcal{A}}$ for $\rightarrow_R$. A tree automaton $\mathcal{A}$ accepts $t \in \mathcal{T}(\Sigma)$ iff there exists a final state q such that $t \xrightarrow{*}_{\mathcal{A}} q$. The language $T(\mathcal{A})$ is the set of terms which are accepted by $\mathcal{A}$. A set T of terms is recognizable if there exists a tree automaton $\mathcal{A}$ such that $T = T(\mathcal{A})$. (see e.g. [6] for more details.)

Lemma 4.19: Let $C[\Omega_{I_V}]$. Then $C[\Omega_{NS_V}]$ iff $C\theta[\Omega_{NT_V}]$ for some θ such that $y\theta \in \mathcal{T}(\mathcal{F}, \{x\})$ for any $y \in C[\,]$.

Proof : It is easily shown using Lemma 4.17. □

By the previous lemma, stability of transitive indices in $t \in \mathcal{T}_\Omega$ only depends on instances of t in $\mathcal{T}(\mathcal{F} \cup \{\Omega\}, \{x\})$. In the Ω_V-reduction, every variable can be considered as constant. Thus we fix $\Sigma = \mathcal{F} \cup \{\Omega, x, z\}$ and after this we restrict the Ω_V-reduction to $\mathcal{T}(\Sigma) \times \mathcal{T}(\Sigma)$. We also write $\mathcal{T}^x_\Omega$ for $\mathcal{T}(\Sigma)$. Let $T_T = \{\, t \in \mathcal{T}^x_\Omega \mid t \xrightarrow{*}_{\Omega_V} s,\ s \uparrow Red^*_V \,\}$. We will show that T_T is recognizable.

Definition 4.20 ([4]): A ground tree transducer G over Σ is a pair (U, D) where $U = (Q, Q_I, R)$ and $U = (Q', Q_I, R')$ are tree automaton over Σ.

The relation $\rightarrow_G$ associated with G is defined on $T(\Sigma)$ by $t \rightarrow_G t'$ iff there exists $s \in \mathcal{T}(\Sigma \cup Q_I)$ such that $t \xrightarrow{*}_U s \xleftarrow{*}_D t'$. A relation associated a ground tree transducer is called a GTT relation.

Lemma 4.21 ([5]): Let T be a recognizable set and let $\rightarrow_G$ be a GTT relation. Then the set $\{\, t \mid t \rightarrow_G s,\ s \in T \,\}$ is recognizable.

Let $T_R = \{\, s \in \mathcal{T}^x_\Omega \mid s \uparrow Red^*_V \,\}$. According to the previous lemma, it suffices to show that T_R is recognizable and $\xrightarrow{*}_{\Omega_V}$ is a GTT relation. t^x denotes the term obtained from t by replacing all variables and Ω's in t by x.

Lemma 4.22: T_R is a recognizable set.

Proof : Let $\mathcal{A} = (Q, Q_f, R)$, where $Q = \{\, q_t \mid t \subseteq s^x,\ s \in Red^*_V \,\} \cup \{\, q_x, q_\Omega \,\}$, $Q_f = \{\, q_t \mid t \equiv s^x, s \in Red^*_V \,\} \cup \{\, q_\Omega \,\}$ and R consists of the following rules:

(*i*) $f(q_{t_1}, \ldots, q_{t_n}) \rightarrow q_t$ where $f \in \mathcal{F}$, $f(t_1, \ldots, t_n) \uparrow t_\Omega$ and $t \not\equiv \Omega$,

(*ii*) $\Omega \rightarrow q_\Omega$, $x \rightarrow q_x$, $z \rightarrow q_x$.

We show that $T(\mathcal{A}) = T_R$.

($\subseteq$) We first prove the following claim: if $s \in \mathcal{T}^x_\Omega$ and $s \xrightarrow{*}_{\mathcal{A}} q_t$ then $s \uparrow t_\Omega$. The proof is by induction on the size of s. *Base step*: Trivial. *Induction step*: Let $s \equiv f(s_1, \ldots, s_n)$. Then there exists a rule $f(q_{t_1}, \ldots, q_{t_n}) \rightarrow q_t$ in R such that $s_i \xrightarrow{*}_{\mathcal{A}} q_{t_i}$ for any i. Note that $f(t_1, \ldots, t_n) \uparrow t_\Omega$. By induction hypothesis, we have $s_i \uparrow t_{i\Omega}$ for any i. If $t_\Omega \equiv \Omega$ then trivially $s \uparrow t_\Omega$. Otherwise, $t_\Omega \equiv f(t'_1, \ldots, t'_n)$ and $t_i \uparrow t'_i$ for any i. We now show that $s_i \uparrow t'_i$ for any i. If $t_i \equiv \Omega$ then $s_i \equiv \Omega$ by construction of $\mathcal{A}$. Therefore $s_i \uparrow t'_i$. If $t_i \not\equiv \Omega$

then we obtain $t_{i\Omega} \geq t'_i$ from $t_i \uparrow t'_i$ because $\Omega \notin t_i$ and t'_i does not contain variables. Hence $s_i \uparrow t'_i$. Thus the claim follows. Assume $s \in \mathcal{T}^x_\Omega$ and $s \xrightarrow{*}_{\mathcal{A}} q_t$ with $q_t \in Q_f$. If $t \equiv \Omega$ then $s \equiv \Omega$ and therefore $s \in T_R$. Otherwise, from the claim, it follows that $s \uparrow t_\Omega$, i.e. $s \uparrow Red^*_V$. Thus $s \in T_R$.

($\supseteq$) It is clear that $\Omega \in T_R$ is accepted by $\mathcal{A}$. If $s \in T_R$ and $s \not\equiv \Omega$ then $s \uparrow t_\Omega$ for some $q_t \in Q_f$ with $t \not\equiv \Omega$. Hence we prove that for any $s \not\equiv \Omega$, if $s \uparrow t_\Omega$ and $q_t \in Q$ with $t \not\equiv \Omega$ then $s \xrightarrow{*}_{\mathcal{A}} q_t$. The proof is by induction on the size of s. *Base step*: Trivial. *Induction step*: Let $s \equiv f(s_1, \ldots, s_n)$. *Case 1.* $t \equiv x$. Let $t'_i \equiv \Omega$ if $s_i \equiv \Omega$; otherwise, let $t'_i \equiv x$. From induction hypothesis, it follows that $s_i \xrightarrow{*}_{\mathcal{A}} q_{t'_i}$ for any i. Since $f(q_{t'_1}, \ldots, q_{t'_n}) \rightarrow q_x \in R$, $s \equiv f(s_1, \ldots, s_n) \xrightarrow{*}_{\mathcal{A}} q_x$. *Case 2.* $t \equiv f(t_1, \ldots, t_n)$. Note that $s_i \uparrow t_{i\Omega}$, $q_{t_i} \in Q$ and $t_i \not\equiv \Omega$ for any i. Let $t'_i \equiv \Omega$ if $s_i \equiv \Omega$; otherwise, let $t'_i \equiv t_i$. From induction hypothesis and the rule $\Omega \rightarrow q_\Omega$, we have $s_i \xrightarrow{*}_{\mathcal{A}} q_{t'_i}$ for any i. Because $f(t'_1, \ldots, t'_n) \uparrow t_\Omega$, there exists $f(q_{t'_1}, \ldots, q_{t'_n}) \rightarrow q_t$ in R. Thus $s \equiv f(s_1, \ldots, s_n) \xrightarrow{*}_{\mathcal{A}} q_t$. □

Lemma 4.23: $\xrightarrow{*}_{\Omega_V}$ is a GTT relation.

Proof : We define the tree automaton U and D as follows. $U = (Q, Q_I, R)$, where $Q = \{ q_t \mid t \underline{\subseteq} s^x, s \in Red \} \cup \{ q_x, q_\Omega \}$, $Q_I = \{ q_t \mid t \equiv s^x, s \in Red \}$ and R consists of the following rules:

(i) $f(q_{t_1}, \ldots, q_{t_n}) \rightarrow q_t$ where $f \in \mathcal{F}$, $f(t_1, \ldots, t_n) \uparrow t_\Omega$ and $t \not\equiv \Omega$,

(ii) $\Omega \rightarrow q_\Omega$, $x \rightarrow q_x$, $z \rightarrow q_x$.

$D = (Q', Q_I, R')$ where $Q' = Q_I \cup \{ q'_t \mid t \underline{\subseteq} r_\Omega, l \rightarrow r \in \mathcal{R} \}$ and R' consists of the following rules:

$(i)'$ $f(q'_{t_1}, \ldots, q'_{t_n}) \rightarrow q'_t$ where $f(t_1, \ldots, t_n) \equiv t$,

$(ii)'$ $q'_s \rightarrow q_t$ where $t \equiv l^x$ and $s \equiv r_\Omega$ for some $l \rightarrow r \in \mathcal{R}$.

We can prove the following claims by a argument similar to that in Lemma 4.22.

(1) Let $s \in \mathcal{T}^x_\Omega$ and $q_t \in Q_I$. Then $s \xrightarrow{*}_U q_t$ iff $s \uparrow t_\Omega$ and $s \not\equiv \Omega$.

(2) Let $s \in \mathcal{T}^x_\Omega$ and $q_t \in Q_I$. Then $s \xrightarrow{*}_D q_t$ iff $s \equiv r_\Omega$ and $t \equiv l^x$ for some $l \rightarrow r \in \mathcal{R}$.

Let $G = (U, D)$. Then it follows from the above claims that $\rightarrow_{\Omega_V} \underline{\subseteq} \rightarrow_G \underline{\subseteq} \xrightarrow{*}_{\Omega_V}$. Because the transitive-reflexive closure of a GTT relation is a GTT relation [4], $\xrightarrow{*}_{\Omega_V}$ is a GTT relation. □

Lemma 4.24: T_T is a recognizable set.

Proof : From Lemmas 4.21, 4.22 and 4.23. □

By the previous lemma, there exists a complete and deterministic automaton $\mathcal{A}_T$ such that $T(\mathcal{A}_T) = T_T$ [6]. The number of states in $\mathcal{A}_T$ is denoted by k_T. The height $\rho(t)$ of t is defined as follows: $\rho(t) = 1 + \max\{ \rho(t_1), \ldots, \rho(t_n) \}$ if $t \equiv f(t_1, \ldots t_n)$ and $n > 0$; otherwise $\rho(t) = 1$.

Lemma 4.25: Let $C[\Omega_{I_V}]$. Then $C[\Omega_{NS_V}]$ iff $C\theta[\Omega_{NT_V}]$ for some θ such that $\rho(y\theta) \leq k_T$ and $y\theta \in \mathcal{T}(\mathcal{F}, \{x\})$ for any $y \in C[\,]$.

Proof :

($\Rightarrow$) From Lemmas 4.17 and 4.19, $C\theta'[z]$ is accepted by $\mathcal{A}_T$ for some θ' such that $y\theta' \in \mathcal{T}(\mathcal{F}, \{x\})$ for any $y \in C[\,]$. Because $\mathcal{A}_T$ is complete and deterministic, for any $y \in C[\,]$ there is exactly one state q of $\mathcal{A}_T$ such that $y\theta' \xrightarrow{*}_{\mathcal{A}_T} q$. Since there exists $s \in \mathcal{T}(\mathcal{F}, \{x\})$ such that $\rho(s) \leq k_T$ and $s \xrightarrow{*}_{\mathcal{A}_T} q$ by pumping lemma [6], we define θ'' by $y\theta'' \equiv s$. Then it is obvious that $C\theta''[z]$ is accepted by $\mathcal{A}_T$. Thus, from Lemma 4.17, $C\theta''[\Omega_{NT_V}]$.

($\Leftarrow$) Trivial. □

Theorem 4.26: It is decidable whether $C[\Omega_{S_V}]$ for a given $C[\Omega]$.

Proof : It is decidable whether $C[\Omega_{I_V}]$ [3], [15]. If $C[\Omega_{NI_V}]$ then $C[\Omega_{NS_V}]$. Otherwise, by Lemma 4.25, it is suffices to check wether $C\theta[\Omega_{T_V}]$ for any θ such that $\rho(y\theta) \leq k_T$ and $y\theta \in \mathcal{T}(\mathcal{F}, \{x\})$ for any $y \in C[\,]$, which is also decidable. □

Example 4.27: Let

$$\mathcal{R} = \begin{cases} f(a, h(x), y) \rightarrow g(h(y), h(x)) \\ g(a, x) \rightarrow a \\ h(a) \rightarrow h(b) \\ b \rightarrow b. \end{cases}$$

The critical pair is only $\langle f(a, h(b), y),\ g(h(y), h(a)) \rangle$. $\mathcal{R}$ is NV-stable balanced joinable because we can show that $f(a, h(b), y) \rightarrow_{S_V} g(h(y), h(b)) \leftarrow_{S_V} g(h(y), h(a))$. Note that $\mathcal{R}$ is not stable balanced joinable. $\mathcal{R}$ is strongly sequential since $\mathcal{R}$ is left-normal system [16]. Thus, from Corollary 4.13, index reduction $\rightarrow_I$ is a normalizing strategy for $\mathcal{R}$.

5. Concluding Remarks

In this paper we show that (1) index reduction is normalizing for the class of stable balanced joinable strongly sequential systems and (2) NV-index reduction is normalizing for the class of NV-stable balanced joinable NV-sequential systems.

Stable and NV-stable balanced joinability properties are undecidable. It remains to indicate decidable subclasses.

It is not easy to generalize our results to more general sequential systems (NVNF-[13], shallow [3] and growing[8] sequential systems). Because index reduction is not balanced weakly Church-Rosser even if the system is orthogonal. For example, consider the following orthogonal NVNF-sequential system:

$$\mathcal{R} = \begin{cases} f(x) \to b \\ b \to g(b) \\ h(a) \to a. \end{cases}$$

Because $f(b)$ and b in $f(b)$ are indices with respect to NVNF-sequentiality, $f(b)$ reduce to b and $f(g(b))$ by index reduction. However b and $f(g(b))$ are not balanced joinable. In NVNF-sequential systems, two indices not being overlapping may nest.

Acknowledgment

This work is partially supported by Grants from Ministry of Education, Science and Culture of Japan, ♯07680350, ♯07680347 and ♯09245212.

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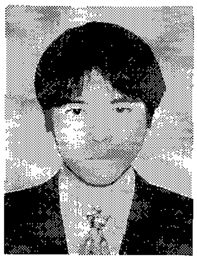

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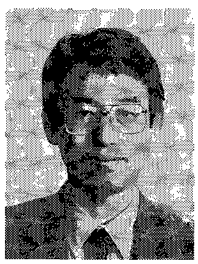

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