

PAPER

Robust On-Line Frequency Identification for a Sinusoid

Xinkai CHEN^{†a)}, Member, Guisheng ZHAI^{††}, and Toshio FUKUDA^{†††}, Nonmembers

SUMMARY This paper discusses the on-line frequency identification problem for a measured sinusoidal signal by using the adaptive method and filter theory. The proposed method is based on an identity between the sinusoidal signal and its second order derivative. For a set of chosen parameters, the proposed method is robust to the initial phase, the amplitude, and the frequency in a wide range. The convergence rate can be adjusted by the chosen parameters. The estimation error mainly depends on the frequency of the sinusoid, the measurement noise and a key design parameter.

key words: on-line frequency identification, sinusoidal signal, adaptive method, filter theory

1. Introduction

In this paper, the on-line frequency identification for a pure sinusoidal signal

$$n(t) = A \sin(\omega t + \phi), \quad (1)$$

is discussed, where $n(t)$ is measurable; the amplitude A , the frequency ω and the initial phase ϕ are all unknown. The *a priori* information about the lower bound of the amplitude, the lower bound of the frequency and upper bound of the frequency is known as A_0 , ω_0 and Ω , respectively.

On-line identification of a measured sinusoidal signal is a very fundamental problem in system theory. Cancelling sinusoidal disturbances in the controller design is discussed recently in [2], [11], [20]. It is well known that the estimation of frequency is the most essential problem in the identification of a sinusoidal signal. In the practical point of view, frequency identification has many applications in, for example, active and vibration control [6], helicopter [7], disk drives [1] and magnetic bearings [8]. In the theoretical point of view, frequency identification is also a challenging problem since the nonlinear dependence on the unknown frequency makes the application of standard well-known techniques very difficult. This paper will focus on the on-line frequency identification problem for a sinusoid.

Several algorithms have been reported until now to estimate the frequency of a sinusoid [2], [9]–[11], [13], [16],

[19]–[21]. To the authors' knowledge, the basic problem of designing a globally convergent estimator still remains open. A pioneer research for estimating the frequency on this subject may refer to [5], [16], [17], in which an adaptive notch filter (ANF) is proposed. Later, it is adapted for continuous-time case in paper [1]. Then, a global convergent estimator based on the ANF is reported in [9]. The estimator takes the following form:

$$\ddot{\zeta}(t) + 2\rho\hat{\omega}(t)\dot{\zeta}(t) + \hat{\omega}^2(t)\zeta(t) = \hat{\omega}^2(t)n(t), \quad (2)$$

$$\dot{\hat{\omega}}(t) = g(t)(2\rho\zeta(t) - \hat{\omega}(t)n(t))\zeta(t)\hat{\omega}(t), \quad (3)$$

$$g(t) = \frac{\epsilon}{\left\{1 + \gamma \left[\zeta^2(t) + \left(\frac{\dot{\zeta}(t)}{\hat{\omega}(t)} \right)^2 \right] \right\} (1 + \mu|\hat{\omega}(t)|^a)}, \quad (4)$$

where $a > 1$ and ϵ , γ , μ and ρ are positive reals. In the result [9], all the signals are globally bounded and the estimated frequency is asymptotically correct for all initial conditions and all frequency values. Furthermore, the tuning procedure for the design parameters is very simple and the transient performance is also enhanced. However, for a set of chosen parameters, the convergence rate is very sensitive to the frequency of the sinusoid. Recently, transient behavior of the ANF and its performance in the presence of noise are further discussed and modified in [4], [13], [19]. Except for the ANF method, several interesting methods based on adaptive observer theory are also reported recently in [11], [12], [14], [21], where the estimation methods of n frequencies are formulated in [12], [14], [21].

In this paper, a different approach of identifying the frequency of a sinusoid is given by using the adaptive method and filter theory. For a set of chosen parameters, the proposed method is robust to the initial phase, the amplitude, and the frequency in a wide range. The convergence rate can be adjusted by the design parameters. The estimation error mainly depends on the frequency of the sinusoid, the measurement noise and a key design parameter. In Sect. 2, the adaptive frequency identification algorithm is introduced. In Sect. 3, the parameter tuning and the robustness of the proposed algorithm are illustrated by computer simulations. Section 4 concludes this paper.

2. Proposed Formulation

First, the "filter" used in this note is defined. Let s denote the differential operator. For a constant $\kappa > 0$ and a signal $\eta(t)$, $\frac{1}{s+\kappa}\eta(t)$ is defined as the solution of the following differential

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[†]The author is with the Department of Electronic and Information Systems, Shibaura Institute of Technology, Saitama-shi, 337-8570 Japan.

^{††}The author is with the Department of Mechanical Engineering, Osaka Prefecture University, Sakai-shi, 599-8531 Japan.

^{†††}The author is with the Department of Mechano-Informatics and Systems, Nagoya University, Nagoya-shi, 464-8603 Japan.

a) E-mail: chen@sic.shibaura-it.ac.jp
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equation

$$\dot{\zeta}(t) + \kappa\zeta(t) = \eta(t), \quad \zeta(0) = 0. \quad (5)$$

Thus, the filter $\frac{h_2(s)}{h_1(s)}\eta(t)$ can be analogously defined, where $h_1(s)$ is a Hurwitz polynomial and $\frac{h_2(s)}{h_1(s)}$ is proper.

2.1 Frequency Identification Algorithm

For the sinusoidal signal $n(t) = A \sin(\omega t + \phi)$, it holds

$$\ddot{n}(t) + \omega^2 n(t) \equiv 0, \quad (6)$$

i.e.

$$(s^2 + \omega^2)n(t) \equiv 0. \quad (7)$$

Let us introduce a monic second order Hurwitz polynomial $(s + \lambda_1)^2$, where $\lambda_1 > 0$ is a design parameter. It can be easily seen that $\left| \frac{s^2 + \omega^2}{(s + \lambda_1)^2} n(t) \right| = \left| \frac{s^2}{(s + \lambda_1)^2} n(t) + \omega^2 \frac{1}{(s + \lambda_1)^2} n(t) \right|$ exponentially decreases to zero, i.e. there exist constants $p_1 \geq 0$ and $p_2 \geq 0$ (which depends on the parameter λ_1) such that

$$\left| \frac{s^2}{(s + \lambda_1)^2} n(t) + \omega^2 \frac{1}{(s + \lambda_1)^2} n(t) \right| \leq p_1 e^{-p_2 t}, \quad (8)$$

Relation (8) means that $-\frac{s^2}{(s + \lambda_1)^2} n(t)$ is the asymptotic estimate of $\omega^2 \frac{1}{(s + \lambda_1)^2} n(t)$.

Now, let us estimate the frequency ω . By using the filtered signals $\frac{s^2}{(s + \lambda_1)^2} n(t)$ and $\frac{1}{(s + \lambda_1)^2} n(t)$, the parameter ω is estimated by an adaptive method based on relation (8). Let the estimate of ω be $\hat{\omega}(t)$ at instant t . Deriving the following partial derivative yields

$$\begin{aligned} & \frac{\partial}{\partial \hat{\omega}(t)} \left(\omega^2 \frac{1}{(s + \lambda_1)^2} n(t) - \hat{\omega}^2(t) \frac{1}{(s + \lambda_1)^2} n(t) \right)^2 \\ &= -4\hat{\omega}(t) \left(\omega^2 \frac{1}{(s + \lambda_1)^2} n(t) - \hat{\omega}^2(t) \frac{1}{(s + \lambda_1)^2} n(t) \right) \frac{1}{(s + \lambda_1)^2} n(t). \end{aligned} \quad (9)$$

By referring to the gradient algorithm [15], [18] and the discontinuous techniques [3] and replacing $\omega^2 \frac{1}{(s + \lambda_1)^2} n(t)$ with $-\frac{s^2}{(s + \lambda_1)^2} n(t)$, the adaptive law of estimating ω is constructed by

$$\begin{aligned} \dot{\hat{\omega}}(t) &= -\Gamma(t) \hat{\omega}(t) \left(\hat{\omega}^2(t) \frac{1}{(s + \lambda_1)^2} n(t) + \frac{s^2}{(s + \lambda_1)^2} n(t) \right) \\ &\quad \times \frac{1}{(s + \lambda_1)^2} n(t), \end{aligned} \quad (10)$$

$$\hat{\omega}(t_k + 0) = \begin{cases} \Omega & \text{if } \hat{\omega}(t_k) \geq 2\Omega \\ \omega_0 & \text{if } \hat{\omega}(t_k) \leq 0.5\omega_0 \end{cases}, \quad (11)$$

where the initial condition $\hat{\omega}(0)$ meets $\Omega \geq \hat{\omega}(0) \geq \omega_0 > 0$; t_k is defined as

$$t_k = \min\{t : t > t_{k-1}, \hat{\omega}(t) \leq 0.5\omega_0 \text{ or } \hat{\omega}(t) \geq 2\Omega\}, \quad (12)$$

and $t_0 = 0$. $\Gamma(t) > 0$ is a time varying tuning parameter which will be given in Sect. 2.2.

Remark 1: Since the algorithm (10)–(12) is constructed based on the exponential convergence of $\left(\frac{s^2}{(s + \lambda_1)^2} n(t) + \omega^2 \frac{1}{(s + \lambda_1)^2} n(t) \right)$ to zero, the parameter λ_1 should not be chosen small in order to get fast convergence of $\hat{\omega}(t)$. Generally, λ_1 should be chosen as $\lambda_1 \geq 1$.

Remark 2: t_k 's defined in (12) are the discontinuous points of the system (10)–(11). By observing (11) and (12), it can be easily seen that $2\Omega \geq \hat{\omega}(t) \geq 0.5\omega_0$.

2.2 Formulation of the Time Varying Parameter $\Gamma(t)$

It is ideal that the proposed frequency identification algorithm (10)–(12) could cope with sinusoids with arbitrary amplitudes. It can be argued that if the amplitude of the considered sinusoid can be estimated, then the amplitude of the sinusoid can be compensated in the frequency identification algorithm. For this purpose, let us estimate the amplitude of the measured sinusoid.

For the signal $n(t) = A \sin(\omega t + \phi)$, it also holds

$$\sqrt{\left(\frac{\dot{n}(t)}{\omega} \right)^2 + n^2(t)} = A. \quad (13)$$

It can be guessed that the parameter A can be estimated by using the estimate of ω and the estimate of the first order derivative of $n(t) = A \sin(\omega t + \phi)$. For this purpose, a method of estimating the first order derivative of $n(t) = A \sin(\omega t + \phi)$ will be firstly considered.

Introduce a Hurwitz polynomial $s + \lambda_2$, where $\lambda_2 > 0$ is a design parameter. From (7), it can be seen that $\left| \frac{s^2 + \omega^2}{s + \lambda_2} n(t) \right| = \left| \dot{n}(t) - \lambda_2 n(t) + \frac{\lambda_2^2 + \omega^2}{s + \lambda_2} n(t) \right|$ exponentially decreases to zero, i.e. there exists a constant $p_3 \geq 0$ such that

$$\left| \dot{n}(t) - \lambda_2 n(t) + \frac{\lambda_2^2 + \omega^2}{s + \lambda_2} n(t) \right| \leq p_3 e^{-\lambda_2 t}. \quad (14)$$

Define

$$w(t) \equiv \lambda_2 n(t) - \left(\lambda_2^2 + \hat{\omega}^2(t) \right) \frac{1}{s + \lambda_2} n(t). \quad (15)$$

It can be easily seen that $w(t)$ and $\frac{w(t)}{\hat{\omega}(t)}$ are uniformly bounded by observing Remark 2. Thus, $\sqrt{\left(\frac{w(t)}{\hat{\omega}(t)} \right)^2 + n^2(t)}$ is also uniformly bounded. If $\hat{\omega}(t)$ is very close to ω , it can be concluded that $w(t)$ is also very close to $\dot{n}(t)$ and $\sqrt{\left(\frac{w(t)}{\hat{\omega}(t)} \right)^2 + n^2(t)}$ can be regarded as the estimate of the amplitude A . Since the measurement noise usually exists, it will directly influence the accuracy of the estimation of the amplitude. Furthermore, $\sqrt{\left(\frac{w(t)}{\hat{\omega}(t)} \right)^2 + n^2(t)}$ may take very small values at the transient stage of the estimation. In order to get a good estimation, the estimate $\hat{A}(t)$ of the amplitude A is given by

$$\hat{A}_1(t) = \lambda_3 \left\{ \sqrt{\left(\frac{w(t)}{\hat{\omega}(t)}\right)^2 + n^2(t)} - \hat{A}_1(t) \right\}, \quad (16)$$

$$\hat{A}(t) = \begin{cases} \hat{A}_1(t) & \text{if } \hat{A}_1(t) \geq 0.5A_0 \\ 0.5A_0 & \text{otherwise} \end{cases}, \quad (17)$$

where $\lambda_3 > 0$ is a design parameter; the initial condition $\hat{A}_1(0)$ meets $\hat{A}_1(0) \geq A_0$.

Remark 3: It can be simply seen that $\hat{A}(t)$ is uniformly bounded and $\hat{A}(t) \geq 0.5A_0$.

Upon the above preparation, let us define the time varying parameter $\Gamma(t)$ as

$$\Gamma(t) = \frac{\alpha_1}{\hat{A}^2(t)} (\hat{\omega}^\beta(t) + \alpha_2), \quad (18)$$

where α_i ($i = 1, 2$) and β are non-negative design parameters.

Remark 4: The parameters $\alpha_1 > 0$ (which is usually chosen to be very large) and $\alpha_2 \geq 0$ are introduced to speed up the convergence rate. $\hat{A}^2(t)$ is introduced in the denominator of $\Gamma(t)$ in order to adjust the convergence rate for different amplitudes. $\hat{\omega}^\beta(t)$ (with $\beta > 0$) is introduced in the numerator of $\Gamma(t)$ in order to adjust the convergence rate for different frequencies.

Remark 5: $\Gamma(t)$ is positive and there exist positive constants $\Gamma_1 > 0$ and $\Gamma_2 > 0$ such that

$$0 < \Gamma_1 \leq \Gamma(t) \leq \Gamma_2. \quad (19)$$

2.3 Main Theoretical Results

Define

$$e_\omega(t) = \hat{\omega}(t) - \omega, \quad e_A(t) = \hat{A}(t) - A. \quad (20)$$

If a noise $\sigma(t)$ is accompanied with the sinusoidal signal, the measured signal can be expressed as $N(t) = n(t) + \sigma(t)$. The signal $n(t)$ in the proposed algorithm should be replaced by $N(t)$. Suppose

$$\left| \left(\frac{s^2}{(s + \lambda_1)^2} \sigma(t) + \omega^2 \frac{1}{(s + \lambda_1)^2} \sigma(t) \right) \right| \leq \delta_1, \quad (21)$$

$$\left| \frac{1}{(s + \lambda_1)^2} \sigma(t) \right| \leq \delta_2, \quad |\sigma(t)| \leq \delta_3, \quad (22)$$

where δ_3 is smaller than A . The next theorem gives the upper bounds of $|e_\omega(t)|$ and $|e_A(t)|$.

Theorem 1: Consider a sinusoid with frequency $\omega \in [\omega_0, \Omega]$ and amplitude $A \geq A_0$ and an additional noise $\sigma(t)$. If δ_1 and δ_2 are small enough, then, for given initial conditions $\hat{\omega}(0) \in [\omega_0, \Omega]$ and $\hat{A}(0) \geq A_0$, and a set of chosen design parameters, there exist positive constants T, K_ω, K_{A_1} and K_{A_2} such that $|e_\omega(t)|$ and $|e_A(t)|$ exponentially decrease as $t > T$ until $|e_\omega(t)| \leq K_\omega \delta_1$ and $|e_A(t)| \leq K_{A_1} \delta_1 + K_{A_2} \delta_3$.

Proof: The proof is given in the Appendix.

Remark 6: By observing the proof of Theorem 1, relation (A·4) should be satisfied for a small T_1 in order to get a fast convergence. This means that $\hat{\omega}(0)$ should not be chosen very small.

Remark 7: Theorem 1 means that the estimation errors of $|e_\omega(t)|$ and $|e_A(t)|$ depend on the noise, the value of λ_1 and the frequency of the sinusoid.

Remark 8: From (A·5) in the proof of Theorem 1, it can be seen that the error $e_\omega(t)$ is dominated by the term $\left(\frac{s^2}{(s + \lambda_1)^2} \sigma(t) + \omega^2 \frac{1}{(s + \lambda_1)^2} \sigma(t) \right)$. Since this term appears in the integral, it can be seen that the high frequency components in the noise can be cut off. Furthermore, for a certain noise $\sigma(t)$, the amplitude of $\left| \left(\frac{s^2}{(s + \lambda_1)^2} \sigma(t) + \omega^2 \frac{1}{(s + \lambda_1)^2} \sigma(t) \right) \right|$ can be reduced by choosing large λ_1 if the frequency ω is large.

Remark 9: By observing (A·14) in the proof of Theorem 1, it can be seen that high frequency components in the noise can be cut off in the estimation of the amplitude.

Remark 10: From Remark 8 and Remark 9, it can be seen that only the low frequency components in the noise affect the estimation precisions of the frequency and amplitude.

Remark 11: In the absence of noise, Theorem 1 also implies that $|e_\omega(t)|$ and $|e_A(t)|$ exponentially decrease to zero as $t > T$.

3. Tuning of the Parameters and Simulations

First of all, let us give a general guideline of the parameters in $\Gamma(t) = \frac{\alpha_1}{\hat{A}^2(t)} (\hat{\omega}^\beta(t) + \alpha_2)$. $\alpha_1 > 0$ should be chosen to be very large and α_2 should be non-negative in order to speed up the convergence rate of $\hat{\omega}(t)$. The parameter $\beta > 0$ is usually chosen as 1 or 0.5. These three parameters will be further discussed in the following.

Generally, the parameter λ_1 should be chosen as $\lambda_1 \geq 1$. Practically, the parameter λ_1 should be determined by observing the amplitude of the noise and the frequency of the sinusoid. This will be illustrated by the following discussions.

Now, let us give a guideline for the choice of the parameters λ_2 and λ_3 .

- In the presence of measurement noise $\sigma(t)$, since $\left(\lambda_2 n(t) - \frac{\lambda_2^2 + \hat{\omega}^2(t)}{s + \lambda_2} n(t) \right) + \left(\lambda_2 \sigma(t) - \frac{\lambda_2^2 + \hat{\omega}^2(t)}{s + \lambda_2} \sigma(t) \right)$ is used to estimate the derivative $\dot{n}(t)$, it is ideal that $\lambda_2 \sigma(t) - \frac{\lambda_2^2 + \hat{\omega}^2(t)}{s + \lambda_2} \sigma(t)$ is very small. Because a large λ_2 may enlarge the amplitude of $\lambda_2 \sigma(t) - \frac{\lambda_2^2 + \hat{\omega}^2(t)}{s + \lambda_2} \sigma(t)$, λ_2 should not be chosen to be large. On the other hand, from (14), in order to get a fast convergence of $\left| \dot{n}(t) - \lambda_2 n(t) + \frac{\lambda_2^2 + \omega^2}{s + \lambda_2} n(t) \right|$, λ_2 should be chosen as $\lambda_2 \geq 1$.
- Since (16) gives a low-pass filter of $\sqrt{\left(\frac{w(t)}{\hat{\omega}(t)}\right)^2 + n^2(t)}$, it is obvious that λ_3 should not be chosen to be large.

On the other hand, in order to get a fast convergence of $\hat{A}_1(t)$, λ_3 should be chosen as $\lambda_3 \geq 1$.

In the following simulations, the *a priori* information of the parameters ω_0 , Ω and A_0 is assumed as $\omega_0 = 0.05$, $\Omega = 500$ and $A_0 = 0.04$. The initial condition is set to $\hat{\omega}(0) = 1$ and $\hat{A}(0) = 0.5$ (see Remark 6). The parameters λ_2 and λ_3 are simply fixed to $\lambda_2 = \lambda_3 = 2$. The simulations are done by using the software Simulink in Matlab. The sampling period is set to 0.0002.

In the simulation study, we are interested in the relative errors $\frac{\hat{\omega}(t)-\omega}{\omega}$ and $\frac{\hat{A}(t)-A}{A}$. For simplicity, define

$$E_\omega(t) = \frac{\hat{\omega}(t) - \omega}{\omega}, \quad E_A(t) = \frac{\hat{A}(t) - A}{A}. \quad (23)$$

3.1 Discussion in the Absence of Noises

For the sinusoid $n(t)$ without noise, simulations are carried out. In this subsection, the convergence is in the sense that there exists an instant $\bar{t} \geq 0$ such that for $|E_\omega(t)| \leq 0.002$ for $t \in [\bar{t}, \infty)$.

First, let us choose $\lambda_2 = 2$. The parameters in $\Gamma(t)$ are chosen as $\alpha_1 = 2 \times 10^4$, $\alpha_2 = 0.2$, $\beta = 1$.

For arbitrary frequencies $0.05 \leq \omega \leq 500$, amplitudes $A \geq 0.04$ and initial phases $\phi \in [0, 2\pi)$, the exponential convergences are observed and the steady errors are zeros for all cases. The convergence rate does not depend on the initial phase.

For $0.05 \leq \omega \leq 120$, the convergence rate almost does not depend on the amplitude A . The convergence rate mainly depends on the frequency ω .

- For $0.05 \leq \omega \leq 0.5$, it is observed that the minimum value of \bar{t} varies from 10 to 7.5.
- For $0.5 \leq \omega \leq 120$, the convergence rates do not differ so much and the convergences are confirmed after $t \geq 7.5$.

If the frequency ω increases from 120, the convergence rate decreases especially for the sinusoids with amplitudes smaller than one. In order to speed up the convergence rate for the sinusoids with high frequencies and small amplitudes, the parameters in $\Gamma(t)$ are changed to $\alpha_1 = 1 \times 10^4$, $\alpha_2 = 120$, $\beta = 0.5$. Simulation results show that, for $14 \leq \omega \leq 500$, the convergence rate almost does not depend on the frequency ω , the amplitude A and initial phase ϕ . The convergences are confirmed after $t \geq 7.5$. However, for this set of chosen parameters, the convergence for the sinusoids with frequencies $\omega < 13$ could not be observed in the time period $[0, 15]$.

Remark 12: For $1 \leq \omega \leq 500$, the convergence rate almost does not depend on the parameter λ_1 . However, for $0.05 \leq \omega < 1$, the convergence rate becomes slower when λ_1 is chosen larger.

Note 1: The characteristic of the convergence rates of the amplitude errors is similar to that of the frequency errors.

Remark 13: As stated in [9], if the parameters of the ANF given in (2)–(4) are chosen as $\rho = 0.4$, $\epsilon = 1.35$, $\gamma = 1$, $\mu = 0.5$, $a = 2$ and the initial conditions are chosen as $[1, 1, 10]$, then, for $n(t) = 100 \sin(5t)$, the convergence is observed after $t \geq 10.5$. However, for $n(t) = 100 \sin(\omega t)$ with frequencies $\omega = 0.05, 0.5, 2, 10, 20, 30, 100, 300$, the corresponding minimum values of \bar{t} are observed as 700, 69, 20, 35, 85, 140, 193, and 578. By choosing a larger ϵ , the convergence rate can be speeded up for high frequency sinusoids. However, the convergence for low frequency sinusoids may not exist (If ϵ is enlarged as $\epsilon = 13.5$, the convergence rates for $\omega \geq 5$ can be significantly speeded up, however the convergence for $\omega \leq 3$ could not be observed during the simulation time interval $[0, 15000]$). Practically, a very good *a priori* knowledge about the range of the frequency may be required in the determination of the parameter ϵ .

3.2 Discussion in the Presence of Noises

In the presence of noises, the steady estimation error may still exist. In this subsection, the convergence is in the sense that there exists an instant $\bar{t} \geq 0$ such that $|E_\omega(t)| \leq 0.05$ for $t \in [\bar{t}, \infty)$.

In order to test the robustness of the proposed algorithm, suppose a random noise $\sigma(t)$ which varies between $-0.3A$ and $0.3A$ (where A is the amplitude of the sinusoid) is accompanied with the measured sinusoid. The simulations are done in the time interval $0 \leq t \leq 15$. The estimation error in the following is derived in the time interval $10 \leq t \leq 15$.

To begin with, the parameters in $\Gamma(t)$ are chosen as $\alpha_1 = 2 \times 10^4$, $\alpha_2 = 0.2$ and $\beta = 1$ (same as those in the absence of noises stated in the beginning of Sect. 3.1).

First, choose $\lambda_1 = 2$. In the occurrence of convergence, it is confirmed by computer simulations that

- the characteristic of the convergence rate is similar to that in the case without noises;
- the estimation error $e_\omega(t)$ almost does not depend on the amplitude A and initial phase ϕ .

In the following, without loss of generality, let us focus on the occurrence of convergence and the estimation error. For the sinusoid in the form $n(t) = \sin(\omega t + 0.3\pi)$, Fig. 1

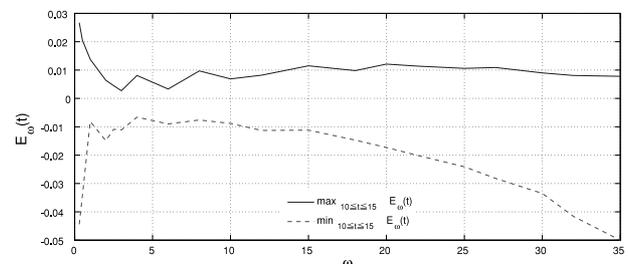


Fig. 1 The variation extent of $E_\omega(t)$ for different frequency ω when $\lambda_1 = 2$.

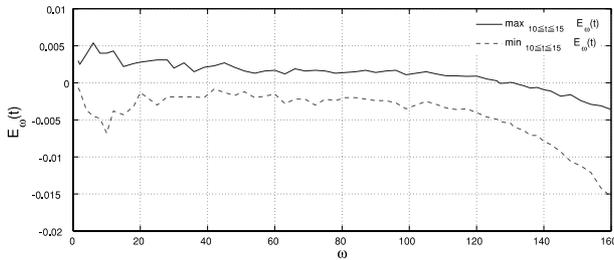


Fig. 2 The variation extent of $E_\omega(t)$ for different frequency ω when $\lambda_1 = 2$.

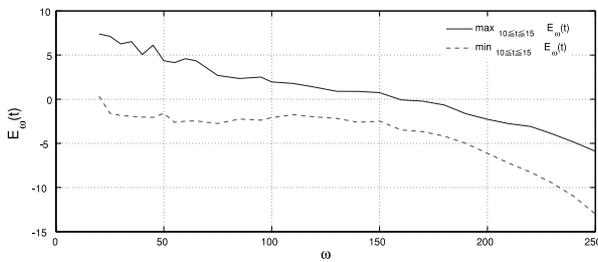


Fig. 3 The variation extent of $E_\omega(t)$ for different frequency ω when $\lambda_1 = 100$.

shows the variation extent of $E_\omega(t)$ for different frequency ω . The upcoming two figures are also plotted for the sinusoids in this form. Figure 1 means that, if λ_1 is chosen as $\lambda_1 = 2$, then a good estimation can be obtained for the sinusoids with frequencies $1.5 \leq \omega \leq 20$. It should be noted that, if the random noise varies within $\pm 1\%$ of the sinusoid amplitude, then the algorithm can also cope with frequencies in the range $0.05 \leq \omega \leq 100$.

Now, let us choose λ_1 as $\lambda_1 = 20$. Figure 2 shows the variation extent of $E_\omega(t)$ for different frequency ω . Figure 2 tells us that, if λ_1 is chosen as $\lambda_1 = 20$, then a very good estimation can be obtained for the sinusoids with frequencies $1.5 \leq \omega \leq 120$. However, it can not cope with the sinusoids with frequencies $\omega \leq 1$.

Remark 14: By observing Fig. 1 and Fig. 2, it can be seen that the performance of the proposed algorithm is improved by choosing a larger λ_1 for the frequency range $1.5 \leq \omega \leq 120$. This also supports Theorem 1 and Remark 7.

If a much better estimation for the frequency in the range $\omega > 120$ is required, the parameter λ_1 should be chosen larger than 20. In order to get a uniform convergence rate, the parameters in $\Gamma(t)$ should be chosen as stated in Sect. 3.1, i.e. $\alpha_1 = 1 \times 10^4$, $\alpha_2 = 120$, $\beta = 0.5$. Figure 3 shows the variation extent of $E_\omega(t)$ for different ω when $\lambda_1 = 100$. Furthermore, by choosing $\lambda_1 = 300$, a very good estimation is obtained for the frequency range $180 \leq \omega \leq 500$. The performance is very similar to that in Fig. 3.

Remark 15: For the low frequencies $0.05 \leq \omega \leq 1.5$, if a much better estimation is required, the convergence rate should be slowed down by considering the trade-off relation between the estimation error and the convergence speed. By

choosing $\lambda_1 = 1$, $\alpha_1 = 2 \times 10^3$, $\alpha_2 = 0$ and $\beta = 1$, a relatively good estimation is confirmed by computer simulations in the time period $0 \leq t \leq 40$.

Remark 16: No matter what value λ_1 is chosen in the range $\lambda_1 \geq 1$, in the occurrence of convergence,

- the characteristic of the convergence rate is similar to that in the case without noises;
- the steady error of $e_\omega(t)$ almost does not depend on the amplitude A and the initial phase ϕ ;
- the steady error of $e_\omega(t)$ is dominated by the frequency ω , the parameter λ_1 and the noise $\sigma(t)$;
- a chosen parameter λ_1 ($\lambda_1 \geq 1$) can cope with a relatively wide range of frequencies;
- there is no steady bias in the estimation error $e_\omega(t)$.

Note 2: The characteristic of the relation between the error $E_A(t)$ and the frequency ω is very similar to that of the relation between $E_\omega(t)$ and ω shown in Figs. 1–3. Corresponding to Fig. 1, $E_A(t)$ (for $10 \leq t \leq 15$) has no steady bias and varies in the interval $[-0.0031, 0.0054]$ for the sinusoids with frequencies in the range $2 \leq \omega \leq 25$. Corresponding to Fig. 2, $E_A(t)$ (for $10 \leq t \leq 15$) has no steady bias and varies in the interval $[-0.0043, 0.0064]$ for the sinusoids with frequencies in the range $1.5 \leq \omega \leq 120$. Corresponding to Fig. 3, $E_A(t)$ (for $10 \leq t \leq 15$) has no steady bias and varies in the interval $[-0.0041, 0.0064]$ for the sinusoids with frequencies in the range $20 \leq \omega \leq 180$. Furthermore, correspondingly, for the sinusoids with frequencies in the range $180 \leq \omega \leq 500$, $E_A(t)$ (for $10 \leq t \leq 15$) has no steady bias and varies in the interval $[-0.0044, 0.0066]$ when the parameters are chosen as $\lambda_1 = 300$, $\alpha_1 = 1 \times 10^4$, $\alpha_2 = 120$ and $\beta = 0.5$.

4. Conclusion

A new method of identifying the frequency of a sinusoidal signal is proposed in this paper based on an identity between the sinusoidal signal and its second order derivative. The amplitude of the sinusoid is simultaneously estimated. Several parameters need to be determined in the application of the proposed algorithm. The choice of the parameters is discussed in Sect. 3. The parameter λ_1 is the key design parameter. In the presence of noises with large amplitudes, the parameter λ_1 should be chosen large for high frequency sinusoids. Furthermore, the parameters in $\Gamma(t)$ should be changed for very high or very low frequency sinusoids in order to adjust the convergence rate. An *a priori* information about the range of the frequency is helpful in the determination of λ_1 and the parameters in $\Gamma(t)$.

For a set of chosen parameters, the proposed method is robust to the initial phase, the amplitude, and the frequency in a wide range. The convergence rate can be adjusted by the chosen parameters. The estimation error mainly depends on the frequency of the sinusoid, the parameter λ_1 and the measurement noise.

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Appendix: Proof of Theorem 1

Step 1: Proof of the convergence of $e_\omega(t)$

Let

$$\begin{aligned} \xi(t) = & \Gamma(t) \left(\hat{\omega}^2(t) + \omega \hat{\omega}(t) \right) \left(\frac{1}{(s + \lambda_1)^2} n(t) \right)^2 \\ & + \Gamma(t) \left(\frac{s^2}{(s + \lambda_1)^2} n(t) + \omega^2 \frac{1}{(s + \lambda_1)^2} n(t) \right) \frac{1}{(s + \lambda_1)^2} n(t). \end{aligned} \quad (\text{A} \cdot 1)$$

From (8), (19) and Remark 2, it is obvious that there exist constants $T_0 \geq 0$, $p_4 > 0$ and $p_5 > 0$ such that $\xi(t) \geq 0$ and

$$\exp \left[- \int_t^t \xi(\tau) d\tau \right] \leq p_4 e^{-p_5(t-t)}, \quad (\text{A} \cdot 2)$$

for $t \geq t \geq T_0$.

On the time interval $[t_{i-1}, t_i]$ defined by (12), $\hat{\omega}(t)$ is differentiable. Now, based on (10), differentiating $e_\omega(t)$ yields

$$\begin{aligned} e'_\omega(t) = & -\Gamma_N(t) \hat{\omega}(t) \\ & \times \left(\frac{s^2}{(s + \lambda_1)^2} N(t) + \hat{\omega}^2(t) \frac{1}{(s + \lambda_1)^2} N(t) \right) \frac{1}{(s + \lambda_1)^2} N(t) \\ = & -\Gamma_N(t) \hat{\omega}(t) \\ & \times \left(\frac{s^2}{(s + \lambda_1)^2} N(t) + \omega^2 \frac{1}{(s + \lambda_1)^2} N(t) \right) \frac{1}{(s + \lambda_1)^2} N(t) \\ & - \Gamma_N(t) \hat{\omega}(t) \left(\hat{\omega}^2(t) - \omega^2 \right) \left(\frac{1}{(s + \lambda_1)^2} N(t) \right)^2 \\ = & \Gamma_N(t) e_\omega(t) \left\{ \left(\hat{\omega}^2(t) + \omega \hat{\omega}(t) \right) \left(\frac{1}{(s + \lambda_1)^2} N(t) \right)^2 \right. \\ & \left. + \left(\frac{s^2}{(s + \lambda_1)^2} N(t) + \omega^2 \frac{1}{(s + \lambda_1)^2} N(t) \right) \frac{1}{(s + \lambda_1)^2} N(t) \right\} \\ & - \Gamma_N(t) \omega \left(\frac{s^2}{(s + \lambda_1)^2} N(t) + \omega^2 \frac{1}{(s + \lambda_1)^2} N(t) \right) \frac{1}{(s + \lambda_1)^2} N(t) \\ = & -\xi_N(t) e_\omega(t) \\ & - \Gamma_N(t) \omega \left(\frac{s^2}{(s + \lambda_1)^2} N(t) + \omega^2 \frac{1}{(s + \lambda_1)^2} N(t) \right) \frac{1}{(s + \lambda_1)^2} N(t), \end{aligned} \quad (\text{A} \cdot 3)$$

where $\Gamma_N(t)$ and $\xi_N(t)$ are respectively obtained from $\Gamma(t)$ and $\xi(t)$ by replacing $n(t)$ with $N(t)$. The uniform boundedness of $\Gamma_N(t)$ can be easily checked. If δ_1 and δ_2 are small enough, then, similar to (A·2), there exist $T_1 \geq 0$, $p_6 > 0$ and $p_7 > 0$ such that $\xi_N(t) \geq 0$ and

$$\exp \left[- \int_t^t \xi_N(\tau) d\tau \right] \leq p_6 e^{-p_7(t-t)}, \quad (\text{A} \cdot 4)$$

for $t \geq t \geq T_1$. Solving (A·3) yields

$$\begin{aligned} e_\omega(t) = & e^{-\int_{T_1}^t \xi_N(\tau) d\tau} e_\omega(T_1) \\ & - \int_{T_1}^t e^{-\int_t^\tau \xi_N(\tau) d\tau} \left(\omega \Gamma_N(t) \left(\frac{s^2}{(s + \lambda_1)^2} N(t) \right) \right. \\ & \left. + \frac{\omega^2}{(s + \lambda_1)^2} N(t) \right) \frac{1}{(s + \lambda_1)^2} N(t) dt. \end{aligned} \quad (\text{A} \cdot 5)$$

On the other hand, from relations (8) and (21), it gives

$$\left| \frac{s^2}{(s + \lambda_1)^2} N(t) + \omega^2 \frac{1}{(s + \lambda_1)^2} N(t) \right| \leq p_1 e^{-p_2 t} + \delta_1. \tag{A·6}$$

Thus, by applying (A·4), (A·6) and the uniform boundedness of $\Gamma_N(t)$ to Eq. (A·5), it can be seen that there exists a constant $p_8 > 0$ such that

$$\begin{aligned} |e_\omega(t)| &\leq p_6 e^{-p_7(t-T_1)} |e_\omega(T_1)| \\ &\quad + p_8 \int_{T_1}^t e^{-p_7(t-l)} (p_1 e^{-p_2 l} + \delta_1) dl \\ &= p_6 e^{-p_7(t-T_1)} |e_\omega(T_1)| + \frac{p_8 \delta_1}{p_7} (1 - e^{-p_7(t-T_1)}) \\ &\quad + \frac{p_8 p_1}{p_7 - p_2} (e^{-p_2 t} - e^{-p_7 t} \cdot e^{(p_7 - p_2) T_1}) \end{aligned} \tag{A·7}$$

Thus, $|e_\omega(t)|$ decreases exponentially as $t \geq T_1$ if it does not satisfy $|e_\omega(t)| \leq \frac{p_8 \delta_1}{p_7}$.

By the boundedness of the right hand side of (10), it follows that $t_i - t_{i-1}$ are greater than a positive constant, say χ , for all i . On every such a time interval $[t_{i-1}, t_i]$, the variable $|e_\omega(t)|$ decreases exponentially and, at t_i , the estimation error $e_\omega(t)$ satisfies $|e_\omega(t_i + 0)| < |e_\omega(t_i - 0)|$. Thus, $|e_\omega(t)|$ is bounded and there exist constants $K_\omega > 0, T > 0, p_9 > 0$ and $p_{10} > 0$ such that

$$|e_\omega(t)| \leq K_\omega \delta_1 + p_9 e^{-p_{10} t}, \tag{A·8}$$

for $t \leq T$. Thus, the result about $|e_\omega(t)|$ is proved.

Step 2: Proof of the convergence of $e_{A_1}(t)$

Define

$$e_{A_1}(t) = \hat{A}_1(t) - A. \tag{A·9}$$

Based on (16), differentiating $e_{A_1}(t)$ yields

$$\begin{aligned} \dot{e}_{A_1}(t) &= \lambda_3 \left(\sqrt{\left(\frac{w_N(t)}{\hat{\omega}(t)} \right)^2 + N^2(t)} - \hat{A}_1(t) \right) \\ &= -\lambda_3 e_{A_1}(t) + \lambda_3 \left(\sqrt{\left(\frac{w_N(t)}{\hat{\omega}(t)} \right)^2 + N^2(t)} - A \right), \end{aligned} \tag{A·10}$$

where $w_N(t)$ is the signal obtained from $w(t)$ (defined in (15)) by replacing $n(t)$ with $N(t)$.

On the other hand, we have

$$\begin{aligned} &\sqrt{\left(\frac{w_N(t)}{\hat{\omega}(t)} \right)^2 + N^2(t)} - A \\ &= \sqrt{\left(\frac{w_N(t)}{\hat{\omega}(t)} \right)^2 + N^2(t)} - \sqrt{\left(\frac{\dot{n}(t)}{\omega} \right)^2 + n^2(t)} \\ &= \frac{1}{\sqrt{\left(\frac{w_N(t)}{\hat{\omega}(t)} \right)^2 + N^2(t)} + A} \end{aligned}$$

$$\begin{aligned} &\times \left((\omega(w_N(t) - \dot{n}(t)) - \dot{n}(t)(\hat{\omega}(t) - \omega)) \frac{1}{\omega \hat{\omega}(t)} \left(\frac{w_N(t)}{\hat{\omega}(t)} + \frac{\dot{n}(t)}{\omega} \right) \right. \\ &\quad \left. + \sigma(t)(N(t) + n(t)) \right). \end{aligned} \tag{A·11}$$

By the definition of $w(t)$ and $w_N(t)$, the difference $w_N(t) - \dot{n}(t)$ can be expressed as

$$\begin{aligned} w_N(t) - \dot{n}(t) &= w_N(t) - w(t) - (\dot{n}(t) - w(t)) \\ &= \lambda_2 N(t) - (\lambda_2^2 + \hat{\omega}^2(t)) \frac{1}{s + \lambda_2} N(t) \\ &\quad - \left(\lambda_2 n(t) - (\lambda_2^2 + \omega^2) \frac{1}{s + \lambda_2} n(t) \right) \\ &\quad - \left(\dot{n}(t) - \lambda_2 n(t) + \frac{\lambda_2^2 + \omega^2}{s + \lambda_2} n(t) \right) \\ &= \lambda_2 \sigma(t) - (\lambda_2^2 + \omega^2) \frac{1}{s + \lambda_2} \sigma(t) \\ &\quad - (\hat{\omega}^2(t) - \omega^2) \frac{1}{s + \lambda_2} N(t) \\ &\quad - \left(\dot{n}(t) - \lambda_2 n(t) + \frac{\lambda_2^2 + \omega^2}{s + \lambda_2} n(t) \right). \end{aligned} \tag{A·12}$$

By substituting (A·12) into (A·11), it can be seen that there exist time-varying bounded functions $P_i(t)$ ($i=1,2,3,4$) such that

$$\begin{aligned} \sqrt{\left(\frac{w_N(t)}{\hat{\omega}(t)} \right)^2 + N^2(t)} - A &= P_1(t) \sigma(t) + P_2(t) \frac{1}{s + \lambda_2} \sigma(t) \\ &\quad + P_3(t) (\hat{\omega}(t) - \omega) + P_4(t), \end{aligned} \tag{A·13}$$

where $P_4(t)$ converges to zero exponentially (see relation (15)). Now, solving Eq. (A·10) and applying (A·13) gives

$$\begin{aligned} e_{A_1}(t) &= e^{-\lambda_3(t-T)} e_{A_1}(T) \\ &\quad + \lambda_3 \int_T^t \left(P_1(\tau) \sigma(\tau) + P_2(\tau) \frac{1}{s + \lambda_2} \sigma(\tau) \right) e^{\lambda_3(\tau-t)} d\tau \\ &\quad + \lambda_3 \int_T^t (P_3(\tau) (\hat{\omega}(\tau) - \omega) + P_4(\tau)) e^{\lambda_3(\tau-t)} d\tau. \end{aligned} \tag{A·14}$$

By applying the results of Step 1 to (A·13), there exist positive constants K_{A_1}, K_{A_2}, p_{11} and p_{12} such that

$$\left| \sqrt{\left(\frac{w_N(t)}{\hat{\omega}(t)} \right)^2 + N^2(t)} - A \right| \leq K_{A_1} \delta_1 + K_{A_2} \delta_3 + p_{11} e^{-p_{12} t}, \tag{A·15}$$

as $t > T$. By observing (A·13) and applying (A·15) to (A·14), it gives

$$\begin{aligned}
|e_{A_1}(t)| &\leq e^{-\lambda_3(t-T)}|e_{A_1}(T)| \\
&\quad + \lambda_3 e^{-\lambda_3 t} \int_T^t (K_{A_1} \delta_1 + K_{A_2} \delta_3 + p_{11} e^{-p_{12} \tau}) e^{\lambda_3 \tau} d\tau \\
&= e^{-\lambda_3(t-T)}|e_{A_1}(T)| + (K_{A_1} \delta_1 + K_{A_2} \delta_3)(1 - e^{\lambda_3(t-T)}) \\
&\quad + \frac{\lambda_3 p_{11}}{\lambda_3 - p_{12}} (e^{-p_{12} t} - e^{-\lambda_3(t-T)} e^{-p_{12} T}). \quad (\text{A} \cdot 16)
\end{aligned}$$

Thus, $|e_{A_1}(t)|$ exponentially decreases as $t \geq T$ until $|e_{A_1}(t)| \leq K_{A_1} \delta_1 + K_{A_2} \delta_3$. Since $|e_A(t)| \leq |e_{A_1}(t)|$, the theorem is proved.



Xinkai Chen received the B.S. and M.S. degrees in mathematics from Hebei University, China, in 1986 and 1989, respectively, the M.E. degree in electronic engineering from Mie University, Japan, in 1995, and the Ph.D. degree in engineering from Nagoya University, Japan, in 1999. He held faculty positions at North China Electric Power University, China (1989–1992), Mie University, Japan (1995–1999), Tokyo Denki University, Japan (1999–2001) and Kinki University, Japan (2001–2004). He is currently an Associate Professor in the Department of Electronic and Information Systems, Shibaura Institute of Technology, Japan. His research interests include robust nonlinear control, adaptive control, motion control and machine vision.

He is currently an Associate Professor in the Department of Electronic and Information Systems, Shibaura Institute of Technology, Japan. His research interests include robust nonlinear control, adaptive control, motion control and machine vision.



Guisheng Zhai received the B.S. degree from Fudan University, China in 1988, and the M.E. and the Ph.D. degrees both in system science, from Kobe University, Kobe, Japan, in 1993 and 1996, respectively. From 1996 to 1998, he worked in the Kansai Laboratory of OKI Electric Industry Co., Ltd., Japan. From 1998 to 2004, he was with the Department of Opto-Mechatronics, Wakayama University, Japan. In April 2004, he joined the faculty of Osaka Prefecture University, Osaka, Japan,

where he is an Associate Professor of Mechanical Engineering. His research interests include large scale and decentralized control systems, robust control, switched systems and switching control, neural networks and signal processing.



Toshio Fukuda graduated from Waseda University in 1971 and received the M.S. and Dr. Eng. from the University of Tokyo in 1973 and 1977, respectively. Meanwhile, he studied at the graduate school of Yale University from 1973 to 1975. In 1977, he joined the National Mechanical Engineering Laboratory and became Visiting Research Fellow at the University of Stuttgart from 1979 to 1980. He joined the Science University of Tokyo in 1982, and then joined Nagoya University in 1989. Currently, he is Professor of Department of Micro System Engineering, Department of Mechano-Informatics and Systems, Nagoya University, Japan, mainly engaging in the research fields of intelligent robotic system, cellular robotic system, mechatronics and micro robotics.

Currently, he is Professor of Department of Micro System Engineering, Department of Mechano-Informatics and Systems, Nagoya University, Japan, mainly engaging in the research fields of intelligent robotic system, cellular robotic system, mechatronics and micro robotics.