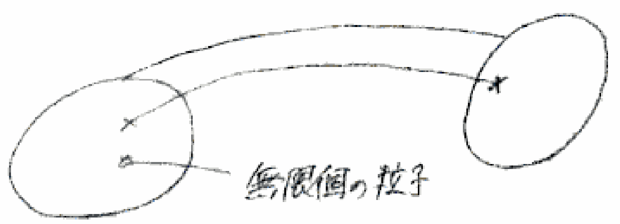
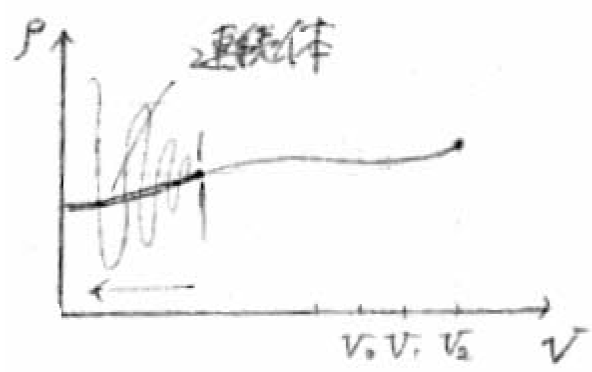
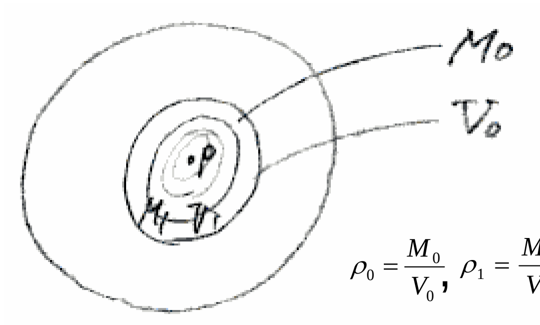
# 4. 変形とひずみ

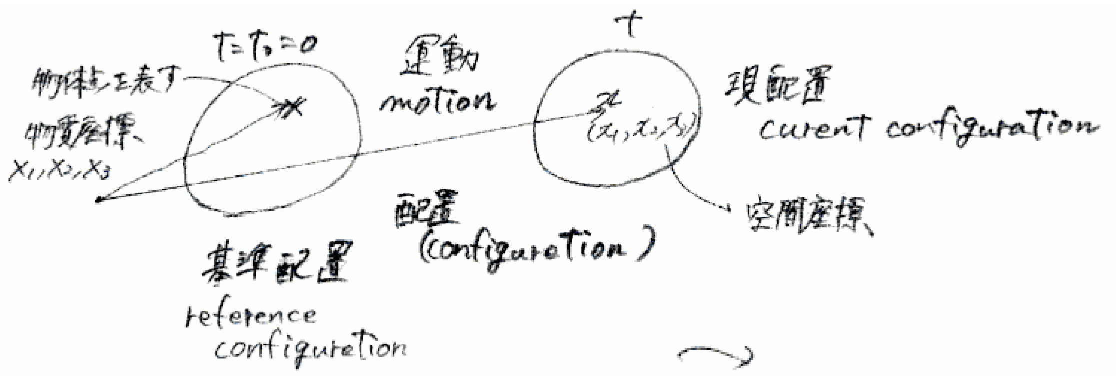
物体：粒子の連続的な集まり




$$\rho_0 = \frac{M_0}{V_0},\ \rho_1 = \frac{M_1}{V_1},\ldots,\ \rho_n = \frac{M_n}{V_n}$$

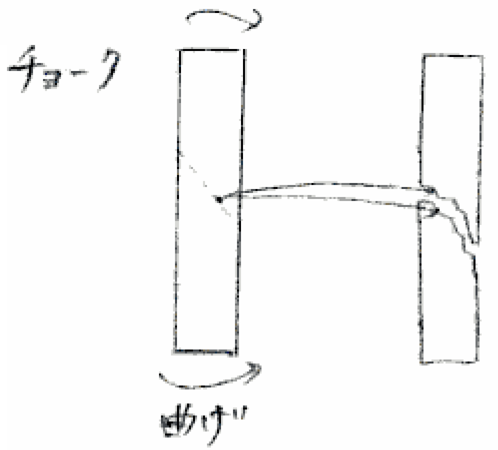



$$\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$$

$$x_i = x_i(x_1, x_2, x_3, t)$$

$t_0$ にて $\mathbf{X} = \mathbf{x}(\mathbf{X}, t_0)$

$$\mathbf{X} = \mathbf{X}(\mathbf{x}, t)$$



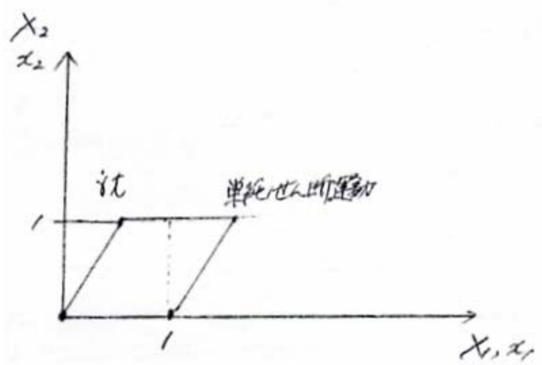
関数行列式
ヤコビアン

$$J = \det\left(\frac{\partial x_i}{\partial x_j}\right) = \begin{vmatrix} \dfrac{\partial x_1}{\partial x_1} & \dfrac{\partial x_1}{\partial x_2} & \dfrac{\partial x_1}{\partial x_3} \\ \dfrac{\partial x_2}{\partial x_1} & \dfrac{\partial x_2}{\partial x_2} & \dfrac{\partial x_2}{\partial x_3} \\ \dfrac{\partial x_3}{\partial x_1} & \dfrac{\partial x_3}{\partial x_2} & \dfrac{\partial x_3}{\partial x_3} \end{vmatrix} \neq 0$$

例 4.1）

時刻 $t$ でのベクトル　$\mathbf{x} = \mathbf{x}_i \mathbf{e}_i$

基準配置　$t = t_0 = 0$ として　$\mathbf{X} = x_a g_a = x_a \mathbf{e}_a$




$$\mathbf{x} = \mathbf{X} + \dot{\gamma} t x_2 \mathbf{e}_1$$

$$\begin{cases} x_1 = X_1 + \dot{\gamma} t X_2 \\ x_2 = X_2 \\ x_3 = X_3 \end{cases}$$

$$(X_1, X_2, X_3) = (0,0,0) \to (x_1, x_2, x_3) = (0,0,0)$$
$$(X_1, X_2, X_3) = (1,0,0) \to (x_1, x_2, x_3) = (1,0,0)$$
$$(X_1, X_2, X_3) = (0,1,0) \to (x_1, x_2, x_3) = (\dot{\gamma} t,1,0)$$
$$(X_1, X_2, X_3) = (1,1,0) \to (x_1, x_2, x_3) = (1+\dot{\gamma} t,1,0)$$

例 4．2）

$$\begin{cases} x_1 = X_1 + (X_1 + X_2)t \\ x_2 = X_2 + (X_1 + X_2)t \\ x_3 = X_3 \end{cases}$$

基準配置　$t_0 = 0$

$t = 2$ のとき　$(x_1, x_2, x_3) = (1,1,0)$　にある物体点の運動の速度を求める

$$v_1 = \left(\frac{\partial x_1}{\partial t}\right)_{X_i 一定} = X_1 + X_2$$

$$v_2 = \left(\frac{\partial x_2}{\partial t}\right)_{X_i 一定} = X_1 + X_2$$

$$v_3 = \left(\frac{\partial x_3}{\partial t}\right)_{X_i 一定} = 0$$

$$\left.\begin{aligned} 1 &= X_1 + (X_1 + X_2)\cdot 2 = 3X_1 + 2X_1 \\ 1 &= X_2 + (X_1 + X_2)\cdot 2 = 2X_1 + 3X_1 \\ 0 &= X_3 \end{aligned}\right\} \quad \begin{aligned} & X_1 = X_2 , X_3 = 0 \\ & 5X_1 = 1 \to X_1 = \frac{1}{5} \end{aligned}$$

$$\therefore \quad v_1 = \frac{2}{5}, \quad v_2 = \frac{2}{5}, \quad v_3 = 0$$

# 4.2　物質表示と空間表示

物質表示法（Lagrange表示法）

$$\phi = \phi(\mathbf{X}, t)$$

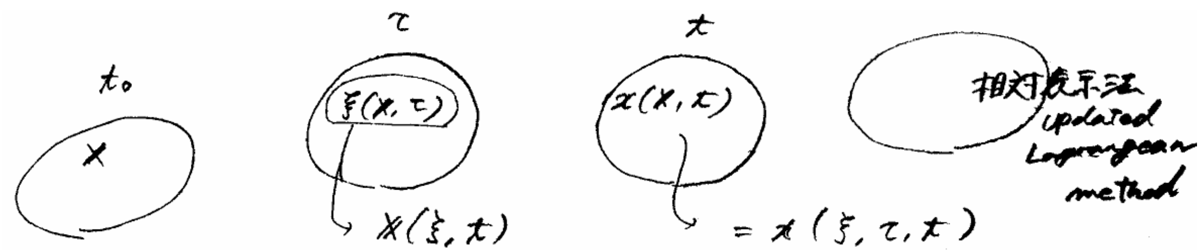
$$\phi = \phi(X_1, X_2, X_3, t)$$

空間表示法( Euler表示法)

$$\phi = \phi(\mathbf{x}, t)$$

$$\phi = \phi(x_1, x_2, x_3, t)$$

流体力学




# 4.3 物質導関数と加速度

物体点 $X$の速度

空間座標 $x$の時間に関する偏微分（物質表示にて）

$$\underline{\mathbf{v}(\mathbf{X}, t)} = \frac{\partial}{\partial t}\mathbf{x}(\mathbf{X}, t)$$

$$\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$$

$$\mathbf{X} = \mathbf{X}(\mathbf{x}, t)$$

$\mathbf{v}(\mathbf{x}, t)$

物体の各点で定義された任意の関数

スカラー（温度，質量，エネルギー，エントロピー）
ベクトル（変位，速度，加速度，熱ベクトル）
テンソル（ひずみ，応力，・・・）

$$\phi(\mathbf{X},t)$$

$$\dot{\phi}(\mathbf{x},t)=\frac{D}{Dt}\phi(\mathbf{x},t)=\frac{\partial}{\partial t}\phi(\mathbf{x},t)$$

物質（時間）導関数
実質（時間）導関数
ラグランジュ微分

$$\frac{D}{Dt}\phi(\mathbf{x},t)=\frac{\partial\phi(\mathbf{x},t)}{\partial t}\frac{\partial t}{\partial t}+\frac{\partial\phi(\mathbf{x},t)}{\partial\mathbf{x}}\underline{\frac{\partial x(\mathbf{x},t)}{\partial t}}$$

$$=\frac{\partial\phi(\mathbf{x},t)}{\partial t}+\underbrace{\mathbf{v}(\mathbf{x},t)\frac{\partial\phi(\mathbf{x},t)}{\partial\mathbf{x}}}_{\text{対流項}}$$

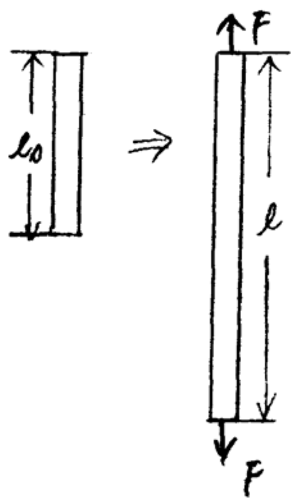
← $\mathbf{v}(\mathbf{x},\mathrm{t})$

空間（時間）導関数
オイラー微分
物体点の加速度

$$\mathbf{a}=\frac{\partial}{\partial t}\mathbf{v}(\mathbf{X},t)=\frac{D}{Dt}\mathbf{v}(\mathbf{x},t)=\frac{\partial\mathbf{v}(\mathbf{x},t)}{\partial t}+\mathbf{v}(\mathbf{x},t)\frac{\partial\mathbf{v}(\mathbf{x},t)}{\partial\mathbf{x}}$$

$$a_i=\frac{\partial v_i}{\partial t}+v_k\frac{\partial v_i}{\partial x_k}$$

## ４．４　変位ベクトル




$\dfrac{l}{l_0}$ ：伸び比

$\varepsilon=\dfrac{l-l_0}{l_0}$ ：公称ひずみ　材力

$\varepsilon'=\dfrac{l-l_0}{l}$

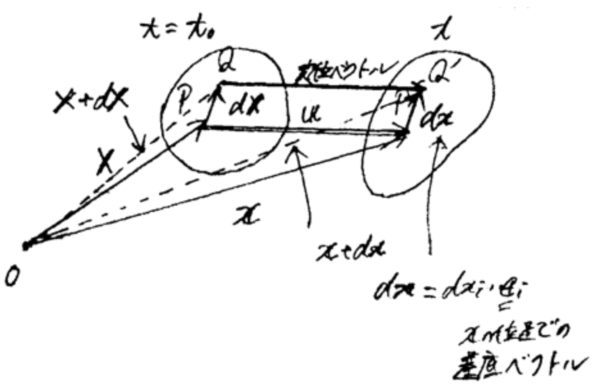
$e=\dfrac{l^2-l_0^2}{2l^2}$

$E=\dfrac{l^2-l_0^2}{2{l_0}^2}$

測度

有限変形

Greenのひずみテンソル
Almansiのひずみテンソル




$$\mathbf{u} = x(\mathbf{X},t) - \mathbf{X} \quad \leftarrow \mathbf{x} = \mathbf{x}(\mathbf{X},t) \quad \mathbf{X} = \mathbf{X}(\mathbf{x},t)$$

$$u_i = x_i - X_i$$

$$u_i = (X_1, X_2, X_3, t) = x_i(x_1, x_2, x_3, t) - X_i$$

↑—— 物質表示（Lagrange表示）

$$\mathbf{u} = \mathbf{x} - \mathbf{X}(\mathbf{x},t)$$

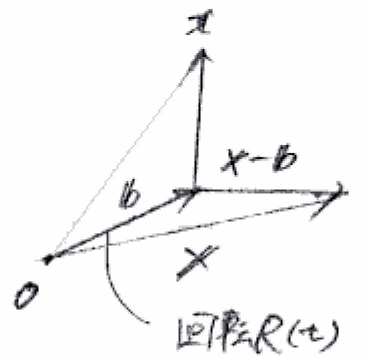
$$u_i(x_1, x_2, x_3, t) = x_i - X_i(x_1, x_2, x_3, t)$$

# 4.5 ひずみと変形勾配

## 4.5.1 剛体運動

$$\mathbf{x} - \mathbf{b} = \underline{\mathbf{R}(\mathbf{t})} \cdot (\mathbf{X} - \underline{\mathbf{b}})$$

直交テンソル　回転軸




例 4.5）

$$\mathbf{X}^{(1)}, \mathbf{X}^{(2)}$$

$$\mathbf{x}^{(1)} - \mathbf{b} = \mathbf{R}(t)(\mathbf{X}^{(1)} - \mathbf{b})$$

$$\mathbf{x}^{(2)} - \mathbf{b} = \mathbf{R}(t)(\mathbf{X}^{(2)} - \mathbf{b})$$

$$\mathbf{x}^{(1)} - \mathbf{x}^{(2)} = \mathbf{R}(t)(\mathbf{X}^{(1)} - \mathbf{X}^{(2)})$$

$$\Delta\mathbf{x} = \mathbf{x}^{(1)} - \mathbf{x}^{(2)}$$

$$\Delta\mathbf{X} = \mathbf{X}^{(1)} - \mathbf{X}^{(2)}$$ より

$$\Delta\mathbf{x} = \mathbf{R}(t)\Delta\mathbf{X}$$

$$\begin{aligned}\Delta\mathbf{x} \cdot \Delta\mathbf{x} &= (\mathbf{R}(t)\Delta\mathbf{X}) \cdot (\mathbf{R}(t)\Delta\mathbf{X}) \\ &= \Delta\mathbf{X} \cdot ({}^t\mathbf{R}(t)\mathbf{R}(t)\Delta\mathbf{X}) \\ &= \Delta\mathbf{X} \cdot \Delta\mathbf{X}\end{aligned}$$

## 4.5.2 変形勾配と変形テンソル

（4.1）

$$x_i = f(X_1, X_2, X_3, t) = x_i(X_1, X_2, X_3, t)$$

$$dx_i = \frac{\partial x_i}{\partial X_\alpha} dX_\alpha = F_{i\alpha} dX_\alpha$$

$$F_{i\alpha} = \frac{\partial x_i}{\partial X_\alpha}$$ ：変形勾配ベクトル

$$d\mathbf{x} = \mathbf{F} \cdot d\mathbf{X}$$

$$\boxed{\mathbf{F} = F_{i\alpha} \mathbf{e}_i \otimes \mathbf{d}_\alpha}$$ ：2重テンソル

$$d\mathbf{x} = \mathbf{F} \cdot d\mathbf{X}$$

$$\begin{aligned} dx_i \mathbf{e}_i &= F_{i\alpha} \mathbf{e}_i \otimes d_\alpha dX_\beta d_\beta \\ &= F_{i\alpha} dX_\beta e_i \delta_{\alpha\beta} \\ &= F_{i\alpha} dX_\alpha e_i \end{aligned}$$

$$d\mathbf{x} \to d\mathbf{X}$$

$$d\mathbf{X} = \mathbf{F}^{-1} d\mathbf{x}$$

$$d\mathbf{x} = \mathbf{F} d\mathbf{X}$$

$$\mathbf{F}^{-1} d\mathbf{x} = d\mathbf{X}$$

$$dX_\alpha = \frac{\partial X_\alpha}{\partial X_i} dx_i = (\mathbf{F}^{-1})_{\alpha i} dx_i$$

$$(\mathbf{F}^{-1}) d_i = \frac{\partial X_\alpha}{\partial X_i}$$

$$\mathbf{F}^{-1} = (\mathbf{F}^{-1})_{\alpha i} d_\alpha \otimes \mathbf{e}_i$$

$\det \mathbf{F} \neq 0$　　正⇸負

2.3.3項の極分解定理を適用

$$\mathbf{F} = \mathbf{R} \cdot \mathbf{U} = \mathbf{V} \cdot \mathbf{R}$$

$\mathbf{R}$：直交テンソル
$\mathbf{U}, \mathbf{V}$：正の定符号テンソル，正定値形

$\det \mathbf{F} > 0 \Rightarrow \det \mathbf{R} > 0 \Rightarrow \mathbf{R}$ ：回転テンソル

$$\begin{aligned} d\mathbf{x} &= \mathbf{F} d\mathbf{X} \\ &= \mathbf{R}\mathbf{U} d\mathbf{X} \\ &= \mathbf{R} \cdot (\mathbf{U} d\mathbf{X}) \end{aligned}$$

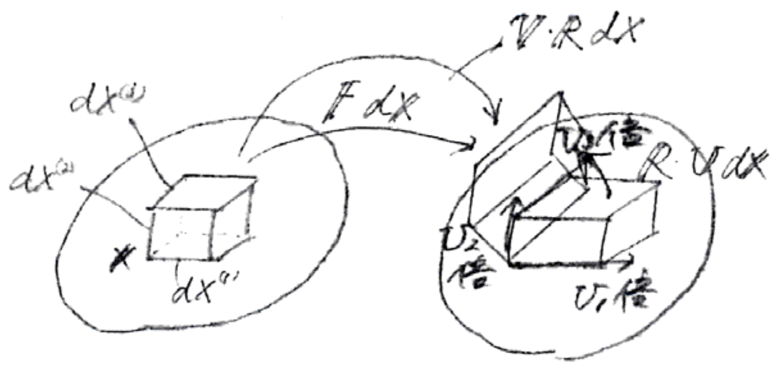
$d\mathbf{X}$ を $\mathbf{U}$ の固有ベクトルの方向に選ぶ

$$\mathbf{U} = \begin{pmatrix} U_{1>0} & 0 & 0 \\ 0 & U_{2>0} & 0 \\ 0 & 0 & U_{3>0} \end{pmatrix}$$

$$\mathbf{U}\cdot d\mathbf{X}^{(1)} = \begin{pmatrix} U_1 & 0 & 0 \\ 0 & U_2 & 0 \\ 0 & 0 & U_3 \end{pmatrix} \begin{pmatrix} d\mathbf{X}^{(1)} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} U_1 d\mathbf{X}^{(1)} \\ 0 \\ 0 \end{pmatrix}$$

$$\mathbf{U}\cdot d\mathbf{X}^{(2)} = \begin{pmatrix} 0 \\ U_2 d\mathbf{X}^{(2)} \\ 0 \end{pmatrix}$$

$$\mathbf{U}\cdot d\mathbf{X}^{(3)} = \begin{pmatrix} 0 \\ 0 \\ U_3 d\mathbf{X}^{(3)} \end{pmatrix}$$




$$d\mathbf{x} = \mathbf{F}d\mathbf{X} = \mathbf{VR}d\mathbf{X} = \mathbf{V}\cdot(\mathbf{R}\cdot d\mathbf{X})$$

$\mathbf{U}$：右ストレッチテンソル　　$\mathbf{F} = \mathbf{RU}$

$\mathbf{V}$：左ストレッチテンソル　　$\mathbf{F} = \mathbf{VR}$

$\mathbf{U}^2 = \mathbf{C} = {}^t\mathbf{FF}$　　（右コーシー・グリーンテンソル）

$${}^t\mathbf{FF} = {}^t(\mathbf{RU})(\mathbf{RU})$$
$$= \mathbf{U}{}^t\mathbf{RRU}$$
$$= \mathbf{UIU} = \mathbf{U}^2$$

$$\mathbf{B} = \mathbf{F}{}^t\mathbf{F} = \mathbf{VR}{}^t(\mathbf{VR}) = \mathbf{V}\underbrace{\mathbf{R}{}^t\mathbf{R}}_{\mathbf{I}}\mathbf{V} = \mathbf{V}^2$$

$$ds^2 = d\mathbf{x}\cdot d\mathbf{x} = (\mathbf{F}\cdot d\mathbf{X})(\mathbf{F}\cdot d\mathbf{X})$$
$$= d\mathbf{X}\cdot{}^t\mathbf{FF}\cdot d\mathbf{X}$$
$$= d\mathbf{X}\cdot\mathbf{C}d\mathbf{X}$$

$$dS^2 = d\mathbf{X}\cdot d\mathbf{X} = (\mathbf{F}^{-1}\cdot d\mathbf{x})(\mathbf{F}^{-1}\cdot d\mathbf{x})$$
$$= d\mathbf{x}\cdot{}^t(\mathbf{F}^{-1})\mathbf{F}^{-1}d\mathbf{x}$$
$$= d\mathbf{x}\cdot(\mathbf{F}{}^t\mathbf{F})^{-1}d\mathbf{x}$$
$$= d\mathbf{x}\cdot\mathbf{B}^{-1}d\mathbf{x}$$